Discrete Event Dynamic Systems
https://doi.org/10.1007/s10626-024-00404-3

Reliable decentralized failure diagnosis of discrete event systems using single-level inference

Takumi Hamada[1] **· Shigemasa Takai**[1]

Received: 31 March 2023 / Accepted: 31 August 2024

Abstract

We consider a reliable decentralized diagnosis problem for discrete event systems in the inference-based framework. This problem requires us to synthesize local diagnosers such that the occurrence of any failure string is correctly detected within a finite number of steps, even if local diagnosis decisions of some local diagnosers are not available. In the case of single-level inference, we introduce a notion of reliable 1-inference-diagnosability and show that reliable 1-inference-diagnosability is a necessary and sufficient condition for the existence of a solution to the reliable decentralized diagnosis problem. Then, we show how to verify reliable 1-inference-diagnosability effectively. When the system to be diagnosed is reliably 1-inference-diagnosable, we compute the delay bound within which the occurrence of any failure string can be detected. Local diagnosers can be constructed using the computed delay bound.

Keywords Discrete event system · Decentralized failure diagnosis · Reliable inference-diagnosability · Delay bound

Nomenclature

$\mathbb{N}$	Set of all nonnegative integers.
$\Sigma^{\geq m}$	Subset of Σ^* defined as $\Sigma^{\geq m} = \{s \in \Sigma^* \mid |s| \geq m\}$.
$\Sigma^{\leq m}$	Subset of Σ^* defined as $\Sigma^{\leq m} = \{s \in \Sigma^* \mid |s| \leq m\}$.
I	Index set $\{1, 2, \ldots, n\}$ of local diagnosers.
$\mathcal{I}^{\geq k}$	Subset of the power set 2^I of I defined as $\mathcal{I}^{\geq k} = \{I' \in 2^I \mid |I'| \geq k\}$.
$\mathcal{I}^k$	Subset of the power set 2^I of I defined as $\mathcal{I}^k = \{I' \in 2^I \mid |I'| = k\}$.
C	Set $\{0, 1, \phi\}$ of diagnosis decisions.
G_K	Automaton that generates the language $K \subseteq L(G)$.
$\widetilde{G}_K$	Augmented automaton that generates Σ^*.
V_F	Finite automaton constructed for verifying $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$.
$\mathcal{V}_{NF}$	Nondeterministic acyclic automaton constructed from V_F.

✉ Shigemasa Takai
takai@eei.eng.osaka-u.ac.jp

Takumi Hamada
t.hamada@is.eei.eng.osaka-u.ac.jp

[1] Division of Electrical, Electronic and Infocommunications Engineering, Osaka University, 2-1 Yamada-Oka, Suita 565-0871, Osaka, Japan

Published online: 20 September 2024

$\widehat{J}_F(p_{\mathcal{V}_{NF}})$	Label of path $p_{\mathcal{V}_{NF}}$ of $\mathcal{V}_{NF}$ defined as Eq. 74.
V_H	Finite automaton constructed for verifying $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$.
$\mathcal{V}_{NH}$	Nondeterministic acyclic automaton constructed from V_H.
$\widehat{J}_H(p_{\mathcal{V}_{NH}})$	Label of path $p_{\mathcal{V}_{NH}}$ of $\mathcal{V}_{NH}$ defined as Eq. 102.
m^*	Delay bound defined as Eq. 114.
m_F^*	Minimum element of $\mathbb{N}_F = \{m \in \mathbb{N} \mid F_2(m) = \emptyset\}$.
m_H^*	Minimum element of $\mathbb{N}_H = \{m \in \mathbb{N} \mid H_2(m) = \emptyset\}$.

1 Introduction

For discrete event systems (DESs), the language-based notion of diagnosability was introduced in Sampath et al. (1995) for the centralized setting, where a single diagnoser diagnoses the system so that the occurrence of any failure string is correctly detected within a finite number of steps. In decentralized failure diagnosis (Cassez 2012; Chakib and Khoumsi 2012; Debouk et al. 2000; Khoumsi 2020; Kumar and Takai 2009; Qiu and Kumar 2006; Su and Wonham 2005; Takai and Kumar 2017; Viana and Basilio 2019; Wang et al. 2011, 2007; Yin and Lafortune 2015), multiple local diagnosers locally diagnose the system. A decentralized diagnoser consists of local diagnosers and issues the diagnosis decision based on the local diagnosis decisions made by local diagnosers. The two kinds of decentralized diagnosis architectures, called the disjunctive architecture (Debouk et al. 2000; Qiu and Kumar 2006) and the conjunctive architecture (Wang et al. 2007), were developed. In the disjunctive architecture, the decentralized diagnoser issues the failure decision if and only if at least one local diagnoser makes the local failure decision. On the other hand, in the conjunctive architecture, the decentralized diagnoser issues the failure decision if and only if all local diagnosers make the local failure decisions. The codiagnosability property plays an important role in characterizing the class of systems which are diagnosable in the decentralized setting. The notions of disjunctive-codiagnosability (Qiu and Kumar 2006) and conjunctive-codiagnosability (Wang et al. 2007) were introduced in the disjunctive and conjunctive architectures, respectively. Interestingly, these two notions are incomparable (Wang et al. 2007).

Inference-based approaches were first introduced for DESs in the setting of decentralized supervisory control (Kumar and Takai 2007; Ricker and Rudie 2007; Yoo and Lafortune 2004). In these approaches, each local supervisor makes a control decision based on inference, which means using the knowledge about control decisions of other local supervisors. In particular, using the knowledge about control decisions issued by other local supervisors unambiguously is referred to as single-level inference. Later, the inference-based approaches were applied to decentralized diagnosis (Khoumsi 2020; Kumar and Takai 2009; Takai and Kumar 2017; Wang et al. 2007). The conditional architecture introduced in Wang et al. (2007) involves single-level inference. The notions of conditional disjunctive-codiagnosability and conditional conjunctive-codiagnosability, both of which are weaker than disjunctive-codiagnosability and conjunctive-codiagnosability, were introduced in the conditional disjunctive and conjunctive architectures, respectively (Wang et al. 2007). The general inference-based frameworks developed in Kumar and Takai (2009); Takai and Kumar (2017) allow multi-level inference. In these general frameworks, a local diagnosis decision is tagged with a nonnegative integer called an ambiguity level. The ambiguity level represents how ambiguous a local diagnoser is about its local diagnosis decision. To correctly detect the occurrence of any failure string within a finite number of steps, any nonfailure string should be distinguished from any sufficiently long failure string or vice versa. The local failure decision with the ambiguity level 0 is taken when a local diagnoser is certain that a failure string has occurred. On the other hand, the local nonfailure decision with the ambiguity level 0 is made when a local diagnoser certainly knows that a sufficiently long failure string has not occurred. In addition, the local failure (respectively, nonfailure) decision with the ambiguity level 1 is taken when a local diagnoser knows that, for any indistinguishable nonfailure (respectively, sufficiently long failure) string, another local diagnoser makes the local nonfailure (respectively, failure) decision with the ambiguity level 0 so that the local failure (respectively, nonfailure) decision with the ambiguity level 1 is overridden. The failure/nonfailure decision with a higher ambiguity level can be similarly explained.

Reliability is a desirable property for safety-critical systems. In most previous work on decentralized diagnosis, it is implicitly assumed that diagnosis decisions of all local diagnosers are available. However, it may be possible that some local decisions are not available, due to some reasons including breakdown of local diagnosers and disconnection of the network. A reliable decentralized diagnosis problem was considered in the disjunctive and conjunctive architectures in Basilio and Lafortune (2009); Nakata and Takai (2013) and Yamamoto and Takai (2014), respectively. Similar problems were considered for decentralized supervisory control in Liu and Lin (2010); Takai and Ushio (2000, 2003) and decentralized prognosis in Yin and Li (2016). Letting n be the number of local diagnosers, the reliable decentralized diagnosis problem requires us to synthesize n local diagnosers such that the occurrence of any failure string is correctly detected within a finite number of steps, as long as at least k ($2 \leq k \leq n$) local diagnosis decisions are available. In other words, the occurrence of any failure string is detected within a finite number of steps, even if diagnosis decisions of at most $n - k$ local diagnosers are not available. In this sense, the decentralized diagnoser consisting of such local diagnosers is reliable. To characterize the existence of a solution to the problem, the notions of (n, k)-reliable disjunctive-codiagnosability and (n, k)-reliable conjunctive-codiagnosability were introduced in Nakata and Takai (2013) and Yamamoto and Takai (2014), respectively.

In this paper, we consider the reliable decentralized diagnosis problem in the inference-based framework of Takai and Kumar (2017). In order to characterize the existence of a solution, we define a notion of (n, k)-reliable 1-inference-diagnosability, which is weaker than (n, k)-reliable disjunctive-codiagnosability and (n, k)-reliable conjunctive-codiagnosability. To do so, the iterative computations over languages introduced in Takai and Kumar (2017) are generalized. We show that (n, k)-reliable 1-inference-diagnosability is a necessary and sufficient condition for the existence of n local diagnosers that solve the reliable decentralized diagnosis problem based on single-level inference. This result generalizes the existing ones of Basilio and Lafortune (2009); Nakata and Takai (2013); Yamamoto and Takai (2014) on reliable decentralized diagnosis without inference. In addition, we present how to verify (n, k)-reliable 1-inference-diagnosability. When the system is (n, k)-reliably 1-inference-diagnosable, we compute the delay bound within which the occurrence of any failure string can be detected, even if diagnosis decisions of at most $n - k$ local diagnosers are not available. The computed delay bound is used to construct local diagnosers.

Recently, the reliable decentralized supervisory control problem has been solved using single-level inference (Takai and Yoshida 2022). In supervisory control, a supervisor is required to always issue a correct control decision for each feasible controllable event. The purpose of failure diagnosis addressed in this paper is to detect the occurrence of a failure string that cannot be directly observed. Due to unobservability of the occurrence of a failure string, the requirement that it should be detected immediately is unrealistic. Therefore, the notion of diagnosability introduced in Sampath et al. (1995) requires that the occurrence of any failure string is correctly detected with a finite delay. Since a delay of detecting the occurrence of a failure string is allowed, the result of Takai and Yoshida (2022) cannot be applied to the reliable decentralized diagnosis problem considered in this paper.

The present paper is an extended version of the authors' conference papers (Hamada and Takai 2022a) and Hamada and Takai (2022b). Its additional contributions are summarized as follows:

- The proofs of the technical results are included
  and
- how to compute the delay bound within which the occurrence of any failure string can be detected is presented.

2 Preliminaries

A DES to be diagnosed is modeled as a finite automaton $G = (Q, \Sigma, \delta, q_0)$, where Q is the finite set of states, Σ is the finite set of events, a partial function $\delta : Q \times \Sigma \to Q$ is the state transition function, and $q_0 \in Q$ is the initial state. A sequence $q_0 \xrightarrow{\sigma_0} q_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{l-1}} q_l$ ($l \geq 1$) of transitions from the initial state q_0 such that $\delta(q_h, \sigma_h) = q_{h+1}$ for each $h \in \{0, 1, \ldots, l-1\}$ is called a path of G. Let Σ^* be the set of all finite strings of elements of Σ, including the empty string ε. The function δ can be generalized to $\delta : Q \times \Sigma^* \to Q$ in the usual manner. For any $q \in Q$ and any $s \in \Sigma^*$, $\delta(q, s)!$ denotes that $\delta(q, s)$ is defined. Let $\mathbb{N}$ be the set of all nonnegative integers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$. For any $s \in \Sigma^*$, $|s| \in \mathbb{N}$ denotes its length and s^m denotes m concatenations of s, where $m \in \mathbb{N}$. Besides, for any $m \in \mathbb{N}$, we let $\Sigma^{\geq m} = \{s \in \Sigma^* \mid |s| \geq m\}$ and $\Sigma^{\leq m} = \{s \in \Sigma^* \mid |s| \leq m\}$.

The generated language of G, denoted by $L(G)$, is defined as $L(G) = \{s \in \Sigma^* \mid \delta(q_0, s)!\}$. For each $s \in L(G)$, the postlanguage of $L(G)$ after s is defined as $L(G)/s = \{t \in \Sigma^* \mid st \in L(G)\}$. For a string $s \in \Sigma^*$, $pr(s)$ denotes the set of all prefixes of s, that is, $pr(s) = \{t \in \Sigma^* \mid \exists u \in \Sigma^*(s = tu)\}$. For a language $K \subseteq \Sigma^*$, $pr(K) = \{s \in \Sigma^* \mid \exists t \in \Sigma^*(st \in K)\}$ is the set of all prefixes of strings in K. K is said to be (prefix-)closed if $K = pr(K)$.

In the setting of decentralized diagnosis, n local diagnosers diagnose the system G based on local event observations. Let $I = \{1, 2, \ldots, n\}$ be the index set of local diagnosers. For the ith local diagnoser ($i \in I$), let $\Sigma_{i,o} \subseteq \Sigma$ be the set of locally observable events and $P_i : \Sigma^* \to \Sigma_{i,o}^*$ be the natural projection from Σ^* to $\Sigma_{i,o}^*$, which is inductively defined as follows:

- $P_i(\varepsilon) = \varepsilon$,
- $\forall s \in \Sigma^*, \forall \sigma \in \Sigma$,

$$P_i(s\sigma) = \begin{cases} P_i(s)\sigma, & \text{if } \sigma \in \Sigma_{i,o} \\ P_i(s), & \text{otherwise.} \end{cases} \tag{1}$$

If a string $s \in L(G)$ is executed in G, then the locally observable event string $P_i(s)$ is observed by the ith local diagnoser. For any language $K \subseteq \Sigma^*$, let $P_i(K) = \{P_i(s) \in \Sigma_{i,o}^* \mid s \in K\}$. Two strings $s, s' \in L(G)$ are said to be indistinguishable (under P_i) if $P_i(s) = P_i(s')$. Then, the inverse projection $P_i^{-1} : \Sigma_{i,o}^* \to 2^{\Sigma^*}$ is defined by $P_i^{-1}(t_i) = \{s \in \Sigma^* \mid P_i(s) = t_i\}$ for any $t_i \in \Sigma_{i,o}^*$. That is, $P_i^{-1}(t_i)$ is the set of strings that are observed as t_i by the ith local diagnoser. We define the globally observable and unobservable event sets $\Sigma_o \subseteq \Sigma$ and $\Sigma_{uo} \subseteq \Sigma$ as $\Sigma_o = \bigcup_{i \in I} \Sigma_{i,o}$ and $\Sigma_{uo} = \Sigma - \Sigma_o$, respectively.

3 Problem formulation

In this section, we formulate a reliable decentralized diagnosis problem in the inference-based framework of Takai and Kumar (2017).

The set of diagnosis decisions is $C = \{0, 1, \phi\}$, where 0 and 1 represent the nonfailure and failure decisions, respectively, while ϕ denotes the unsure decision. Due to local event observations, a local diagnoser is possibly ambiguous whether a failure string has occurred. To represent the degree of ambiguity of a local diagnosis decision, a nonnegative integer, which is called the ambiguity level, is attached to it. If a local diagnoser is unambiguous about its diagnosis decision, 0 is attached as its ambiguity level. For each $i \in I$, an inference-based local diagnoser D_i is defined as a function $D_i : P_i(L(G)) \to C \times \mathbb{N}$. For each $s \in L(G)$, $D_i(P_i(s))$ is denoted by

$$D_i(P_i(s)) = (c_i(P_i(s)), n_i(P_i(s))), \tag{2}$$

where, for the locally observed event string $P_i(s) \in P_i(L(G))$, $c_i(P_i(s)) \in C$ denotes the local diagnosis decision of D_i and $n_i(P_i(s)) \in \mathbb{N}$ is the ambiguity level of $c_i(P_i(s))$.

In this paper, we consider a situation where diagnosis decisions of some local diagnosers are not available with some reasons. Let k be a nonnegative integer such that $2 \leq k \leq n$, which represents the minimum number of local diagnosers whose diagnosis decisions are available.

Remark 1 In the case of $k = 1$, it is possible that the diagnosis decision of only one local supervisor is available. Therefore, each local diagnoser has to be synthesized as a centralized diagnoser that works alone, and the reliable decentralized diagnosis problem considered in this paper can be simply solved by the existing results on centralized diagnosis. This is the reason why we exclude the case of $k = 1$.

We define two subsets $\mathcal{I}^{\geq k}$ and $\mathcal{I}^k$ of the power set 2^I of I as $\mathcal{I}^{\geq k} = \{I' \in 2^I \mid |I'| \geq k\}$ and $\mathcal{I}^k = \{I' \in 2^I \mid |I'| = k\}$. For each $I' \in \mathcal{I}^{\geq k}$, it is possible that diagnosis decisions of only local diagnosers D_i with $i \in I'$ are available. The decentralized diagnoser consisting of local diagnosers D_i with $i \in I'$ is defined as a function $D_{I'} : L(G) \to C$ such that, for each $s \in L(G)$, the diagnosis decision $D_{I'}(s)$ is given as

$$D_{I'}(s) = \begin{cases} 1, & \text{if } \forall i \in I'(n_i(P_i(s)) = n_{I'}(s) \Rightarrow c_i(P_i(s)) = 1) \\ 0, & \text{if } \forall i \in I'(n_i(P_i(s)) = n_{I'}(s) \Rightarrow c_i(P_i(s)) = 0) \\ \phi, & \text{otherwise,} \end{cases} \tag{3}$$

where $n_{I'}(s)$ is the minimum ambiguity level of local decisions, i.e.,

$$n_{I'}(s) = \min\{n_i(P_i(s)) \in \mathbb{N} \mid i \in I'\}. \tag{4}$$

Unlike a local diagnoser $D_i : P_i(L(G)) \to C \times \mathbb{N}$, the decentralized diagnoser $D_{I'} : L(G) \to C$ issues the diagnosis decision $D_{I'}(s) \in C$ for a string $s \in L(G)$ without attaching its ambiguity level. When a string $s \in L(G)$ is executed in G, each local diagnoser makes the local diagnosis decision $c_i(P_i(s))$ with the ambiguity level $n_i(P_i(s))$. Then, the diagnosis decision $D_{I'}(s)$ of the decentralized diagnoser $D_{I'}$ is taken to be the same as the local diagnosis decision whose ambiguity level is minimal. The value $n_{I'}(s)$ can be considered as the ambiguity level of the diagnosis decision $D_{I'}(s)$. Since the subject of failure diagnosis is detecting the occurrence of a failure string, the value $n_{I'}(s)$ is not issued by the decentralized diagnoser $D_{I'}$.

For the sake of simplicity, we assume in the remainder of the paper that the system G to be diagnosed is deadlock free, that is, for any $s \in L(G)$, there exists $\sigma \in \Sigma$ such that

$s\sigma \in L(G)$. Besides, we assume that the nonfailure behavior of the system G is described by a nonempty closed regular sublanguage $K \subseteq L(G)$. That is, K is the set of all strings without failures. For any $s \in K$, a failure is modeled by the occurrence of an event $\sigma \in \Sigma$ such that $s\sigma \in L(G) - K$. Any string in $L(G) - K$ (respectively, K) is called a failure (respectively, nonfailure) string.

Remark 2 When the system G is not deadlock free, G has to be modified by adding a self-loop transition by a new unobservable event $\sigma_{uo} \notin \Sigma$ at any deadlock state reached by a string $s \in L(G)$ such that $s\sigma \notin L(G)$ for any $\sigma \in \Sigma$. The modified system is deadlock free, and the results of the paper are applicable to it.

Given a nonnegative integer $k \in \mathbb{N}$ with $2 \leq k \leq n$ and a nonnegative integer $N \in \mathbb{N}$, a notion of (n, k)-N-inferring local diagnosers is defined as follows:

Definition 1 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$, where $n \in \mathbb{N}$ is the number of local diagnosers and k is the minimum number of local diagnosers whose diagnosis decisions are available, and $N \in \mathbb{N}$ be a nonnegative integer that represents an upper bound of the ambiguity level of a diagnosis decision. For a nonempty closed regular sublanguage $K \subseteq L(G)$, n local diagnosers $D_i : P_i(L(G)) \to C \times \mathbb{N}$ $(i = 1, 2, \ldots, n)$ are said to be (n, k)-N-inferring if the following two conditions hold:*

- *It holds that*

$$\forall I' \in \mathcal{I}^{\geq k}, \forall s \in L(G) - K(D_{I'}(s) = 1 \Rightarrow n_{I'}(s) \leq N) \tag{5}$$

or

$$\forall I' \in \mathcal{I}^{\geq k}, \forall s \in K(D_{I'}(s) \neq 1 \Rightarrow n_{I'}(s) \leq N), \tag{6}$$

- *there exists $m \in \mathbb{N}$ such that*

$$\forall I' \in \mathcal{I}^{\geq k}, \forall s \in (L(G) \cap (L(G) - K)\Sigma^{\geq m}) \cup K(n_{I'}(s) \leq N \Rightarrow D_{I'}(s) \neq \phi). \tag{7}$$

If n local diagnosers D_i $(i = 1, 2, \ldots, n)$ are (n, k)-N-inferring, then, for any $I' \in \mathcal{I}^{\geq k}$,

- the ambiguity level of the failure decision of the decentralized diagnoser $D_{I'}$ for any failure string or that of the nonfailure or unsure decision of $D_{I'}$ for any nonfailure string is bounded by N, and
- for any nonfailure or sufficiently long failure string, the decision of $D_{I'}$ is not unsure if its ambiguity level is less than or equal to N.

