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MEASURED HAUSDORFF CONVERGENCE OF RIEMANNIAN MANIFOLDS AND LAPLACE OPERATORS

dedicated to Prof. Hideki Ozeki on his 60th birthday

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0. Introduction

K. Fukaya introduced in [11] a topology on a set of metric spaces equipped with Borel measures, called the measured Hausdorff topology and discussed the continuity of the eigenvalues of the Laplace operators of Riemannian manifolds with uniformly bounded curvature. The purpose of the present paper is to study more closely the Laplace operators of Riemannian manifolds which collapse in this topology to a space of lower dimension while keeping their curvature bounded.

0.1. According to [16], a map $h: X \to Y$ of metric spaces is said to be an ε-Hausdorff approximation if $|\mathrm{dis}(x, x')-\mathrm{dis}(h(x), h(x'))| \leqq \varepsilon$ for all $x, x' \in X$ and the ε-neighborhood of the image $h(X)$ coincides with Y. A sequence of compact metric spaces $\{X_i\}$ converges, by definition, to a compact metric space Y in the Hausdorff distance if there are a sequence of positive numbers $\{\varepsilon(i)\}$ going to zero as i tends to infinity and $\varepsilon(i)$-Hausdorff approximations $h_i: X_i \to Y$ of X_i into Y. Moreovre when each metric space X_i is equipped with a Borel measure μ_i of unit mass, according to [11], we say that $\{(X_i, \mu_i)\}$ converges to Y with a Borel measure μ_∞ of unit mass in the measured Hausdorff topology, if in addition, these maps $h_i: X_i \to Y$ are Borel measurable and the push-forward measure $h_{i*}\mu_i$ converges to μ_∞ in the weak* topology.

Now we shall consider a sequence of compact Riemannian manifolds $\{(M_i, g_i)\}$ of dimension m whose sectional curvature K_{M_i} is bounded uniformly in its absolute value by a constant, say 1, and assume that tihs sequence converges to a compact metric space M_∞ in the Hausdorff distance. When the volume of M_i is bounded uniformly away from zero by a positive constant, Gromov's

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convergence theorem says that the Lipschitz distance between M_i and M_∞ actually goes to zero as i tends to infinity (cf. [16]). Moreover, we see that M_∞ is a smooth manifold with a metric g_∞ of class $C^{1,\alpha}$ $(0<\alpha<1)$ and in fact of the Sobolev spaces $W^{2,p}\,(1<p<\infty)$), and there exists a diffeomorphism $f_i: M_i \to M_\infty$ (for large i) such that $f_{i*}(g_i)$ converges to g_∞ with respect to the $C^{1,\alpha}$ topology (cf. [27], [15], [21], and also [3]). In this case, we will be able to say that the Laplace operator of M_i is sufficiently close to that of M_∞ for large i. We remark that harmonic coordinates with certain properties (cf. §1.1) play an important role in a proof of this convergence theorem.

On the other hand, when the volume of M_i goes to zero as i tends to infinity, the geometric situation is more complicated. We recall here some parts of the fibration theorem by Fukaya [10, 12, 13]. We first mention that there exists a smooth manifold F_∞ with a metric of class $C^{1,\alpha}$ (and in fact of the Sobolev spaces $W^{2,p}$) on which the orthogonal group $O(m)$ acts by isometries in such a way that M_∞ is isometric to the quotient space $F_\infty/O(m)$. Secondly if M_∞ becomes a smooth manifold of positive dimension n $(<m)$, then there exists a fibration $f_i: M_i \to M_\infty$ (for large i) such that (i) for all $z \in M_\infty$, the diameter of $f_i^{-1}(z) \leqq \varepsilon(i)$; (ii) f_i is an $\varepsilon(i)$-almost Riemannian submersion, that is, for all $z \in M_\infty$, $x \in f_i^{-1}(z)$ and $X \in T_x M_i$ normal to $f_i^{-1}(z)$, $(1-\varepsilon(i))\,|df_i(X)| \leqq |X| \leqq (1+\varepsilon(i))\,|df_i(X)|$; (iii) the second fundamental form Ddf_i of f_i is bounded uniformly in i, namely, $|Ddf_i| \leqq C$, where $\{\varepsilon(i)\}$ is a sequence of positive numbers going to zero as i tends to infinity, and C is a posotive constant independent of i. See also a recent paper of Cheeger, Fukaya and Gromov [6: §2] for a new proof of the fibration theorem. In particular, we refer to it for the last assertion. At this stage, we equip each manifold M_i with the canonical probability measure μ_i (namely, μ_i=the volume element divided by the volume), and we assume in addition that $\{M_i\}$ converges to M_∞ with a Borel measure μ_∞ with respect to the measured Hausdorff topology. Then we may assume that the push-forward measure $f_{i*}(\mu_i)$ of μ_i by the above fibration f_i converges weakly to μ_∞ as i tends to infinity. According to Fukaya [11, 13], the Laplace operator Δ_i (acting on functions) of M_i is closely related to an operator $\mathcal{L}_\infty$ on M_∞ defined by

$$\mathcal{L}_\infty h = \frac{1}{\chi_\infty}\operatorname{div}(\chi_\infty \operatorname{grad} h) = \Delta_\infty h + \operatorname{grad}(\log \chi_\infty) h\,,$$

where χ_∞ denotes the density of the limit measure μ_∞ which turns out to be a positive function of class $C^{1,\alpha}$ and in fact of the Sobolev spaces $W^{2,p}$ (cf. §1.4 of this paper). A main result in [11] says that for all $k=1, 2, \cdots$, the k-th eigenvalue of Δ_i conevrges to that of $\mathcal{L}_\infty$ as i goes to infinity. See also [13]. In this paper, we want to study more closely relationships between the Laplace operator of M_i and the operator $\mathcal{L}_\infty$ on M_∞.

0.2. Let us now state our main results.

Theorem A. *Let $\{M_i\}_{i=1,2,\cdots}$ be a sequence of m-dimensional compact Riemannian manifolds the sectioanl curvature of which is bounded in its absolute value by* 1. *Suppose M_i with the canonical probability measure μ_i converges to a compact metric space M_∞ with a Borel measure μ_∞ of unit mass with respect to the measured Hausdorff topology, and further that M_∞ is a smooth manifold of dimension n $(0<n<m)$ (with a metric of class $C^{1,\alpha}$ (any $\alpha\in(0,1)$). Then for large i, there exists a fibration $\Phi_i\colon M_i\to M_\infty$ with the following properties:*

(I) *for all $x\in M_i$,*

$$\operatorname{dis}(f_i(x),\ \Phi_i(x))\leqq\varepsilon(i)\,,$$

where $f_i\colon M_i\to M_\infty$ is the fibration mentioned as above and $\{\varepsilon(i)\}$ is a sequence of positive constants which tends to zero as i goes to infinity;

(II) *given $\alpha\in[0,1)$, there exist a sequence $\{\varepsilon'(i)\}$ of positive constants tending to zero as i goes to infinity and a positive constant C depending only on α, m, n and M_∞ such that for all $h\in C^\infty(M_\infty)$,*

$$\text{(i)}\qquad (1-\varepsilon'(i))\Phi_i^*(|dh|)\leqq|d\Phi_i^*(h)|\leqq(1+\varepsilon'(i))\Phi_i^*(|dh|)\,,$$
$$(1-\varepsilon'(i))\Phi_i^*(|Ddh|)-\varepsilon'(i)\Phi_i^*(|dh|)$$
$$\leqq|Dd\Phi_i^*(h)|\leqq(1+\varepsilon'(i))\Phi_i^*(|Ddh|)+C\Phi_i^*(|dh|)$$

on M_i,

$$\text{(ii)}\qquad |\Phi_i^*(h)|_{C^{k,\alpha}(M_i)}\leqq C|h|_{C^{k,\alpha}(M_\infty)}\ (k=0,1,2)\,,$$

$$\text{(iii)}\qquad |\Delta_i\Phi_i^*(h)-\Phi_i^*(\mathcal{L}_\infty h)|\leqq\varepsilon'(i)\Phi_i^*(|Ddh|+|dh|)\,,$$

on M_i, where Δ_i stands for the Laplace operator of M_i and $\mathcal{L}_\infty$ is the operator defined in 0.1.

We remark that the constant C in this theorem actually can be chosen in such a way that it depends only on α, m, n, the "curvature" (in generalized sense) of M_∞(cf. Fact 1.5 in §1.3) and the injectivity radius of M_∞.

Let M_i, M_∞ and Φ_i be as in Theorem A. For $h\in C^\infty(M_i)$, we define a function $\Theta_i(h)$ on M_∞ by

$$\Theta_i(h)(z)=\frac{1}{\operatorname{Vol}(\Phi_i^{-1}(z))}\int_{\Phi_i^{-1}(z)}h$$

$(z\in M_\infty)$. Then we have the following

Theorem B. *Let M_i, Δ_i, M_∞, $\mathcal{L}_\infty$, Φ_i and $\Theta_i\colon C^\infty(M_i)\to C^\infty(M_\infty)$ be as above. Given $\alpha\in(0,1)$ and $p\in(1,\infty)$, there exists a sequence $\{\varepsilon(i)\}$ of positive constants tending to zero as i goes to infinity such that for large i and all $h\in C^\infty(M_i)$,*

$$\text{(i)}\qquad |\Theta_i(h)|_{C^1(M_\infty)}\leqq(1+\varepsilon(i))|h|_{C^1(M_i)}\,,$$

(ii) $\|\Theta_i(h)\|_{W^{2,p}(M_\infty,\mu_\infty)} \leqq (1+\varepsilon(i))\|h\|_{W^{2,p}(M_i,\mu_i)} + \varepsilon(i)|h|_{C^{1,\alpha}(M_i)}$,

(iii) $|\mathcal{L}_\infty\Theta_i(h) - \Theta_i(\Delta_i h)| \leqq \varepsilon(i)\{\Theta_i(|Ddh|) + (1+\Theta_i(|D\eta_i|))|h|_{C^{1,\alpha}(M_i)}\}$,

on M_∞, *where* η_i *denotes the mean curvature vector field along the fibers of* Φ_i *on* M_i,

(iv) $\|\mathcal{L}_\infty\Theta_i(h) - \Theta_i(\Delta_i h)\|_{L^p(M_\infty,\mu_\infty)} \leqq \varepsilon(i)(\|Ddh\|_{L^p(M_i,\mu_i)} + |h|_{C^{1,\alpha}(M_i)})$,

(v) $|h - \Phi_i^* \circ \Theta_i(h)|_{C^0(M_i)} \leqq \varepsilon(i)|h|_{C^{0,\alpha}(M_i)}$,

(vi) $|h - \Phi_i^* \circ \Theta_i(h)|_{C^1(M_i)} \leqq \varepsilon(i)|h|_{C^{1,\alpha}(M_i)}$,

(vii) $\|h - \Phi_i^* \circ \Theta_i(h)\|_{W^{2,p}(M_i,\mu_i)} \leqq \varepsilon(i)(|h|_{C^0(M_i)} + |\Delta_i h|_{C^{0,\alpha}(M_i)})$.

We should here explain the notations used in this theorem. For a smooth function h on a Riemannian manifold M with a Borel measure μ, we set

$$\|h\|_{L^p(M,\mu)} := \left[\int_M |h|^p \, d\mu\right]^{1/p} \; (<p<\infty);$$

$$\|h\|_{W^{k,p}(M,\mu)} := \left[\int_M \sum_{j=0}^k |D^j h|^p \, d\mu\right]^{1/p}.$$

When μ is the Riemannian voluem element of M, we write simply $\|h\|_{W^{k,p}(M)}$ for $\|h\|_{W^{k,p}(M,\mu)}$.

Theorems A and B will be proved, respectively, in Sections 2 and 3, after some preliminaries in Section 1. As mentioned before, in case the injectivity radii of the given manifolds in the theorems are bounded away from zero uniformly, we can cover each of them with harmonic coordinates of certain uniform geometric estimates. This fact has proved useful as the literature shows. However this is not the case in our theorems. The main idea of proving them is to construct local fibrations by harmonic functions with geometric data, instead of harmonic coordinate systems or local diffeomorphisms by harmonic functions.

0.3. Let $\{M_i\}$ be a sequence of compact Riemannian manifold as in Theorem A which converges to a metric space M_∞ in the Hausdorff distance. According to [12], we consider the frame bundle FM_i of M_i equipped with the canonical Riemannian metric so that the fibers of FM_i are totally geodesic and the sectional curvature of FM_i remains bounded by a constant depending only on m. Let F_∞ be a metric space to which a subsequence of $\{FM_i\}$, denoted again by $\{FM_i\}$, converges with respect to the Hausdorff distance. Then F_∞ turns out to be a smooth manifold with a metric of class $C^{1,\alpha}$ and the orthogonal group $O(m)$ acts on F_∞ as isometries in such a way that M_∞ is isometric to the quotient space $F_\infty/(O(m)$. Moreover there exists an $O(m)$ equivariant fibration $\tilde{f}_i: FM_i \to F_\infty$ with the same properties as the fibration $f_i: M_i \to M_\infty$ mentioned in 0.1 when M_∞ is a manifold. See [12, 13] and also [6: §2] for details. When FM_i with the canonical probability measure $\tilde{\mu}_i$ converges, with respect to the measured

Hausdorff topology, to F_∞ with a Borel measure $\tilde{\mu}_\infty$ of unit mass (which is clearly $O(m)$-invariant), we can apply Theorems A and B to FM_i, F_∞ and $\hat{f}_i$: $FM_i \to F_\infty$, and we have a fibration $\tilde{\Phi}_i$: $FM_i \to F_\infty$ which approximates $\hat{f}_i$ in the C^0 topology (in fact, $C^{1,\alpha}$ topology). We remark that in this case, the constant C in Theorem A depends only on α, m, dim F_∞ and the injectivity radius of F_∞ and that M_i with the measure μ_i converges, in the measured Hausdorff topology, to M_∞ with the push-formward measure of $\tilde{\mu}_\infty$ by the projection of F_∞ onto M_∞. Through this fibration $\tilde{\Phi}_i$, we are able to compare approximately the space of $O(m)$-invariant (smooth) functions of F_∞ with that of $O(m)$-invariant (smooth) functions of the frame bundle FM_i which can be identified with the space of (smooth) functions of M_i.

0.4. As an application, we can generalize the main result in [11] and show that if a sequence of compact Riemannian m-manifolds with uniformly bounded curvature and diameter converges with respect to the measured Hausdorff topology, then not only the eigenvalues but their eigenfunctions converge in a stronger topology than that of [11]. See [22] for details. Moreover in Section 4, we investigate the energy spectrum of harmonic mappings into nonpositively curved manifolds and show certain continuity of the energy spectrum in the topology of measured Hausdorff convergence (cf. Theorem 4.1).

Finally we would like to explain briefly a primary motivation of studying the relationships between the Laplace operator of M_i and the operator $\mathcal{L}_\infty$ on M_∞ as in Theorems A and B. Let $\boldsymbol{R}^m/\Gamma$ be a flat noncompact Riemannian manifold of nontrivial fundamental group. This space can be seen as the total space of a flat vector bundle over a compact flat Riemannian manifold, say Σ_0, of dimension $k>0$. Let $\Sigma_r(r>0)$ be the hypersurfaces of the points which have equidistance r to the base Σ_0, and consider a family $\{\tilde{\Sigma}_r\}$ of compact Riemannian manifolds by setting $\tilde{\Sigma}_r = \frac{1}{r}\Sigma_r$. Then this family provides us a typical example of Riemannian manifolds collapsing to a lower dimensional space while keeping their curvatures and diameters bounded as r goes to infinity. In fact as r goes to infinity, $\tilde{\Sigma}_r$ converges, with respect to the measured Hausdorff topology, to the quotient space Σ_∞ of the unit sphere $S^{m-k-1}(1)$ by a closed subgroup K of $O(m-k)$, where K is determined by the action of Γ and the limit measure is the purh-forward of the canonical measure on $S^{m-k-1}(1)$ by the projection of $S^{m-k-1}(1)$ onto Σ_∞. On the other hand, the space of harmonic functions on $\boldsymbol{R}^m/\Gamma$ (and its perturbation) is closely related to the eigenvalues and the eigenfunctions of the operator associated with the measure on the quotient space Σ_∞. In view of this example, we expect that Theorems A and B (and their local forms) would be of some use in investigating function theoretic properties of noncompact Riemannian manifolds. Subsequently, based on the results of this paper, we want to discuss harmonic functions of polynomial growth on certain open Riemannian manifolds (cf. [22], [23, II]).

During the preparation of the first version of this paper, the author received the preprint of [6], Theorem 2.6 in which allowed him to make the original arguments of this paper clearer. He is grateful to Kenji Fukaya for sending it. The last section is concerned with harmonic mappings and it has been added in this revised one. The author would like to thank Hisashi Naito for helpful conversations on this subject.

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1. Preliminaries

We will recall here some known results and give a few lemmas on manifolds of bounded curvature, to prove Theorems A and B.

1.1. Harmonic coordinates

Let (M, g) be a complete Riemannian manifold of dimension m such that the sectional curvature K_M satisfies: $|K_M| \leqq \Lambda_0^2$ for a positive constant Λ_0. Let x be a point of M and $\exp_x\colon T_xM \to M$ the exponential map of M at x. We will identify the tangent space T_xM with Euclidean space $\boldsymbol{R}^m$ by a linear isometry. Then for a positive number r less than π/Λ_0, the exponential map induces a local diffeomorphism from the Euclidean ball $B^m(r)$ onto the geodesic ball $B(x, r)$ of M with radius r around x. Let g^* be the pull-back metric on $B^m(r)$. If we take r so small that r is less than a positive constant C_0 depending only on m and Λ_0, then we can find a (harmonic) coordinate system $\varphi=(x_1, \cdots, x_m)$ on $B^m(r)$ which has the following properties (see [19], [20] for details):

(i) All components x_j are harmonic functions with respect to g^*.

(ii) If we write the metric g^* in this coordinate system as

$$g^* = \sum_{j,k=1}^m g^*_{jk}(x)\, dx_j dx_k\,,$$

then the coefficients $g^*_{jk}(x)$ satisfy:

$$\frac{1}{C_1}\,|\xi|^2 \leqq \sum_{j,k=1}^m g^*_{jk}(x)\, \xi^j \xi^k \leqq C_1 |\xi|^2$$

$$|g^*_{jk}(x) - \delta_{jk}| \leqq C_2 r^2$$

for all $x\in\varphi(B^m(r))$ and $\xi=(\xi^1, \cdots, \xi^m)\in \boldsymbol{R}^m$, where C_1 and C_2 are positive constants depending only on m and Λ_0.

(iii) Given $\alpha\in(0, 1)$, there exists a positive constant C_3 depending only on m, Λ_0 and α such that the Hölder norms of g^*_{jk} satisfy

$$|g^*_{jk}|_{C^{1,\alpha}(\varphi(B^m(r)))}\leqq C_3\,,$$

and given $p\in(1, \infty)$, there exists a positive constant C_4 depending only on m, Λ_0 and p such that the Sobolev norms satisfy

$$\|g^*_{jk}\|_{W^{2,p}(\varphi(B^m(r)))}\leqq C_4\,.$$

(iv) If we assume that the n-th covariant derivative of the Ricci tensor Ric_M of M satisfies:

$$|D^n\mathrm{Ric}_M|\leqq\Lambda_n$$

for a constant Λ_n, then there is a constant $C_{3,n}$ depending only on m, n, α, Λ_0, and Λ_n such that

$$|g^*_{jk}|_{C^{1+n,\alpha}(\varphi(B^m(r)))}\leqq C_{3,n}\,.$$

Here and after when we consider the Hölder and Sobolev norms of functions on a fixed coordinate neighborhood as above, we follow the usual notaions for them (cf. [14]).

In relation with the regularity properties of Riemannian metrics in terms of the harmonic coordinates described as in (i)~(iii), we recall here the following

Fact 1.1. *On the set of complete Riemannian manifolds $M=(M,g)$ of dimension m with $|K_M|\leqq1$, there exists for all $\varepsilon>0$, a smoothing operator, $g\to S_\varepsilon(g)=g'$, such that*

(i) $$e^{-\varepsilon}g\leqq g'\leqq e^{\varepsilon}g$$

(ii) $$|D-D'|\leqq\varepsilon$$

(iii) $$|D'^nR'|\leqq\Lambda_n(m,\varepsilon)\quad(n=1, 2, \cdots)\,,$$

where D (resp., D') denotes the Riemannian connection of g (resp., g'), R' stands for the curvature tensor of g', and $\{\Lambda_n(m,\varepsilon)\}$ is a sequnece of positive constants depending only on m and ε. Moreover at $x\in M$, the value of g' depends only on $g|B(x, \frac{1}{4})$. Finally any isometry of g is also an isometry of g'.

This theorem is due to Abresch [1]. See also [4], [6; §1].

1.2. A regularity estimate on manifolds of bounded curvature

Let $M=(M,g)$ be a complete Riemannian manifold of dimension m with $|K_M|\leqq\Lambda_0^2$. We will use the same notations as in 1.1, and in addition, we denote by dv^* the volume element of the Riemannian metric g^* on $B^m(r)$. Let us first show the following

Lemma 1.2. *For a positive number a less than $\pi/4\Lambda_0$ and for a nonnegative continuous function f on a geodesic ball $B(x, 4a)$, it holds:*

$$\frac{1}{V^*(4a)}\int_{B^m(a)} f^*\, dv^* \leqq \frac{1}{\mathrm{Vol}(B(x,a))}\int_{B(x,a)} f \leqq \frac{1}{V^*(a)}\int_{B^m(4a)} f^*\, dv^*,$$

where $f^*=f\circ\exp_x$ and $V^*(a)$ stands for the volume of a ball $B^m(a)$ with respect to the pull-back metric g^*.

Proof. We first observe that for a positive number a less than $\pi/2\Lambda_0$ and for all $y\in B(x, a)$,

$$\#\{v\in B^m(2a)\colon \exp_x v = y\} \geqq \#\{v\in B^m(a)\colon \exp_x v = x\}, \tag{1.1}$$

$$\#\{v\in B^m(2a)\colon \exp_x v = x\} \geqq \#\{v\in B^m(a)\colon \exp_x v = y\}. \tag{1.2}$$

Then it follows from (1.1) that

$$\begin{aligned}\int_{B^m(2a)} f^*\, dv^* &= \int_{B(x,2a)} f(y)\, \#\{v\in B^m(2a)\colon \exp_x v = y\}\\ &\geqq \int_{B(x,a)} f(y)\, \#\{v\in B^m(2a)\colon \exp_x v = y\}\\ &\geqq \#\{v\in B^m(a)\colon \exp_x v = x\}\int_{B(x,a)} f.\end{aligned}$$

In particular, we have

$$V^*(2a)\geqq\#\{v\in B^m(a)\colon \exp_x v = x\}\ \mathrm{Vol}(B(x,a)).$$

Similarly by (1.2), we have

$$\begin{aligned}&\int_{B^m(a)} f^*\, dv^* \leqq \#\{v\in B^m(2a)\colon \exp_x v = x\}\int_{B(x,a)} f;\\ &V^*(a)\leqq\#\{v\in B^m(2a)\colon \exp_x v = x\}\ \mathrm{Vol}(B(x,a)).\end{aligned}$$

Thus it is easy to see that the lemma holds.

Now using harmonic coordinates with the properties described in 1.1 (i)~(iii), and combining Lemma 1.2 together with the elliptic regularity theory (cf. [14]) and the standard covering argument, we have the following

Lemma 1.3. *Let M be a complete Riemannian manifold of dimension m such that the sectional curvature K_M is bounded in its absolute value by a positive constant Λ_0. Then for given positive numbers $p\in(1, \infty)$, d_1 and d_2, there is a positive constant C depending only on m, Λ_0, p, d_1 and d_2 such that for all smooth functions f on* *a geodesic ball $B(x, b)$ around a point x with radius $b \leqq d_1$ and for all a with $0<a<b$ and $b-a \geqq d_2$,*

$$\|f\|_{W^{2,p}(B(x,a))} \leqq C \frac{\mathrm{Vol}(B(x,a))^{1/p}}{\mathrm{Vol}(B(x,b))^{1/p}} \{\|f\|_{L^p(B(x,b))} + \|\Delta f\|_{L^p(B(x,b))}\},$$

where Δf stands for the Laplacian of f.

1.3. Smooth Hausdorff approximations

In this section, we recall some basic results, that we need in this paper, on collapsing Riemannian manifolds while keeping their curvature bounded. See [10, 12, 13] and [6: §2] for details.

Let $\{M_i\}$ be a sequence of compact Riemannian manifolds of dimension m with $|K_M| \leqq 1$ and assume that $\{M_i\}$ converges to a metric space M_∞ in the Hausdorff topology as i goes to infinity. We first recall the following

Fact 1.4 (Fukaya [12,13]). *Let $\{M_i\}$ and M_∞ be as above. Then there exists a smooth manifold F_∞ with a metric of class $C^{1,\alpha}$ $(0<\alpha<1)$ on which the orthogonal group $O(m)$ acts by isometries in such a way that M_∞ is isometric to the quotient space $F_\infty/O(m)$.*

When M_∞ becomes a manifold, M_∞ is a smooth manifold with a metric g_∞ of class $C^{1,\alpha}(0<\alpha<1)$ and in fact of the Sobolev spaces $W^{2,p}(1<p<\infty)$. To show Fact 1.4, Fukaya [12] considered the frame bundle FM_i of M_i equipped with the canonical Riemannian metric so that the sectional curvature of FM_i remains bounded by a constant depending only on m. F_∞ is a limit of $\{FM_i\}$ with respect to the Hausdorff distance. Although the smoothness of the metrics of the limit spaces can not be expected, we are able to make use of Fact 1.1. Actually, we have

Fact 1.5 (Fukaya [13]). *Let $\{M_i\}$ and M_∞ be as above. Suppose M_∞ is a smooth manifold. Then there is a sequence of smooth approximations $\{g_\infty^{(\delta)}\}$ $(0<\delta)$ for the metric g_∞ of M_∞ such that they have the same properties as in (i) and (ii) of Fact 1.1 and also*

$$|\textit{the sectional curvature of } g_\infty^{(\delta)}| \leqq \Lambda_0$$

for some positive constant Λ_0 which is independent of (small) δ.

We should mention further the following

Fact 1.6 (Fukaya [10,12,13]). *Let $\{M_i\}$ and M_∞ be as above. Suppose M_∞ is a smooth manifold of positive dimension n $(0<n<m)$. Then there exists a fibration $f_i\colon M_i \to M_\infty$ (for every large i) satisfying*

(i) *for all $z \in M_\infty$, the diameter of $f_i^{-1}(z) \leqq \varepsilon(i)$;*

(ii) *f_i is an $\varepsilon(i)$-almost Riemannian submersion, that is, for all $z\in M_\infty$, $x\in f_i^{-1}(z)$, and $X\in T_xM_i$ normal to $f_i^{-1}(z)$,*

$$(1-\varepsilon(i))\,|df_i(X)|\leqq|X|\leqq(1+\varepsilon(i))\,|df_i(X)|\ ;$$

(iii) *the second fundamental form Ddf_i of f_i is bounded uniformly in i, namely,*

$$|Ddf_i|\leqq C\,,$$

where $\{\varepsilon(i)\}$ is a sequence of positive constants going to zero as i tends to infinity, and C is a positive constant depending only on m, n, Λ_0 as in Fact 1.5 and the lower bound of the injectivity radius of M_∞.

See Cheeger, Fukaya and Gromov [6: §2] and also the arguments therein for a new proof of this fact and in particular the last assertion in this form.

REMARK 1.7. Consider the case M_∞ is not a smooth manifold, and assume that a sequence of the frame bundles $\{FM_i\}$ converges to a metric space F_∞ with respect to the Hausdorff distance. As we mentioned above, F_∞ is in fact a smooth manifold with a metric of class $C^{1,\alpha}$ and the orthogonal group $O(m)$ acts on it as isometries in such a way that M_∞ is isometric to the quotient space $F_\infty/O(m)$. Moreover there exists an $O(m)$ equivariant fibration $\tilde{f}_i\colon FM_i\to F_\infty$ (for large i) which satisfies the assertions of Fact 1.6 (the constant C there depends in this case only on m, n and the lower bound of the injectivity radius of F_∞). See the references cited above for these facts.

1.4. Regularity of limit measure in measured Hausdorff convergence

Let $\{M_i\}$, M_∞ and f_i be as in Fact 1.6, and let M_i be equipped with the canonical probability measure μ_i (=the volume element/Vol(M_i)). Suppose in addition that $\{M_i\}$ converges to M_∞ with a Borel measure μ_∞ in the measured Hausdorff topology, that is, the push-forward measure $f_{i*}\mu_i$ converges weakly to μ_∞. We denote by g_i (resp., $g_i^{(\delta)}$) the Riemannian metric of M_i (resp., the approximations of g_i as in Fact 1.1). When we consider M_i with $g_i^{(\delta)}$ (δ being fixed), taking a subsequence if neccesarily, we may assume that $(M_i, g_i^{(\delta)})$ converges to the smooth Riemannian manifold $(M_\infty, g_\infty^{(\delta)})$ in the measured Hausdorff topology. Then we have a fibration $f_i^{(\delta)}\colon (M_i, g_i^{(\delta)})\to(M_\infty, g_\infty^{(\delta)})$ as in Fact 1.6 and the limit measure $\mu_\infty^{(\delta)}$ to which the push-forward measure $f_{i*}^{(\delta)}\mu_i^{(\delta)}$ of the cannonical probability measure $\mu_i^{(\delta)}$ converges weakly. In this case, Fukaya [13] shows that the density $\chi_\infty^{(\delta)}$ of the limit measure $\mu_\infty^{(\delta)}$ is a smooth positive function on M_∞, and as for the regularity of the limit measure μ_∞, the arguments there suggest that the density χ_∞ is of class $C^{1,\alpha}$ $(0<\alpha<1)$ and actually of the Sobolev spaces $W^{2,p}$ $(1<p<\infty)$. Since this does not seem to be clear from the arguments there and we need some estimates on χ_∞ in the proof of Theorem A, we will prove it. In what follows, we assume that the injectivity radius of M_∞ is bounded from below by a positive constant ι.

Since what we need are mainly some local properties of the limit measure, we first discuss locally on the problem. Consider M_∞ equipped with the metric $g_\infty^{(\delta)}$ for a while. Let us take a point p_∞ of M_∞ and choose a (harmonic) coordinate system $\psi=(z_1, \cdots, z_n)$ which has the properties described in 1.1 (i)$\sim$(iii). We may assume that ψ is defined on a geodesic ball $B(p_\infty, C_5)$ and $\psi(B(p_\infty, C_5))$ contains a Euclidean ball $B^n(C_6)$, where C_5 and C_5 depend only on n, Λ_0 and ι. Here Λ_0 is as in Fact 1.5. Note that the constants $C_1\sim C_4$ in 1.1 (ii) and (iii) depend also on ι in this case. We write the metric $g_\infty^{(\delta)}$ in this coordinate system $(z_1, \cdots, z_n)$ as

$$g_\infty^{(\delta)} = \textstyle\sum_{j,k=1}^n g_\infty^{(\delta)}{}_{jk}\, dz_j dz_k\,.$$

We want ot obtain certain regularity estimates for the limit measures $\mu_\infty^{(\delta)}$ (and then μ_∞) on the ball $B^n(C_6)$, using the regulairty properties of $g_\infty^{(\delta)}{}_{jk}$ $(j, k=1, \cdots, n)$.