A reliable decentralized diagnosis problem considered in this paper is formulated as follows:

Problem 1 *(Reliable Decentralized Diagnosis Problem) Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. A reliable decentralized diagnosis problem is a problem of synthesizing (n, k)-N-inferring local diagnosers $D_i : P_i(L(G)) \to C \times \mathbb{N}$ $(i = 1, 2, \ldots, n)$ that satisfy*

$$\exists m \in \mathbb{N}, \forall I' \in \mathcal{I}^{\geq k}, \forall s \in L(G) \cap (L(G) - K)\Sigma^{\geq m}(D_{I'}(s) = 1) \tag{8}$$

and

$$\forall I' \in \mathcal{I}^{\geq k}, \forall s \in K(D_{I'}(s) \neq 1). \tag{9}$$

The conditions Eqs. 8 and 9 guarantee that there exists a nonnegative integer $m \in \mathbb{N}$ such that if diagnosis decisions of at least k local diagnosers are available, the occurrence of any failure string can be correctly detected within m steps. In this sense, the decentralized diagnoser consisting of local diagnosers D_i $(i = 1, 2, \ldots, n)$ that satisfy the conditions Eqs. 8 and 9 is reliable.

4 Solvability of reliable decentralized diagnosis problem

We consider the reliable decentralized diagnosis problem formulated in the last section. For the solvability of the problem, we focus on the case of $N = 1$ and present a necessary and sufficient condition for the existence of (n, k)-1-inferring local diagnosers that solve the problem.

4.1 Reliable 1-inference-diagnosability

To characterize the existence of (n, k)-1-inferring local diagnosers that solve the reliable decentralized diagnosis problem, we introduce a notion of (n, k)-reliable 1-inference-diagnosability.

For a nonempty closed regular sublanguage $K \subseteq L(G)$ that models the nonfailure behavior of G, we introduce a sequence $\{(F_h(m), H_h(m))\}_{h\in\mathbb{N}}$ of language pairs, where $m \in \mathbb{N}$ is an arbitrary nonnegative integer, to define a notion of (n, k)-reliable 1-inference-diagnosability and synthesize (n, k)-1-inferring local diagnosers. Initially, $F_0(m)$ and $H_0(m)$ are defined as

$$F_0(m) = L(G) \cap (L(G) - K)\Sigma^{\geq m}, \tag{10}$$

$$H_0(m) = K. \tag{11}$$

$F_0(m)$ is the set of failure strings such that at least m events have occurred after the occurrences of the corresponding failures, while $H_0(m)$ is the set of all nonfailure strings. Then, for any $h \in \mathbb{N}$, $F_{h+1}(m)$ and $H_{h+1}(m)$ are defined as

$$F_{h+1}(m) = F_h(m) \cap \left\{ \bigcup_{I'\in\mathcal{I}^k} \left(\bigcap_{i\in I'} P_i^{-1} P_i(H_h(m)) \right) \right\}, \tag{12}$$

$$H_{h+1}(m) = H_h(m) \cap \left\{ \bigcup_{I'\in\mathcal{I}^k} \left(\bigcap_{i\in I'} P_i^{-1} P_i(F_h(m)) \right) \right\}. \tag{13}$$

By Eq. 12 (respectively, Eq. 13), $F_{h+1}(m)$ (respectively, $H_{h+1}(m)$) is a sublanguage of $F_h(m)$ (respectively $H_h(m)$), which means that as the value of h increases, $F_h(m)$ (respectively $H_h(m)$) becomes smaller. For each string in $F_{h+1}(m)$ (respectively, $H_{h+1}(m)$), there exist an index set $I' \in \mathcal{I}^k$ and an indistinguishable string in $H_h(m)$ (respectively, $F_h(m)$) for each $i \in I'$.

Remark 3 In the case of $k = n$, that is, diagnosis decisions of all n local diagnosers are available, the languages $F_{h+1}(m)$ and $H_{h+1}(m)$ are given by

$$F_{h+1}(m) = F_h(m) \cap \left(\bigcap_{i\in I} P_i^{-1} P_i(H_h(m)) \right), \tag{14}$$

$$H_{h+1}(m) = H_h(m) \cap \left(\bigcap_{i\in I} P_i^{-1} P_i(F_h(m)) \right) \tag{15}$$

for any $h \in \mathbb{N}$, which are the same as those defined in Takai and Kumar (2017).

For each string in $F_0(m) - F_1(m)$ (respectively, $H_0(m) - H_1(m)$), at least one local diagnoser can distinguish it from strings in $H_0(m)$ (respectively, $F_0(m)$) and make the failure (respectively, nonfailure) decision whose ambiguity level is 0, even if diagnosis decisions of at most $n-k$ local diagnosers are not available. We consider any string $s \in F_1(m) - F_2(m)$ (respectively, $s \in H_1(m) - H_2(m)$). There exists at least one local diagnoser that can distinguish it from strings in $H_1(m)$ (respectively, $F_1(m)$). In addition, for each string in $H_0(m) - H_1(m)$ (respectively, $F_0(m) - F_1(m)$), the nonfailure (respectively, failure) decision is issued by another local diagnoser unambiguously. Therefore, based on single-level inference, a local diagnoser that can distinguish $s \in F_1(m) - F_2(m)$ (respectively, $s \in H_1(m) - H_2(m)$) from strings in $H_1(m)$ (respectively, $F_1(m)$) is able to make the failure (respectively, nonfailure) decision whose ambiguity level is 1. Moreover, if $F_2(m) = \emptyset$ (respectively, $H_2(m) = \emptyset$), the failure (respectively, nonfailure) decision whose ambiguity level is less than or equal to 1 can be made for any string in $F_0(m)$ (respectively, $H_0(m)$). This observation motivates us to introduce a notion of (n, k)-reliable 1-inference-diagnosability as follows:

Definition 2 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \le k \le n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$, the system G is said to be (n, k)-reliably 1-inference-diagnosable if*

$$\exists m \in \mathbb{N}(F_2(m) = \emptyset \vee H_2(m) = \emptyset). \tag{16}$$

Remark 4 When $k = n$, (n, k)-reliable 1-inference-diagnosability is reduced to the 1-inference-diagnosability condition of Takai and Kumar (2017).

The two notions of reliable codiagnosability were defined in Nakata and Takai (2013); Yamamoto and Takai (2014). For a nonempty closed regular sublanguage $K \subseteq L(G)$, the system G is

- (n, k)-reliably disjunctive-codiagnosable (Nakata and Takai 2013) if

$$\exists m \in \mathbb{N}, \forall s \in L(G) \cap (L(G) - K)\Sigma^{\ge m}(|I_D(s)| \ge n - k + 1), \tag{17}$$

  where $I_D(s) = \{i \in I \mid P_i^{-1} P_i(s) \cap L(G) \subseteq L(G) - K\}$,
- (n, k)-reliably conjunctive-codiagnosable (Yamamoto and Takai 2014) if

$$\exists m \in \mathbb{N}, \forall s \in K(|I_C(s)| \ge n - k + 1), \tag{18}$$

  where $I_C(s) = \{i \in I \mid P_i^{-1} P_i(s) \cap L(G) \subseteq K\Sigma^{\le m}\}$.

By the definition of $F_1(m)$ (respectively, $H_1(m)$), G is (n, k)-reliably disjunctive-codiagnosable (respectively, (n, k)-reliably conjunctive-codiagnosable) if and only if there exists $m \in \mathbb{N}$ such that $F_1(m) = \emptyset$ (respectively, $H_1(m) = \emptyset$). Thus, the following proposition is obtained.

Proposition 1 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \le k \le n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$, if the system G is (n, k)-reliably disjunctive-codiagnosable or (n, k)-reliably conjunctive-codiagnosable, then it is (n, k)-reliably 1-inference-diagnosable.*

Proposition 1 shows that (n, k)-reliable 1-inference-diagnosability is weaker than (n, k)-reliable disjunctive-codiagnosability and (n, k)-reliable conjunctive-codiagnosability. As shown in Example 1 later, the reverse relation does not hold.

4.2 Synthesis of local diagnosers

We assume that, for a nonempty closed regular sublanguage $K \subseteq L(G)$, the system G is (n,k)-reliably 1-inference-diagnosable. Then, there exists $m \in \mathbb{N}$ such that $F_h(m) = H_h(m) = \emptyset$ for any $h \in \mathbb{N}$ with $h \geq 3$. For such $m \in \mathbb{N}$, according to the synthesis method developed in Takai and Kumar (2017), we synthesize an inference-based local diagnoser $D_i : P_i(L(G)) \to C \times \mathbb{N}$ for each $i \in I$. Since $F_h(m) = H_h(m) = \emptyset$ for any $h \in \mathbb{N}$ with $h \geq 3$, the language pairs $(F_h(m), H_h(m))$ $(h = 4, 5, 6, \dots)$ are redundant. Therefore, the four language pairs $(F_h(m), H_h(m))$ $(h = 0, 1, 2, 3)$ are used for this purpose. For each $s \in L(G)$, $n_i^f(P_i(s)) \in \mathbb{N}$, which we call the ambiguity level of the failure decision, and $n_i^h(P_i(s)) \in \mathbb{N}$, which we call the ambiguity level of the nonfailure decision, are computed by

$$n_i^f(P_i(s)) = \min\{h \in \{0,1,2,3\} \mid P_i(s) \notin P_i(H_h(m))\}, \tag{19}$$

$$n_i^h(P_i(s)) = \min\{h \in \{0,1,2,3\} \mid P_i(s) \notin P_i(F_h(m))\}. \tag{20}$$

$n_i^f(P_i(s))$ (respectively, $n_i^h(P_i(s))$) is the minimum integer h such that s can be distinguished from strings in $H_h(m)$ (respectively, $F_h(m)$) under P_i. It follows from $F_3(m) = H_3(m) = \emptyset$ that $n_i^f(P_i(s))$ and $n_i^h(P_i(s))$ are well-defined. Using $n_i^f(P_i(s))$ and $n_i^h(P_i(s))$, the local diagnosis decision $c_i(P_i(s)) \in C$ and its ambiguity level $n_i(P_i(s)) \in \mathbb{N}$ are determined as follows:

$$c_i(P_i(s)) = \begin{cases} 1, & \text{if } n_i^f(P_i(s)) < n_i^h(P_i(s)) \\ 0, & \text{if } n_i^h(P_i(s)) < n_i^f(P_i(s)) \\ \phi, & \text{otherwise,} \end{cases} \tag{21}$$

$$n_i(P_i(s)) = \min\{n_i^f(P_i(s)), n_i^h(P_i(s))\}. \tag{22}$$

The local diagnosis decision $c_i(P_i(s))$ is determined by comparing $n_i^f(P_i(s))$ and $n_i^h(P_i(s))$. If $n_i^f(P_i(s))$ (respectively, $n_i^h(P_i(s))$) is smaller than $n_i^h(P_i(s))$ (respectively, $n_i^f(P_i(s))$), then the local failure (respectively, nonfailure) decision is made.

The following proposition is obtained in the same way as Lemma 2 of Takai and Kumar (2017), which shows that if the system G is (n,k)-reliably 1-inference-diagnosable for a given nonempty closed regular sublanguage $K \subseteq L(G)$, then the n local diagnosers D_i $(i = 1, 2, \dots, n)$ synthesized by Eqs. 19–22 solve the reliable decentralized diagnosis problem in the case of $N = 1$.

Proposition 2 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$, if the system G is (n,k)-reliably 1-inference-diagnosable, then the n local diagnosers $D_i : P_i(L(G)) \to C \times \mathbb{N}$ $(i = 1, 2, \dots, n)$ synthesized by* Eqs. 19–22 *for any $m \in \mathbb{N}$ such that $F_2(m) = \emptyset$ or $H_2(m) = \emptyset$ are (n,k)-1-inferring and satisfy the conditions*

$$\forall I' \in \mathcal{I}^{\geq k}, \forall s \in L(G) \cap (L(G) - K)\Sigma^{\geq m}(D_{I'}(s) = 1) \tag{23}$$

and Eq. 9.

Proposition 2 shows that the n local diagnosers $D_i : P_i(L(G)) \to C \times \mathbb{N}$ $(i = 1, 2, \dots, n)$ synthesized by Eqs. 19–22 for $m \in \mathbb{N}$ with $F_2(m) = \emptyset$ or $H_2(m) = \emptyset$ can detect the occurrence of any failure string correctly within m steps. How to compute such $m \in \mathbb{N}$ will be presented in Section 6.

4.3 Existence of solution

By Proposition 2, (n,k)-reliable 1-inference-diagnosability is a sufficient condition for the existence of a solution to the reliable decentralized diagnosis problem in the case of $N=1$. In the following theorem, we show that this condition is also necessary.

Theorem 3 *Let $k\in\mathbb{N}$ be a nonnegative integer such that $2\le k\le n$. For a nonempty closed regular sublanguage $K\subseteq L(G)$, there exist (n,k)-1-inferring local diagnosers $D_i : P_i(L(G))\to C\times\mathbb{N}$ $(i=1,2,\dots,n)$ that satisfy the conditions* Eqs. 8 *and* 9 *if and only if the system G is (n,k)-reliably 1-inference-diagnosable.*

Proof To prove the sufficiency part, we suppose that G is (n,k)-reliably 1-inference-diagnosable. By Proposition 2, there exist (n,k)-1-inferring local diagnosers D_i $(i=1,2,\dots,n)$ that satisfy the conditions Eqs. 8 and 9.

We prove the necessity part. We consider arbitrary (n,k)-1-inferring local diagnosers D_i $(i=1,2,\dots,n)$ that satisfy the conditions Eqs. 8 and 9. Since they satisfy Eq. 8, there exists $m\in\mathbb{N}$ such that Eq. 23 holds. For the sake of contradiction, we suppose that G is not (n,k)-reliably 1-inference-diagnosable. Then, we have $F_2(m)\ne\emptyset$ and $H_2(m)\ne\emptyset$. Since D_i $(i=1,2,\dots,n)$ are (n,k)-1-inferring, the condition Eqs. 5 or 6 in the first condition of Definition 1 holds for $N=1$.

We first consider the case where Eq. 5 holds. For any $s\in F_2(m)\ne\emptyset$, there exists $I'\in\mathcal{I}^k$ such that

$$s\in F_1(m)\cap\left(\bigcap_{i\in I'}P_i^{-1}P_i(H_1(m))\right). \tag{24}$$

Since $s\in F_1(m)\subseteq F_0(m)=L(G)\cap(L(G)-K)\Sigma^{\ge m}$ and $I'\in\mathcal{I}^k\subseteq\mathcal{I}^{\ge k}$, by Eq. 23, we have $D_{I'}(s)=1$. By Eq. 5, we have $n_{I'}(s)\le 1$. We consider any $j\in I'$ such that $n_{I'}(s)=n_j(P_j(s))$. By $D_{I'}(s)=1$, we have $c_j(P_j(s))=1$. Since $s\in P_j^{-1}P_j(H_1(m))$ by Eq. 24, there exists $s'\in H_1(m)$ such that $P_j(s)=P_j(s')$. It follows from $s'\in H_1(m)$ and $s'\in P_j^{-1}P_j(s)\subseteq P_j^{-1}P_j(F_0(m))$ that there exists $I''_j\in\mathcal{I}^k$ such that

$$s'\in H_0(m)\cap\left(\bigcap_{i\in I''_j}P_i^{-1}P_i(F_0(m))\right) \tag{25}$$

and $j\in I''_j$. Since $s'\in H_0(m)=K$ and $I''_j\in\mathcal{I}^k\subseteq\mathcal{I}^{\ge k}$, by Eq. 9, we have $D_{I''_j}(s')\ne 1$. In addition, since $P_j(s)=P_j(s')$ and $n_j(P_j(s))=n_{I'}(s)\le 1$, we have $n_{I''_j}(s')\le n_j(P_j(s'))=n_j(P_j(s))\le 1$. It follows from the second condition of Definition 1 that $D_{I''_j}(s')=0$. We consider any $j'\in I''_j$ such that $n_{I''_j}(s')=n_{j'}(P_{j'}(s'))$. By $D_{I''_j}(s')=0$, we have $c_{j'}(P_{j'}(s'))=0$. Since $c_j(P_j(s'))=c_j(P_j(s))=1$, we have $n_{I''_j}(s')<n_j(P_j(s'))=n_j(P_j(s))=n_{I'}(s)\le 1$, which implies $n_{I''_j}(s')=0$. By Eq. 25 and $j'\in I''_j$, we have $s'\in P_{j'}^{-1}P_{j'}(F_0(m))$. Then, there exists $s''\in F_0(m)=L(G)\cap(L(G)-K)\Sigma^{\ge m}$ such that $P_{j'}(s')=P_{j'}(s'')$. By Eq. 23, we have $D_{I''_j}(s'')=1$. Since $n_{j'}(P_{j'}(s''))=n_{j'}(P_{j'}(s'))=n_{I''_j}(s')=0$, we have $n_{I''_j}(s'')=n_{j'}(P_{j'}(s''))=0$, which implies together with $c_{j'}(P_{j'}(s''))=c_{j'}(P_{j'}(s'))=0$ that $D_{I''_j}(s'')\ne 1$. This contradicts $D_{I''_j}(s'')=1$.

We next consider the case where Eq. 6 holds. For any $s \in H_2(m) \neq \emptyset$, there exists $I' \in \mathcal{I}^k$ such that

$$s \in H_1(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(F_1(m)) \right). \tag{26}$$

Since $s \in H_1(m) \subseteq H_0(m) = K$ and $I' \in \mathcal{I}^k \subseteq \mathcal{I}^{\geq k}$, by Eq. 9, we have $D_{I'}(s) \neq 1$. In addition, by Eq. 6, we have $n_{I'}(s) \leq 1$. It follows from the second condition of Definition 1 that $D_{I'}(s) = 0$. We consider any $j \in I'$ such that $n_{I'}(s) = n_j(P_j(s))$. By $D_{I'}(s) = 0$, we have $c_j(P_j(s)) = 0$. Since $s \in P_j^{-1} P_j(F_1(m))$ by Eq. 26, there exists $s' \in F_1(m)$ such that $P_j(s) = P_j(s')$. It follows from $s' \in F_1(m)$ and $s' \in P_j^{-1} P_j(s) \subseteq P_j^{-1} P_j(H_0(m))$ that there exists $I''_j \in \mathcal{I}^k$ such that

$$s' \in F_0(m) \cap \left(\bigcap_{i \in I''_j} P_i^{-1} P_i(H_0(m)) \right) \tag{27}$$

and $j \in I''_j$. Since $s' \in F_0(m) = L(G) \cap (L(G) - K)\Sigma^{\geq m}$ and $I''_j \in \mathcal{I}^k \subseteq \mathcal{I}^{\geq k}$, by Eq. 23, we have $D_{I''_j}(s') = 1$. We consider any $j' \in I''_j$ such that $n_{I''_j}(s') = n_{j'}(P_{j'}(s'))$. By $D_{I''_j}(s') = 1$, we have $c_{j'}(P_{j'}(s')) = 1$. Since $c_j(P_j(s')) = c_j(P_j(s)) = 0$, we have $n_{I''_j}(s') < n_j(P_j(s')) = n_j(P_j(s)) = n_{I'}(s) \leq 1$, which implies $n_{I''_j}(s') = 0$. By Eq. 27 and $j' \in I''_j$, we have $s' \in P_{j'}^{-1} P_{j'}(H_0(m))$. Then, there exists $s'' \in H_0(m) = K$ such that $P_{j'}(s') = P_{j'}(s'')$. By Eq. 9, we have $D_{I''_j}(s'') \neq 1$. In addition, by Eq. 6, we have $n_{I''_j}(s'') \leq 1$. It follows from the second condition of Definition 1 that $D_{I''_j}(s'') = 0$. Since $n_{j'}(P_{j'}(s'')) = n_{j'}(P_{j'}(s')) = n_{I''_j}(s') = 0$, we have $n_{I''_j}(s'') = n_{j'}(P_{j'}(s'')) = 0$, which implies together with $c_{j'}(P_{j'}(s'')) = c_{j'}(P_{j'}(s')) = 1$ that $D_{I''_j}(s'') \neq 0$. This contradicts $D_{I''_j}(s'') = 0$. □

Example 1 We consider a DES modeled by the finite automaton G shown in Fig. 1, where $\Sigma = \{a, b, c, d, e, f, g\}$. We assume that $n = 4$, that is, there are four local diagnosers. Let the locally observable event sets be $\Sigma_{1,o} = \{a, e, g\}$, $\Sigma_{2,o} = \{b, e, g\}$, $\Sigma_{3,o} = \{c, e, g\}$, and $\Sigma_{4,o} = \{d, e, g\}$. We assume that diagnosis decisions of at least three local diagnosers are

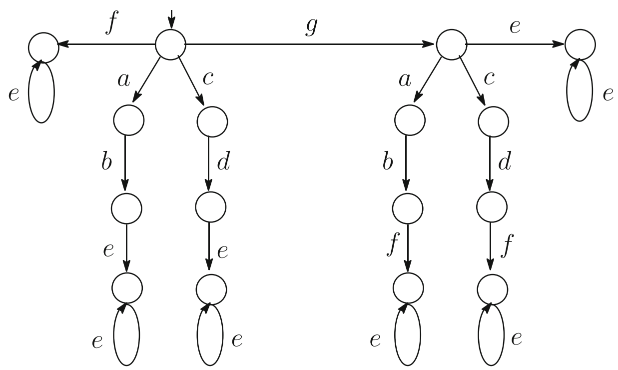


Fig. 1 Finite automaton G for Example 1

available, that is, $k = 3$. As the nonfailure behavior of G, we consider a nonempty closed regular sublanguage $K \subseteq L(G)$ that is generated by the finite automaton G_K shown in Fig. 2. That is, the occurrence of a failure is modeled by the event f.