Let $f_i^{(\delta)}\colon (M_i, g_i^{(\delta)})\to(M_\infty, g_\infty^{(\delta)})$ be a fibration as above and define a function $\theta_i^{(\delta)}$ on M_∞ by $\theta_i^{(\delta)}(z)=\mathrm{Vol}((f_i^{(\delta)})^{-1}(z))$. Then $\theta_i^{(\delta)}$ satisfies

$$|d\log\theta_i^{(\delta)}|\leqq C_7 \tag{1.3}$$

because of Fact 1.1 (for $f_i^{(\delta)}$), where C_7 is a positive constant depending only on m, n, Λ_0 and ι. Moreover we define a function $\zeta_i^{(\delta)}$ on $B^n(C_6)$ by

$$\zeta_i^{(\delta)} = \frac{\theta_i^{(\delta)}\circ\psi^{-1}}{\mathrm{Vol}((f_i^{(\delta)})^{-1}(B(p_\infty, C_5))}\,.$$

Then by (1.3) and the fact that $f_i^{(\delta)}$ is an $\varepsilon(i)$-almost Riemannian submersion with $\varepsilon(i)$ tending to zero as i goes to infinity, we have

$$\begin{aligned} &\frac{1}{C_8}\leqq\zeta_i^{(\delta)}\leqq C_8\\ &|d\zeta_i^{(\delta)}|\leqq C_9\end{aligned} \tag{1.4}$$

for some positive constants C_8 and C_9 depending only on m, n, Λ_0 and ι. Let us here define an elliptic differential operator $\mathcal{L}_i^{(\delta)}$ of second order by

$$\mathcal{L}_i^{(\delta)} = \textstyle\sum_n^{j,k=1} \dfrac{1}{\zeta_i^{(\delta)}G^{(\delta)}}\dfrac{\partial}{\partial z_j}\left[\zeta_i^{(\delta)}G^{(\delta)}g_\infty^{(\delta)jk}\dfrac{\partial}{\partial z_k}\right],$$

where $G^{(\delta)}=\det[g_\infty^{(\delta)}{}_{jk}]^{1/2}$ and $[g_\infty^{(\delta)jk}]$ is the inverse matrix of $[g_\infty^{(\delta)}{}_{jk}]$. Given $r\in(0, C_0)$, we solve the following Dirichlet problems:

$$\begin{aligned} \mathcal{L}_i^{(\delta)}h_i^{(\delta)j} &= 0 &&\text{in } B^n(r)\\ h_i^{(\delta)j} &= z_j &&\text{on } \partial B^n(r)\end{aligned} \tag{1.5}$$

$(j=1, \cdots, n)$. Then it turns out from (1.4) and the standard elliptic regularity theory (cf. [14] and also [17]) that for a positive constant C_{10} depending only on m, n, Λ_0 and ι, the following assertions (1.6) (i)$\sim$(iii) hold on $B^n(r_0)$ for $r_0 \leqq C_{10}$:

(i) $H_i^{(\delta)}=(h_i^{(\delta)1}, \cdots, h_j^{(\delta)n})$ defines a diffeomorphism of $B^n(r_0)$ onto itself;

(ii) given $\alpha \in (0, 1)$,

$$|H_i^{(\delta)}|_{C^{1,\alpha}(B^n(r_0))} \leqq C_{11} \tag{1.6}$$

for a positive constant C_{11} depending only on m, n, Λ_0, ι and α;

(iii) for all $j=1, \cdots, n$,

$$|dh_i^{(\delta)j}-dz_j| \leqq C_{12} r_0$$

for a positive constant C_{12} depending only on m, n, Λ_0 and ι.

In addition, we may assume that for all $j, k=1, \cdots, n$,

$$|g_\infty^{(\delta)}(\operatorname{grad} h_i^{(\delta)j}, \operatorname{grad} h_i^{(\delta)k})-\delta_{jk}| \leqq \frac{1}{10n} \tag{1.7}$$

(cf. (1.6) (iii) and 1.1 (ii)). We assume now that the measure $\zeta_i^{(\delta)} dz_1 \wedge \cdots \wedge dz_n$ converges weakly to a measure, as i goes to infinity. Then the density $\zeta_\infty^{(\delta)}$ of the limit measure coincides with $b\chi_\infty^{(\delta)}$ and it satisfies (1.4), where b is a positive number given by $b=\lim_{i\to\infty} \operatorname{Vol}(M_i)/\operatorname{Vol}(f_i^{(\delta)})^{-1}(B(p_\infty, C_5))$. We remark that by Bishop-Gromov's inequality,

$$C_{13} \leqq b \leqq 1,$$

where C_{13} is a positive constant depending only on m, C_5 and $\operatorname{diam}(M_\infty)$. Moreover we may assume thta $H_i^{(\delta)}$ converges, as i goes to infinity, to a diffeomorphism $H_\infty^{(\delta)}$ of $B^n(r_0)$ onto itself in the $C^{1,\alpha}$ topology $(0<\alpha<1)$. Each component $h_\infty^{(\delta)j}$ of $H_\infty^{(\delta)}$ is the solution of the Dirichlet problem:

$$\begin{aligned} \mathcal{L}_\infty^{(\delta)} h_\infty^{(\delta)j} &= 0 \qquad \text{in } B^n(r_0) \\ h_\infty^{(\delta)j} &= z_j \qquad \text{on } \partial B^n(r_0), \end{aligned} \tag{1.8}$$

where $\mathcal{L}_\infty^{(\delta)}$ is the operator given by

$$\mathcal{L}_\infty^{(\delta)} = \sum_{j,k=1}^n \frac{1}{\zeta_\infty^{(\delta)} G^{(\delta)}} \frac{\partial}{\partial z_j}\left[\zeta_\infty^{(\delta)} G^{(\delta)} g_\infty^{(\delta)jk} \frac{\partial}{\partial z_k}\right].$$

Let us consider in turn the manifold M_i equipped with the metric $g_i^{(\delta)}$. Choose a point p_i in such a way that $f_i^{(\delta)}(p_i)=p_\infty$. Let $\exp_i\colon T_{p_i}M_i \to M_i$ be the exponential mapping at p_i. As in 1.1, we will identify the tangent space $T_{p_i}M_i$ with Euclidean space $\boldsymbol{R}^n$ by a linear isometry and use a harmonic coordinate system $\varphi=(x_1, \cdots, x_m)$ with the propeties described in 1.1. Then taking a subsequence if neccesarily, we may assume that on $B^m(r)$ $(r \leqq C_0)$, the pull-back metric $g_i^{(\delta)*}$ converges to a (smooth) metric $g_\infty^{(\delta)*}$. We set $f_i^{(\delta)*}=f_i^{(\delta)}\circ\exp_i$. Then as i goes to infinity, $f_i^{(\delta)}$ converges to a submersion $\Pi_\infty^{(\delta)}$ from $B^m(r)$ onto a neighborhood $U_\infty^{(\delta)}$ of p_∞. By choosing r_0' and r_1 appropriately, we may assume that $B^n(r_0')\subset U_\infty^{(\delta)}=\Pi_\infty^{(\delta)}(B^m(r_1))\subset B^n(r_0)$, where r_0' and r_1 are positive constants depending only on m, n, Λ_0 and ι. We observe that $\Pi_\infty^{(\delta)}$ is actually a Riemannian submersion with respect to the metrics $g_\infty^{(\delta)*}$ and $g_\infty^{(\delta)}$, and that the mean curvature vector field $\eta_\infty^{(\delta)*}$ of the fibers is basic, namely, horizontal and projectable, since $\Pi_\infty^{(\delta)}$ coincides with the projection of $B^m(r_1)$ onto $U_\infty^{(\delta)}$ by the isometric action of a Lie group germ (cf. [16], [12, 13]). Moreover if we denote by $\tau(\Pi_\infty^{(\delta)})$ the tension field of $\Pi_\infty^{(\delta)}$, then $\tau(\Pi_\infty^{(\delta)})=-d\Pi_\infty^{(\delta)}(\eta_\infty^{(\delta)*})=(\mathrm{grad}\log\zeta_\infty^{(\delta)})\circ\Pi_\infty^{(\delta)}$. This implies that each component of $H_\infty^{(\delta)}\circ\Pi_\infty^{(\delta)}$ is harmonic (with respect to the metric $g_\infty^{(\delta)*}$). Hence given $\alpha\in(0,1)$, we have

$$|H_\infty^{(\delta)}\circ\Pi_\infty^{(\delta)}|_{C^{3,\alpha}(B^m(r_1))}\leqq C_{14}$$

for a positive constant C_{14} depending only on m, n, Λ_0 and ι (cf. 1.1(i)$\sim$(iii)). Thus taking account of (1.7), we see that given $p\in(1,\infty)$,

$$\|\eta_\infty^{(\delta)*}\|_{W^{1,p}(B^m(r_1))}\leqq C_{15}$$

from which it follows that

$$\begin{aligned}&\|z_j\circ\Pi_\infty^{(\delta)}\|_{W^{3,p}(R^m(r_1))}\leqq C_{16}\,,\\&\|\log\zeta_\infty^{(\delta)}\|_{W^{2,p}(B^n(r_0))}\leqq C_{17}\,,\\&\|H_\infty^{(\delta)}\|_{W^{3,p}(B^n(r_0))}\leqq C_{18}\,.\end{aligned}$$

Here $C_{14}\sim C_{18}$ are positive constants depending only on m, n, Λ_0, ι and p.

We have discussed so far the sequence of Riemannian manifolds $(M_i, g_i^{(\delta)})$ and its limit $(M_\infty, g_\infty^{(\delta)})$ with respect to the measured Hausdorff topology, while keeping a (small) positive number δ fixed. However as we have shown, all constants in the above estimates are able to be taken independently of δ. Thus the following results hold.

Lemma 1.8. *Let $g_\infty^{(\delta)*}, g_\infty^{(\delta)}$ and $\Pi_\infty^{(\delta)}: B^m(r_1)\to U_\infty^{(\delta)}$ ($p_\infty\in U_\infty^{(\delta)}\subset M_\infty$) be as above. Suppose that for a sequence $\{\delta_j\}$ going to zero, $g_\infty^{(\delta_j)*}$ (resp., $g_\infty^{(\delta_j)}$) converges in the $C^{1,\alpha}$ topology to a metric g_∞^* (resp., g_∞) of class $C^{1,\alpha}$ ($0<\alpha<1$) and further $\Pi_\infty^{(\delta_j)}$ converges to a map $\Pi_\infty: B^m(r_1)\to U_\infty$ of $B^m(r_1)$ onto a neighborhood U_∞ of p_∞. Then Π_∞ is a Riemannian submersion of class $C^{2,\alpha}$ with respect to the limit metrics g_∞^* and g_∞ such that the mean curvature vector field η_∞^* along the fibres of Π_∞ is basic and the tension field $\tau(\Pi_\infty)$ is given by $\tau(\Pi_\infty)=-d\Pi_\infty(\eta_\infty^*)=(\mathrm{grad}\log\chi_\infty)\circ\Pi_\infty$. Moreover given $p\in(1,\infty)$, the covariant derivative $D\eta_\infty^*$ (with respect to g_∞^*) satisfies*

$$\|D\eta_\infty^*\|_{L^p(B^m(r_1))}\leqq C\,,$$

where C is a positive constant depending only on m ($=\dim M_i$), n ($=\dim M_\infty$),

Λ_0 *(as in Fact* 1.5), ι (= *a lower bound of the injectivity radius of* (M_∞, g_∞)) *and* p.

Lemma 1.9. *Let* M_∞ *be as in Theorem A and let* m, n, Λ_0 *and* ι *be as above.*
(i) *The density* χ_∞ *of the limit measure* μ_∞ *on* M_∞ *satisfies*

$$|\log \chi_\infty|_{C^0(M_\infty)} \leqq C$$
$$|d \log \chi_\infty|_{C^0(M_\infty)} \leqq C',$$

where C is a positive constant depending only on m, n, Λ_0, ι *and* d (=*an upper bound of the diameter of* M_∞), *and also* C' *is a positive constant depending only on* m, n, Λ_0 *and* ι, *Furthermore given* $p \in (1, \infty)$,

$$||Dd \log \chi_\infty||_{L^p(M_\infty, \mu_\infty)} \leqq C'',$$

where C'' *is a positive constant depending only on* m, n, Λ_0, ι, d *and* p.

(ii) *For all points* p_∞ *of* M_∞ *there are a positive constant* r_0 $(<\iota)$ *depending only on* m, n, Λ_0 *and* ι, *and a coordinate system* $\phi=(h_1, \cdots, h_n)$: $B(p_\infty, r_0) \to \phi(B(p_\infty, r_0))$ *such that any component* h_j *of* ϕ *satisfies*

$$\mathcal{L}_\infty h_j := (\Delta_\infty + \mathrm{grad} \log \chi_\infty) h_j = 0$$

where Δ_∞ *stands for the Laplace operator of* M_∞. *Moreover given* $p \in (1, \infty)$,

$$||h_j||_{W^{3,p}(B(p_\infty, r_0), \mu_\infty)} \leqq C^{(3)},$$

where $C^{(3)}$ *is a positive constant depending only on* m, n, Λ_0, ι *and* p.

2. Construction of Fibration and Proof of Theorem A

Let $\{(M_i, \mu_i)\}$, (M_∞, μ_∞) and $\mathcal{L}_\infty (=\Delta_\infty + \mathrm{grad} \log \chi_\infty)$ be as in Theorem *A* and let $f_i: (M_i, g_i) \to (M_\infty, g_\infty)$ be a fibration as in Fact 1.6. In order to prove Theorem A, we first perturb this fibration locally and then construct a submersion with certain additional properties, making use of a center-of-mass technique.

2.1. Local fibrations by harmonic functions

We begin with the construction of local fibrations, using certain harmonic functions. We fix a sufficiently small positive number r as in Lemma 1.9 (ii). Let p_∞ be a point of M_∞ and p_i a point of M_i such that $f_i(p_i)=p_\infty$. Set $U_i := f_i^{-1}(B(p_\infty, r))$ $(\subset M_i)$ for simplicity. For a given function $h \in C^{2,\alpha}(\overline{B(p_\infty, r)})$ which satisfies: $\mathcal{L}_\infty h := (\Delta_\infty + \mathrm{grad} \log \chi_\infty) h = 0$ in $B(p_\infty, r)$, consider a unique solution h_i of the Dirichlet problem:

$$\begin{aligned} \Delta_i h_i &= 0 \quad &&\text{in} \quad U_i \\ h_i &= h \circ f_i \quad &&\text{on} \quad \partial U_i . \end{aligned}$$

Then applying a gradinet estimate by Cheng-Yau [7] to h_i, we have

$$|dh_i|(x) \leqq C_1\left(1+\frac{1}{\operatorname{dis}(z, \partial B(p_\infty, r))}\right) \{\max_{B(p_\infty,r)} h - \min_{B(p_\infty,r)} h\}$$

for all $z \in B(p_\infty, r)$ and all $x \in f_i^{-1}(z)$, where C_1 is a constant depending only on the dimension m of M_i. This implies that for all $z \in B(p_\infty, r)$,

$$\begin{aligned} &\max_{f_i^{-1}(z)} h_i - \min_{f_i^{-1}(z)} h_i \leqq \\ (2.1) \quad &C_1\left(1+\frac{1}{\operatorname{dis}(z, \partial B(p_\infty, r))}\right) \operatorname{diam}(f_i^{-1}(z)) \{\max_{B(r_\infty,r)} h - \min_{B(p_\infty,r)} h\}. \end{aligned}$$

As in Section 1.1, let $\exp_i: T_{p_i}M_i \to M_i$ denote the exponential map of M_i at p_i, and identify the tangent space $T_{p_i}M_i$ with Euclidean m-space $\boldsymbol{R}^m$ by choosing an orthonormal basis of $T_{p_i}M_i$. Moreover we use a harmonic coordinate system $(x_1, \cdots, x_m)$ defined on the ball $B^m(3r)$ in $\boldsymbol{R}^m$ which has the properties described in 1.1 (i)~(iii) (r is asumed to be small enough). We consider a family of functions $h_i^* := h_i \circ \exp_i$ (for large i) on domains $U_i^* := \exp_i^{-1}(U_i) \cap B^m(2r)$. We first observe that the $C^{0,\alpha}$ norm of h_i^* on $\overline{U_i^*}$ and the $C^{3,\alpha}$ norm of h_i^* on a fixed compact subdomain V^* of U_i^* are bounded uniformly in i. More precisely, given $\alpha \in (0, 1)$, there is a positive constant C_2 depending only on m, the constant C in Fact 1.6 (iii), α and the $C^{0,\alpha}$ norm of the given solution h such that

$$(2.2) \qquad |h_i^*|_{C^{0,\alpha}(\overline{U_i^*})} \leqq C_2,$$

and in addition, there is a positive constant C_3 depending only on m, α, $\operatorname{dis}(V^*, \partial U_i^*)$ and the C^0 norm of h on $B(p_\infty, r)$ such that

$$(2.3) \qquad |h_i^*|_{C^{3,\alpha}(V^*)} \leqq C_3.$$

These estimates are derived from the standard elliptic regularity theory (cf. [14]) and the regularity properties of the pull-back metrics $g_i^* = \exp_i^* g_i$ in terms of the harmonic coordinates as above.

To make our arguments here clearer, we will descuss under the following hypothesis: as i goes to infinity, a sequence of the pull-back metrics $\{g_i^*\}$ converges in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ metric g_∞^* on $B^m(3r)$ and moreover a family of the mappings $\{f_i^*\}$ defined by $f_i^* := f_i \circ \exp_i: B^m(3r) \to M_\infty$ converges in the $C^{1,\alpha}$ topology to a fibration $\Pi_\infty: B^m(3r) \to M_\infty$ which is a Riemannina submersion of class $C^{2,\alpha}$ with respect to the limit metric g_∞^*. As noted at the end of 1.4, the mean curvature vector field η_∞^* of the fibres of Π_∞ is basic and satisfies:

$$\|D\eta_\infty^*\|_{L^p(B^m(3r))} \leqq C_4,$$

where C_4 is a positive constant depending only on m and r, and furthermore the tension field $\tau(\Pi_\infty)$ satisfies:

$$\tau(\Pi_\infty) = -d\Pi_\infty(\eta_\infty^*) = (\operatorname{grad} \log \chi_\infty) \circ \Pi_\infty$$

Obviously $\{U_i^*\}$ converges to $U^*:=\Pi_\infty^{-1}(B(p_\infty, r))\cap B^m(2r)$. Now we claim that $\{h_i^*\}$ converges to the harmonic function $h_\infty^*:=h\circ\Pi_\infty$ (with respect to the limit metric g_∞^*) in such a way that for any compact subdomain V^* of U^*.

$$\lim_{i\to\infty} |h_i^*-h_\infty^*|_{C^{3,\alpha}(V^*)} = 0\,. \tag{2.4}$$

Indeed, for a given subsequence, say $\{h_j^*\}$, of $\{h_i^*\}$, by (2.2) and (2.3), we can find a subsequence of $\{h_j^*\}$ which converges to some $h'\in C^0(\overline{U^*})\cap C^{3,\alpha}(U^*)$ in the same manner as in (2.4). It is clear that h' is harmonic with respect to the limit metric g_∞^*, and $h'=h_\infty^*$ on $\partial U^*\cap B^m(2r)$. Moreover by (2.1), we see that h' is constant along the fibers of the projection Π_∞, namely there is $h''\in C^0(\overline{B(p_\infty, r)})$ $\cap\, C^{2,\alpha}(B(p_\infty, r))$ such that $h'=h''\circ\Pi_\infty$ and $h''=h$ on $\partial B(p_\infty, r)$. Since h'' also satisfies: $\mathcal{L}_\infty h''=0$ on $B(p_\infty, r)$, the uniqueness of solution of Dirichlet problem shows that h'' coincides with h. Thus our claim is clear.

Based on what we have just observed, we shall next construct a local fibration of a neighborhood of p_i over that of p_∞. By Lemma 1.9(ii), we have a coordinate system $H=\{h_1, \cdots, h_n\}$ defined on $B_\infty(p_\infty, r)$ whose components are solutions of equation: $\mathcal{L}_\infty h_\beta=0$. We may assume that $H(B(p_\infty, r))=B^n(1)\,(\subset \boldsymbol{R}^n)$ and all components h_β belong to $C^{2,\alpha}(B(p_\infty, r))$, and further that $H(B(p_\infty, r/5))\subset B^n(2/5)\subset B^n(3/5)\subset H(B(p_\infty, 4r/5))$, by taking r small enough. Let U_i and U_i^* as above. Let $h_{i,\beta}(\beta=1, \cdots, n)$ be solutions of the following Dirichlet problems:

$$\begin{aligned} \Delta_i h_{i,\beta} &= 0 \qquad \text{in} \quad U_i \\ h_{i,\beta} &= h_\beta\circ f_i \qquad \text{on} \quad \partial U_i\,. \end{aligned}$$

Moreover we put $H_i:=\{h_{i,1}, \cdots, h_{i,n}\}\colon U_i\to B^n(1)$, $F_i:=H^{-1}\circ H_i\colon U_i\to B(p_\infty, r)$ and $F_i^*:=F_i\circ\exp_i\colon U_i^*\to B(p_\infty, r)$. Then $\{H\circ F_i^*\}$ converges to $H\circ\Pi_\infty$ in the same manner as in (2.4). In particular, we have

$$\begin{aligned} &|H\circ F_i^*|_{C^{3,\alpha}(B^m(4r/5))}\leqq C_5 \\ &\lim_{i\to\infty} |H\circ F_i^*-H\circ\Pi_\infty|_{C^{3,\alpha}(B^m(4r/5))}=0\,, \end{aligned} \tag{2.5}$$

where C_5 depends only on m and r. This implies that if we set $W_i:=F_i^{-1}(B(p_\infty, 4r/5))$ for simplicity, $F_i\colon W_i\to B(p_\infty, 4r/5)$ becomes a submersion satisfying

$$\max\,\{\mathrm{dis}(f_i(x), F_i(x))\colon x\in W_i\}\leqq\varepsilon(i)\,,$$

where $\{\varepsilon(i)\}$ denotes a sequence of positive numbers which goes to zero as i tends to infinity, and moreover it will be kept to stand for such a sequence in what follows. To describe some other properties of the submersions F_i, it is convenient to fix several notations here. For a point $x\in W_i$ and a tangent vector $E\in T_xM_i$, we denote by $\mathcal{V}_i(x)$, $\mathcal{H}_i(x)$, $\mathcal{V}_iE$, and $\mathcal{H}_iE$, respectively, the subspace of T_xM_i which consists of vectors tangent to the fiber $F_i^{-1}(F_i(x))$ through

x, the orthogonal complement of $\mathcal{V}_i(x)$ in T_xM_i, the $\mathcal{V}_i(x)$-component of E, the $\mathcal{H}_i(x)$-component of E. We may call $\mathcal{V}_i(x)$ and $\mathcal{H}_i(x)$, respectively, the vertical subspace at x and the horizontal subspace at x (of the fibration $F_i\colon W_i \to B(p_\infty, 4r/5)$). Correspondingly we have two distributions $\mathcal{V}_i$ and $\mathcal{H}_i$ called the vertical distribution and the horizontal distribution. $\mathcal{V}_i$ and $\mathcal{H}_i$ also denote the projections onto them. For a vector field X on $B(p_\infty, 4r/5)$, we denote by $\hat{X}$ the vector field on W_i satisfying

$$\hat{X}(x) \in \mathcal{H}_i(x),\ dF_i(\hat{X}(x)) = X \circ F_i(x) \quad (x \in W_i).$$

As in [26], we define (2,1) tensor fields $A(F_i)$ and $T(F_i)$ on W_i whose values on vector fields E_1, E_2 are, respectively, given by

$$A(F_i)(E_1, E_2) := \mathcal{H}D_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}D_{\mathcal{H}E_1}\mathcal{H}E_2\,,$$
$$T(F_i)(E_1, E_2) := \mathcal{H}D_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}D_{\mathcal{V}E_1}\mathcal{H}E_2\,.$$

Moreover it is also convenient to define a (2, 1) tensor field $B(F_i)$ on W_i by

$$B(F_i)(E_1, E_2) := \mathcal{H}D_{E_1}E_2 - [D_{dF_i(E_1)}dF_i(E_2)]^\wedge\,.$$

We observe that

$$dF_i(B(F_i)(E_1, E_2)) = -DdF_i(E_1, E_2)\,.$$

We remark here that the tensor $B(F)$ of a submersion $F\colon N \to M$ between Riemannian manifolds N and M vanishes on the horizontal distribution if it is a Riemannain submersion (cf. [26]). In particular, since the projection $\Pi_\infty\colon (U^*, g^*_\infty) \to B(p_\infty, r)$ is a Riemannian submersion (of class $C^{2,\alpha}$), the tensor field $B(\Pi_\infty)$ of this fibration vanishes on the horizontal distributions of Π_∞. Namely we have

$$B(\Pi_\infty)(X^*, Y^*) = 0$$

for all horizontal vectors X^* and Y^* of the fibration Π_∞.

We are now in a position to summarize what we have observed so far in the following assertions (i)$\sim$(vi):

(i) $\max\,\{\mathrm{dis}(f_i(x), F_i(x))\colon x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq \varepsilon(i)$;

(ii) F_i is an $\varepsilon(i)$-almost Riemannian submersion;

(iii) if we denote by $\eta_i(x)$ the mean curvature at a point x of the fiber $F_i^{-1}(F_i(x))$ in M_i,

$$\max\{|dF_i(\eta_i(x)) + (\mathrm{grad}\log \chi_\infty)\circ F_i(x)|\colon x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq \varepsilon(i);$$

(iv) $\max\{(|B(F_i)(X, Y)|\colon X, Y \in \mathcal{H}_i(x),\quad |X| = |Y| = 1,\quad x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq \varepsilon(i)$;

(v) $\max\{|DdF_i|(x)\colon x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq C$; in particular,

$\max\{|A(F_i)|(x)\colon x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq C$;

$\max\{|T(F_i)|(x)\colon x \in F_i^{-1}(B(p_\infty, 4r/5))\} \leqq C$, where C is a positive constant depending only on m and r. In addition, by Lemma 1.2, we see that given $p\in(1, \infty)$,

$$\text{(vi)} \qquad \frac{1}{\mathrm{Vol}(B(p_i, 4r/5))}\int_{B(p_i,4r/5)} |D\eta_i|^p \leqq C' ;$$
$$\frac{1}{\mathrm{Vol}(B(p_i, 4r/5))}\int_{B(p_i,4r/5)} |DDdF_i|^p \leqq C'$$

where C' is a positive constant dependign only on m, r and p.

2.2. Construction of global fibration

In this section, we will construct a global fibration of M_i onto M_∞ by applying a center-of-mass technique to the local fibrations constructed in 2.1. In general, the metric g_∞ of M_∞ may not be smooth. In order to carry out our construction, we will use a family of smooth Riemannian metrics $\{g_\infty^{(\delta)}\}$ on M_∞ which has the properties described in Fact 1.5. Let us fix a small positive constant r as in 2.1 and take a (finite) family of points $\{p_{\infty\beta}\}$ of M_∞ in such a way that $\mathrm{dis}(p_{\infty\beta}, p_{\infty\gamma})\geqq r/10$ if $p_{\infty\beta}\neq p_{\infty\gamma}$ and the union of the geodesic balls $B(p_{\infty\beta}, r/5)$ covers M_∞. Let $H_\beta=(h_1^\beta, \cdots, h_n^\beta)\colon B(p_{\infty\beta}, r)\to B^n(1)$ be a coordinate system such that the components of H_β are the solutions of equation: $\mathcal{L}_\infty h_\nu^\beta=0$ on $B(p_{\infty\beta}, r)$ $(\nu=1, \cdots, n)$. As in 2.2, if we set $U_{i,\beta}:=f_i^{-1}(B(p_{\infty\beta}, r))\,(\subset M_i)$ and define maps $F_{i,\beta}\colon U_{i,\beta}\to B(p_{\infty\beta}, r)$ in such a way that $\Delta_i(H_\beta\circ F_{i,\beta})=0$ on $U_{i,\beta}$ and $F_{i,\beta}=f_i$ on $\partial U_{i,\beta}$, then $F_{i,\beta}$ gives a fibration over $B(p_{\infty\beta}, 4r/5)$ (for large i) and has the properties (i)$\sim$(vi) described at the end of 2.1. Let us now fix a smooth function $\xi(s)$ on $\boldsymbol{R}$ which satisfies: $\xi(s)=1$ if $|s|\leqq 2/5$, $\xi(s)=0$ if $|s|\geqq 3/5$ and $0\leqq\xi(s)\leqq 1$ for all s. Define smooth functions $\{\xi_{i,\beta}\}$ on M_i by

$$\xi_{i,\beta} := \frac{\xi(|H_\beta\circ F_{i,\beta}|)}{\Sigma_\nu\xi(|H_\nu\circ F_{i,\nu}|)}\,.$$

Then by (2.3), we get a partition of unity $\{\xi_{i,\beta}\}_\beta$ on M_i subordinate to the covering $\{W_{i,\beta}:=F_i^{-1}(B(p_{\infty\beta}, 4r/5))\}$ such that given $\alpha\in(0, 1)$ and $p\in(1, \infty)$,

$$\text{(2.6)} \qquad \begin{aligned} &|\xi_{i,\beta}|_{C^{2,\alpha}(M_i)}\leqq C_6\\ &\|\xi_{i,\beta}\|_{W^{3,p}(M_i,\mu_i)}\leqq C_7\,, \end{aligned}$$

where C_6 (resp., C_7) is a positive constant depending only on m, r and α (resp, m, r, diam M_∞ and p). We fix sufficiently large i and small δ for a while. Then it is not hard to see that for any point $x\in W_{i,\beta}\cap W_{i,\gamma}$ and tangent vectors $X, Y\in T_xM_i$ of unit lnegth,

$$\text{(2.7)} \qquad |dF_{i,\beta}(X)-P_{\beta\gamma}^{(\delta)}(dF_{i,\gamma}(X))|\leqq\varepsilon(i)+\tau(\delta)$$

$$\text{(2.8)} \qquad |D^{(\delta)}dF_{i,\beta}(X, Y)-P_{\beta\gamma}^{(\delta)}(D^{(\delta)}dF_{i,\gamma}(X, Y))|\leqq\varepsilon(i)+\tau(\delta)\,.$$

Here $\{\tau(\delta)\}$ denotes a sequence of positive numbers which goes to zero as δ tends to zero. Since $F_{i,\beta}$ has the property (i) at the end of 2.1, there is a unique minimal geodesic (with respect to $g_\infty^{(\delta)}$) joining $F_{i,\gamma}(x)$ with $F_{i,\beta}(x)$ for any $x\in W_{i,\beta}\cap W_{i,\gamma}$. $P_{\beta\gamma}^{(\delta)}$ denotes the parallel displacement along it. Moreover $D^{(\delta)}$ stands for the Riemannian connection with respect to $g_\infty^{(\delta)}$.

Now for sufficiently large i and small δ, we have a smooth map $\Phi_{i,\delta}$ of M_i onto M_∞ which is the center-of-mass with respect to $\{F_{i,\beta}, \xi_{i,\beta}\}$ and $g_\infty^{(\delta)}$. Namely, $\Phi_{i,\delta}$ is defined by

$$\textstyle\sum_\beta \xi_{i,\beta}(x) \exp^{(\delta)}_{\Phi_{i,\delta}(x)}{}^{-1} F_{i,\beta}(x) = 0$$

($x\in M_i$), where $\exp^{(\delta)}$ denotes the exponential map with respect to $g_\infty^{(\delta)}$ (cf. [5]). It is obvious that $\Phi_{i,\delta}$ satisfies

$$\max\{\operatorname{dis}(f_i(x), \Phi_{i,\delta}(x)) : x\in M_i\} \leqq \varepsilon(i)\,. \tag{2.9}$$

We denote by $\Lambda_1(\delta)$ and $\Lambda_2(\delta)$, respectively, the upper bounds of the norms of the covariant derivatives $D^{(\delta)}R_\delta$ and $D^{(\delta)}D^{(\delta)}R_\delta$ of the curvature tensor R_δ of $g_\infty^{(\delta)}$. Let us here take a sequence $\{\delta_i\}$ in such a way that both $\Lambda_1(\delta_i)\varepsilon(i)$ and $\Lambda_2(\delta_i)\varepsilon(i)^2$ go to zero as i tends to infinity. We write Φ_i for Φ_{i,δ_i}. Then the properties of $\{F_{i,\beta}\}$ stated at the end of 2.1 and the standard comparison arguments applied to the center-of-mass Φ_i (cf. Lemmas A.1$\sim$A.3 in Appendix) together with (2.6)$\sim$(2.9) yield the following

Theorem A′. *Let $\{M_i\}$ and M_∞ be as in Theorem A. Then for large i, there exists a fibration $\Phi_i\colon M_i\to M_\infty$ such that*

(i) $\max\{\operatorname{dis}(f_i(x), \Phi_i(x)) : x\in M_i\} \leqq \varepsilon(i)$;

(ii) *Φ_i is an $\varepsilon(i)$-almost Riemannian submersion;*

(iii) *if η_i denotes the mean curvature vector field along the fibers of Φ_i, it holds that*

$$\max\{|d\Phi_i(\eta_i(x)) + (\operatorname{grad} \operatorname{long} \chi_\infty)\circ\Phi_i(x)| : x\in M_i\} \leqq \varepsilon(i);$$

(iv) *the tensor field $B(\Phi_i)$ satisfies*

$$\max\{|B(\Phi_i)|(x) : x\in M_i\} \leqq \varepsilon(i);$$

(v) *the second fundamental form $Dd\Phi_i$ of Φ_i satisfies*

$$|Dd\Phi_i| \leqq C\,,$$

and in particular it holds that

$$|A(\Phi_i)| \leqq C\,, \qquad |T(\Phi_i)| \leqq C\,.$$

Here $B(\Phi_i)$, $A(\Phi_i)$ and $T(\Phi_i)$ are respectively the (2, 1) *tensor fields on M_i defined as in* 2.1, *$\{\varepsilon(i)\}$ is a sequence of positive numgers going to zero as i tends to infinity,* *and C is a positive constant depending only on m* ($=\dim M_i$), $n(=\dim M_\infty)$, *"the curvature bound"* Λ_0 *for* M_∞ *as in Fact* 1.5 *and a lower bound* ι *of the injectivity radius of* M_∞. *In addition, given* $p\in(1,\infty)$, Φ_i *satisfies:*

$$\text{(vi)} \qquad \| |D\eta_i| \|_{L^p(M_i,\mu_i)} \leqq C', \quad \| |DDd\Phi_i| \|_{L^p(M_i,\mu_i)} \leqq C',$$

where C' *is a positive constant depending only on* m, n, Λ_0, ι, diam M_∞ *and* p.