We show that G is (4, 3)-reliably 1-inference-diagnosable for the sublanguage $K \subseteq L(G)$. We have $\mathcal{I}^3 = \{I_1, I_2, I_3, I_4\}$, where $I_1 = \{1, 2, 3\}$, $I_2 = \{1, 2, 4\}$, $I_3 = \{1, 3, 4\}$, and $I_4 = \{2, 3, 4\}$. For any $m \in \mathbb{N}$ with $m \geq 1$, initially, we have

$$F_0(m) = fe^m e^* + g(abfe^m e^* + cdfe^m e^*), \tag{28}$$
$$H_0(m) = pr((abe^* + cde^*) + g(e^* + ab + cd)). \tag{29}$$

Since

$$P_1(F_0(m)) = e^m e^* + g(ae^m e^* + e^m e^*), \tag{30}$$
$$P_2(F_0(m)) = e^m e^* + g(be^m e^* + e^m e^*), \tag{31}$$
$$P_3(F_0(m)) = e^m e^* + g(e^m e^* + ce^m e^*), \tag{32}$$
$$P_4(F_0(m)) = e^m e^* + g(e^m e^* + de^m e^*), \tag{33}$$
$$P_1(H_0(m)) = pr((ae^* + e^*) + g(e^* + a)), \tag{34}$$
$$P_2(H_0(m)) = pr((be^* + e^*) + g(e^* + b)), \tag{35}$$
$$P_3(H_0(m)) = pr((e^* + ce^*) + g(e^* + c)), \tag{36}$$
$$P_4(H_0(m)) = pr((e^* + de^*) + g(e^* + d)), \tag{37}$$

we have

$$F_0(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(H_0(m)) \right) = fe^m e^*, \tag{38}$$

$$H_0(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(F_0(m)) \right) = ge^m e^* \tag{39}$$

for each $I' \in \mathcal{I}^3$. It follows that $F_1(m) = fe^m e^*$ and $H_1(m) = ge^m e^*$, which imply that G is neither (4, 3)-reliably disjunctive-codiagnosable nor (4, 3)-reliably conjunctive-

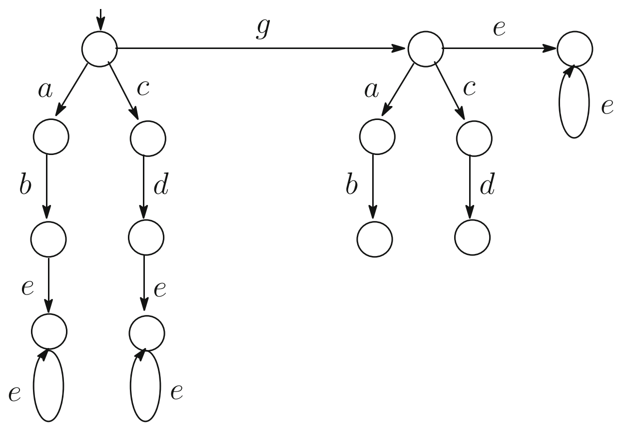


Fig. 2 Finite automaton G_K for Example 1

codiagnosable. In addition, since $P_i(F_1(m)) = e^m e^*$ and $P_i(H_1(m)) = ge^m e^*$ for each $i \in I$, we have

$$F_1(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(H_1(m)) \right) = \emptyset, \tag{40}$$

$$H_1(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(F_1(m)) \right) = \emptyset \tag{41}$$

for each $I' \in \mathcal{I}^3$. It follows that $F_2(m) = H_2(m) = \emptyset$, that is, G is (4, 3)-reliably 1-inference-diagnosable. Note that (n, k)-reliable 1-inference-diagnosability requires that there exists $m \in \mathbb{N}$ such that $F_2(m) = \emptyset \vee H_2(m) = \emptyset$. This example is a special case where both $F_2(m)$ and $H_2(m)$ are empty for any $m \in \mathbb{N}$ with $m \geq 1$.

Since G is (4, 3)-reliably 1-inference-diagnosable, by Proposition 2, the local diagnosers D_i $(i = 1, 2, 3, 4)$ synthesized by Eqs. 19–22 for any $m \in \mathbb{N}$ with $m \geq 1$ solve the reliable decentralized diagnosis problem in the case of $N = 1$. We let $m = 1$. The diagnosis decisions

Table 1 Local decisions of D_i $(i = 1, 2, 3, 4)$ and their ambiguity levels

$t \in P_1(L(G))$	$n_1^f(t)$	$n_1^h(t)$	$c_1(t)$	$n_1(t)$
$t \in \varepsilon + a$	1	0	0	0
$t \in ee^*$	1	2	1	1
$t \in aee^*$	1	0	0	0
$t \in g + ga$	1	0	0	0
$t \in gee^*$	2	1	0	1
$t \in gaee^*$	0	1	1	0
$t \in P_2(L(G))$	$n_2^f(t)$	$n_2^h(t)$	$c_2(t)$	$n_2(t)$
$t \in \varepsilon + b$	1	0	0	0
$t \in ee^*$	1	2	1	1
$t \in bee^*$	1	0	0	0
$t \in g + gb$	1	0	0	0
$t \in gee^*$	2	1	0	1
$t \in gbee^*$	0	1	1	0
$t \in P_3(L(G))$	$n_3^f(t)$	$n_3^h(t)$	$c_3(t)$	$n_3(t)$
$t \in \varepsilon + c$	1	0	0	0
$t \in ee^*$	1	2	1	1
$t \in cee^*$	1	0	0	0
$t \in g + gc$	1	0	0	0
$t \in gee^*$	2	1	0	1
$t \in gcee^*$	0	1	1	0
$t \in P_4(L(G))$	$n_4^f(t)$	$n_4^h(t)$	$c_4(t)$	$n_4(t)$
$t \in \varepsilon + d$	1	0	0	0
$t \in ee^*$	1	2	1	1
$t \in dee^*$	1	0	0	0
$t \in g + gd$	1	0	0	0
$t \in gee^*$	2	1	0	1
$t \in gdee^*$	0	1	1	0

of D_i $(i = 1, 2, 3, 4)$ together with their ambiguity levels are shown in Table 1. For example, for $ge \in P_1(L(G))$, $D_1(ge) = (c_1(ge), n_1(ge))$ is computed as follows. By Eqs. 19 and 20, we have $n_1^f(ge) = 2$ and $n_1^h(ge) = 1$. Since $1 = n_1^h(ge) < n_1^f(ge) = 2$, $c_1(ge) = 0$ and $n_1(ge) = 1$ are obtained by Eqs. 21 and 22, respectively, which implies that D_1 issues the nonfailure decision whose ambiguity level is 1 after observing ge.

For example, for $I_1 = \{1, 2, 3\} \in \mathcal{I}^3 \subseteq \mathcal{I}^{\geq 3} = \mathcal{I}^3 \cup \{I\}$, the diagnosis decisions of D_{I_1} are shown in Tables 2. For example, for $gcdfe \in L(G) - K$, $D_{I_1}(gcdfe)$ is computed as follows. By $0 = n_3(P_3(gcdfe)) < n_1(P_1(gcdfe)) = n_2(P_2(gcdfe)) = 1$, we have $n_{I_1}(gcdfe) = n_3(P_3(gcdfe)) = 0$ and $D_{I_1}(gcdfe) = 1$, which imply that D_{I_1} detects the occurrence of f with the ambiguity level 0 after the occurrence of $gcdfe$. We can verify that D_i $(i = 1, 2, 3, 4)$ are (n, k)-1-inferring and satisfy the conditions

$$\forall I' \in \mathcal{I}^{\geq 3}, \forall s \in L(G) \cap (L(G) - K)\Sigma^{\geq 1}(D_{I'}(s) = 1) \tag{42}$$

and Eq. 9 for $k = 3$. That is, they correctly detect the any occurrence of the event f within one step, even if a diagnosis decision of one local diagnoser is not available.

Remark 5 In Takai and Yoshida (2022), the reliable decentralized supervisory control problem has been solved using single-level inference. To solve the problem for a nonempty regular sublanguage $K \subseteq L(G)$ given as a control specification, four language pairs $(D_h(\sigma), E_h(\sigma))$ $(h = 0, 1, 2, 3)$ per controllable event $\sigma \in \Sigma_c$ Ramadge and Wonham (1987), where $\Sigma_c \subseteq \Sigma$ is the set of controllable events, are defined. Initially, $D_0(\sigma)$ and $E_0(\sigma)$ are defined as

$$D_0(\sigma) = \{s \in pr(K) \mid s\sigma \in L(G) - pr(K)\}, \tag{43}$$

$$E_0(\sigma) = \{s \in pr(K) \mid s\sigma \in pr(K)\}. \tag{44}$$

Then, for $h = 0, 1, 2$, $D_{h+1}(\sigma)$ and $E_{h+1}(\sigma)$ are defined as

$$D_{h+1}(\sigma) = D_h(\sigma) \cap \left\{ \bigcup_{I' \in \mathcal{I}^k} \left(\bigcap_{i \in I'} P_i^{-1} P_i(E_h(\sigma)) \right) \right\}, \tag{45}$$

$$E_{h+1}(\sigma) = E_h(\sigma) \cap \left\{ \bigcup_{I' \in \mathcal{I}^k} \left(\bigcap_{i \in I'} P_i^{-1} P_i(D_h(\sigma)) \right) \right\}. \tag{46}$$

Then, the notion of (n, k)-reliable 1-inference-observability is defined as

$$\forall \sigma \in \Sigma_c(D_2(\sigma) = \emptyset \vee E_2(\sigma) = \emptyset). \tag{47}$$

In addition, the language pairs $(D_h(\sigma), E_h(\sigma))$ $(h = 0, 1, 2, 3)$ are used to synthesize local supervisors. Since the number $|\Sigma_c|$ of controllable events is finite, $(D_h(\sigma), E_h(\sigma))$ $(h = 0, 1, 2, 3)$ are effectively computable for all controllable events $\sigma \in \Sigma_c$ using the standard

Table 2 Decisions of D_{I_1}

$s \in L(G)$	$n_{I_1}(s)$	$D_{I_1}(s)$
$s \in f + pr(abe^* + cde^*)$	0	0
$s \in fee^*$	1	1
$s \in pr(g(abf + cdf))$	0	0
$s \in g(abfee^* + cdfee^*)$	0	1
$s \in gee^*$	1	0

operations over finite automata. On the other hand, the language pairs $(F_h(m), H_h(m))$ $(h = 0, 1, 2, 3)$ introduced to solve the reliable decentralized diagnosis problem in this paper involve a nonnegative integer $m \in \mathbb{N}$, which represents a delay of detecting a failure string. Since $\mathbb{N}$ is an infinite set, it is impossible to compute $(F_h(m), H_h(m))$ $(h = 0, 1, 2, 3)$ for all nonnegative integers $m \in \mathbb{N}$. This is the reason why the result of Takai and Yoshida (2022) cannot be applied to the reliable decentralized diagnosis problem considered in this paper.

5 Verification of reliable 1-inference-diagnosability

In this section, we develop a method for verifying (n, k)-reliable 1-inference-diagnosability effectively. Given a nonnegative integer $m \in \mathbb{N}$, we can verify whether $F_2(m) = \emptyset \vee H_2(m) = \emptyset$ holds using the standard operations over finite automata. However, to verify (n, k)-reliable 1-inference-diagnosability, we need to test the existence of $m \in \mathbb{N}$ such that $F_2(m) = \emptyset \vee H_2(m) = \emptyset$. For this purpose, we develop a verification method in this section.

For a nonempty closed regular sublanguage $K \subseteq L(G)$ that models the nonfailure behavior of the system G, there exists a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$ that generates it. That is, it holds that $L(G_K) = K$. We augment the automaton G_K by adding a dump state $q_d \notin Q_K$. The augmented automaton is defined as $\widetilde{G}_K = (\widetilde{Q}_K, \Sigma, \widetilde{\delta}_K, q_{K,0})$, where the state set is $\widetilde{Q}_K = Q_K \cup \{q_d\}$, and the state transition function $\widetilde{\delta}_K : \widetilde{Q}_K \times \Sigma \to \widetilde{Q}_K$ is given as

$$\widetilde{\delta}_K(\widetilde{q}_K, \sigma) = \begin{cases} \delta_K(\widetilde{q}_K, \sigma), & \text{if } \widetilde{q}_K \in Q_K \wedge \delta_K(\widetilde{q}_K, \sigma)! \\ q_d, & \text{otherwise} \end{cases} \tag{48}$$

for each $\widetilde{q}_K \in \widetilde{Q}_K$ and each $\sigma \in \Sigma$. It follows from the definition of the state transition function $\widetilde{\delta}_K$ that $L(\widetilde{G}_K) = \Sigma^*$. Let

$$G \parallel \widetilde{G}_K = (Q \times \widetilde{Q}_K, \Sigma, \alpha, (q_0, q_{K,0})) \tag{49}$$

be the synchronous composition of G and $\widetilde{G}_K$, where the state transition function $\alpha : (Q \times \widetilde{Q}_K) \times \Sigma \to (Q \times \widetilde{Q}_K)$ is given as

$$\alpha((q, \widetilde{q}_K), \sigma) = \begin{cases} (\delta(q, \sigma), \widetilde{\delta}_K(\widetilde{q}_K, \sigma)), & \text{if } \delta(q, \sigma)! \\ \text{undefined}, & \text{otherwise} \end{cases} \tag{50}$$

for each $(q, \widetilde{q}_K) \in Q \times \widetilde{Q}_K$ and each $\sigma \in \Sigma$. Then, it holds that $L(G \parallel \widetilde{G}_K) = L(G) \cap L(\widetilde{G}_K) = L(G) \cap \Sigma^* = L(G)$. For each $s \in L(G)$, $s \in L(G) - K$ if and only if the second element of the state reached by the execution of $s \in L(G)$ in $G \parallel \widetilde{G}_K$ is the dump state q_d.

By Definition 2, G is not (n, k)-reliably 1-inference-diagnosable if and only if

$$\forall m \in \mathbb{N}(F_2(m) \neq \emptyset \wedge H_2(m) \neq \emptyset). \tag{51}$$

In the section, we show how to verify $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$ and $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$, separately.

Remark 6 Unlike the verification approach of Sampath et al. (1995), we do not use a diagnoser automaton that generates the projection $P_i(L(G))$ $(i \in I)$ to verify (n, k)-reliable 1-inference-diagnosability. We adopt the approach, called the verifier approach, introduced in Jiang et al. (2001); Yoo and Lafortune (2002). The advantage of the verifier approach is that constructing a diagnoser automaton whose computational complexity is exponential in $|Q \times \widetilde{Q}_K|$, where Q and $\widetilde{Q}_K$ are the state sets of G and $\widetilde{G}_K$, respectively, is not necessary.

5.1 Verification of $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$

First, we show how to verify whether $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$. By composing $n(n-1)+1$ copies of $G \parallel \widetilde{G}_K$, which are used to trace failure strings, and n copies of $\widetilde{G}_K$, which are used to trace nonfailure strings, we construct a finite automaton

$$V_F = (R_F, \Sigma_V, \delta_F, r_{F,0}) \tag{52}$$

as follows:

- The state set R_F is given by

$$R_F = (Q \times \widetilde{Q}_K) \times \left(\prod_{i=1}^{n} \prod_{j=1}^{n} Q_{Fij} \right), \tag{53}$$

  where

$$Q_{Fij} = \begin{cases} \widetilde{Q}_K, & \text{if } i = j \\ Q \times \widetilde{Q}_K, & \text{otherwise} \end{cases} \tag{54}$$

  for each $i, j \in I$.
- The initial state $r_{F,0} \in R_F$ is given by

$$r_{F,0} = ((q_0, q_{K,0}), q_{F11,0}, \dots, q_{F1n,0}, q_{F21,0}, \dots, q_{Fnn,0}), \tag{55}$$

  where

$$q_{Fij,0} = \begin{cases} q_{K,0}, & \text{if } i = j \\ (q_0, q_{K,0}), & \text{otherwise} \end{cases} \tag{56}$$

  for each $i, j \in I$.
- The event set Σ_V is given by

$$\begin{aligned} \Sigma_V = \{ & (\sigma, \overline{\sigma}_{11}, \dots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \dots, \overline{\sigma}_{nn}) \in \overline{\Sigma}^{n^2+1} \mid \\ & \sigma \in \Sigma_o \wedge [\forall i, j \in I(\overline{\sigma}_{ij} = e_{\Sigma_o}(i, j, \sigma))]\} \\ & \cup \{\sigma, \overline{\sigma}_{11}, \dots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \dots, \overline{\sigma}_{nn}) \in \overline{\Sigma}^{n^2+1} \mid \\ & \sigma \in \Sigma_{uo} \wedge [\forall i, j \in I(\overline{\sigma}_{ij} = \varepsilon)]\} \\ & \cup \{(\varepsilon, \overline{\sigma}_{11}, \dots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \dots, \overline{\sigma}_{nn}) \in \overline{\Sigma}^{n^2+1} \mid \\ & \exists i' \in I, \exists \sigma \in \Sigma - \Sigma_{i',o} \\ & [\forall i, j \in I(\overline{\sigma}_{ij} = e_{\Sigma_{i',uo}}(i, j, \sigma))]\} \\ & \cup \{(\varepsilon, \overline{\sigma}_{11}, \dots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \dots, \overline{\sigma}_{nn}) \in \overline{\Sigma}^{n^2+1} \mid \\ & \exists i' \in I, \exists j' \in I - \{i'\}, \exists \sigma \in \Sigma - \Sigma_{j',o} \\ & [\forall i, j \in I(\overline{\sigma}_{ij} = e_{\Sigma_{i'j',uo}}(i, j, \sigma))]\}, \end{aligned} \tag{57}$$

  where

$$\overline{\Sigma}^{n^2+1} = \underbrace{(\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \times \cdots \times (\Sigma \cup \{\varepsilon\})}_{n^2+1 \text{ times}} \tag{58}$$

  and

$$e_{\Sigma_o}(i, j, \sigma) = \begin{cases} \sigma, & \text{if } \sigma \in \Sigma_{i,o} \cap \Sigma_{j,o} \\ \varepsilon, & \text{otherwise,} \end{cases} \tag{59}$$

$$e_{\Sigma_{i',uo}}(i,j,\sigma)=\begin{cases}\sigma, \text{ if } i=i' \wedge [j=i \vee \sigma \in \Sigma_{j,o}] \\ \varepsilon, \text{ otherwise,}\end{cases} \tag{60}$$

$$e_{\Sigma_{i'j',uo}}(i,j,\sigma)=\begin{cases}\sigma, \text{ if } i=i' \wedge j=j' \\ \varepsilon, \text{ otherwise}\end{cases} \tag{61}$$

for each $i, j \in I$ and each $\sigma \in \Sigma$. Then, any element of Σ_V, denoted by $(\overline{\sigma}, \overline{\sigma}_{11}, \ldots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{nn})$, satisfies the following three conditions:

- $\overline{\sigma} \neq \varepsilon \vee [\exists i, j \in I(\overline{\sigma}_{ij} \neq \varepsilon)]$,
- $\forall i \in I(P_i(\overline{\sigma}) = P_i(\overline{\sigma}_{ii}))$,
- $\forall i \in I, \forall j \in I - \{i\}(P_j(\overline{\sigma}_{ii}) = P_j(\overline{\sigma}_{ij}))$.

- For any $r_F = ((q, \widetilde{q}_K), q_{F11}, \ldots, q_{F1n}, q_{F21}, \ldots, q_{Fnn}) \in R_F$ and any $\sigma_V = (\overline{\sigma}, \overline{\sigma}_{11}, \ldots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{nn}) \in \Sigma_V$, $\delta_F(r_F, \sigma_V)!$ if and only if the following two conditions hold:
  - $\overline{\sigma} \neq \varepsilon \Rightarrow \alpha((q, \widetilde{q}_K), \overline{\sigma})!$,
  - $\forall i \in I, \forall j \in I - \{i\}(\overline{\sigma}_{ij} \neq \varepsilon \Rightarrow \alpha(q_{Fij}, \overline{\sigma}_{ij})!)$.