It is not hard to derive Theorem A from this.

3. Proof of Theorem B

Let $\Phi_i: M_i \to M_\infty$ be as in Theorem A (or A′). For a smooth function h on M_i. we define a smooth function $\Theta_i(h)$ on M_∞ by

$$\Theta_i(h)(z) = \frac{1}{\mathrm{Vol}(\Phi_i^{-1}(z))} \int_{\Phi_i^{-1}(z)} h .$$

In the following, we denote as before by η_i the mean curvature vector field along the fibers of Φ_i, and also by $\varepsilon(i)$ a sequence of positive numbers tending to zero as i goes to infinity.

3.1. Proofs of (i), (v) and (vi) in Theorem B

Let X be a vector field on M_∞ and $\hat{X}$ the horizontal lift of X with respect to the fibration $\Phi_i: M_i \to M_\infty$. Then a direct computation shows that

$$d\Theta_i(h)(X) = \Theta_i(dh(\hat{X})) - \Theta_i(hg_i(\hat{X}, \eta_i)) + \Theta_i(h)\Theta_i(g_i(\hat{X}, \eta_i)) . \tag{3.1}$$

Hence we have by Theorem A′ (i) and (ii) that

$$|d\Theta_i(h)(X) - \Theta_i(dh(\hat{X}))| \leqq \varepsilon(i) |\Theta_i(h)| \tag{3.2}$$

from which the first assertion of the theorem follows easily. The estimate (v) is a direct consequence of the fact that $\mathrm{diam}(\Phi_i^{-1}(z)) \leqq \varepsilon(i)$ for all $z \in M_\infty$. To prove the estimate (vi), we need the following

Lemma 3.1. *Let h be a smooth function on* M_i. *Then:*
(i) *for all vertical vector* $V \in \mathcal{V}_i(x)$ *at a point* $x \in M_i$,

$$|dh(V)| \leqq \varepsilon(i)\ |h|_{C^{1,\alpha}(M_i)} |V| ;$$

(ii) *for all vectors X tangent to* M_∞ *at a point* z, *and for all points* $x, y \in \Phi_i^{-1}(z)$,

$$|dh_x(\hat{X}) - dh_y(\hat{X})| \leqq \varepsilon(i) |h|_{C^{1,\alpha}(M_i)} |X| .$$

This lemma will be proved later. Now we want to complete the proof of the estimate (vi). By virtue of the first assertion of Lemma 3.1, it suffices to show that

$$|d\Theta_i(h)(X)-dh_x(\hat{X})|\leqq\varepsilon(i)|h|_{C^{1,\alpha}(M_i)}|X|$$

for a tangent vector X of M_∞ at a point z and a point $x\in\Phi_i^{-1}(z)$. This follows from (3.2) and the second assertion (ii) of Lemma 3.1. Indeed,

$$\begin{aligned}|d\Theta_i(h)(X)-dh_x(\hat{X})|&\leqq|d\Theta_i(h)(X)-\Theta_i(dh(\hat{X}))|+|\Theta_i(dh(\hat{X}))-dh_x(\hat{X})|\\&\leqq\varepsilon(i)|\Theta_i(h)|\,|X|+|\Theta_i(dh(\hat{X})-dh_x(\hat{X}))|\\&\leqq\varepsilon(i)|\Theta_i(h)|\,|X|+\varepsilon(i)|h|_{C^{1,\alpha}(M_i)}|X|\\&\leqq\varepsilon(i)|h|_{C^{1,\alpha}(M_i)}|X|.\end{aligned}$$

Proof of Lemma 3.1. Let $B(p_i,r)$ be a geodesic ball around a point p_i with radius r, which is less than $\frac{1}{2}C_0$, where C_0 is a positive constant as in 1.1 ($\Lambda_0=1$). As before, we will identify the tangent space at p_i with $\boldsymbol{R}^m$, denoted by $\exp_i$ the exponential mapping at p_i and use a harmonic coordinate system $(x_1,\cdots,x_m)$ with the properties described in 1.1. For a point x of $B(p_i,r)$ and a unit vertical vector V at x, we take $x^*\in B^m(r)$ with $\exp_i x^*=x$, and then denote by V^* the vector at x^* such that $d\exp_i(V^*)=V$. Let $\xi^*(s)$ be a unique goedesic (with respect to the pull-back metric g_i^*) starting at x^* to the direction of V^*. In terms of the coordinates $(x_1,\cdots,x_m)$, the components $\xi_l^*(s)$ of $\xi^*(s)$ satisfy:

$$\xi_l^{*\prime\prime}+\sum_{j,k=1}^m\xi_j^{*\prime}\xi_k^{*\prime}\,{}^i\Gamma_{jk}^l(\xi^*)=0\qquad(l=1,\cdots,m),$$

where $\{{}^i\Gamma_{jk}^l\}$ stands for the Christoffel symbols of the metric g_i^*. Clearly there is a positive constant C_1 scuh that

$$|\xi_l^{*\prime}|\leqq C_1,\quad|\xi_l^{*\prime\prime}|\leqq C_1,\tag{3.3}$$

$(l=1,\cdots,m)$, where C_1 is a constant depending only on m. We set $h^*:=h\circ\exp_i$. Then we have

$$\begin{aligned}&|(h^*\circ\xi^*)'(s)-(h^*\circ\xi^*)'(0)|\\&\leqq\sum_{l=1}^m|\frac{\partial h^*}{\partial x_l}(\xi^*(s))-\frac{\partial h^*}{\partial x_l}(x^*)|\,|\xi_l^{*\prime}(s)|+|\frac{\partial h^*}{\partial x_l}(x^*)|\,|\xi_l^{*\prime}(s)-\xi_l^{*\prime}(0)|\\&\leqq C_2(s^\alpha+s)|h|_{C^{1,\alpha}(M_i)},\end{aligned}\tag{3.4}$$

where C_2 is a constant depending only on m. Suppose $|dh(V)|$ does not vanish, and take two positive numbers d and d', respectively, such that

$$|dh(V)|\geqq d\ |h|_{C^{1,\alpha}(M_i)},$$

and

$$C_2(d'^\alpha+d')\leqq\frac{1}{2}d.$$

Then we get

$$\begin{aligned}|(h^*\circ\xi^*)'(s)| &\geqq |dh(V)| - |(h^*\circ\xi^*)'(s) - (h^*\circ\xi^*)'(0)| \\ &\geqq \{d - C_2(s^\alpha + s)\}\, |h|_{C^{1,\alpha}(M_i)} \\ &\geqq \frac{1}{2} d\, |h|_{C^{1,\alpha}(M_i)},\end{aligned}$$

for any $s \in [0, d']$. Therefore it follows that

$$|(h^*\circ\xi^*)(s) - h^*(x^*)| \geqq \frac{1}{2} d\, s\, |h|_{C^{1,\alpha}(M_i)} \quad (0 \leqq s \leqq d'). \tag{3.5}$$

On the other hand, we see that

$$\begin{aligned}|(h^*\circ\xi^*)(s) - h^*(x^*)| &= |h(\xi(s)) - h(x)| \quad (\xi(s) := \exp_i \xi^*(s)) \\ &\leqq |h|_{C^{1,\alpha}(M_i)} \operatorname{dis}(\xi(s), x) \\ &\leqq |h|_{C^{1,\alpha}(M_i)} \{\operatorname{diam}(\mathcal{F}_i(x)) + \operatorname{dis}(\xi(s), \mathcal{F}_i(x))\},\end{aligned} \tag{3.6}$$

where $\mathcal{F}_i(x)$ stands for the fiber of the fibration $\Phi_i: M_i \to M_\infty$ through x. Observe that

$$\operatorname{dis}(\xi(s), \mathcal{F}_i(x)) \leqq C_3 s^2 \tag{3.7}$$

because of Theorem A' (v), where C_3 is a constant depending only on the constant C in Theorem A′ (v). Thus it follows from (3.5), (3.6) and (3.7) that

$$C_3 s^2 - \frac{1}{2} d\, s + \delta_i \geqq 0 \qquad \text{for any} \quad s \in [0, d'], \tag{3.8}$$

where we have set $\delta_i := \max\{\operatorname{diam}(\mathcal{F}_i(x)) : x \in M_i\}$. Therefore we can find positive constants C_4 and C_5 depending only on C_3 in such a way that if d is less than C_4, then d must be less than $C_5 \delta_i^{1/2}$. This implies that

$$|dh(V)| \leqq 2C_5 \delta_i^{1/2}\, |h|_{C^{1,\alpha}(M_i)},$$

which proves the first assertion of the lemma. As for the second one, we claim that when $|X| = 1$, the covariant derivative of $\hat{X}$ in the direction of a unit vertical vector V is uniformly bounded by a constant C_6 depending only on the constant C as above. Indeed, for any unit vertical vector W, we have

$$\begin{aligned}|g_i(D_V\hat{X}, W)| = |g_i(\hat{X}, \mathcal{H}(D_V W)| &\leqq (1 + \varepsilon(i))\, |d\Phi_i(D_V W)| \\ &\leqq (1 + \varepsilon(i))\, |Dd\Phi_i(V, W)| \\ &\leqq C_6.\end{aligned}$$

Similarly, for any unit horizontal vector Y, we have

$$\begin{aligned}|g_i(D_V\hat{X}, Y)| &\leqq (1 + \varepsilon(i))\, |g_\infty(d\Phi_i(D_V\hat{X}), d\Phi_i(Y))| \\ &= (1 + \varepsilon(i))\, |g_\infty(Dd\Phi_i(V, \hat{X}), d\Phi_i(Y))| \\ &\leqq C_6.\end{aligned}$$

These show our claim. Now it is easy to see that the second assertion of the lemma holds to be ture. This completes the proof of Lemma 3.1.

3.2. Proof of (ii), (iii), (iv) and (vii) in Theorem B

We first carry out direct computations of the hessain $Dd\Theta_i(h)$ of $\Theta_i(h)$ and get

$$Dd\Theta_i(h)(X,X)=\Theta_i(Ddh(\hat{X},\hat{X}))+\Pi_2(h;X,X)+\Pi_3(h;X,X)+\Pi_4(h;X,X)\,, \tag{3.9}$$

where we have put

$$\begin{aligned}\Pi_2(h;X,X)&=\Theta_i(dh(B[\Phi_i](\hat{X},\hat{X})))+\Theta_i(hg_i(B[\Phi_i](\hat{X},\hat{X}),\eta_i))+\\&\quad\Theta_i(h)\Theta_i(g_i(B[\Phi_i](\hat{X},\hat{X}),\eta_i))+\Theta_i(dh(A[\Phi_i](\hat{X},\hat{X})))\,,\\ \Pi_3(h;X,X)&=2\Theta_i(dh(\hat{X})g_i(\hat{X},\eta_i))-2\Theta_i(dh(\hat{X}))\Theta_i(g_i(\hat{X},\eta_i))+\Theta_i(hg_i(\hat{X},\eta_i)^2)+\\&\quad 2\Theta_i(h)\Theta_i(g_i(\hat{X},\eta_i))^2-2\Theta_i(hg_i(\hat{X},\eta_i))\Theta_i(g_i(\hat{X},\eta_i))-\Theta_i(h)\Theta_i(g(\hat{X},\eta_i)^2),\\ \Pi_4(h;X,X)&=\Theta_i(h)\Theta_i(g_i(\hat{X},D_{\hat{X}}\eta_i))-\Theta_i(hg_i(\hat{X},D_{\hat{X}}\eta_i))\,.\end{aligned}$$

Applying Theorme A' and Lemma 3.1(i) to $\Pi_2(h;X,X)$, we have

$$|\Pi_2(h;X,X)|\leqq\varepsilon(i)\;|h|_{C^{1,\alpha}(M_i)}|X|^2\,.$$

Moreover using Theorem A$'$ as in (3.2), we see that

$$|\Pi_3(h;X,X)|\leqq\varepsilon(i)\;|h|_{C^1(M_i)}|X|^2\,.$$

Since $\Pi_4(h;X,X)=\Pi_4(h-c;X,X)$ for any constant c, it is clear that

$$|\Pi_4(h;X,X)|\leqq\varepsilon(i)\;|h|_{C^{0,\alpha}(M_i)}\Theta_i(|D\eta_i|)|X|^2\,.$$

Hence it follows from these inequalities and (3.9) that

$$|Dd\Theta_i(h)(X,X)-\Theta_i(Ddh(\hat{X},\hat{X})|\leqq\varepsilon(i)(1+\Theta_i(|D\eta_i|))|h|_{C^{1,\alpha}(M_i)}|X|^2. \tag{3.10}$$

In particular, we have

$$\begin{aligned}||Dd\Theta_i(h)||_{L^p(M_i,\mu_i)}&\leqq(1+\varepsilon(i))||\Theta_i(|Ddh|)||_{L^p(M_i,\rho_i)}+\\&\qquad\varepsilon(i)\,\{1+||\Theta_i(|D\eta|)||_{L^p(M_i,\mu_i)}\}\;|h|_{C^{1,\alpha}(M_i)}\\&\leqq(1+\varepsilon(i))||\,|Ddh|\,||_{L^p(M_i,\mu_i)}+\varepsilon(i)\;|h|_{C^{1,\alpha}(M_i\mu_i)}\,,\end{aligned}$$

where we have used Theorem A$'$ and the fact that for any $f\in C^\infty(M_i)$,

$$||\Theta_i(f)||_{L^p(M_i,\mu_i)}\leqq(1+\varepsilon(i))||f||_{L^p(M_i,\mu_i)}\,.$$

This can be also derived from Theorem A$'$. Thus we have shown the estimate (ii) in theorem B.