  If $\delta_F(r_F, \sigma_V)!$, then

$$\delta_F(r_F, \sigma_V) = ((q', \widetilde{q}'_K), q'_{F11}, \ldots, q'_{F1n}, q'_{F21}, \ldots, q'_{Fnn}), \tag{62}$$

  where

$$(q', \widetilde{q}'_K) = \begin{cases}\alpha((q, \widetilde{q}_K), \overline{\sigma}), & \text{if } \overline{\sigma} \neq \varepsilon \\ (q, \widetilde{q}_K), & \text{otherwise}\end{cases} \tag{63}$$

  and, for each $i, j \in I$,

$$q'_{Fij} = \begin{cases}\widetilde{\delta}_K(q_{Fij}, \overline{\sigma}_{ij}), & \text{if } i = j \wedge \overline{\sigma}_{ij} \neq \varepsilon \\ \alpha(q_{Fij}, \overline{\sigma}_{ij}) & \text{if } i \neq j \wedge \overline{\sigma}_{ij} \neq \varepsilon \\ q_{Fij}, & \text{otherwise.}\end{cases} \tag{64}$$

The finite automaton V_F traces $n^2 + 1$ strings $s_0 \in L(G)$, $s_{ij} \in L(G)$ $(i, j = 1, 2, \ldots, n)$ such that

$$\forall i \in I[P_i(s_0) = P_i(s_{ii}) \wedge [\forall j \in I - \{i\}(P_j(s_{ii}) = P_j(s_{ij}))]]. \tag{65}$$

For each state $r_F = ((q, \widetilde{q}_K), q_{F11}, \ldots, q_{F1n}, q_{F21}, \ldots, q_{Fnn}) \in R_F$ of V_F, we let $\pi_0(r_F) = (q, \widetilde{q}_K)$ and $\pi_{ij}(r_F) = q_{Fij}$ for each $i, j \in I$. Similarly, for each event $\sigma_V = (\overline{\sigma}, \overline{\sigma}_{11}, \ldots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{nn}) \in \Sigma_V$, we let $\pi_0(\sigma_V) = \overline{\sigma}$ and $\pi_{ij}(\sigma_V) = \overline{\sigma}_{ij}$ for each $i, j \in I$. Then, for each $s_V \in \Sigma_V^*$, $\pi_0(s_V)$ and $\pi_{ij}(s_V)$ for any $i, j \in I$ are defined as

$$\pi_0(s_V) = \begin{cases}\varepsilon, & \text{if } s_V = \varepsilon \\ \pi_0(\sigma_{V,1})\pi_0(\sigma_{V,2}) \cdots \pi_0(\sigma_{V,|s_V|}), & \text{otherwise,}\end{cases} \tag{66}$$

and

$$\pi_{ij}(s_V) = \begin{cases}\varepsilon, & \text{if } s_V = \varepsilon \\ \pi_{ij}(\sigma_{V,1})\pi_{ij}(\sigma_{V,2}) \cdots \pi_{ij}(\sigma_{V,|s_V|}), & \text{otherwise,}\end{cases} \tag{67}$$

where s_V is denoted by $s_V = \sigma_{V,1}\sigma_{V,2} \cdots \sigma_{V,|s_V|}$ if $s_V \neq \varepsilon$.

For any $m \in \mathbb{N}$, we assume that there exist $s_V \in L(V_F)$ and $I' \in \mathcal{I}^k$ such that $\pi_0(s_V) \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$ and, for each $i \in I'$, the following two conditions hold:

- $\pi_{ii}(s_V) \in K = H_0(m)$,
- $\exists I''_i \in \mathcal{I}^k[i \in I''_i \wedge [\forall j \in I''_i - \{i\}(\pi_{ij}(s_V) \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m))]]$.

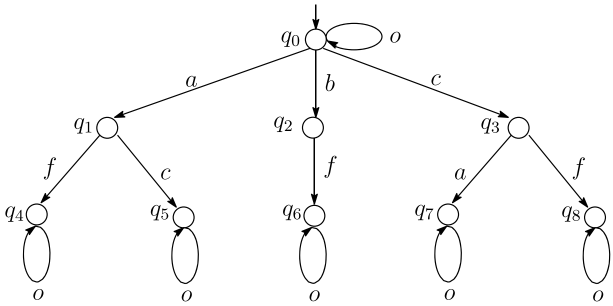


Fig. 3 Finite automaton G for Example 2

Since $P_i(\pi_0(s_V)) = P_i(\pi_{ii}(s_V))$ for any $i \in I'$ and $P_j(\pi_{ii}(s_V)) = P_j(\pi_{ij}(s_V))$ for any $i \in I'$ and any $j \in I''_i - \{i\}$, we have

$$\pi_0(s_V) \in F_0(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(H_0(m)) \right) \subseteq F_1(m) \tag{68}$$

and, for any $i \in I'$,

$$\pi_{ii}(s_V) \in H_0(m) \cap \left(\bigcap_{j \in I''_i} P_j^{-1} P_j(F_0(m)) \right) \subseteq H_1(m). \tag{69}$$

Again, since $P_i(\pi_0(s_V)) = P_i(\pi_{ii}(s_V))$ for any $i \in I'$, we have

$$\pi_0(s_V) \in F_1(m) \cap \left(\bigcap_{i \in I'} P_i^{-1} P_i(H_1(m)) \right) \subseteq F_2(m) \neq \emptyset. \tag{70}$$

Therefore, for any $m \in \mathbb{N}$, the existence of such $s_V \in L(V_F)$ and $I' \in \mathcal{I}^k$ implies $F_2(m) \neq \emptyset$. The verification method developed in this paper is based on this reasoning.

Example 2 For a DES modeled by the finite automaton G shown in Fig. 3, where $\Sigma = \{a, b, c, f, o\}$, we let $n = 3$, that is, there are three local diagnosers. Let the locally observable

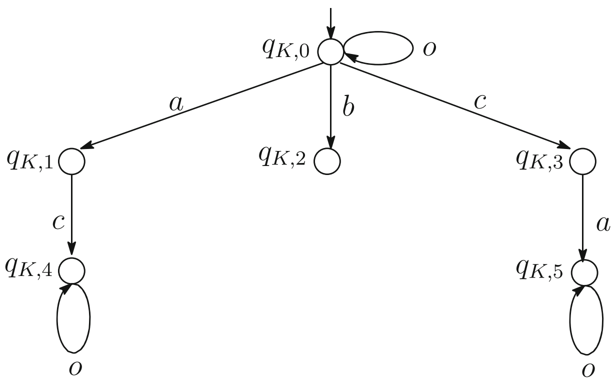


Fig. 4 Finite automaton G_K for Example 2

event sets be $\Sigma_{1,o} = \{a, b, o\}$, $\Sigma_{2,o} = \{b, c, o\}$, and $\Sigma_{3,o} = \{a, c, o\}$. We assume that diagnosis decisions of at least two local diagnosers are available, that is, $k = 2$. As the nonfailure behavior of G, we consider a nonempty closed regular sublanguage $K \subseteq L(G)$ that is generated by the finite automaton G_K shown in Fig. 4. That is, the event f represents the occurrence of a failure.

A part of the finite automaton V_F is shown in Fig. 5. This part of V_F shows that, for any $m \in \mathbb{N}$, the string $s_{V,m} = \sigma_{V,1}\sigma_{V,2}\cdots\sigma_{V,12}\sigma_{V,13}^m$ can be generated by V_F, where

$$\sigma_{V,1} = (\varepsilon, c, c, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\ \sigma_{V,2} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),$$
$$\sigma_{V,3} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, b, \varepsilon, \varepsilon, \varepsilon),\ \sigma_{V,4} = (\varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),$$
$$\sigma_{V,5} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\ \sigma_{V,6} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon),$$

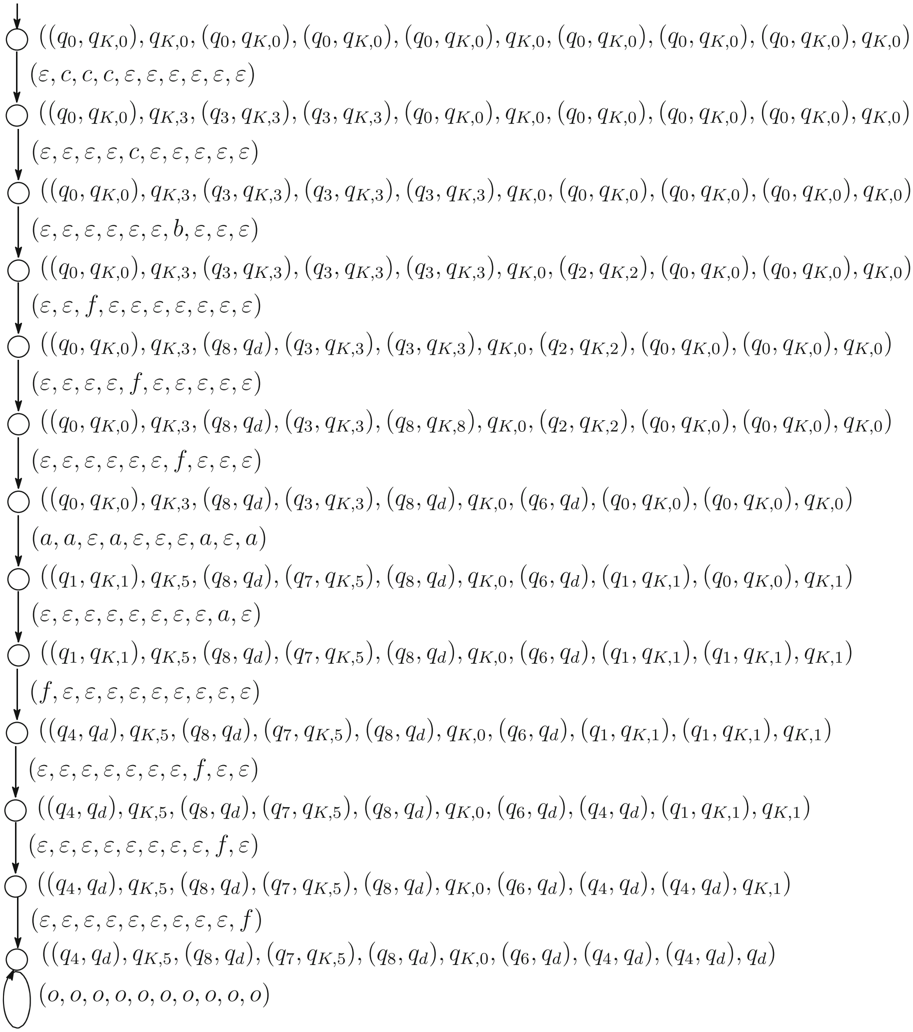


Fig. 5 A part of finite automaton V_F for Example 2

$$\sigma_{V,7} = (a, a, \varepsilon, a, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon, a),\ \sigma_{V,8} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon),$$
$$\sigma_{V,9} = (f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\ \sigma_{V,10} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon),$$
$$\sigma_{V,11} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon),\ \sigma_{V,12} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f),$$
$$\sigma_{V,13} = (o, o, o, o, o, o, o, o, o, o).$$

We consider $\{1, 2\} \in \mathcal{I}^2$. Then, we have $\pi_0(s_{V,m}) = afo^m \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$, $\pi_{11}(s_{V,m}) = cao^m \in K = H_0(m)$, $\pi_{22}(s_{V,m}) = o^m \in K = H_0(m)$, and $\pi_{12}(s_{V,m}) = \pi_{21}(s_{V,m}) = cfo^m \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$. In addition, we have $P_1(\pi_0(s_{V,m})) = P_1(\pi_{11}(s_{V,m})) = ao^m$, $P_2(\pi_0(s_{V,m})) = P_2(\pi_{22}(s_{V,m})) = o^m$, $P_2(\pi_{11}(s_{V,m})) = P_2(\pi_{12}(s_{V,m})) = co^m$, and $P_1(\pi_{22}(s_{V,m})) = P_1(\pi_{21}(s_{V,m})) = o^m$. It follows that $\pi_0(s_{V,m}) \in F_2(m) \neq \emptyset$ for any $m \in \mathbb{N}$.

To effectively verify whether $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$, based on the finite automaton V_F, a nondeterministic acyclic finite automaton

$$\mathcal{V}_{NF} = (\mathcal{R}_{NF}, \Sigma_V, \delta_{NF}, R_{NF,0}), \tag{71}$$

which has the same event set Σ_V as V_F, is constructed as follows:

- The state set $\mathcal{R}_{NF}$ is the set of all maximal strongly connected components of V_F.
- The initial state $R_{NF,0} \in \mathcal{R}_{NF}$ is a maximal strongly connected component of V_F such that $r_{F,0} \in R_{NF,0}$.
- The nondeterministic state transition function $\delta_{NF} : \mathcal{R}_{NF} \times \Sigma_V \to 2^{\mathcal{R}_{NF}}$ is given as

$$\begin{aligned}\delta_{NF}(R_{NF}, \sigma_V) = \{&R'_{NF} \in \mathcal{R}_{NF} \mid R_{NF} \neq R'_{NF} \\ &\wedge [\exists r_F \in R_{NF}, \exists r'_F \in R'_{NF}(\delta_F(r_F, \sigma_V) = r'_F)]\}\end{aligned} \tag{72}$$

  for any $R_{NF} \in \mathcal{R}_{NF}$ and any $\sigma_V \in \Sigma_V$.

We define a labeling function $J_F : \mathcal{R}_{NF} \times I \to 2^I$ as

$$\begin{aligned}&J_F(R_{NF}, i) \\ &= \{j \in I - \{i\} \mid [\exists r_F \in R_{NF}(\pi_{ij}(r_F) \in Q \times \{q_d\})] \\ &\quad \wedge [\exists r_F, r'_F \in R_{NF}, \exists \sigma_V \in \Sigma_V(\delta_F(r_F, \sigma_V) = r'_F \wedge \pi_{ij}(\sigma_V) \neq \varepsilon)]\}\end{aligned} \tag{73}$$

for each $R_{NF} \in \mathcal{R}_{NF}$ and each $i \in I$. For any path $R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ $(l \geq 1)$ of $\mathcal{V}_{NF}$, denoted by $p_{\mathcal{V}_{NF}}$, its label $\widehat{J}_F(p_{\mathcal{V}_{NF}}) \in 2^I$ is given as

$$\begin{aligned}\widehat{J}_F(p_{\mathcal{V}_{NF}}) = \Bigg\{&i \in I \mid \Bigg|\bigcup_{h \in \{1,2,\ldots,l\}} J_F(R_{NF,h}, i)\Bigg| \geq k - 1 \\ &\wedge [\exists r_F \in R_{NF,l}(\pi_{ii}(r_F) \in Q_K)]\Bigg\}.\end{aligned} \tag{74}$$

Example 3 We consider the setting of Example 2. For the part of the finite automaton V_F shown in Fig. 5, the corresponding part of $\mathcal{V}_{NF}$ is shown in Fig. 6. Note that this part is a special case where each state that is a subset of the state set of V_F is singleton.

As shown in Fig. 6, a singleton state

$$\begin{aligned}\{&((q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), \\ &(q_8, q_d), q_{K,0}, (q_6, q_d), (q_4, q_d), (q_4, q_d), q_d)\} \in \mathcal{R}_{NF},\end{aligned}$$

$$
\begin{array}{l}
\downarrow \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, c, c, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_3, q_{K,3}), (q_3, q_{K,3}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_3, q_{K,3}), (q_3, q_{K,3}), (q_3, q_{K,3}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, b, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_3, q_{K,3}), (q_3, q_{K,3}), (q_3, q_{K,3}), q_{K,0}, (q_2, q_{K,2}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_8, q_d), (q_3, q_{K,3}), (q_3, q_{K,3}), q_{K,0}, (q_2, q_{K,2}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_8, q_d), (q_3, q_{K,3}), (q_8, q_{K,8}), q_{K,0}, (q_2, q_{K,2}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_0, q_{K,0}), q_{K,3}, (q_8, q_d), (q_3, q_{K,3}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \\
\downarrow (a, a, \varepsilon, a, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon, a) \\
\bigcirc\ \{((q_1, q_{K,1}), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_1, q_{K,1}), (q_0, q_{K,0}), q_{K,1})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon) \\
\bigcirc\ \{((q_1, q_{K,1}), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_1, q_{K,1}), (q_1, q_{K,1}), q_{K,1})\} \\
\downarrow (f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_1, q_{K,1}), (q_1, q_{K,1}), q_{K,1})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon) \\
\bigcirc\ \{((q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_4, q_d), (q_1, q_{K,1}), q_{K,1})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon) \\
\bigcirc\ \{((q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_4, q_d), (q_4, q_d), q_{K,1})\} \\
\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f) \\
\bigcirc\ \{((q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}), (q_8, q_d), q_{K,0}, (q_6, q_d), (q_4, q_d), (q_4, q_d), q_d)\}
\end{array}
$$

Fig. 6 A part of nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$ for Example 3

denoted by $\{r_F\}$, is reachable in $\mathcal{V}_{NF}$. We consider any path $p_{\mathcal{V}_{NF}}$ that ends with $\{r_F\}$. For $1 \in I$, we have $\pi_{11}(r_F) = q_{K,5} \in Q_K$. In addition, we have $\pi_{12}(r_F) = (q_8, q_d) \in Q \times \{q_d\}$, $\delta_F(r_F, \sigma_{V,13}) = r_F$, and $\pi_{12}(\sigma_{V,13}) = o \neq \varepsilon$ for $\sigma_{V,13} = (o, o, o, o, o, o, o, o, o, o) \in \Sigma_V$. It follows that $2 \in J_F(\{r_F\}, 1)$. Thus, we have $1 \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$. Similarly, we can show that $2 \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$.

Then, we have the following theorem, which shows how to verify whether $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$.

Theorem 4 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$,*

$\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$ *if and only if there exists a path* $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ $(l \geq 1)$ *of the nondeterministic acyclic finite automaton* $\mathcal{V}_{NF}$ *that satisfies*

$$\begin{aligned}&\exists h \in \{1, 2, \ldots, l\}\\ &\quad[\exists r_F \in R_{NF,h}(\pi_0(r_F) \in Q \times \{q_d\})]\\ &\quad\wedge[\exists r_F, r'_F \in R_{NF,h}, \exists \sigma_V \in \Sigma_V(\delta_F(r_F, \sigma_V) = r'_F \wedge \pi_0(\sigma_V) \neq \varepsilon)]\end{aligned} \tag{75}$$

and

$$|\widehat{J}_F(p_{\mathcal{V}_{NF}})| \geq k. \tag{76}$$

Proof ($\Leftarrow$) We suppose that there exists a path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ $(l \geq 1)$ of $\mathcal{V}_{NF}$ that satisfies Eqs. 75 and 76. We consider any $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$ and any $j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i)$. By the definition of $J_F(R_{NF,h}, i)$, we have $j \in I - \{i\}$. In addition, there exists $h \in \{1, 2, \ldots, l\}$ such that

$$\begin{aligned}&[\exists r_F \in R_{NF,h}(\pi_{ij}(r_F) \in Q \times \{q_d\})]\\ &\wedge[\exists r_F, r'_F \in R_{NF,h}, \exists \sigma_V \in \Sigma_V(\delta_F(r_F, \sigma_V) = r'_F \wedge \pi_{ij}(\sigma_V) \neq \varepsilon)].\end{aligned} \tag{77}$$

By Eqs. 75, 77, and the definition of $\widehat{J}_F(p_{\mathcal{V}_{NF}})$, there exists a path $p_{V_F} : r_{F,0} \xrightarrow{\sigma_V^{(0)}} r_{F,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_F-1)}} r_{F,l_F}$ $(l_F \geq 1)$ of V_F that satisfies the following three conditions:

- $\exists h_{01}, h_{02}, h_{03} \in \mathbb{N}$

$$\begin{aligned}&0 \leq h_{01} \leq h_{02} < h_{03} \leq l_F\\ &\quad\wedge r_{F,h_{01}} = r_{F,h_{03}} \wedge \pi_0(r_{F,h_{02}}) \in Q \times \{q_d\} \wedge \pi_0(\sigma_V^{(h_{02})}) \neq \varepsilon,\end{aligned} \tag{78}$$

- $\forall i \in \widehat{J}_F(p_{\mathcal{V}_{NF}}), \forall j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i), \exists h_{ij1}, h_{ij2}, h_{ij3} \in \mathbb{N}$

$$\begin{aligned}&0 \leq h_{ij1} \leq h_{ij2} < h_{ij3} \leq l_F\\ &\quad\wedge r_{F,h_{ij1}} = r_{F,h_{ij3}} \wedge \pi_{ij}(r_{F,h_{ij2}}) \in Q \times \{q_d\} \wedge \pi_{ij}(\sigma_V^{(h_{ij2})}) \neq \varepsilon,\end{aligned} \tag{79}$$

- $\forall i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})(\pi_{ii}(r_{F,l_F}) \in Q_K)$.

We consider any $m \in \mathbb{N}$. By Algorithm 1, we construct $s_V, t_{V,0}, t_{V,ij} \in \Sigma_V^*$ for each $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$ and each $j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i)$. By Algorithm 1, we have $s_V, t_{V,0}, t_{V,ij} \in L(V_F)$ and $t_{V,0}, t_{V,ij} \in pr(s_V)$ for each $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$ and each $j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i)$. Since $t_{V,0} \in pr(s_V)$, there exists $u_{V,0} \in \Sigma_V^*$ such that $s_V = t_{V,0}u_{V,0}$. Let $t_0 = \pi_0(t_{V,0})$ and $u_0 = \pi_0(u_{V,0})$. By the construction of $t_{V,0}$ in Algorithm 1, we have $\alpha((q_0, q_{K,0}), t_0) = \pi_0(r_{F,h_{01}}) = \pi_0(r_{F,h_{03}})$. Since $\pi_0(r_{F,h_{02}}) \in Q \times \{q_d\}$, we have $\alpha((q_0, q_{K,0}), t_0) \in Q \times \{q_d\}$, which implies $t_0 \in L(G) - K$. In addition, since $|u_0| \geq m$ by $\pi_0(\sigma_V^{(h_{02})}) \neq \varepsilon$, we have $t_0u_0 \in F_0(m)$. Let $s_{ii} = \pi_{ii}(s_V)$ for each $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$. Then, we have $\widetilde{\delta}_K(q_{K,0}, s_{ii}) = \pi_{ii}(r_{F,l_F}) \in Q_K$, which implies $s_{ii} \in K = H_0(m)$. In addition, we have $P_i(t_0u_0) = P_i(s_{ii})$, which implies together with $s_{ii} \in H_0(m)$ that $t_0u_0 \in P_i^{-1}P_i(H_0(m))$. By Eq. 76, we have $t_0u_0 \in F_1(m)$. For each $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$ and each $j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i)$, since $t_{V,ij} \in pr(s_V)$, there exists $u_{V,ij} \in \Sigma_V^*$ such that $s_V = t_{V,ij}u_{V,ij}$. Let $t_{ij} = \pi_{ij}(t_{V,ij})$ and $u_{ij} = \pi_{ij}(u_{V,ij})$. By the construction of $t_{V,ij}$ in Algorithm 1, we have $\alpha((q_0, q_{K,0}), t_{ij}) = \pi_{ij}(r_{F,h_{ij1}}) = \pi_{ij}(r_{F,h_{ij3}})$. Since $\pi_{ij}(r_{F,h_{ij2}}) \in Q \times \{q_d\}$, we have $\alpha((q_0, q_{K,0}), t_{ij}) \in Q \times \{q_d\}$, which implies $t_{ij} \in L(G) - K$. In addition, since $|u_{ij}| \geq m$ by $\pi_{ij}(\sigma_V^{(h_{ij2})}) \neq \varepsilon$, we have $t_{ij}u_{ij} \in F_0(m)$.

Algorithm 1 Constructions of $s_V, t_{V,0}, t_{V,ij} \in \Sigma_V^*$ $(\forall i \in \widehat{J}_F(p_{\mathcal{V}_{NF}}), \forall j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i))$

Require: $\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_F-1)}, h_{01}, h_{03}, h_{ij1}, h_{ij3}$ $(\forall i \in \widehat{J}_F(p_{\mathcal{V}_{NF}}), \forall j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i))$, m

$s_V \leftarrow \varepsilon, t_{V,0} \leftarrow \varepsilon, t_{V,ij} \leftarrow \varepsilon$ $(\forall i \in \widehat{J}_F(p_{\mathcal{V}_{NF}}), \forall j \in \bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i))$
$h \leftarrow 0$
while $h \le l_F - 1$ **do**
  if $h = h_{01}$ **then**
    $t_{V,0} \leftarrow s_V$
    $s_V \rightarrow s_V(\sigma_V^{(h_{01})}\sigma_V^{(h_{01}+1)}\cdots\sigma_V^{(h_{03}-1)})^m$
  end if
  $\Phi(h) \leftarrow \{(i, j) \mid h = h_{ij1}\}$
  while $\Phi(h) \neq \emptyset$ **do**
    Pick any $(i, j) \in \Phi(h)$
    $t_{V,ij} \leftarrow s_V$
    $s_V \leftarrow s_V(\sigma_V^{(h_{ij1})}\sigma_V^{(h_{ij1}+1)}\cdots\sigma_V^{(h_{ij3}-1)})^m$
    $\Phi(h) \leftarrow \Phi(h) - \{(i, j)\}$
  end while
  $s_V \leftarrow s_V\sigma_V^{(h)}$
  $h \leftarrow h + 1$
end while

Furthermore, we have $P_j(s_{ii}) = P_j(t_{ij}u_{ij})$, which implies together with $t_{ij}u_{ij} \in F_0(m)$ that $s_{ii} \in P_j^{-1}P_j(F_0(m))$. Since $|\bigcup_{h\in\{1,2,\ldots,l\}} J_F(R_{NF,h}, i)| \ge k - 1$ and $s_{ii} \in P_i^{-1}P_i(t_0u_0) \subseteq P_i^{-1}P_i(F_0(m))$, we have $s_{ii} \in H_1(m)$.

By $t_0u_0 \in F_1(m)$, Eq. 76, and $t_0u_0 \in P_i^{-1}P_i(s_{ii}) \subseteq P_i^{-1}P_i(H_1(m))$ for each $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$, we have $t_0u_0 \in F_2(m) \neq \emptyset$.

($\Leftarrow$) For any $m \in \mathbb{N}$ with $m > |R_F|$, where R_F is the finite state set of V_F, we consider any $s_0 \in F_2(m) \neq \emptyset$. Then, s_0 can be written as $s_0 = t_0u_0$ such that $t_0 \in L(G) - K$ and $|u_0| \ge m$. There exists $I' \in \mathcal{I}^k$ such that $t_0u_0 \in P_i^{-1}P_i(H_1(m))$ for each $i \in I'$. For each $i \in I'$, there exists $s_{ii} \in H_1(m)$ such that $s_{ii} \in P_i^{-1}P_i(t_0u_0) \subseteq P_i^{-1}P_i(F_0(m))$. Since $s_{ii} \in H_1(m)$ and $s_{ii} \in P_i^{-1}P_i(F_0(m))$, there exists $I_i'' \in \mathcal{I}^k$ such that $i \in I_i''$ and $s_{ii} \in P_j^{-1}P_j(F_0(m))$ for each $j \in I_i''$. For each $j \in I_i'' - \{i\}$, there exists $s_{ij} \in F_0(m)$ such that $P_j(s_{ij}) = P_j(s_{ii})$. Then, s_{ij} can be written as $s_{ij} = t_{ij}u_{ij}$ such that $t_{ij} \in L(G) - K$ and $|u_{ij}| \ge m$. Thus, there exists $\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_F-1)} \in L(V_F)$ $(l_F \ge 1)$ that satisfies the following three conditions:

- $\pi_0(s_V) = t_0u_0$,
- $\forall i \in I'(\pi_{ii}(s_V) = s_{ii})$,
- $\forall i \in I', \forall j \in I_i'' - \{i\}(\pi_{ij}(s_V) = t_{ij}u_{ij})$,

where $s_V = \sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_F-1)}$. There exists $\widetilde{h}_0 \in \mathbb{N}$ with $0 \le \widetilde{h}_0 \le l_F - 1$ such that $\pi_0(\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(\widetilde{h}_0)}) = t_0 \in L(G) - K$. In addition, for each $i \in I'$ and each $j \in I_i'' - \{i\}$, there exists $\widetilde{h}_{ij} \in \mathbb{N}$ with $0 \le \widetilde{h}_{ij} \le l_F - 1$ such that $\pi_{ij}(\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(\widetilde{h}_{ij})}) = t_{ij} \in L(G) - K$.

We consider the path $p_{V_F} : r_{F,0} \xrightarrow{\sigma_V^{(0)}} r_{F,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_F-1)}} r_{F,l_F}$ ($l_F \geq 1$) obtained by executing s_V in V_F. For the path p_{V_F}, there exists the path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_V^{(h_{F,0})}} R_{NF,1} \xrightarrow{\sigma_V^{(h_{F,1})}} \cdots \xrightarrow{\sigma_V^{(h_{F,l-1})}} R_{NF,l}$ ($l \geq 1$) of $\mathcal{V}_{NF}$ such that

$$\{r_{F,0}, \ldots, r_{F,h_{F,0}}\} = R_{NF,0}, \tag{80}$$

$$\{r_{F,h_{F,0}+1}, \ldots, r_{F,h_{F,1}}\} = R_{NF,1}, \tag{81}$$

$$\vdots$$

$$\{r_{F,h_{F,l-1}+1}, \ldots, r_{F,l_F}\} = R_{NF,l} \tag{82}$$

for some $h_{F,0}, h_{F,1}, \ldots, h_{F,l-1} \in \{0, 1, \ldots, l_F - 1\}$ such that $0 \leq h_{F,0} < h_{F,1} < \cdots < h_{F,l-1} < l_F$. Since $u_0 = \pi_0(\sigma_V^{(\widetilde{h}_0+1)} \cdots \sigma_V^{(l_F-1)})$ and $|u_0| \geq m > |R_F|$, there exist $h_{01}, h_{02}, h_{03} \in \mathbb{N}$ that satisfy $\widetilde{h}_0 < h_{01} \leq h_{02} < h_{03} \leq l_F$, $r_{F,h_{01}} = r_{F,h_{03}}$, and $\pi_0(\sigma_V^{(h_{02})}) \neq \varepsilon$. In addition, since $t_0 \in L(G) - K$ and

$$t_0 \in pr(\pi_0(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{01}-1)})) \subseteq pr(\pi_0(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{02}-1)})), \tag{83}$$

we have

$$\pi_0(r_{F,h_{02}}) = \alpha((q_0, q_{K,0}), \pi_0(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{02}-1)})) \in Q \times \{q_d\}. \tag{84}$$

Thus, the path $p_{\mathcal{V}_{NF}}$ satisfies Eq. 75.

We show that the path $p_{\mathcal{V}_{NF}}$ satisfies Eq. 76. For each $i \in I'$, since $\pi_{ii}(s_V) = s_{ii} \in H_1(m) \subseteq K$, we have

$$\pi_{ii}(r_{F,l_F}) = \widetilde{\delta}_K(q_{K,0}, \pi_{ii}(s_V)) \in Q_K \tag{85}$$

for $r_{F,l_F} \in R_{NF,l}$. For each $j \in I_i'' - \{i\}$, since $u_{ij} = \pi_{ij}(\sigma_V^{(\widetilde{h}_{ij}+1)} \cdots \sigma_V^{(l_F-1)})$ and $|u_{ij}| \geq m > |R_F|$, there exist $h_{ij1}, h_{ij2}, h_{ij3} \in \mathbb{N}$ that satisfy $\widetilde{h}_{ij} < h_{ij1} \leq h_{ij2} < h_{ij3} \leq l_F$, $r_{F,h_{ij1}} = r_{F,h_{ij3}}$, and $\pi_{ij}(\sigma_V^{(h_{ij2})}) \neq \varepsilon$. In addition, since $t_{ij} \in L(G) - K$ and

$$t_{ij} \in pr(\pi_{ij}(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{ij1}-1)})) \subseteq pr(\pi_{ij}(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{ij2}-1)})), \tag{86}$$

we have

$$\pi_{ij}(r_{F,h_{ij2}}) = \alpha((q_0, q_{K,0}), \pi_{ij}(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{ij2}-1)})) \in Q \times \{q_d\}. \tag{87}$$

It follows that $j \in J_F(R_{NF,h}, i)$ for some $h \in \{1, 2, \ldots, l\}$. Since $|I_i'' - \{i\}| = k - 1$, we have

$$\left| \bigcup_{h \in \{1,2,\ldots,l\}} J_F(R_{NF,h}, i) \right| \geq k - 1. \tag{88}$$

Thus, we have $i \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$. Since $|I'| = k$, $p_{\mathcal{V}_{NF}}$ satisfies Eq. 76. □

The result of Theorem 4 can be explained as follows. We consider a string $s_V \in L(V_F)$ that can be generated along a path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,(l-1)}} R_{NF,l}$ ($l \geq 1$) of $\mathcal{V}_{NF}$ that satisfies Eqs. 75 and 76. By Eq. 76, there exists an index set $I' \in \mathcal{I}^k$ such that $I' \subseteq \widehat{J}_F(p_{\mathcal{V}_{NF}})$. By Eq. 75, $\pi_0(s_V)$ is a failure string in $L(G) - K$ that can be arbitrarily extended such that $\pi_0(s_V) \in F_0(m)$ for any $m \in \mathbb{N}$. It follows from the definition of $\widehat{J}_F(p_{\mathcal{V}_{NF}})$ that $\pi_{ii}(s_V) \in K = H_0(m)$ for each $i \in I'$. By the construction of V_F, we have

$P_i(\pi_0(s_V)) = P_i(\pi_{ii}(s_V))$. In addition, by the definitions of $J_F(R_{NF}, i)$ and $\widehat{J}_F(p_{\mathcal{V}_{NF}})$, there exists $I_i'' \in \mathcal{I}^k$ with $i \in I_i''$ such that, for each $j \in I_i'' - \{i\}$, $\pi_{ij}(s_V)$ is a failure string in $L(G) - K$ that can be arbitrarily extended such that $\pi_{ij}(s_V) \in F_0(m)$. Moreover, we have $P_j(\pi_{ii}(s_V)) = P_j(\pi_{ij}(s_V))$. It follows that $\pi_0(s_V) \in F_1(m)$ and $\pi_{ii}(s_V) \in H_1(m)$ for each $i \in I'$, which implies $\pi_0(s_V) \in F_2(m) \neq \emptyset$.

Remark 7 The number $|R_F|$ of states of the finite automaton V_F is at most $|Q|^{n(n-1)+1} \times (|Q_K|+1)^{n^2+1}$. The number $|\Sigma_V|$ of events of V_F is at most $(n^2+1)|\Sigma|$. Thus, the complexity of constructing V_F is $O(|Q|^{n(n-1)+1} \times |Q_K|^{n^2+1} \times n^2|\Sigma|)$. To construct the nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$, we need to find all maximal strongly connected components of V_F. Its computational complexity is $O(|Q|^{n(n-1)+1} \times |Q_K|^{n^2+1} \times n^2|\Sigma|)$. Note that the value of k is irrelevant to the construction of V_F and $\mathcal{V}_{NF}$.

Then, $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$ can be verified by exploring paths of $\mathcal{V}_{NF}$ as shown in Theorem 4. To verify the existence of a path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ ($l \geq 1$) that satisfies Eqs. 75 and 76, we have to identify all states $R_{NF} \in \mathcal{R}_{NF}$ such that

$$\begin{aligned}&[\exists r_F \in R_{NF}(\pi_0(r_F) \in Q \times \{q_d\})]\\&\wedge[\exists r_F, r_F' \in R_{NF}, \exists \sigma_V \in \Sigma_V(\delta_F(r_F, \sigma_V) = r_F' \wedge \pi_0(\sigma_V) \neq \varepsilon)]\end{aligned} \tag{89}$$

and construct the labeling function $J_F : \mathcal{R}_{NF} \times I \to 2^I$ defined by Eq. 73. The computational complexity of identifying all states $R_{NF} \in \mathcal{R}_{NF}$ that satisfy Eq. 89 is $O(|\mathcal{R}_{NF}| \times |R_F| \times |\Sigma_V|)$, where $|\mathcal{R}_{NF}|$ is at most $|R_F|(\leq |Q|^{n(n-1)+1} \times (|Q_K|+1)^{n^2+1})$. In addition, the computational complexity of constructing the labeling function $J_F : \mathcal{R}_{NF} \times I \to 2^I$ is $O(|\mathcal{R}_{NF}| \times n^2 \times |R_F| \times |\Sigma_V|)$. Since $\mathcal{V}_{NF}$ is acyclic and events of a path of $\mathcal{V}_{NF}$ are not relevant to Eqs. 75 and 76, the number of paths that have to be explored is at most $\sum_{r=1}^{|\mathcal{R}_{NF}|-1} {}_{|\mathcal{R}_{NF}|-1}\mathrm{P}_r$. Once all states $R_{NF} \in \mathcal{R}_{NF}$ that satisfy Eq. 89 are identified and the labeling function $J_F : \mathcal{R}_{NF} \times I \to 2^I$ is constructed, the computational complexity for verifying the existence of a path $p_{\mathcal{V}_{NF}}$ that satisfies Eqs. 75 and 76 is $O(n \times (\sum_{r=1}^{|\mathcal{R}_{NF}|-1} {}_{|\mathcal{R}_{NF}|-1}\mathrm{P}_r))$.

Example 4 Again, we consider the setting of Example 2. We verify whether $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$ holds by Theorem 4. For this purpose, we construct the nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$ based on the finite automaton V_F.

As in Example 3, we consider any path $p_{\mathcal{V}_{NF}}$ that ends with $\{r_F\}$, where

$$\begin{aligned}r_F = (&(q_4, q_d), q_{K,5}, (q_8, q_d), (q_7, q_{K,5}),\\&(q_8, q_d), q_{K,0}, (q_6, q_d), (q_4, q_d), (q_4, q_d), q_d).\end{aligned} \tag{90}$$

Since $\pi_0(r_F) = (q_4, q_d) \in Q \times \{q_d\}$, $\delta_F(r_F, \sigma_{V,13}) = r_F$, and $\pi_0(\sigma_{V,13}) = o \neq \varepsilon$ for $\sigma_{V,13} = (o, o, o, o, o, o, o, o, o, o) \in \Sigma_V$, $p_{\mathcal{V}_{NF}}$ satisfies Eq. 75. As shown in Example 3, it holds that $1, 2 \in \widehat{J}_F(p_{\mathcal{V}_{NF}})$. Then, we have $|\widehat{J}_F(p_{\mathcal{V}_{NF}})| \geq 2 = k$, which implies that $p_{\mathcal{V}_{NF}}$ satisfies Eq. 76. By Theorem 4, we can conclude that $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$ holds.

5.2 Verification of $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$

Next, we present a method for verifying whether $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$. For this purpose, we construct a finite automaton

$$V_H = (R_H, \Sigma_V, \delta_H, r_{H,0}) \tag{91}$$

by composing n copies of $G \parallel \widetilde{G}_K$, which are used to trace failure strings, G_K, and $n(n-1)$ copies of $\widetilde{G}_K$, which are used to trace nonfailure strings, as follows:

- The state set R_H is given by

$$R_H = Q_K \times \left(\prod_{i=1}^{n} \prod_{j=1}^{n} Q_{Hij} \right), \tag{92}$$

  where

$$Q_{Hij} = \begin{cases} Q \times \widetilde{Q}_K, & \text{if } i = j \\ \widetilde{Q}_K, & \text{otherwise} \end{cases} \tag{93}$$

  for each $i, j \in I$.
- The initial state $r_{H,0} \in R_H$ is given by

$$r_{H,0} = (q_{K,0}, q_{H11,0}, \ldots, q_{H1n,0}, q_{H21,0}, \ldots, q_{Hnn,0}), \tag{94}$$

  where

$$q_{Hij,0} = \begin{cases} (q_0, q_{K,0}), & \text{if } i = j \\ q_{K,0}, & \text{otherwise} \end{cases} \tag{95}$$

  for each $i, j \in I$.
- The event set Σ_V is the same as that of V_F.
- For any $r_H = (q_K, q_{H11}, \ldots, q_{H1n}, q_{H21}, \ldots, q_{Hnn}) \in R_H$ and any $\sigma_V = (\overline{\sigma}, \overline{\sigma}_{11}, \ldots, \overline{\sigma}_{1n}, \overline{\sigma}_{21}, \ldots, \overline{\sigma}_{nn}) \in \Sigma_V$, $\delta_H(r_H, \sigma_V)!$ if and only if the following two conditions hold:

  - $\overline{\sigma} \neq \varepsilon \Rightarrow \delta_K(q_K, \overline{\sigma})!$,
  - $\forall i \in I(\overline{\sigma}_{ii} \neq \varepsilon \Rightarrow \alpha(q_{Hii}, \overline{\sigma}_{ii})!)$.

  If $\delta_H(r_H, \sigma_V)!$, then

$$\delta_H(r_H, \sigma_V) = (q'_K, q'_{H11}, \ldots, q'_{H1n}, q'_{H21}, \ldots, q'_{Hnn}), \tag{96}$$

  where

$$q'_K = \begin{cases} \delta_K(q_K, \overline{\sigma}), & \text{if } \overline{\sigma} \neq \varepsilon \\ q_K, & \text{otherwise} \end{cases} \tag{97}$$

  and, for each $i, j \in I$,

$$q'_{Hij} = \begin{cases} \alpha(q_{Hij}, \overline{\sigma}_{ij}), & \text{if } i = j \wedge \overline{\sigma}_{ij} \neq \varepsilon \\ \widetilde{\delta}_K(q_{Hij}, \overline{\sigma}_{ij}) & \text{if } i \neq j \wedge \overline{\sigma}_{ij} \neq \varepsilon \\ q_{Hij}, & \text{otherwise.} \end{cases} \tag{98}$$

For any $m \in \mathbb{N}$, we assume that there exist $s_V \in L(V_H)$ and $I' \in \mathcal{I}^k$ such that $\pi_0(s_V) \in K = H_0(m)$ and, for each $i \in I'$, the following two conditions hold:

- $\pi_{ii}(s_V) \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$,
- $\exists I''_i \in \mathcal{I}^k[i \in I''_i \wedge [\forall j \in I''_i - \{i\}(\pi_{ij}(s_V) \in K = H_0(m))]]$.