We are now in a position of verify the third estimate (iii) in Theorem B. For this, we denote by $\mathrm{tr}_{\mathcal{H}}Ddh$ (resp. $\mathrm{tr}_{\mathcal{V}}Ddh$) the trace of Ddh restricted to the horizontal subspace $\mathcal{H}$ (resp. the vertical subspace $\mathcal{V}$) with resepct to the fibration $\Phi_i\colon M_i\to M_\infty$. Then we observe that along each fiber of Φ_i. $\mathrm{tr}_{\mathcal{V}}Ddh+dh(\eta_i)$ coincides with the Laplacina of the restriction of h to the fiber, and hence it holds that

$$\Theta_i(\mathrm{tr}_{\mathcal{V}}Ddh+dh(\eta_i))=0\,.$$

Let $\{X_a\}_{a=1,\cdots,n}$ be orthonormal frame fields in an open set of M_∞. Then it follows from (3.9) that

$$\begin{aligned}&\mathcal{L}_\infty\Theta_i(h)-\Theta_i(\Delta_ih)=\\&\quad\Theta_i(\textstyle\sum_{a=1}^n Ddh(\hat{X}_a,\hat{X}_a)-\mathrm{tr}_{\mathcal{H}}Ddh)+\Theta_i(dh(\eta_i))+d\Theta_i(h)(\tau_\infty)+\Pi_5(h)\,,\end{aligned}$$

where we put

$$\begin{aligned}&\tau_\infty=\mathrm{grad}\log\chi_\infty\,,\\&\Pi_5(h)=\textstyle\sum_{a=1}^n\Pi_2(h;X_a,X_a)+\Pi_3(h;X_a,X_a)+\Pi_4(h;X_a,X_a)\,.\end{aligned}$$

Theorem A′(i) implies that

$$|\Theta_i(\textstyle\sum_{a=1}^n Ddh(\hat{X}_a,\hat{X}_a)-\mathrm{tr}_{\mathcal{H}}Ddh)|\leqq\varepsilon(i)\,\Theta_i(|Ddh|)\,,$$

and also Theorem A′(iii) and (3.1) show that

$$\begin{aligned}|\Theta_i(dh(\eta_i))+d\Theta_i(h)(\tau_\infty)|&\leqq|\Theta_i(dh(\eta_i+\hat{\tau}_\infty))|+\\&\qquad|\Theta_i(hg_i(\hat{\tau}_\infty,\eta_i))-\Theta_i(h)\Theta_i(g_i(\hat{\tau}_\infty,\eta_i))|\\&\leqq\varepsilon(i)\,\{\Theta_i(|h|)+\Theta_i(|dh|)\}\,.\end{aligned}$$

Moreover by (3.10), we see that

$$|\Pi_5(h)|\leqq\varepsilon(i)\,(1+\Theta_i(|D\eta_i|))\,|h|_{C^{1,\alpha}(M_i)}\,.$$

Thus we have

$$(3.11)\quad|\mathcal{L}_\infty\Theta_i(h)-\Theta_i(\Delta_ih)|\leqq\varepsilon(i)\,\{\Theta_i(|Ddh|)+(1+\Theta_i(|D\eta_i|))\,|h|_{C^{1,\alpha}(M_i)}\}\,,$$

from which Theorem B (iii) follows. Combining this with Theorem A′ (vi), we can derive the fourth assertion (iv) of the theorem.

It remains to prove the last inequality (vii) of the theorem. For this, we express the Laplacian of $(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(h)$ as follows:

$$\begin{aligned}\Delta_i(Id.-\Phi_i^*\circ\Theta_i)(h))&=(Id.-\Phi_i^*\circ\Theta_i)(\Delta_ih)+\Phi_i^*\circ(\Theta_i\circ\Lambda_i-\mathcal{L}_\infty\circ\Theta_i)(h)+\\&\qquad(\Phi_i^*\circ\mathcal{L}_\infty-\Lambda_i\circ\Phi_i^*)(\Theta_i(h))\,.\end{aligned}$$

Applying teh estimate (v) of this theorem to $(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(\Lambda_ih)$, we get

$$|(\mathrm{Id.}-\Phi_i\circ\Theta_i)(\Delta_i h)|\leqq\varepsilon(i)|\Delta_i h|_{C^{0,\alpha}(M_i)}$$

By (3.11) and Theorem A′ (ii), (vi), we can see that

$$\|\Phi_i^*\circ(\Theta_i\circ\Delta_i-\mathcal{L}_\infty\circ\Theta_i)(h)\|_{L^p(M_i,\mu_i)}\leqq\varepsilon(i)\ \{\|Ddh\|_{L^p(M_i,\mu_i)}+|h|_{C^{1,\alpha}(M_i)}\}.$$

Moreover it turns out from Theorem A′ (iii) and (3.10) that

$$\begin{aligned}|(\Phi_i^*\circ\mathcal{L}_\infty-\Delta_i\circ\Phi_i^*)(\Theta_i(h))|&\leqq\varepsilon(i)\ \{|Dd\Theta_i(h)|+|d\Theta_i(h)|\}\circ\Phi_i\\&\leqq\varepsilon(i)\ \{\Theta_i(|Ddh|)+(1+\Theta_i(|D\eta_i|))|h|_{C^{1,\alpha}(M_i)}\}\circ\Phi_i\,,\end{aligned}$$

and hence it follows from Theorem A′(ii), (vi) again that

$$\|(\Phi_i^*\circ\mathcal{L}_\infty-\Delta_i\circ\Phi_i^*)(\Theta_i h))\|_{L^p(M_i\mu_i)}\leqq\varepsilon(i)\ \{\|Ddh\|_{L^p(M_i,\mu_i)}+|h|_{C^{1.\alpha}(M_i)}\}.$$

Therefore we have

$$\|\Delta_i(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(h)\|_{L^p(M_i,\mu_i)}\leqq\varepsilon(i)\ \{|h|_{C^0(M_i)}+|\Delta_i h|_{C^{0.\alpha}(M_i)}\}.$$

Finally applying Lemma 1.2 to $(\mathrm{Id.}-\Phi_i\circ\Theta_i)(h)$, we get

$$\begin{aligned}&\|(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(h)\|_{L^p(M_i,\mu_i)}\\&\quad\leqq C\,\{(\|(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(h)\|_{L^p(M_i,\mu_i)}+\|\Delta_i(\mathrm{Id.}-\Phi_i^*\circ\Theta_i)(h)\|_{L^p(M_i,\mu_i)}\}\\&\quad\leqq\varepsilon(i)\ \{|h|_{C^0(M_i)}+|\Delta_i h|_{C^0(M_i)}\}.\end{aligned}$$

This completes the proof of Theorem B.

3.3. Remark

Let M_i be a compact Riemannian manifold as in Theorem A. Let ω_i be a positive smooth function on M_i such that

$$\int_{M_i}\omega\,d\mu_i=1\,.$$

and

$$|Dd\omega_i|\leqq c\,,$$

on M_i, where c is a positive constant (independent of i). Suppose that M_i equipped with the measure $\omega_i\mu_i$ converges as i goes to infinity to a compact metric space M_∞ with a measure ξ_∞ with respect to the measured Hausdorff topology, and suppose that M_∞ is a smooth manifold. Then the density ω_∞ of the limit measure ξ_∞ is of class $C^{1,\alpha}$ (for any $\alpha\in(0,1)$). Moreover if the Laplace operator Δ_i of M_i and the operator $\mathcal{L}_\infty$ on M_∞ considered so far are respectively replaced with the following operators:

$$\begin{aligned}\mathcal{L}_i f&=\Delta_i f+\mathrm{grad}\,(\log\omega_i)\,(f)\ ;\\\mathcal{L}_\infty h&=\Delta_\infty h+\mathrm{grad}\,(\log\omega_\infty)\,(h)\,,\end{aligned}$$

then the same assertions as in Theorems A and B hold. This fact will be used in [23, II].

4. Energy spectrum of harmonic mappings into nonpositively curved manifolds

In this section, we study the energy spectrum of harmonic mappings into manifolds with nonpositive sectional curvature, and as an application of Theorems A and B, we shall show certain continuity of the energy spectrum in the topology of measured Hausdorff convergence.

4.1. Review on some basic results on harmonic mappings

Let $M=(M, g)$ and $N=(N, h)$ be two compact Riemannian manifold. Given a smooth mapping ϕ of M into N, the energy density $e(\phi)$ is a function defined by the trace of the induced tensor ϕ^*h with respect to the metric g, and the energy $E(\phi)$ of ϕ is given by

$$E(\phi) = \int_M e(\phi)\, \mathrm{dvol}_g .$$

A smooth mapping $\phi: M \to N$ is said to be harmonic if the energy functional E is stationary at ϕ, or equivalently if the tension filed $\tau(\phi)$ vanishes.

From now on we assume that N has nonpositive sectional curvature. A fundamental theorme due to Eells and Sampson [8] asserts that any smooth mapping $\phi: M \to N$ is homotopic to a harmonic mapping which has minimum energy in its homotopy class. In addition, Hartman [18] showed a uniqueness theorem saying that if ϕ_0 and ϕ_1 are homotopic harmonic mappings, then they are smoothly homotopic through harmonic mappings; and the energy is constant on any arcwise connected set of harmonic mappings (in fact, if $\{\phi_s : s \in [0, 1]\}$ is a smooth family of harmonic mappings joining ϕ_0 with ϕ_1, then the energy density $e(\phi_s)(x)$ at a point $x \in M$ is independent of s). We denote by $\mathcal{H}(M, N)$ the set of all harmonic mappings of M into N, and consider the energy spectrum $\{E(\phi): \phi \in \mathcal{H}(M, N)\}$. In view of the above results, we may set $E(C) = E(\phi)$ for a component C of $\mathcal{H}(M, N)$, where ϕ belongs to C. According to Adachi and Sunada [2], there are explicit positive constants C_1 and C_2 depending only on the diameters $\mathrm{diam}(M)$, $\mathrm{diam}(N)$, the volumes $\mathrm{Vol}(M)$, $\mathrm{Vol}(N)$, and the lower bounds on the Ricci curvature of M, N such that

$$\sharp\{C \subset \mathcal{H}(M, N): E(C) \leqq \lambda^2\} \leqq C_1 \exp C_2 \lambda$$

for any λ. For the latter purpose, we put the connected components $\{C_0, C_1, C_2, \cdots\}$ of $\mathcal{H}(M, N)$ in order as follows:

$$E(C_\nu) \leqq E(C_\nu), \qquad \text{if} \quad \nu \leqq \nu'$$

(hence C_0 consists of constant mappings). In addition, we set

$$\sigma_\nu(M, N) = E_{\mu_M}(\mathcal{C}_\nu),$$

where μ_M stands for the canonical probability measure $\mathrm{Vol}(M)^{-1}\,\mathrm{dvol}_g$ of M and

$$E_{\mu_M}(\phi) = \int_M e(\phi)\, d\mu_M .$$

Here for convenience, we understand $\sigma_\nu(M, N)=\infty$ if $\nu-1$ is greater than the number of the components of $\mathcal{H}(M, N)$.

Before stating our result in this section, we shall recall briefly the Eells -Sampson's method to prove the above existence theorem. Given a C^1 mapping $\phi_0\colon M\to N$, we consider solutions of the heat equation

$$\begin{aligned} \frac{\partial \phi_t}{\partial t} &= \tau(\phi_t) \\ \phi_t|_{t=0} &= \phi_0 . \end{aligned} \tag{4.1}$$

Then they showed that this equation has a unique solution (ϕ_t) defined for all $t\geqq 0$; ϕ_t is continuous at $t=0$, along with its first order space derivatives, and moreover that the limit

$$\phi_\infty(x) = \lim_{t\to\infty} \phi_t(x)$$

exists C^k uniformly (for all $k\geqq 0$) and defines a harmonic mapping $\phi_\infty\colon M\to N$ homotopic to ϕ_0. An important observation in their proof of this result is that the energy density $e(\phi_t)$ of a solution (ϕ_t) satisfies

$$\frac{\partial e(\phi_t)}{\partial t} - \Delta e(\phi_u) \leqq 2(m-1)\rho^2\, e(\phi_t),$$

where a positive constant ρ is taken in such a way that the Ricci curvature of M is bounded below by $-(m-1)\rho^2$. This derives the following estimates:

$$\begin{aligned} &\sup e(\phi_t) \leqq C_3 \sup e(\phi_0) &&\text{for}\quad t\in[0, 1], \\ &\sup e(\phi_t) \leqq C_4 \int_M e(\phi_0)\,\mathrm{dvol}_g &&\text{for}\quad t\geqq 1, \end{aligned}$$

where C_3 and C_4 are positive constants depending only on M and N. In fact, when the diameter $\mathrm{diam}(M)$ of M is bounded from above by a constant D, we have a explicit positive constant $C(m, \rho, D)$ depending only on m, ρ and D such that $C_3\leqq C(m, \rho, D)$ and $C_4\leqq C(m, \rho, D)/\mathrm{Vol}(M)$. This can be verified by the same arguments as in [8, §9] together with the heat kernel estimate due to Li and Yau [24, Corollary 3.1] and the Bishop-Gromov's volume comparison theorem [16, 5.3. bis Lemme]. Hence we have

$$\begin{aligned} &\sup e(\phi_t) \leqq C(m, \rho, D) \sup e(\phi_0) &&\text{for}\quad t\in[0, 1], \\ &\sup e(\phi_t) \leqq C(m, \rho, D)\, E_{\mu_M}(\phi_0) &&\text{for}\quad t\geqq 1 . \end{aligned} \tag{4.2}$$

This implies in particular that for a harmonic mapping $\phi\in\mathcal{H}(M, N)$,

$$\sup e(\phi)\leqq C(m, \rho, D)\, E_{\mu_M}(\phi)\,, \tag{4.3}$$

which is observed also in [2].

4.2. Statements of results

We are now in a position to state

Theorem 4.1. *Let $\{M_i\}_{i=1,2,\dots}$ be a sequence of m-dimensional compact Riemannian manifolds the sectional curvature of which is bounded in the absolute value by 1. Suppose M_i with the canonical measure $\mu_i=\mu_{M_i}$ converges to a compact metric space M_∞ with a Borel measure μ_∞ of unit mass with respect to the measured Hausdorff topology. Let N be a compact Riemannian manifold of nonpositive sectional curvature. Then for any ν, the limit of $\sigma_\nu(M_i, N)$ as i goes to infinity exists if $\omega=\infty$ or if $\omega<\infty$ and $\nu<\omega$; and $\sigma_\nu(M_i, N)$ diverges otherwise, where ω stands for the number of the arcwise connected components of continuous mappings of M_∞ into N if it is finite, and $\omega=\infty$ otherwise.*

Before the proof of the theorem, we make some remarks. Let M_∞, μ_∞ and N be as above and let g_∞ be the Riemannian metric of class $C^{1,\alpha}(\alpha\in(0, 1))$ on $M_\infty\backslash\Sigma$, where Σ denotes the singular set of M_∞. Then for any Lipschitz mapping $\phi\in C^{0,1}(M_\infty, N)$, the energy density $e(\phi)$ can be defined almost everywhere on $M_\infty\backslash\Sigma$ and it is bounded there. Hence we can define the energy of ϕ with respect to the measure μ_∞ by

$$E_{\mu_\infty}(\phi)=\int_{M_\infty\backslash\Sigma} e(\phi)\mu_\infty\,,$$

which may be written as

$$E_{\mu_\infty}(\phi)=\int_{M_\infty} e(\phi)\mu_\infty\,,$$

since $\mu_\infty(\Sigma)=0$. Then correspondingly to the Eells-Sampson's and Hartman's results, we have the following

Assertion 4.2. *There exists a Lipschitz mapping ϕ in a given homotopy class θ of $C^{0,1}(M_\infty, N)$ such that*

$$E_{\mu_\infty}(\phi)=\inf\{E_{\mu_\infty}(\phi)\colon \phi\in\theta\}\,. \tag{4.4}$$

Such a mapping ϕ is of class $C^{2,\alpha}(\alpha\in(0, 1))$ on $M_\infty\backslash\Sigma$, and it satisfies there

$$\tau_\infty(\phi)+d\phi(\operatorname{grad}\log \chi_\infty)=0\,, \tag{4.5}$$

where $\tau_\infty(\phi)$ denotes the tension field of the mapping $\phi\colon M_\infty\backslash\Sigma\to N$, and χ_∞ stands for the density of the measure μ_∞ with respect to the Riemannian measure $dvol_{g_\infty}$ as in §1.4; *in addiiton,*

$$e(\phi)\leqq C\, E_{\mu_\infty}(\phi)\,, \tag{4.6}$$

on $M_\infty\backslash\Sigma$, where C is a positive constant depending only on m and the diameter of M_∞. Moreover the set of Lipschitz mappings satisfying (4.4) *is arcwise connected.*

When the limit space M_∞ is a smooth manifold, we may assume that the metric g_∞ and the density χ_∞ of the measure μ_∞ are also smooth, because we can approximate them respectively by smooth ones as mentioned in 1.3 and 1.4. Hence the above assertion (except that the constant C in (4.6) depends only on the dimension m and the diameter of M_∞) can be derived from the Eells-Sampson's and Hartman's results. See [25] or [9, (10.20)] for details. Letting $G_\varepsilon(\varepsilon>0)$ be Riemannian metrics on the product manifold $M_\infty\times S^1$ of M_∞ and the unit circle $S^1=(S^1, d\theta^2)$ defined by

$$G_\varepsilon = g_\infty+\varepsilon\,\chi_\infty^2 d\theta^2\,,$$

we note that a map $\phi\colon M_\infty\to N$ satisfies equation (4.5) if and only if $\phi\circ\pi\colon M_\infty\times S^1\to N$ is harmonic with respect to G_ε for any $\varepsilon>0$, where $\pi\colon M_\infty\times S^1\to M_\infty$ denotes the canonical projection. Assertion 4.2 in the general case will be verified in the proof of Theorem 4.1.

In view of Assertion 4.2, we set

$$E_{\mu_\infty}(\theta) = \inf\{E_{\mu_\infty}(\psi)\colon \psi\in\theta\}$$

for a connected component θ of $C^{0,1}(M_\infty, N)$. Then we can put as before the components $\{\theta_0, \theta_1, \theta_2, \cdots\}$ in order so that

$$E_{\mu_\infty}(\theta_\nu)\leqq E_{\mu_\infty}(\theta_{\nu'}) \qquad \text{if} \quad \nu\leqq\nu'\,.$$

Theorem 4.1 implies actually that for each ν,

$$\lim_{i\to\infty} \sigma_\nu(M_i, N) = \sigma_\nu(M_\infty, \mu_\infty, N)\,,$$

where we set

$$\sigma_\nu(M_\infty, \mu_\infty, N) = E_{\mu_\infty}(\theta_\nu)\,.$$

4.3. Proof of Theorem 4.1: the case M_∞ is smooth

Let (M_i, μ_i) and (M_∞, μ_∞) be as in Theorem 4.1. We shall first consider the case the limit space M_∞ is a smooth manifold. Let $\Phi_i\colon M_i\to M_\infty$ be a smooth submersion over M_∞ as in Theorem A, and let $\Phi_i^*\colon [M_\infty, N]\to[M_i, N]$ be the induced map of the set of homotopy classes $[M_\infty, N]$ of Lipschitz mappings $C^{0,1}(M_\infty, N)$ into the set of homotopy classes $[M_i, N]$ of Lipschitz mappings $C^{0,1}(M_i, N)$ defined by $\Phi_i^*([\phi])=[\Phi_i^*(\phi)]$ for $\phi\in C^{0,1}(M_\infty, N)$.

We take first a sequence $\{\varepsilon(i)\}$ converging to zero in such a way that the diameter of the fiber $\Phi_i^{-1}(z)$ over a point z of M_∞ is bounded from above by $\varepsilon(i)$.

Let ψ be a smooth mapping of M_t into N, and let ψ_t be the solution of equation (4.1) with $\psi_0=\psi$. Then it follows from (4.2) that for every $t\in(0, \infty]$, the energy density $e(\psi_t)$ satisfies

$$\begin{aligned} e(\psi_t)&\leqq C_5 \sup e(\psi) &\quad \text{if} \quad t\in(0, 1] \\ e(\psi_t)&\leqq C_5\, E_{\mu_i}(\psi) &\quad \text{if} \quad t\in[1, \infty] \end{aligned}$$

on M_i, where C_0 is a positive constant depending only on m and an upper bound, say D, for the diameters of M_i. Therefore for all $t\geqq 1$ and for every $z\in M_\infty$ and all $x, y\in\Phi_i^{-1}(z)$, we have

$$\mathrm{dis}_N(\psi_t(x),\ \psi_t(y))\leqq \varepsilon(i)C(m, D)E_{\mu_i}(\psi)^{1/2}\,.$$

Let $w\colon N\to \boldsymbol{R}^q$ be an isometric embedding of N into some Euclidean pace $\boldsymbol{R}^q$. Then we have a smooth family of mappongs $\Theta_i(w\circ\psi_i)$ of M_∞ into $\boldsymbol{R}^q$ satisfying

$$|\Theta_i(w\circ\psi_t)\circ\Phi_i - w\circ\psi_t|\leqq\begin{cases}\varepsilon(i)\, C_5 \sup e(\psi)^{1/2} & \text{if} \quad t\in(0, 1]\,,\\ \varepsilon(i)\, C_5\, E_{\mu_i}(\psi)^{1/2} & \text{if} \quad t\in[1, \infty]\,.\end{cases}$$

Let N_r be a tubular r-neighborhood of $w(N)$ in $\boldsymbol{R}^q$ such that the projection map $\Pi\colon N_r\to N$ can be defined. In what follows, we fix a positive constant K and we assume that $E_{\mu_i}(\psi)$ is less than K. Take large i so that $\varepsilon(i)C(m, D)K^{1/2}$ is less than r, then we can define a smooth family of mappings $A_i(\psi_t)$ $(t\in[1, \infty])$ of M_∞ into N by

$$A_i(\psi_t) := \Pi\circ\Theta_i(\psi_t)\,.$$

We observe first that

$$\mathrm{dis}_N(A_i(\psi_t)\circ\Phi_i(x),\ \psi_t(x))\leqq\varepsilon(i)\tau C_5\, E_{\mu_i}(\psi)^{1/2} \tag{4.7}$$

for all $x\in M_i$, where τ is a positive constant depending only on the embedding $w\colon N\to\boldsymbol{R}^q$. Here we assume that $\varepsilon(i)\tau C(m, D)K^{1/2}$ is less than the injectivity radius of N. Then (4.7) implies that for each $t\in[1, \infty]$, $\Phi_i^*(A_i(\psi_t))=A_i(\psi_t)\circ\Phi_i$ is homotopic to ψ_t and hence to ψ, namely,

$$\Phi_i^*([A_i(\psi_t)]) = [\psi]\,. \tag{4.8}$$

Secondly we notice that

$$E_{\mu_\infty}(A_i(\psi_\infty))\leqq(1+\varepsilon'(i))E_{\mu_i}(\psi_\infty)+\varepsilon'(i)\,, \tag{4.9}$$

and hence

$$E_{\mu_\infty}([A_i(\psi_\infty)])\leqq(1+\varepsilon'(i))E_{\mu_i}([\psi])+\varepsilon'(i)\,, \tag{4.10}$$

where $\varepsilon'(i)$ is a positive constant which goes to zero as i tends to infinity. Indeed, it follows from Theorem B (vi) that

$$|\Phi_i^*(\Theta_i(w\circ\psi_\infty))-w\circ\psi_\infty|_{C^1(M_i)}\leqq\varepsilon''(i)\,|w\circ\psi_\infty|_{C^{1,\alpha}(M_i)}\,,$$

where $\varepsilon''(i)$ is as in Theorem B and $\alpha\in(0,1)$. Since ψ_∞ is harmonic, it satisfies

$$\Delta_i w\circ\psi_\infty=\operatorname{tr} S(dw\circ\psi_\infty,\ dw\circ\psi_\infty)\,,$$

where S stands for the second fundamental form of the embedding $w\colon N\to\boldsymbol{R}^q$. Therefore it turns out from the elliptic regularity estimates that

$$|w\circ\psi_\infty|_{C^{2,\alpha}(M_i)}\leqq C_6(E_{\mu_i}(\psi)^{1/2}+1) \tag{4.11}$$

for some constant $C_6>0$ depending only on $m, D, \alpha\in(0,1)$ and the embedding $w\colon N\to\boldsymbol{R}^q$, and hence

$$|\Phi_i^*(\Theta_i(w\circ\psi_\infty))-w\circ\psi_\infty|_{C^1(M_i)}\leqq\varepsilon''(i)\,C_6(E_{\mu_i}(\psi)^{1/2}+1)\,,$$

Now (4.9) is clear, because of the properties of the projection Π and the fibration Φ_i.

Given a homotopy class $\theta\in[M_\infty,N]$, let $\phi\colon M_\infty\to N$ be a smooth mapping of M_∞ into N in this class. Then by Theorem A (i) ,we see that

$$E_{\mu_i}(\Phi_i^*\phi)\leqq(1+\varepsilon^{(3)}(i))E_{\mu_i}(\phi)\,,$$

where $\varepsilon^{(3)}(i)$ is a positive constant as in Theorem A, and hence we have

$$E_{\mu_i}(\Phi_i^*\theta)\leqq(1+\varepsilon^{(3)}(i))\,E_{\mu_\infty}(\theta)\,. \tag{4.12}$$

Now we set $[M_i,N]_K:=\{\theta\in[M_i,N]\colon E_{\mu_i}(\theta)<K\}$ and also $[M_\infty,N]_K:=\{\theta\in[M_\infty,N]\colon E_{\mu_\infty}(\theta)<K\}$. Let $\phi\colon M_\infty\to N$ be a mapping in a class $\theta\in[M_\infty,N]_K$ satisfying $E_{\mu_\infty}(\phi)=E_{\mu_\infty}(\theta)$. Let $(\Phi_i^*\phi)_t$ be the solution of equation (4.1). Since the energy density $e(\phi)$ is bounded by $C_7E_{\mu_\infty}(\phi)$ for some constant $C_7>0$ depending only on M_∞, taking sufficiently large i, we may assume that $A_i((\Phi_i^*\phi)_t)$ can be defined for all $t\in(0,\infty]$. We notice that in this case, $A_i((\Phi_i^*\phi)_t)$ converges to the given mapping ϕ as t goes to zero. This implies that $\theta=[\phi]=[A_i((\Phi_i^*\phi)_\infty)]$. Thus by (4.12), we obtain

$$\begin{aligned}E_{\mu_i}(\Phi_i^*\theta)&=E_{\mu_i}([(\Phi_i^*\phi)_\infty])\\&\geqq(1+\varepsilon'(i))^{-1}\{E_{\mu_\infty}([A_i((\Phi_i^*\phi)_\infty)]+\varepsilon'(i)\}\\&=(1+\varepsilon'(i))^{-1}(E_{\mu_\infty}(\theta)+\varepsilon'(i))\,.\end{aligned} \tag{4.13}$$

Hence we have by (4.12) and (4.13)

$$|E_{\mu_\infty}(\theta)-E_{\mu_i}(\Phi_i^*(\theta))|\leqq\delta(i)\,(E_{\mu_\infty}(\theta)+1) \tag{4.14}$$

for all $\theta\in[M_\infty,N]_K$ and for large i, where $\{\delta(i)\}$ is a sequence of positive constants which converges to zero as i goes to infinity and which is independnet of K.

Moreover for two classes θ and $\theta'\in[M_\infty, N]_K$ take mappings ϕ and ϕ' of M_∞ into N satisfying $E_{\mu_\infty}(\phi)=E_{\mu_\infty}(\theta)$ and $E_{\mu_\infty}(\phi')=E_{\mu_\infty}(\theta')$ respectively. Suppose $\Phi_i^*\phi$ is homotopic to $\Phi_i^*\phi'$. Then we have a smooth family of harmonic mappings $\psi_s(0\leqq s\leqq 1)$ joining $\psi_0=(\Phi_i^*\phi)_\infty$ to $\psi_1=(\Phi_i^*\phi')_\infty$ which gives rise to a homotopy of the smooth mappings $A_i(\psi_s)$ between $A_i((\Phi_i^*\phi)_\infty)$ and $A_i((\Phi_i^*\phi')_\infty)$ for large i. This shows that ϕ is homotopic to ϕ', namely $\theta=\theta'$. Thus in view of this observation, (4.8) and (4.14), we can deduce that if K does not belong to the energy spectrum $\{\sigma_\nu(M_\infty, \mu_\infty, N)\}$, then for large i, Φ_i^* actually induces a bijection between $[M_\infty, N]_K$ and $[M_i, N]_K$ satisfying (4.14). This shows the assertion of Theorem 4.1 in the case M_∞ is smooth.

We shall here make an observation on the convergence of harmonic mappings considered. Fix first a positive integer ν and set

$$C_\nu^{(i)} := \{\phi\in\mathcal{H}(M_i, N)\colon E_{\mu_i}(\phi) = \sigma_\nu(M_i, N)\}$$
$$C_\nu^{(\infty)} := \{\phi\in C^{2,\alpha}(M_\infty, N)\colon E_{\mu_\infty}(\phi) = \sigma_\nu(M_\infty, \mu_\infty, N)\}$$

Here it is assumed that $C_\nu^{(i)}$ corresponds to $C_\nu^{(\infty)}$ via the fibration $\Phi_i\colon M_i\to M_\infty$. Then for large i and for any $\phi\in C_\nu^{(i)}$, we have a smooth mapping $A_i(\phi)\colon M_\infty\to N$ $(\subset \boldsymbol{R}^q)$. It follows from (4.9) and Theorem B (ii) that

$$||w\circ A_i(\phi)||_{W^{2,p}(M_\infty)}\leqq C_8$$

for some constant $C_8>0$, where $p\in(1, \infty)$. Hence taking account of (4.9), we see that for any neighborhood of $C_\nu^{(\infty)}$ in the $C^{1,\alpha}$ topology, $A_i(C_\nu^{(i)})$ lies in the neighborhood for large i.

4.4. Proof of Theorem 4.1: the case M_∞ is not smooth

In this case, as mentioned in 0.3, we consider the frame bundle FM_i of M_i equipped with the canonical Riemannian metric. We first remark that for a harmonic mapping $\phi\in\mathcal{H}(M_i, N)$, the pull-back $\pi_i^*\phi$ of ϕ by the bundle projection $\pi_i\colon FM_i\to M_i$ is also harmonic, namely $\pi_i^*\phi\in\mathcal{H}(FM_i, N)$, and moreover for two harmonic mappings ϕ and $\psi\in\mathcal{H}(M_i, N)$, they are homotopic if so are the pull-backs $\pi_i^*\phi$ and $\pi_i^*\psi$, because all the fibers of FM_i are totally geodesic. In this sense, the space $\mathcal{H}(M_i, N)$ and its connected components $\{C_\nu^{(i)}\}$ can be identified respectively with the subspace of $\mathcal{H}(FM_i, N)$ which consists of $O(m)$-invariant harmonic mappings and the subset of the connected components of $\mathcal{H}(FM_i, N)$.

Now we assume that FM_i with its canoncal probability measure $\tilde{\mu}_i$ converges to a metric space F_∞ with a Borel measure $\tilde{\mu}_i$ with respect to the measured Hausdorff topology. Then F_∞ is actually a smooth manifold with $C^{1,\alpha}$ metric on which the orthogonal group $O(m)$ acts as isometries in such a way that M_∞ is isometric to the quotient space $FM_\infty/O(m)$. In what follows, we will identify the space of Lipschitz mappings $C^{0,1}(M_\infty, N)$ of M_∞ into N with $O(m)$-invariant

Lipschitz mappings in the space $C^{0,1}(F_\infty, N)$. Let $f_i\colon M_i \to N$, $\hat{f}_i\colon FM_\infty \to N$ and $\tilde{\Phi}_i\colon FM_i \to N$ be as in 0.3. Then as we have just shown, for a given constant $K>0$ which does not belongs to the energy spectrum $\{\sigma_\nu(FM_\infty, \mu_\infty, N)\}$, $\tilde{\Phi}_i$ induces a bijection of $[F_\infty, N]_K$ onto $[FM_i, N]_K$ for large i. Let θ be a homotopy class in $[F_\infty, N]_K$ which contains an $O(m)$-invariant Lipschitz mapping ϕ. Then for all $z \in FM_i$ and $a \in O(m)$, we have

$$\begin{aligned}
\mathrm{dis}_N(\tilde{\Phi}_i^*\phi(z), \tilde{\Phi}_i^*\phi(az)) &= \mathrm{dis}_N(\phi(\tilde{\Phi}_i(z)), \phi(\tilde{\Phi}_i(az))) \\
&\leqq \mathrm{dis}_N(\phi(\tilde{\Phi}_i(z)), \phi(\tilde{f}_i(z))) + \mathrm{dis}_N(\phi(\tilde{f}_i(z)), \phi(\tilde{f}_i(az))) \\
&\quad + \mathrm{dis}_N(\phi(\tilde{f}_i(az)), \phi(\tilde{\Phi}_i(az))) \\
&\leqq \mathrm{dil}\,\phi \cdot \mathrm{dis}_{F_\infty}(\tilde{\Phi}_i(z), \tilde{f}_i(z)) + \mathrm{dis}_N(\phi(\tilde{f}_i(z)), \phi(a\tilde{f}_i(z))) \\
&\quad + \mathrm{dil}\,\phi \cdot \mathrm{dis}_{F_\infty}(\tilde{f}_i(az), \tilde{\Phi}_i(az)) \\
&\leqq 2\,\mathrm{dil}\,\phi \cdot \delta'(i)\,,
\end{aligned}$$

where dil ϕ denotes the dilatation of ϕ and $\{\delta'(i)\}$ is a sequence of positive constants with $\lim_{i\to\infty} \delta'(i)=0$ such that

$$\mathrm{dis}_{F_\infty}(\tilde{\Phi}_i(z), \tilde{f}_i(z)) \leqq \delta'(i)$$

for all $z \in FM_i$. Hence for large i so that $2\,\mathrm{dil}\,\phi \cdot \delta'(i)$ is less than the injectivity radius of N, $\tilde{\Phi}_i^*\phi$ is homotopic to some $O(m)$-invariant mapping. This implies in particular that the harmonic mappings belonging to the homotopy class $\tilde{\Phi}_i^*\theta$ are all $O(m)$-invariant. On the other hand, let ϕ be an $O(m)$-invariant harmonic mapping in $[FM_i, N]_K$. Then as in 4.3, for large i, we can define a smooth mapping $A_i(\phi)$ of F_∞ into N and we see that $A_i(\phi)$ is almost $O(m)$-invariant, namely,

$$\mathrm{dis}_N(A_i(\phi)(z), A_i(\phi)(az)) \leqq \delta''(i)$$

for all $z \in F_\infty$ and $a \in O(m)$, where $\delta''(\mathrm{i})$ is a sequence of positive constants with $\lim_{i\to\infty} \delta''(\mathrm{i})=0$. Therefore the homotopy class θ to which $A_i(\phi)$ belongs contains some $O(m)$-invariant Lipschitz mappong, if i is so large that $\delta''(\mathrm{i})$ is less than the injectivity radius of N. Moreover as we noted at the end of 4.3, $A_i(\phi)$ is close to the space of mappings ψ in θ such that $E_{\tilde{\mu}_\infty}(\psi)=E_{\tilde{\mu}_\infty}(\theta)$.