Then, we have $H_2(m) \neq \emptyset$ in a similar way to the reasoning about $F_2(m) \neq \emptyset$.

Example 5 For a DES modeled by the finite automaton G shown in Fig. 7, where $\Sigma = \{a, b, c, f, o\}$, we let $n = 3$, that is, there are three local diagnosers. Let the locally observable event sets be $\Sigma_{1,o} = \{a, b, o\}$, $\Sigma_{2,o} = \{b, c, o\}$, and $\Sigma_{3,o} = \{a, c, o\}$. We assume that diagnosis decisions of at least two local diagnosers are available, that is, $k = 2$. As the

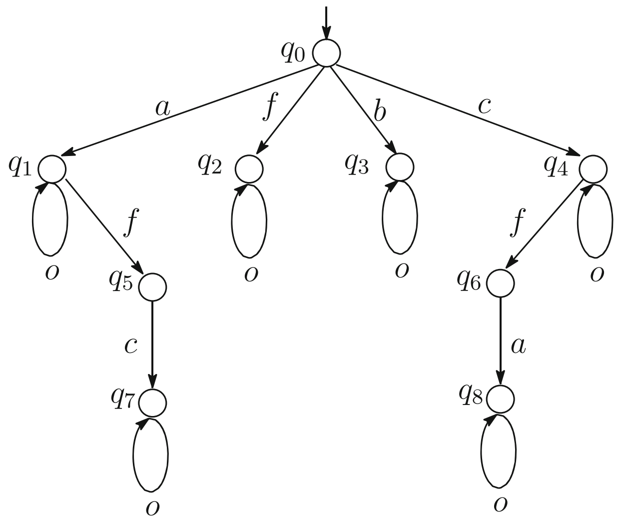


Fig. 7 Finite automaton G for Example 5

nonfailure behavior of G, we consider a nonempty closed regular sublanguage $K \subseteq L(G)$ that is generated by the finite automaton G_K shown in Fig. 8. That is, the event f represents the occurrence of a failure.

A part of the finite automaton V_H is shown in Fig. 9. This part of V_H shows that, for any $m \in \mathbb{N}$, the string $s_{V,m} = \sigma_{V,1}\sigma_{V,2} \cdots \sigma_{V,8}\sigma_{V,9}^m$ can be generated by V_H, where

$$\begin{aligned}
&\sigma_{V,1} = (a, a, \varepsilon, a, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon, a), \ \sigma_{V,2} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, a, \varepsilon),\\
&\sigma_{V,3} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, b, \varepsilon, \varepsilon, \varepsilon), \ \sigma_{V,4} = (\varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\\
&\sigma_{V,5} = (\varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon), \ \sigma_{V,6} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\\
&\sigma_{V,7} = (\varepsilon, c, c, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon), \ \sigma_{V,8} = (\varepsilon, \varepsilon, \varepsilon, \varepsilon, c, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),\\
&\sigma_{V,9} = (o, o, o, o, o, o, o, o, o, o).
\end{aligned}$$

We consider $\{1, 2\} \in \mathcal{I}^2$. Then, we have $\pi_0(s_{V,m}) = ao^m \in K = H_0(m)$, $\pi_{11}(s_{V,m}) = afco^m \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$, $\pi_{22}(s_{V,m}) = fo^m \in L(G) \cap (L(G) - K)\Sigma^{\geq m} = F_0(m)$, and $\pi_{12}(s_{V,m}) = \pi_{21}(s_{V,m}) = co^m \in K = H_0(m)$. In addition, we have $P_1(\pi_0(s_{V,m})) = P_1(\pi_{11}(s_{V,m})) = ao^m$, $P_2(\pi_0(s_{V,m})) = P_2(\pi_{22}(s_{V,m})) = o^m$,

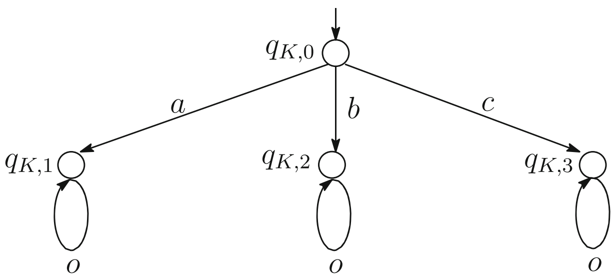


Fig. 8 Finite automaton G_K for Example 5

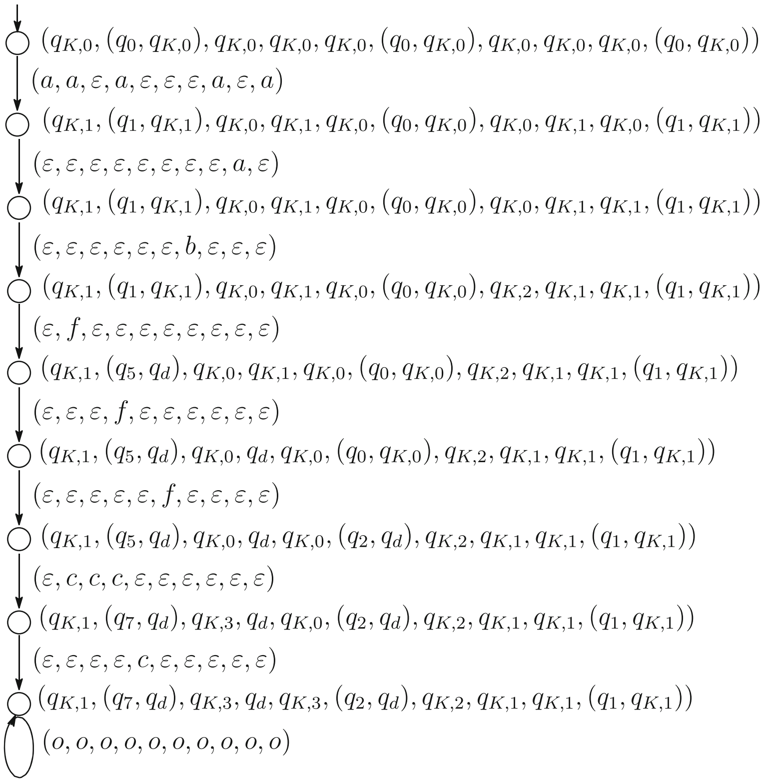


Fig. 9 A part of finite automaton V_H for Example 5

$P_2(\pi_{11}(s_{V,m})) = P_2(\pi_{12}(s_{V,m})) = co^m$, and $P_1(\pi_{22}(s_{V,m})) = P_1(\pi_{21}(s_{V,m})) = o^m$. It follows that $\pi_0(s_{V,m}) \in H_2(m) \neq \emptyset$ for any $m \in \mathbb{N}$.

For each state $r_H = (q_K, q_{H11}, \ldots, q_{H1n}, q_{H21}, \ldots, q_{Hnn}) \in R_H$ of V_H, we let $\pi_0(r_H) = q_K$ and $\pi_{ij}(r_H) = q_{Hij}$ for each $i, j \in I$. To effectively verify whether $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$, based on the finite automaton V_H, we construct a nondeterministic acyclic finite automaton

$$\mathcal{V}_{NH} = (\mathcal{R}_{NH}, \Sigma_V, \delta_{NH}, R_{NH,0}) \tag{99}$$

as follows:

- The state set $\mathcal{R}_{NH}$ is the set of all maximal strongly connected components of V_H.
- The initial state $R_{NH,0} \in \mathcal{R}_{NH}$ is a maximal strongly connected component of V_H such that $r_{H,0} \in R_{NH,0}$.
- The nondeterministic state transition function $\delta_{NH} : \mathcal{R}_{NH} \times \Sigma_V \to 2^{\mathcal{R}_{NH}}$ is given as

$$\begin{aligned}\delta_{NH}(R_{NH}, \sigma_V) = \{&R'_{NH} \in \mathcal{R}_{NH} \mid R_{NH} \neq R'_{NH} \\ &\wedge[\exists r_H \in R_{NH}, \exists r'_H \in R'_{NH}(\delta_H(r_H, \sigma_V) = r'_H)]\}\end{aligned} \tag{100}$$

for any $R_{NH} \in \mathcal{R}_{NH}$ and any $\sigma_V \in \Sigma_V$.

A labeling function $J_H : \mathcal{R}_{NH} \to 2^I$ is defined as

$$\begin{aligned} &J_H(R_{NH}) \\ &= \{i \in I \mid [\exists r_H \in R_{NH}(\pi_{ii}(r_H) \in Q \times \{q_d\})] \\ &\quad \wedge [\exists r_H, r'_H \in R_{NH}, \exists \sigma_V \in \Sigma_V(\delta_H(r_H, \sigma_V) = r'_H \wedge \pi_{ii}(\sigma_V) \neq \varepsilon)]\} \end{aligned} \tag{101}$$

for each $R_{NH} \in \mathcal{R}_{NH}$. For any path $R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ of $\mathcal{V}_{NH}$, denoted by $p_{\mathcal{V}_{NH}}$, its label $\widehat{J}_H(p_{\mathcal{V}_{NH}}) \in 2^I$ is given as

$$\begin{aligned} &\widehat{J}_H(p_{\mathcal{V}_{NH}}) \\ &= \Bigg\{ i \in I \mid i \in \bigcup_{h \in \{1,2,\ldots,l\}} J_H(R_{NH,h}) \\ &\qquad \wedge |\{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}| \geq k - 1 \Bigg\}. \end{aligned} \tag{102}$$

Example 6 We consider the setting of Example 5. For the part of the finite automaton V_H shown in Fig. 9, the corresponding part of $\mathcal{V}_{NH}$ is shown in Fig. 10. As shown in Fig. 10, a singleton state

$$\{(q_{K,1}, (q_7, q_d), q_{K,3}, q_d, q_{K,3}, (q_2, q_d), q_{K,2}, q_{K,1}, q_{K,1}, (q_1, q_{K,1}))\} \in \mathcal{R}_{NH},$$

denoted by $\{r_H\}$, is reachable in $\mathcal{V}_{NH}$. We consider any path $p_{\mathcal{V}_{NH}}$ that ends with $\{r_H\}$. For $1 \in I$, we have $\pi_{11}(r_H) = (q_7, q_d) \in Q \times \{q_d\}$, $\delta_H(r_H, \sigma_{V,9}) = r_H$, and $\pi_{11}(\sigma_{V,9}) = o \neq \varepsilon$

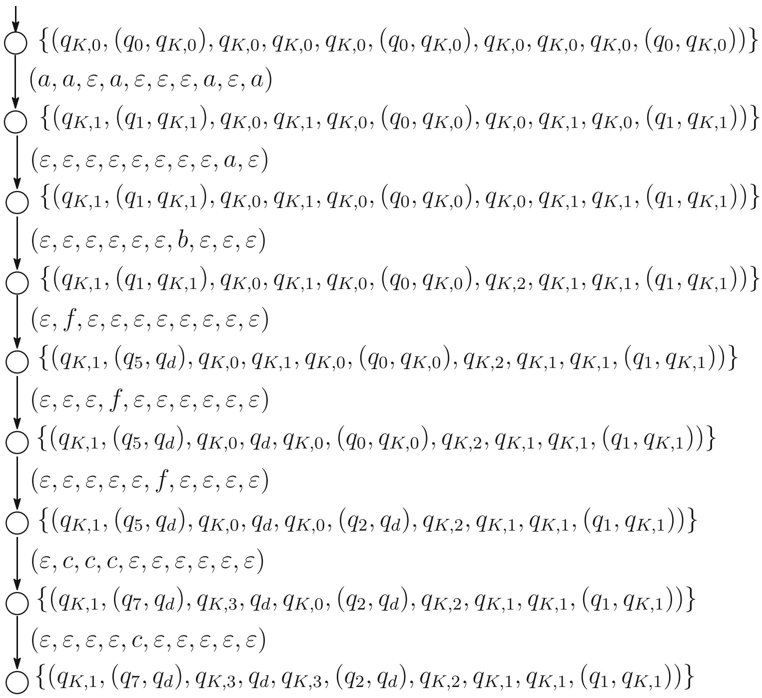


Fig. 10 A part of nondeterministic acyclic finite automaton $\mathcal{V}_{NH}$ for Example 6

for $\sigma_{V,9} = (o, o, o, o, o, o, o, o, o, o) \in \Sigma_V$, which imply $1 \in J_H(\{r_H\})$. In addition, we have $\pi_{12}(r_H) = q_{K,3} \in Q_K$. It follows that $1 \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Similarly, we can show that $2 \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$.

The following theorem shows how to verify whether $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$.

Theorem 5 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$, $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$ if and only if there exists a path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ of the nondeterministic acyclic finite automaton $\mathcal{V}_{NH}$ that satisfies*

$$|\widehat{J}_H(p_{\mathcal{V}_{NH}})| \geq k. \tag{103}$$

Proof ($\Leftarrow$) We suppose that there exists a path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ of $\mathcal{V}_{NH}$ that satisfies Eq. 103. We consider any $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Then, there exists $h \in \{1, 2, \ldots, l\}$ such that $i \in J_H(R_{NH,h})$. By the definition of $J_H(R_{NH,h})$, we have

$$\begin{aligned}
&[\exists r_H \in R_{NH,h}(\pi_{ii}(r_H) \in Q \times \{q_d\})] \\
&\quad \wedge [\exists r_H, r'_H \in R_{NH,h}, \exists \sigma_V \in \Sigma_V(\delta_H(r_H, \sigma_V) = r'_H \wedge \pi_{ii}(\sigma_V) \neq \varepsilon)].
\end{aligned} \tag{104}$$

By Eq. 104 and the definition of $\widehat{J}_H(p_{\mathcal{V}_{NH}})$, there exists a path $p_{V_H} : r_{H,0} \xrightarrow{\sigma_V^{(0)}} r_{H,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_H-1)}} r_{H,l_H}$ $(l_H \geq 1)$ of V_H that satisfies the following two conditions:

- $\forall i \in \widehat{J}_H(p_{\mathcal{V}_{NH}}), \exists h_{ii1}, h_{ii2}, h_{ii3} \in \mathbb{N}$

$$\begin{aligned}
&0 \leq h_{ii1} \leq h_{ii2} < h_{ii3} \leq l_H \\
&\quad \wedge r_{H,h_{ii1}} = r_{H,h_{ii3}} \wedge \pi_{ii}(r_{F,h_{ii2}}) \in Q \times \{q_d\} \wedge \pi_{ii}(\sigma_V^{(h_{ii2})}) \neq \varepsilon,
\end{aligned} \tag{105}$$

- $\forall i \in \widehat{J}_H(p_{\mathcal{V}_{NH}}), \forall j \in \{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}(\pi_{ij}(r_{H,l_H}) \in Q_K)$.

We consider any $m \in \mathbb{N}$. By Algorithm 2, we construct $s_V, t_{V,ii} \in \Sigma_V^*$ for each $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. By Algorithm 2, we have $s_V, t_{V,ii} \in L(V_H)$ and $t_{V,ii} \in pr(s_V)$ for each $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Let $s_0 = \pi_0(s_V)$. Then, by the construction of V_H, we have $s_0 \in K = H_0(m)$. For each $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$, since $t_{V,ii} \in pr(s_V)$, there exists $u_{V,ii} \in \Sigma_V^*$ such that $s_V = t_{V,ii}u_{V,ii}$. Let $t_{ii} = \pi_{ii}(t_{V,ii})$ and $u_{ii} = \pi_{ii}(u_{V,ii})$. By the construction of $t_{V,ii}$ in Algorithm 2, we have $\alpha((q_0, q_{K,0}), t_{ii}) = \pi_{ii}(r_{H,h_{ii1}}) = \pi_{ii}(r_{H,h_{ii3}})$. Since $\pi_{ii}(r_{H,h_{ii2}}) \in Q \times \{q_d\}$, we have $\alpha((q_0, q_{K,0}), t_{ii}) \in Q \times \{q_d\}$, which implies $t_{ii} \in L(G) - K$. In addition, since $|u_{ii}| \geq m$ by $\pi_{ii}(\sigma_V^{(h_{ii2})}) \neq \varepsilon$, we have $t_{ii}u_{ii} \in F_0(m)$. Furthermore, we have $P_i(s_0) = P_i(t_{ii}u_{ii})$, which implies together with $t_{ii}u_{ii} \in F_0(m)$ that $s_0 \in P_i^{-1}P_i(F_0(m))$. By Eq. 103, we have $s_0 \in H_1(m)$. For each $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$ and each $j \in \{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}$, let $s_{ij} = \pi_{ij}(s_V)$. Then, we have $\widetilde{\delta}(q_{K,0}, s_{ij}) = \pi_{ij}(r_{H,l_H}) \in Q_K$, which implies $s_{ij} \in K = H_0(m)$. In addition, we have $P_j(t_{ii}u_{ii}) = P_j(s_{ij})$, which implies together with $s_{ij} \in H_0(m)$ that $t_{ii}u_{ii} \in P_j^{-1}P_j(H_0(m))$. Since $|\{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}| \geq k - 1$ and $t_{ii}u_{ii} \in P_i^{-1}P_i(s_0) \subseteq P_i^{-1}P_i(H_0(m))$, we have $t_{ii}u_{ii} \in F_1(m)$.

By $s_0 \in H_1(m)$, Eq. 103, and $s_0 \in P_i^{-1}P_i(t_{ii}u_{ii}) \subseteq P_i^{-1}P_i(F_1(m))$ for each $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$, we have $s_0 \in H_2(m) \neq \emptyset$.

($\Leftarrow$) For any $m \in \mathbb{N}$ with $m > |R_H|$, where R_H is the finite state set of V_H, we consider any $s_0 \in H_2(m) \neq \emptyset$. There exists $I' \in \mathcal{I}^k$ such that $s_0 \in P_i^{-1}P_i(F_1(m))$ for each $i \in I'$. For each $i \in I'$, there exists $s_{ii} \in F_1(m)$ such that $s_{ii} \in P_i^{-1}P_i(s_0) \subseteq P_i^{-1}P_i(H_0(m))$.

Algorithm 2 Constructions of $s_V, t_{V,ii} \in \Sigma_V^*$ $(\forall i \in \widehat{J}_H(p_{\mathcal{V}_{NH}}))$

Require: $\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_H-1)}, h_{ii1}, h_{ii3}$ $(\forall i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})), m$

$s_V \leftarrow \varepsilon, t_{V,ii} \leftarrow \varepsilon$ $(\forall i \in \widehat{J}_H(p_{\mathcal{V}_{NH}}))$
$h \leftarrow 0$
while $h \leq l_H - 1$ **do**
  $\Phi(h) \leftarrow \{i \mid h = h_{ii1}\}$
  while $\Phi(h) \neq \emptyset$ **do**
    Pick any $i \in \Phi(h)$
    $t_{V,ii} \leftarrow s_V$
    $s_V \leftarrow s_V(\sigma_V^{(h_{ii1})}\sigma_V^{(h_{ii1}+1)}\cdots\sigma_V^{(h_{ii3}-1)})^m$
    $\Phi(h) \leftarrow \Phi(h) - \{i\}$
  end while
  $s_V \leftarrow s_V\sigma_V^{(h)}$
  $h \leftarrow h+1$
end while

Then, s_{ii} can be written as $s_{ii} = t_{ii}u_{ii}$ such that $t_{ii} \in L(G) - K$ and $|u_{ii}| \geq m$. Since $t_{ii}u_{ii} \in F_1(m)$ and $t_{ii}u_{ii} \in P_i^{-1}P_i(H_0(m))$, there exists $I_i'' \in \mathcal{I}^k$ such that $i \in I_i''$ and $t_{ii}u_{ii} \in P_j^{-1}P_j(H_0(m))$ for each $j \in I_i''$. For each $j \in I_i'' - \{i\}$, there exists $s_{ij} \in H_0(m)$ such that $P_j(s_{ij}) = P_j(s_{ii})$. Thus, there exists $\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_H-1)} \in L(V_H)$ $(l_H \geq 1)$ that satisfies the following three conditions:

- $\pi_0(s_V) = s_0$,
- $\forall i \in I'(\pi_{ii}(s_V) = t_{ii}u_{ii})$,
- $\forall i \in I', \forall j \in I_i'' - \{i\}(\pi_{ij}(s_V) = s_{ij})$,

where $s_V = \sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_H-1)}$. For each $i \in I'$, there exists $\widetilde{h}_{ii} \in \mathbb{N}$ with $0 \leq \widetilde{h}_{ii} \leq l_H - 1$ such that $\pi_{ii}(\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(\widetilde{h}_{ii})}) = t_{ii} \in L(G) - K$.

We consider the path $p_{V_H} : r_{H,0} \xrightarrow{\sigma_V^{(0)}} r_{H,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_H-1)}} r_{H,l_H}$ $(l_H \geq 1)$ obtained by executing s_V in V_H. For the path p_{V_H}, there exists the path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_V^{(h_{H,0})}} R_{NH,1} \xrightarrow{\sigma_V^{(h_{H,1})}} \cdots \xrightarrow{\sigma_V^{(h_{H,l-1})}} R_{NH,l}$ $(l \geq 1)$ of $\mathcal{V}_{NH}$ such that

$$\{r_{H,0}, \ldots, r_{H,h_{H,0}}\} = R_{NH,0}, \tag{106}$$

$$\{r_{H,h_{H,0}+1}, \ldots, r_{H,h_{H,1}}\} = R_{NH,1}, \tag{107}$$

$$\vdots$$

$$\{r_{H,h_{H,l-1}+1}, \ldots, r_{H,l_H}\} = R_{NH,l} \tag{108}$$

for some $h_{H,0}, h_{H,1}, \ldots, h_{H,l-1} \in \{0, 1, \ldots, l_H - 1\}$ such that $0 \leq h_{H,0} < h_{H,1} < \cdots < h_{H,l-1} < l_H$.

We show that the path $p_{\mathcal{V}_{NH}}$ satisfies Eq. 103. For each $i \in I'$, since $u_{ii} = \pi_{ii}(\sigma_V^{(\widetilde{h}_{ii}+1)}\cdots\sigma_V^{(l_H-1)})$ and $|u_{ii}| \geq m > |R_H|$, there exist $h_{ii1}, h_{ii2}, h_{ii3} \in \mathbb{N}$ that satisfy $\widetilde{h}_{ii} < h_{ii1} \leq h_{ii2} < h_{ii3} \leq l_H$, $r_{H,h_{ii1}} = r_{H,h_{ii3}}$, and $\pi_{ii}(\sigma_V^{(h_{ii2})}) \neq \varepsilon$. In addition, since $t_{ii} \in L(G) - K$ and

$$t_{ii} \in pr(\pi_{ii}(\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(h_{ii1}-1)})) \subseteq pr(\pi_{ii}(\sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(h_{ii2}-1)})), \tag{109}$$

we have

$$\pi_{ii}(r_{H,h_{ii2}}) = \alpha((q_0, q_{K,0}), \pi_{ii}(\sigma_V^{(0)}\sigma_V^{(1)} \cdots \sigma_V^{(h_{ii2}-1)})) \in Q \times \{q_d\}. \tag{110}$$

It follows that $i \in J_H(R_{NH,h})$ for some $h \in \{1, 2, \ldots, l\}$. In addition, for each $j \in I_i'' - \{i\}$, since $\pi_{ij}(s_V) = s_{ij} \in H_0(m) = K$, we have

$$\pi_{ij}(r_{H,l_H}) = \widetilde{\delta}_K(q_{K,0}, \pi_{ij}(s_V)) \in Q_K \tag{111}$$

for $r_{H,l_H} \in R_{NH,l}$. Since $|I_i'' - \{i\}| = k - 1$, we have

$$|\{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}| \geq k - 1. \tag{112}$$

Thus, we have $i \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Since $|I'| = k$, $p_{\mathcal{V}_{NH}}$ satisfies Eq. 103. □

Remark 8 The number $|R_H|$ of states of the finite automaton V_H is at most $|Q|^n \times |Q_K| \times (|Q_K|+1)^{n^2}$. The number $|\Sigma_V|$ of events of V_H is at most $(n^2+1)|\Sigma|$. Thus, the complexity of construction V_H is $O(|Q|^n \times |Q_K|^{n^2+1} \times n^2|\Sigma|)$. For constructing the nondeterministic acyclic finite automaton $\mathcal{V}_{NH}$, we need to find all maximal strongly connected components of V_H. Its computational complexity is $O(|Q|^n \times |Q_K|^{n^2+1} \times n^2|\Sigma|)$. Similar to V_F and $\mathcal{V}_{NF}$, the value of k is irrelevant to the construction of V_H and $\mathcal{V}_{NH}$.

Then, $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$ can be verified by exploring paths of $\mathcal{V}_{NH}$ as shown in Theorem 5. To verify whether there exists a path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ that satisfies Eq. 103, we have to construct the labeling function $J_H : \mathcal{R}_{NH} \rightarrow 2^I$ defined by Eq. 101. Its computational complexity is $O(|\mathcal{R}_{NH}| \times n \times |R_H| \times |\Sigma_V|)$, where $|\mathcal{R}_{NH}|$ is at most $|R_H|(\leq |Q|^n \times |Q_K| \times (|Q_K|+1)^{n^2})$. Since $\mathcal{V}_{NH}$ is acyclic and events of a path of $\mathcal{V}_{NH}$ are not relevant to Eq. 103, the number of paths that have to be explored is at most $\sum_{r=1}^{|\mathcal{R}_{NH}|-1} {}_{|\mathcal{R}_{NH}|-1}\mathrm{P}_r$. Once the labeling function $J_H : \mathcal{R}_{NH} \rightarrow 2^I$ is constructed, the computational complexity for verifying the existence of a path $p_{\mathcal{V}_{NH}}$ that satisfies Eq. 103 is $O(n^2 \times (\sum_{r=1}^{|\mathcal{R}_{NH}|-1} {}_{|\mathcal{R}_{NH}|-1}\mathrm{P}_r))$.

Example 7 Again, we consider the setting of Example 5. We verify whether $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$ holds by Theorem 5. For this purpose, we construct the nondeterministic acyclic finite automaton $\mathcal{V}_{NH}$ based on the finite automaton V_H.

As in Example 6, we consider any path $p_{\mathcal{V}_{NH}}$ that ends with $\{r_H\}$, where

$$r_H = (q_{K,1}, (q_7, q_d), q_{K,3}, q_d, q_{K,3}, (q_2, q_d), q_{K,2}, q_{K,1}, q_{K,1}, (q_1, q_{K,1})). \tag{113}$$

As shown in Example 6, it holds that $1, 2 \in \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Then, we have $|\widehat{J}_H(p_{\mathcal{V}_{NH}})| \geq 2 = k$, which implies that $p_{\mathcal{V}_{NH}}$ satisfies Eq. 103. By Theorem 5, we can conclude that $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$ holds.

6 Computation of delay bound

When the system G to be diagnosed is (n, k)-reliably 1-inference-diagnosable for a nonempty closed regular sublanguage $K \subseteq L(G)$, there exists $m \in \mathbb{N}$ such that $F_2(m) = \emptyset \vee H_2(m) = \emptyset$. Let $m^* \in \mathbb{N}$ be the such minimum integer, that is,

$$m^* = \min\{m \in \mathbb{N} \mid F_2(m) = \emptyset \vee H_2(m) = \emptyset\}. \tag{114}$$

As shown in Proposition 2, the n local diagnosers $D_i : P_i(L(G)) \rightarrow C \times \mathbb{N}$ $(i = 1, 2, \ldots, n)$ synthesized by Eqs. 19–22 for any $m \in \mathbb{N}$ with $F_2(m) = \emptyset$ or $H_2(m) = \emptyset$ are

(n, k)-1-inferring and satisfy Eqs. 23 and 9. That is, if diagnosis decisions of at least k local diagnosers are available, the occurrence of any failure string can be correctly detected within m steps. Hence, m^* can be considered as the delay bound and can be used to synthesize local diagnosers using Eqs. 19–22. Letting $\mathbb{N}_F = \{m \in \mathbb{N} \mid F_2(m) = \emptyset\}$ and $\mathbb{N}_H = \{m \in \mathbb{N} \mid H_2(m) = \emptyset\}$, it holds that

$$m^* = \begin{cases} \min\{m_F^*, m_H^*\}, & \text{if } \mathbb{N}_F \neq \emptyset \wedge \mathbb{N}_H \neq \emptyset \\ m_F^*, & \text{if } \mathbb{N}_F \neq \emptyset \wedge \mathbb{N}_H = \emptyset \\ m_H^*, & \text{if } \mathbb{N}_F = \emptyset \wedge \mathbb{N}_H \neq \emptyset, \end{cases} \tag{115}$$

where

$$m_F^* = \begin{cases} \min \mathbb{N}_F, & \text{if } \mathbb{N}_F \neq \emptyset \\ \text{undefined}, & \text{otherwise}, \end{cases} \tag{116}$$

$$m_H^* = \begin{cases} \min \mathbb{N}_H, & \text{if } \mathbb{N}_H \neq \emptyset \\ \text{undefined}, & \text{otherwise}. \end{cases} \tag{117}$$

To compute m^*, we develop methods for computing m_F^* and m_H^* when $\mathbb{N}_F \neq \emptyset$ and $\mathbb{N}_H \neq \emptyset$, respectively.

6.1 Computation of m_F^*

To compute m_F^* in the case of $\mathbb{N}_F \neq \emptyset$, we use the nondeterministic acyclic finite automaton $\mathcal{V}_{NF} = (\mathcal{R}_{NF}, \Sigma_V, \delta_{NF}, R_{NF,0})$, which is constructed for verifying whether $\forall m \in \mathbb{N}(F_2(m) \neq \emptyset)$. Let

$$I_F = \{0\} \cup \{ij \mid [i, j \in I] \wedge [i \neq j]\}. \tag{118}$$

Each element of I_F indicates the corresponding component of the finite automaton $V_F = (R_F, \Sigma_V, \delta_F, r_{F,0})$ that traces a failure string. For each $i_F \in I_F$, we introduce a weight of each transition of $\mathcal{V}_{NF}$ by a function $w_{F,i_F} : \mathcal{R}_{NF} \times \Sigma_V \times \mathcal{R}_{NF} \to \{0, 1\}$ defined as

$$w_{F,i_F}(R_{NF}, \sigma_V, R'_{NF}) = \begin{cases} 1, \text{ if } [R'_{NF} \in \delta_{NF}(R_{NF}, \sigma_V)] \wedge [\pi_{i_F}(\sigma_V) \neq \varepsilon] \\ \qquad \wedge [\exists r'_F \in R'_{NF}(\pi_{i_F}(r'_F) \in Q \times \{q_d\})] \\ 0, \text{ otherwise} \end{cases} \tag{119}$$

for each $R_{NF}, R'_{NF} \in \mathcal{R}_{NF}$ and each $\sigma_V \in \Sigma_V$. This weight of a transition of $\mathcal{V}_{NF}$ is used to count the number of occurrences of events after failure. For any path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ ($l \geq 1$) of $\mathcal{V}_{NF}$, its weight with respect to $i_F \in I_F$ is defined as

$$w_{F,i_F}(p_{\mathcal{V}_{NF}}) = \sum_{h=0}^{l-1} w_{F,i_F}(R_{NF,h}, \sigma_{V,h}, R_{NF,h+1}). \tag{120}$$

We consider any index set $I' \in \mathcal{I}^k$. Then, we define the set, denoted by $\Upsilon_{I'}$, of all functions $\upsilon_{I'} : I' \to \mathcal{I}^k$ such that $i \in \upsilon_{I'}(i)$ for any $i \in I'$. For example, in the case of $n = 3$ and $k = 2$, for $I' = \{1, 2\} \in \mathcal{I}^2$, a function $\upsilon_{\{1,2\}} : \{1, 2\} \to \mathcal{I}^2$ such that $\upsilon_{\{1,2\}}(1) = \{1, 2\}$ and $\upsilon_{\{1,2\}}(2) = \{2, 3\}$ is an element of $\Upsilon_{\{1,2\}}$, since $1 \in \upsilon_{\{1,2\}}(1)$ and $2 \in \upsilon_{\{1,2\}}(2)$. We consider any path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ ($l \geq 1$) of $\mathcal{V}_{NF}$. For any pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$, a subset $\Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \subseteq I_F$ whose elements are involved to compute m_F^* is defined as follows:

- $0 \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ if and only if Eq. 75 in Theorem 4 does not hold,
- for any $ij \in I_F - \{0\}$, $ij \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ if and only if $i \in I'$, $j \in \upsilon_{I'}(i) - \{i\}$, and $j \notin J_F(R_{NF,h}, i)$ for any $h \in \{1, 2, \ldots, l\}$, where the function $J_F : \mathcal{R}_{NF} \times I \to 2^I$ is defined by Eq. 73.

Intuitively, for each $i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, the number of occurrences of events after failure can be computed as $w_{F,i_F}(p_{\mathcal{V}_{NF}})$.

The following lemma shows the nonemptyness of $\Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$.

Lemma 1 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \le k \le n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$, we assume that $\mathbb{N}_F \neq \emptyset$. We consider any path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_{V,0}} R_{NF,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NF,l}$ ($l \ge 1$) of the nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$. For any pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$, if*

$$\forall i \in I', \exists r_F \in R_{NF,l}(\pi_{ii}(r_F) \in Q_K), \tag{121}$$

then $\Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \neq \emptyset$.

Proof Since $\mathbb{N}_F \neq \emptyset$, there exists $m \in \mathbb{N}$ such that $F_2(m) = \emptyset$. By Theorem 4, we first consider the case where Eq. 75 does not hold for $p_{\mathcal{V}_{NF}}$. Then, by the definition of $\Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, we have $0 \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \neq \emptyset$. We next consider the case where Eq. 76 does not hold for $p_{\mathcal{V}_{NF}}$. Since $|\widehat{J}_F(p_{\mathcal{V}_{NF}})| < k$, we have $I' - \widehat{J}_F(p_{\mathcal{V}_{NF}}) \neq \emptyset$. We consider any $i \in I' - \widehat{J}_F(p_{\mathcal{V}_{NF}})$. By Eq. 121 and the definition of $\widehat{J}_F(p_{\mathcal{V}_{NF}})$, we have

$$\left| \bigcup_{h \in \{1,2,\ldots,l\}} J_F(R_{NF,h}, i) \right| < k - 1. \tag{122}$$

Therefore, there exists $j \in \upsilon_{I'}(i) - \{i\}$ such that $j \notin J_F(R_{NF,h}, i)$ for any $h \in \{1, 2, \ldots, l\}$. It follows that $ij \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \neq \emptyset$. □

We define the weight $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ of $p_{\mathcal{V}_{NF}}$ with respect to a pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$ as

$$\begin{aligned} & w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \\ &= \begin{cases} \min\{w_{F,i_F}(p_{\mathcal{V}_{NF}}) \in \mathbb{N} \mid i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})\}, & \text{if Eq. 121 holds} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{123}$$

By Lemma 1, $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ is well-defined. Then, we define the weight $w_F(p_{\mathcal{V}_{NF}})$ of $p_{\mathcal{V}_{NF}}$ as

$$w_F(p_{\mathcal{V}_{NF}}) = \max\{w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \in \mathbb{N} \mid I' \in \mathcal{I}^k \wedge \upsilon_{I'} \in \Upsilon_{I'}\}. \tag{124}$$

Finally, letting $Path(\mathcal{V}_{NF})$ be the set of all paths of $\mathcal{V}_{NF}$, we define w_F as the maximum weight

$$w_F = \max\{w_F(p_{\mathcal{V}_{NF}}) \in \mathbb{N} \mid p_{\mathcal{V}_{NF}} \in Path(\mathcal{V}_{NF})\} \tag{125}$$

over $Path(\mathcal{V}_{NF})$. Since $\mathcal{V}_{NF}$ is acyclic, $Path(\mathcal{V}_{NF})$ is finite, which implies that w_F is effectively computable.

The following theorem shows that m_F^* can be computed as $m_F^* = w_F$.

Theorem 6 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$, if $\mathbb{N}_F \neq \emptyset$ then $m_F^* = w_F$.*

Proof First, we prove that $m_F^* \leq w_F$. For the sake of contradiction, we suppose that $w_F < m_F^*$. By the definition of m_F^*, we have $F_2(w_F) \neq \emptyset$. As shown in the proof of the ($\Leftarrow$) part of Theorem 4, there exist $I' \in \mathcal{I}^k$, $I_i'' \in \mathcal{I}^k$ with $i \in I_i''$ for each $i \in I'$, and $s_V \in L(V_F)$ such that

- $\pi_0(s_V) \in F_0(w_F)$,
- $\forall i \in I' : \pi_{ii}(s_V) \in H_0(w_F)$,
- $\forall i \in I', \forall j \in I_i'' - \{i\} : \pi_{ij}(s_V) \in F_0(w_F)$.

Then, we have $s_V \neq \varepsilon$, and s_V can be written as $s_V = \sigma_V^{(0)}\sigma_V^{(1)}\cdots\sigma_V^{(l_F-1)}$ ($l_F \geq 1$). We consider the path $p_{V_F} : r_{F,0} \xrightarrow{\sigma_V^{(0)}} r_{F,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_F-1)}} r_{F,l_F}$ obtained by executing s_V in V_F. For the path p_{V_F}, there exists the path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_V^{(h_{F,0})}} R_{NF,1} \xrightarrow{\sigma_V^{(h_{F,1})}} \cdots \xrightarrow{\sigma_V^{(h_{F,l-1})}} R_{NF,l}$ ($l \geq 1$) in $Path(\mathcal{V}_{NF})$ such that

$$\{r_{F,0}, \ldots, r_{F,h_{F,0}}\} = R_{NF,0}, \tag{126}$$

$$\{r_{F,h_{F,0}+1}, \ldots, r_{F,h_{F,1}}\} = R_{NF,1}, \tag{127}$$

$$\vdots$$

$$\{r_{F,h_{F,l-1}+1}, \ldots, r_{F,l_F}\} = R_{NF,l} \tag{128}$$

for some $h_{F,0}, h_{F,1}, \ldots, h_{F,l-1} \in \{0, 1, \ldots, l_F - 1\}$ such that $0 \leq h_{F,0} < h_{F,1} < \cdots < h_{F,l-1} < l_F$. For I', we consider the function $\upsilon_{I'} : I' \to \mathcal{I}^k$ such that $\upsilon_{I'}(i) = I_i''$ for each $i \in I'$. Since $i \in \upsilon_{I'}(i)$ for each $i \in I'$, we have $\upsilon_{I'} \in \Upsilon_{I'}$. For each $i \in I'$, since $\pi_{ii}(s_V) \in H_0(w_F) = K$, we have $\pi_{ii}(r_{F,l_F}) = \widetilde{\delta}_K(q_{K,0}, \pi_{ii}(s_V)) \in Q_K$ for $\delta_F(r_{F,0}, s_V) = r_{F,l_F} \in R_{NF,l}$. It follows that Eq. 121 holds.

By the definition of w_F and $w_F(p_{\mathcal{V}_{NF}})$, we have

$$w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \leq w_F(p_{\mathcal{V}_{NF}}) \leq w_F. \tag{129}$$

Since Eq. 121 holds, by the definition of $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, there exists $i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ such that $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) = w_{F,i_F}(p_{\mathcal{V}_{NF}})$. By $i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, we have $\pi_{i_F}(r_{F,h_{i_F}}) \notin Q \times \{q_d\}$, $\pi_{i_F}(r_{F,h_{i_F}+1}) \in Q \times \{q_d\}$, and

$$|\pi_{i_F}(\sigma_V^{(h_{i_F})}\sigma_V^{(h_{i_F}+1)}\cdots\sigma_V^{(l_F-1)})| = w_{F,i_F}(p_{\mathcal{V}_{NF}}) \tag{130}$$

for some $h_{i_F} \in \{0, 1, \ldots, l_F - 1\}$. It follows that $\pi_{i_F}(s_V) \in K\Sigma^{\leq w_{F,i_F}(p_{\mathcal{V}_{NF}})} \subseteq K\Sigma^{\leq w_F}$. Besides, since $i_F = 0$ or $i_F = ij$ with $i \in I'$ and $j \in I_i'' - \{i\}$, we have $\pi_{i_F}(s_V) \in F_0(w_F) = L(G) \cap (L(G) - K)\Sigma^{\geq w_F}$, which contradicts $\pi_{i_F}(s_V) \in K\Sigma^{\leq w_F}$. Thus, we have $m_F^* \leq w_F$.

We next prove that $m_F^* \geq w_F$. For the sake of contradiction, we suppose that $m_F^* < w_F$. By the definition of w_F, there exists a path $p_{\mathcal{V}_{NF}} : R_{NF,0} \xrightarrow{\sigma_V^{(h_{F,0})}} R_{NF,1} \xrightarrow{\sigma_V^{(h_{F,1})}} \cdots \xrightarrow{\sigma_V^{(h_{F,l-1})}} R_{NF,l}$ ($l \geq 1$) in $Path(\mathcal{V}_{NF})$ such that $w_F = w_F(p_{\mathcal{V}_{NF}})$. In addition, by the definition of

$w_F(p_{\mathcal{V}_{NF}})$, there exist $I' \in \mathcal{I}^k$ and $\upsilon_{I'} \in \Upsilon_{I'}$ such that $w_F(p_{\mathcal{V}_{NF}}) = w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$. Since $0 \le m_F^* < w_F$, by the definition of $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, we have Eq. 121 and

$$w_F = w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}}) \le w_{F,i_F}(p_{\mathcal{V}_{NF}}) \tag{131}$$

for any $i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$. Besides, for any $i_F \in (\{0\} \cup \{ij \mid i \in I' \wedge j \in \upsilon_{I'}(i) - \{i\}\}) - \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, there exists $h_{i_F} \in \{1, 2, \ldots, l\}$ such that

$$\begin{aligned}&[\exists r_F \in R_{NF,h_{i_F}} : \pi_{i_F}(r_F) \in Q \times \{q_d\}]\\ &\wedge[\exists r_F, r'_F \in R_{NF,h_{i_F}}, \exists \sigma_V \in \Sigma_V : \delta_F(r_F, \sigma_V) = r'_F \wedge \pi_{i_F}(\sigma_V) \neq \varepsilon].\end{aligned} \tag{132}$$

Then, there exists a path $p_{V_F} : r_{F,0} \xrightarrow{\sigma_V^{(0)}} r_{F,1} \xrightarrow{\sigma_V^{(1)}} \cdots \xrightarrow{\sigma_V^{(l_F-1)}} r_{F,l_F}$ of V_F such that, for some $\widehat{h}_{F,0}, \widehat{h}_{F,1}, \ldots, \widehat{h}_{F,l-1} \in \{0, 1, \ldots, l_F - 1\}$ with $\widehat{h}_{F,0} < \widehat{h}_{F,1} < \cdots < \widehat{h}_{F,l-1}$,

$$\{r_{F,0}, \ldots, r_{F,\widehat{h}_{F,0}}\} = R_{NF,0}, \tag{133}$$

$$\{r_{F,\widehat{h}_{F,0}+1}, \ldots, r_{F,\widehat{h}_{F,1}}\} = R_{NF,1}, \tag{134}$$

$$\vdots$$

$$\{r_{F,\widehat{h}_{F,l-1}+1}, \ldots, r_{F,l_F}\} = R_{NF,l}, \tag{135}$$

$\sigma_V^{(\widehat{h}_{F,p})} = \sigma_V^{(h_{F,p})}$ for each $p \in \{0, 1, \ldots, l-1\}$, and

$$\begin{aligned}&\pi_{i_F}(r_{F,\widehat{h}_{F,h_{i_F}-1}+1}) \in Q \times \{q_d\}\\ &\wedge|\pi_{i_F}(\sigma_V^{(\widehat{h}_{F,h_{i_F}-1}+1)}\sigma_V^{(\widehat{h}_{F,h_{i_F}-1}+2)}\cdots\sigma_V^{(l_F-1)})| \ge w_F - 1\end{aligned} \tag{136}$$

for each $i_F \in (\{0\} \cup \{ij \mid i \in I' \wedge j \in \upsilon_{I'}(i) - \{i\}\}) - \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$. In addition, for each $i_F \in \Psi_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$, since $w_F \le w_{F,i_F}(p_{\mathcal{V}_{NF}})$, there exists $h_{i_F} \in \{1, 2, \ldots, l\}$ such that Eq. 136 holds. Let $s_V = \sigma_V^{(0)}\sigma_V^{(1)}\ldots\sigma_V^{(l_F-1)}$. Then, we have $\pi_0(s_V) \in L(G) \cap (L(G) - K)\Sigma^{\ge w_F - 1} \subseteq L(G) \cap (L(G) - K)\Sigma^{\ge m_F^*} = F_0(m_F^*)$ and $\pi_{ij}(s_V) \in L(G) \cap (L(G) - K)\Sigma^{\ge w_F - 1} \subseteq L(G) \cap (L(G) - K)\Sigma^{\ge m_F^*} = F_0(m_F^*)$ for any $i \in I'$ and any $j \in \upsilon_{I'}(i) - \{i\}$. In addition, for any $i \in I'$, by Eq. 121, we have $\widetilde{\delta}(q_{K,0}, \pi_{ii}(s_V)) = \pi_{ii}(r_{F,l_F}) \in Q_K$, which implies $\pi_{ii}(s_V) \in K = H_0(m_F^*)$. Since $P_i(\pi_0(s_V)) = P_i(\pi_{ii}(s_V))$ for each $i \in I'$ and $P_j(\pi_{ii}(s_V)) = P_j(\pi_{ij}(s_V))$ for each $i \in I'$ and each $j \in \upsilon_{I'}(i) - \{i\}$, we have $\pi_0(s_V) \in F_2(m_F^*)$, which contradicts $F_2(m_F^*) = \emptyset$. Thus, it holds that $m_F^* \ge w_F$. $\square$

Remark 9 Since the number of transitions of a path in $Path(\mathcal{V}_{NF})$ is at most $|\mathcal{R}_{NF}| - 1$, by Theorem 6, we have $m_F^* \le |\mathcal{R}_{NF}| - 1$.

Example 8 We consider the setting of Example 1. A part of the nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$ is shown in Fig. 11. Let $p_{\mathcal{V}_{NF}}$ denote the path in $Path(\mathcal{V}_{NF})$ shown in Fig. 11.

We consider an index set $\{1, 2, 3\} \in \mathcal{I}^3$ and a function $\upsilon_{\{1,2,3\}} : \{1, 2, 3\} \to \mathcal{I}^3$ such that $\upsilon_{\{1,2,3\}}(i) = \{1, 2, 3\}$ for each $i \in \{1, 2, 3\}$. It follows that $\upsilon_{\{1,2,3\}} \in \Upsilon_{\{1,2,3\}}$. As shown in Fig. 11, a singleton state

$$\begin{aligned}&\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), q_{K,0}, (q_1, q_d), (q_0, q_{K,0}),\\ &\quad (q_1, q_d), (q_1, q_d), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\} \in \mathcal{R}_{NF},\end{aligned}$$

where $q_1 = \delta(q_0, f)$, is reached by the path $p_{\mathcal{V}_{NF}}$. Let $\{r_F\}$ denote this singleton state. Since $\pi_{ii}(r_F) = q_{K,0} \in Q_K$ for any $i \in \{1, 2, 3\}$, Eq. 121 holds. For $\{1, 2, 3\}$ and $\upsilon_{\{1,2,3\}}$, we have

$$\Psi_{F,(\{1,2,3\},\upsilon_{\{1,2,3\}})}(p_{\mathcal{V}_{NF}}) = \{0, 12, 13, 21, 23, 31, 32\}. \tag{137}$$

It holds that, for each $i_F \in \Psi_{F,(\{1,2,3\},\upsilon_{\{1,2,3\}})}(p_{\mathcal{V}_{NF}})$, $w_{F,i_F}(p_{\mathcal{V}_{NF}}) = 1$. Then, we have $w_{F,(\{1,2,3\},\upsilon_{\{1,2,3\}})}(p_{\mathcal{V}_{NF}}) = 1$. By computing $w_{F,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NF}})$ for all pairs $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^3$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$, we obtain $w_F(p_{\mathcal{V}_{NF}}) = 1$. Finally, by examining all paths in $Path(\mathcal{V}_{NF})$, we have $w_F^* = 1$.

6.2 Computation of m_H^*

To compute m_H^* in the case of $\mathbb{N}_H \neq \emptyset$, we use the nondeterministic acyclic finite automaton $\mathcal{V}_{NH} = (\mathcal{R}_{NH}, \Sigma_V, \delta_{NH}, R_{NH,0})$, which is constructed for verifying whether $\forall m \in \mathbb{N}(H_2(m) \neq \emptyset)$. Let

$$I_H = \{ii \mid i \in I\}. \tag{138}$$

$$\{((q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), q_{K,0}, (q_1, q_d), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), q_{K,0}, (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), (q_0, q_{K,0}), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

$$\Big\downarrow (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, f, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

$$\{((q_1, q_d), q_{K,0}, (q_1, q_d), (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), q_{K,0}, (q_1, q_d), (q_0, q_{K,0}), (q_1, q_d), (q_1, q_d), q_{K,0}, (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), (q_0, q_{K,0}), q_{K,0})\}$$

Fig. 11 A part of nondeterministic acyclic finite automaton $\mathcal{V}_{NF}$ for Example 8

Each element of I_H indicates the corresponding component of the finite automaton $V_H = (R_H, \Sigma_V, \delta_H, r_{H,0})$ that traces a failure string. For each $i_H \in I_H$, we introduce a weight of each transition of $\mathcal{V}_{NH}$ by a function $w_{H,i_H} : \mathcal{R}_{NH} \times \Sigma_V \times \mathcal{R}_{NH} \to \{0, 1\}$ defined as

$$w_{H,i_H}(R_{NH}, \sigma_V, R'_{NH}) = \begin{cases} 1, \text{ if } [R'_{NH} \in \delta_{NH}(R_{NH}, \sigma_V)] \wedge [\pi_{i_H}(\sigma_V) \neq \varepsilon] \\ \quad \wedge [\exists r'_H \in R'_{NH}(\pi_{i_H}(r'_H) \in Q \times \{q_d\})] \\ 0, \text{ otherwise} \end{cases} \tag{139}$$

for each $R_{NH}, R'_{NH} \in \mathcal{R}_{NH}$ and each $\sigma_V \in \Sigma_V$. This weight of a transition of $\mathcal{V}_{NH}$ is used to count the number of occurrences of events after failure. We consider any path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ of $\mathcal{V}_{NH}$. Its weight with respect to $i_H \in I_H$ is defined as

$$w_{H,i_H}(p_{\mathcal{V}_{NH}}) = \sum_{h=0}^{l-1} w_{H,i_H}(R_{NH,h}, \sigma_{V,h}, R_{NH,h+1}). \tag{140}$$

For any pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$, a subset $\Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) \subseteq I_H$ whose elements are involved to compute m^*_H is defined as

$$\Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) = \{ii \in I_H \mid \forall h \in \{1, 2, \ldots, l\}(i \notin J_H(R_{NH,h}))\}, \tag{141}$$

where the function $J_H : \mathcal{R}_{NH} \to 2^I$ is defined by Eq. 101. Intuitively, for each $i_H \in \Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}})$, the number of occurrences of events after failure can be computed as $w_{H,i_H}(p_{\mathcal{V}_{NH}})$.

The following lemma shows the nonemptyness of $\Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}})$.

Lemma 2 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \leq k \leq n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$, we assume that $\mathbb{N}_H \neq \emptyset$. We consider any path $p_{\mathcal{V}_{NH}} : R_{NH,0} \xrightarrow{\sigma_{V,0}} R_{NH,1} \xrightarrow{\sigma_{V,1}} \cdots \xrightarrow{\sigma_{V,l-1}} R_{NH,l}$ $(l \geq 1)$ of the nondeterministic acyclic finite automaton $\mathcal{V}_{NH}$. For any pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$, if*

$$\forall i \in I', \forall j \in \upsilon_{I'} - \{i\}[\exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)], \tag{142}$$

then $\Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) \neq \emptyset$.

Proof Since $\mathbb{N}_H \neq \emptyset$, there exists $m \in \mathbb{N}$ such that $H_2(m) = \emptyset$. By Theorem 5, Eq. 103 does not hold for $p_{\mathcal{V}_{NH}}$. Since $|\widehat{J}_H(p_{\mathcal{V}_{NH}})| < k$, we have $I' - \widehat{J}_H(p_{\mathcal{V}_{NH}}) \neq \emptyset$. We consider any $i \in I' - \widehat{J}_H(p_{\mathcal{V}_{NH}})$. Since $|\upsilon'_I(i) - \{i\}| = k - 1$, by Eq. 142, we have

$$|\{j \in I - \{i\} \mid \exists r_H \in R_{NH,l}(\pi_{ij}(r_H) \in Q_K)\}| \geq k - 1. \tag{143}$$

By the definition of $\widehat{J}_H(p_{\mathcal{V}_{NH}})$, we have $i \notin J_H(R_{NH,h})$ for any $h \in \{1, 2, \ldots, l\}$. It follows that $ii \in \Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) \neq \emptyset$. □

We define the weight $w_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}})$ of $p_{\mathcal{V}_{NH}}$ with respect to a pair $(I', \upsilon_{I'})$ of an index set $I' \in \mathcal{I}^k$ and a function $\upsilon_{I'} \in \Upsilon_{I'}$ as

$$w_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) = \begin{cases} \min\{w_{H,i_H}(p_{\mathcal{V}_{NH}}) \in \mathbb{N} \mid i_H \in \Psi_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}})\}, & \text{if Eq. 142 holds} \\ 0, & \text{otherwise.} \end{cases} \tag{144}$$

By Lemma 2, $w_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}})$ is well-defined. Then, we define the weight $w_H(p_{\mathcal{V}_{NH}})$ of $p_{\mathcal{V}_{NH}}$ as

$$w_H(p_{\mathcal{V}_{NH}}) = \max\{w_{H,(I',\upsilon_{I'})}(p_{\mathcal{V}_{NH}}) \in \mathbb{N} \mid I' \in \mathcal{I}^k \wedge \upsilon_{I'} \in \Upsilon_{I'}\}. \tag{145}$$

Finally, letting $Path(\mathcal{V}_{NH})$ be the set of all paths of $\mathcal{V}_{NH}$, we define w_H as the maximum weight

$$w_H = \max\{w_H(p_{\mathcal{V}_{NH}}) \in \mathbb{N} \mid p_{\mathcal{V}_{NH}} \in Path(\mathcal{V}_{NH})\} \tag{146}$$

over $Path(\mathcal{V}_{NH})$. Since $\mathcal{V}_{NH}$ is acyclic, $Path(\mathcal{V}_{NH})$ is finite, which implies that w_H is effectively computable.

The following theorem, which can be proved in a similar way to Theorem 6, shows that m_H^* can be computed as $m_H^* = w_H$.

Theorem 7 *Let $k \in \mathbb{N}$ be a nonnegative integer such that $2 \le k \le n$. For a nonempty closed regular sublanguage $K \subseteq L(G)$ generated by a finite automaton $G_K = (Q_K, \Sigma, \delta_K, q_{K,0})$, if $\mathbb{N}_H \neq \emptyset$ then $m_H^* = w_H$.*

Remark 10 Since the number of transitions of a path in $Path(\mathcal{V}_{NH})$ is at most $|\mathcal{R}_{NH}| - 1$, by Theorem 7, we have $m_H^* \le |\mathcal{R}_{NH}| - 1$.

7 Conclusion

We considered the reliable decentralized diagnosis problem for DESs. We introduced a notion of reliable 1-inference-diagnosability and showed that reliable 1-inference-diagnosability is a necessary and sufficient condition for the existence of local diagnosers that solve the reliable decentralized diagnosis problem using single-level inference. We presented a method for effectively verifying reliable 1-inference-diagnosability. Moreover, we computed the delay bound within which the occurrence of any failure string can be detected. The computed delay bound is used to synthesize local diagnosers. Reliable decentralized diagnosis using multi-level inference is a subject of future work.

Funding Open Access funding provided by Osaka University.

Declarations

Conflicts of interest The authors have no conflict of interest to declare that are relevant to this article.

References

Basilio JC, Lafortune S (2009) Robust codiagnosability of discrete event systems. In: Proceedings of the 2009 American control conference. St. Louis, MO, USA, pp 2202–2209

Cassandras CG, Lafortune S (2021) Introduction to discrete event systems, 3rd edn. Springer
Cassez F (2012) The complexity of codiagnosability for discrete event and timed systems. IEEE Trans Autom Control 57(7):1752–1764
Chakib H, Khoumsi A (2012) Multi-decision diagnosis: decentralized architectures cooperating for diagnosing the presence of faults in discrete event systems. Discrete Event Dyn Syst 22(3):333–380
Cormen TH, Leiserson CE, Rivest RL (1990) Introduction to algorithms. MIT Press
Debouk R, Lafortune S, Teneketzis D (2000) Coordinated decentralized protocols for failure diagnosis of discrete event systems. Discrete Event Dyn Syst 10(1&2):33–86
Hamada T, Takai S (2022a) Reliable diagnosability for decentralized diagnosis of discrete event systems with single-level inference. In: Proceedings of the 2022 American control conference. Atlanta, GA, USA, pp 3746–3751
Hamada T, Takai S (2022b) Verification of reliable inference-diagnosability for decentralized diagnosis with single-level inference. In: Proceedings of the 16th international workshop on discrete event systems. Prague, Czechia, pp 244–249
Jiang S, Huang Z, Chandra V, Kumar R (2001) A polynomial algorithm for testing diagnosability of discrete-event systems. IEEE Trans Autom Control 46(8):1318–1321
Khoumsi A (2020) Arborescent architecture for decentralized diagnosis of discrete event systems. Discrete Event Dyn Syst 30(2):335–375
Kumar R, Takai S (2007) Inference-based ambiguity management in decentralized decision-making: Decentralized control of discrete event systems. IEEE Trans Autom Control 52(10):1783–1794
Kumar R, Takai S (2009) Inference-based ambiguity management in decentralized decision-making: decentralized diagnosis of discrete-event systems. IEEE Trans Autom Sci Eng 6(3):479–491
Liu F, Lin H (2010) Reliable supervisory control for general architecture of decentralized discrete event systems. Automatica 46(9):1510–1516
Nakata S, Takai S (2013) Reliable decentralized failure diagnosis of discrete event systems. SICE J Control Meas Syst Integr 6(5):353–359
Qiu W, Kumar R (2006) Decentralized failure diagnosis of discrete event systems. IEEE Trans Syst Man Cybern Part A: Syst Human 36(2):384–395
Ramadge PJ, Wonham WM (1987) Supervisory control of a class of discrete event processes. SIAM J Control Optim 25(1):206–230
Ricker SL, Rudie K (2007) Knowledge is a terrible thing to waste: using inference in discrete-event control problems. IEEE Trans Autom Control 52(3):428–441
Sampath M, Sengupta R, Lafortune S, Sinnamohideen K, Teneketzis D (1995) Diagnosability of discrete-event systems. IEEE Trans Autom Control 40(9):1555–1575
Su R, Wonham WM (2005) Global and local consistencies in distributed fault diagnosis for discrete-event systems. IEEE Trans Autom Control 50(12):1923–1935
Takai S, Kumar R (2017) A generalized framework for inference-based diagnosis of discrete event systems capturing both disjunctive and conjunctive decision-making. IEEE Trans Autom Control 62(6):2778–2793
Takai S, Kumar R (2018) Implementation of inference-based diagnosis: computing delay bound and ambiguity levels. Discrete Event Dyn Syst 28(2):315–348
Takai S, Ushio T (2000) Reliable decentralized supervisory control of discrete event systems. IEEE Trans Syst Man Cybern Part B: Cybern 30(5):661–667
Takai S, Ushio T (2003) Reliable decentralized supervisory control of discrete event systems with the conjunctive and disjunctive fusion rules. IEICE Trans Fundam E86-A(11):2731–2738
Takai S, Yoshida S (2022) Reliable decentralized supervisory control of discrete event systems with single-level inference. IEICE Trans Fundam E105-A(5):799–807
Viana GS, Basilio JC (2019) Codiagnosability of discrete event systems revisited: a new necessary and sufficient condition and its applications. Automatica 101:354–364
Wang Y, Yoo T-S, Lafortune S (2007) Diagnosis of discrete event systems using decentralized architectures. Discrete Event Dyn Syst 17(2):233–263
Wang W, Girard AR, Lafortune S, Lin F (2011) On codiagnosability and coobservability with dynamic observations. IEEE Trans Autom Control 56(7):1551–1566
Yamamoto T, Takai S (2014) Reliable decentralized diagnosis of discrete event systems using the conjunctive architecture. IEICE Trans Fundam E97-A(7):1605–1614
Yin X, Lafortune S (2015) Codiagnosability and coobservability under dynamic observations: transformation and verification. Automatica 61:241–252
Yin X, Li Z (2016) Reliable decentralized fault prognosis of discrete-event systems. IEEE Trans Syst Man Cybern: Syst 46(11):1598–1603

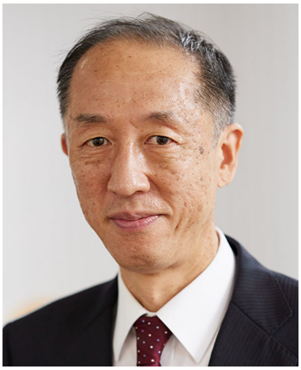
Yokota S, Yamamoto T, Takai S (2017) Computation of the delay bounds and synthesis of diagnosers for decentralized diagnosis with conditional decisions. Discrete Event Dyn Syst 27(1):45–84

Yoo T-S, Garcia HE (2008) Diagnosis of behaviors of interest in partially-observed discrete-event systems. Syst Control Lett 57(12):1023–1029

Yoo T-S, Lafortune S (2002) Polynomial-time verification of diagnosability of partially observed discrete-event systems. IEEE Trans Autom Control 47(9):1491–1495

Yoo T-S, Lafortune S (2004) Decentralized supervisory control with conditional decisions: Supervisor existence. IEEE Trans Autom Control 49(11):1886–1904

Takumi Hamada received the B.E. and M.E. degrees in 2019 and 2021, respectively, from Osaka University, Suita, Osaka, Japan. His research interests include fault diagnosis of discrete event systems.



Shigemasa Takai received the B.E. and M.E. degrees from Kobe University, Kobe, Japan, in 1989 and 1991, respectively, and the Ph.D degree from Osaka University, Suita, Japan, in 1995. From 1992 to 1998, he was a Research Associate at Osaka University. In 1998, he joined Wakayama University, Wakayama, Japan, as a Lecturer, and became an Associate Professor in 1999. From 2004 to 2009, he was an Associate Professor at Kyoto Institute of Technology, Kyoto, Japan. Since 2009, he has been a Professor at Osaka University, Suita, Osaka, Japan. His research interests include supervisory control and fault diagnosis of discrete event systems.