Now it is not hard to see from these observations that the assertions of Theorem 4.1 and also Assertion 4.2 hold even in the case the singular set Σ of M_∞ is not empty.

4.5. Remark

In Theorem 4.1, we fix the target manifold N. However, it is possible to relax this condition as follows:

Let $\{N_j\}_{j=1,2,\dots}$ be a sequence of compact Riemannian manifolds of dimension n such that

(i) $-a^2 \leqq$ the sectional curvature of $N_j \leqq 0$;

(ii) the diameter of $N_j \leqq b$;

(iii) the injectivity radius of $N_j \geqq c$

for positive constants a, b and c. Suppose N_j converges to a metric space N_∞ with respect to the Gromov-Hausdorff distance. Let M_i be as in Theorem 4.1. Then for each ν,

$$\lim_{i,j\to\infty} \sigma_\nu(M_i, N_j) = \sigma_\nu(M_\infty, \mu_\infty, N_\infty) .$$

Here we recall that N_∞ is actually a smooth manifold of dimention n with metric of class $C^{1,\alpha}$ (for any $\alpha \in (0, 1)$) and there exist $C^{1,\alpha}$ diffeomorphisms $h_j: N_\infty \to N_j$ such that the metrics $h_j^* N_j$ converge to the metric of N_∞ in the $C^{1,\beta}$ topology $(0<\beta<\alpha)$. Hence we can define the energy spectrum $\{\sigma_\nu(M_\infty, \mu_\infty, N_\infty)\}$ as before.

Appendix

We consider a certain family of local smooth maps defined on open subsets of a complete Riemannian manifold into another manifold and provide some estimates for its center-of-mass up to the third derivatives in Lemmas A.1$\sim$A.3. which are used to prove Theorem A$'$ in Section 2. The results in these lemmas are able to be derived from the standard results on the derivatives of exponential mappings of Riemannian manifolds. We will prove them here for the completeness.

Let M and N be complete Riemannian manifolds. Suppose we have a locally finite open covering $\{U_\alpha\}_{\alpha\in A}$ of N and a family of smooth maps $f_\alpha: U_\alpha \to M$. Let us take a partition of unity $\{\xi_\alpha\}_{\alpha\in A}$ subordinate to the covering $\{U_\alpha\}_{\alpha\in A}$. In what follows, we will assume that the conditions stated below hold:

(C.1) the sectional curvature of M is bounded by a positive constant Λ_0^2 in its absolute value, and the curvature tensor R of M satisfies: $|DR| \leqq \Lambda_1$ and $|DDR| \leqq \Lambda_2$ for some positive constants Λ_1 and Λ_2;

(C.2) the injectivity radius of M is greater than or equal to a positive constant ι;

(C.3) if we denote by δ the supremum of the distance between $f_\alpha(x)$ and $f_\beta(x)$ when $x \in N$ and indices α, β range so that $x \in U_\alpha \cap U_\beta$, then $\delta \leqq r_0$, where $r_0 := \min\{\iota, \pi/4\Lambda_0\}$;

(C.4) the multiplicity of the covering $\{U_\alpha\}_{\alpha\in A}$ is less than or equal to a positive integer ν;

(C.5) there are positive constants Γ_1 and Γ_2 such that for each α, $|d\xi_\alpha| \leqq \Gamma_1$ and $|Dd\xi_\alpha| \leqq \Gamma_2$.

Now by (C.2), (C.3) and (C.4), we have uniquely a smooth map $F: N \to M$, called a center-of-mass of $\{f_\alpha\}$ with weight $\{\xi_\alpha\}$, such that for each $x \in N$,

$$\sum_{\alpha \in A} \xi_\alpha(x) \exp^{-1}_{F(x)} f_\alpha(x) = 0 ,$$

where $\exp_q$ stands for the exponential map of M at a point q (cf. e.g., [5: §8]). For the sake of convenience, we set

$$E(p, q) = \exp_p^{-1} q \in T_p M$$

for $(p, q) \in M \times M$ with $d(p, q) < r_0$. Then the definition of the map F reads

$$\sum_{\alpha \in A} \xi_\alpha(x)\, E(F(x), f_\alpha(x)) = 0 . \tag{A.1}$$

In what follows, we will identify the tangent space of the product manifold $M \times M$ at a point (p, q) with the direct sum $T_p M \oplus T_q M$ of the tangent spaces $T_p M$ and $T_q M$ of M at the points p and q.

To begin with, we want to estimate the differential dF of F. Let us first fix a point x_0 of N and then take a neighborhood V of x_0 in N and a neighborhood B of $F(x_0)$ in M in such a way that $F(V) \subset B$ and there are orthonormal frame fields $\{e_1, \cdots, e_m\}$ $(m := \dim M)$ over B. Define smooth functions $E_i: B \times B \to \boldsymbol{R}$ $(i = 1, \cdots, m)$ by

$$E_i(p, q) = \langle E(p, q), e_i(p) \rangle .$$

To give estimates for the derivatives dE_i and DdE_i, we define one-forms ω_i and covariant tensors Ω_i of degree 2 $(i = 1, \cdots, m)$ on $B \times B$ by

$$\begin{aligned} \omega_i(T \oplus V) &= \langle -T + P_{qp}(V), e_i(p) \rangle + \langle E(p, q), D_T e_i(p) \rangle , \\ \Omega_i(T \oplus V, U \oplus W) &= \langle -T + P_{qp}(V), D_U e(p) \rangle + \langle -U + P_{qp}(W), D_T e(p) \rangle + \\ &\quad \langle E(p, q), \frac{1}{2}\{DDe_i(T, U) + DDe_i(U, T)\} \rangle , \end{aligned}$$

where $(p, q) \in B \times B$, $T, U \in T_p M$, $V, W \in T_q M$, and P_{qp} denotes the parallel displacement from $T_q M$ to $T_p M$ along a unique minimal geodesic joining q with p. Then for all $p, q \in B$, tangent vectors $T, U \in T_p M$ and tangent vectors $V, W \in T_q M$, we have

$$|dE_i(T \oplus V) - \omega_i(T \oplus V)| \leqq C\, \Lambda_0^2 \operatorname{dis}(p, q)^2 \, (|T| + |V|) , \tag{A.2}$$

$$\begin{aligned} &|DdE_i(T \oplus V, U \oplus W) - \Omega_i(T \oplus V, U \oplus W)| \leqq \\ &\quad C\{\Lambda_0^2 \operatorname{dis}(p, q)^2 \, |De_i(p)| (|T||U| + |U||V| + |T||W|) + \\ &\quad (\Lambda_1 \operatorname{dis}(p, q) + \Lambda_0^2)(1 + \Lambda_0^2 \operatorname{dis}(p, q)^2) \times \\ &\qquad \operatorname{dis}(p, q)(|T||U| + |U||V| + |T||W| + |V||W|)\} , \end{aligned} \tag{A.3}$$

where C is a positive constant depending only on $r_0 \Lambda_0$. We omit the proofs for these estimates, since they can be verified by the standard comparison arguments. Now we return to (A.1), from which it follows that

$$\sum_{\alpha\in A}\xi_\alpha E_i(F,f_\alpha)=0 \qquad (i=1,\cdots,m) \tag{A.4}$$

on V, and hence by differentiating (A.4), we obtain

$$\sum_{\alpha\in A} d\xi_\alpha(X)\,E_i(F,f_\alpha)+\xi_\alpha\,dE_i(dF(X)\oplus df_\alpha(X))=0$$

for any $x\in V$ and $X\in T_xN$. Then applying (A.2) to dE_i, and using (A.4), (C.3), (C.4) and (C.5), we see that

$$\begin{aligned}|\langle dF(X)-\textstyle\sum_{\alpha\in A}\xi_\alpha(x)P_\alpha(df_\alpha(X)),\ e_i(F(x))\rangle|\leqq\\ \nu\{\Gamma_1\delta+C\Lambda_0^2\delta(|dF|(x)+\textstyle\sum_\alpha'|df_\alpha|(x))\}\,|X|,\end{aligned} \tag{A.5}$$

for $X\in T_xN$ $(x\in V)$. Here and after, $P_\alpha(df_\alpha(X))$ denotes the parallel displacement of $df_\alpha(X)$ along a unique minimal geodesic joining $f_\alpha(x)$ to $F(x)$, and C's stand for positive constants depending only on $r_0\Lambda_0$ unless otherwise stated. Moreover for simplicity, we denote by $\sum_\alpha'$ the summation over the indices α such that $\xi_\alpha(x)\neq 0$. Now the following lemma is clear from (A.5).

Lemma A.1. *Under the above notations, it holds:*

$$|dF-\textstyle\sum_{\alpha\in A}\xi_\alpha P_\alpha\circ df_\alpha|\leq C\,\nu\{\Gamma_1\delta+\Lambda_0^2\delta(|dF|+\sum_\alpha'|df_\alpha|)\}$$

on N.

In order to estimate the second fundamental form DdF of the map F, we differentiate twice the left side of (A.4) and get

$$\sum_{\alpha\in A}Dd(\xi_\alpha E_i(F,f_\alpha))=0$$

$(i=1,\cdots,m)$. We will assume in this paragraph that

$$|De_i|(F(x_0))=0 \qquad (i=1,\cdots,m)\,. \tag{A.6}$$

Then for unit tangent vectors X and Y at x_0, we obtain the following identity:

$$\begin{aligned}\textstyle\sum_{\alpha\in A}\Big[&Dd\xi_\alpha(X,Y)\,E_i(F,f_\alpha)+\\ &d\xi_\alpha(X)\{\langle -dF(Y)+P_\alpha(df_\alpha(Y)),\,e_i(F)\rangle+dE_i(dF(Y)\oplus df_\alpha(Y))-\\ &\qquad\qquad\omega_i(dF(Y)\oplus df_\alpha(Y))\}+\\ &d\xi_\alpha(Y)\{\langle -dF(X)+P_\alpha(df_\alpha(X)),\,e_i(F)\rangle+dE_i(dF(X)\oplus df_\alpha(X))-\\ &\qquad\qquad\omega_i(dF(X)\oplus df_\alpha(X))\}+\\ &\xi_\alpha\{\langle -DdF(X,Y)+P_\alpha(Ddf_\alpha(X,Y)),\,e_i(F)\rangle+\\ &\quad dE_i(DdF(X,Y)\oplus Ddf_\alpha(X,Y))-\omega_i(DdF(X,Y)\oplus Ddf_\alpha(X,Y))\}+\\ &\xi_\alpha DdE_i(dF(X)\oplus df_\alpha(X),\,dF(Y)\oplus df_\alpha(Y))\Big]=0\,.\end{aligned}$$

Then it follows from this identity together with (A.3) that

$$\begin{aligned}(\text{A.7})\qquad &|DdF(X, Y)-\textstyle\sum_{\alpha\in A}\{\xi_\alpha P_\alpha(Ddf_\alpha(X, Y))-d\xi_\alpha(X)P_\alpha(df_\alpha(Y))-\\ &d\xi_\alpha(Y)P_\alpha(df_\alpha(X))\}|\leqq C\nu\{\Gamma_2\delta+\Gamma_1\Lambda_0^2\delta^2(|dF|+\textstyle\sum'_\alpha|df_\alpha|)+\\ &\Lambda_0^2\delta^2(|DdF|+\textstyle\sum'_\alpha|Ddf_\alpha|)\}+\Pi_1(X, Y)\,.\end{aligned}$$

Here we set

$$\Pi_1(X, Y)=\textstyle\sum_{i=1}^m|\sum_{\alpha\in A}\xi_\alpha DdE_i(dF(X)\oplus df_\alpha(X),\ dF(Y)\oplus df_\alpha(Y))|\,.$$

Observe that $\Pi_1(X, Y)$ can be written as

$$\Pi_1(X, Y)=\textstyle\sum_{i=1}^m|\sum_{\alpha\in A}\xi_\alpha(DdE_i-\Omega_i)(dF(X)\oplus df_\alpha(X),\ dF(Y)\oplus df_\alpha(Y))|\,,$$

where we have used the fact that for all $i=1, \cdots, m$,

$$\textstyle\sum_{\alpha\in A}\xi_\alpha\Omega_\alpha=0\qquad\text{at}\quad F(x_0)\,,$$

because of (A.3) and (A.5). Hence it turns out from (A.3) that

$$(\text{A.8})\qquad |\Pi_1(X, Y)|\leqq C\nu(\Lambda_1\delta+\Lambda_0^2)(1+\Lambda_0^2\delta^2)\delta(|dF|+\textstyle\sum'_\alpha|df_\alpha|)\,.$$

Thus by (A.7) and (A.8), we have the following

Lemma A.2. *Under the above notations, it holds on N:*

$$\begin{aligned}&|DdF-\textstyle\sum_{\alpha\in A}\xi_\alpha P_\alpha\circ Ddf_\alpha-\sum_{\alpha\in A}d\xi_\alpha\odot(P_\alpha\circ df_\alpha)|\leqq\\ &\quad C\nu\{\Gamma_2\delta+\Gamma_1\Lambda_0^2\delta^2(|dF|+\textstyle\sum'_\alpha|df_\alpha|)+\Lambda_0^2\delta^2(|DdF|+\sum'_\alpha|Ddf_\alpha|)+\\ &\qquad\qquad(\Lambda_1\delta+\Lambda_0^2)(1+\Lambda_0^2\delta^2)\delta(|dF|+\textstyle\sum'_\alpha|df_\alpha|)^2\},\end{aligned}$$

where

$$d\xi_\alpha\odot(P_\alpha\circ df_\alpha)(X, Y):=d\xi_\alpha(X)P_\alpha(df_\alpha(Y))+d\xi_\alpha(Y)P_\alpha(df_\alpha(X))\,.$$

We are now in a position to give an estimate for the covariant derivative $DDdF$ of the second fundamental form DdF. So far as $DDdF$ is concerned, we need apparently to carry out direct but longer calculations for estimating it. However, since it is sufficient for our purpose to get locally a rough estimate for $DDdF$, we will omit the details. To state our result on $DDdF$, we will assume in addition to (C.1$\sim$C.5) that

(C.6) there is a positive constant λ such that on M, $|dF|\leqq\lambda$, $|DdF|\leqq\lambda$, $|df_\alpha|\leqq\lambda$, and $|Ddf_\alpha|\leqq\lambda$ for any $\alpha\leqq A$;

Moreover let us fix a coordinate system $(x_1, \cdots, x_m)$ on B and use the orthonormal frame fields $\{e_1, \cdots, e_m\}$ obtained from the natural vector fields $\{\partial/\partial x_1, \cdots, \partial/\partial x_m\}$ by the Gram-Schmidt's procedure, and let us assume that

(C.7) there is a positive constant Γ_3 such that $|De_i| \leqq \Gamma_3$ $(i=1, \cdots, m)$ on B.

Then it turns out from direct computations that on $B \times B$,

$$(A.9) \qquad |DDdE_i(p, q) - \langle E(p, q), DDDe_i(p)\rangle| \leqq C^*(\mathrm{dis}(p, q)\Lambda_1, \mathrm{dis}(p, q)^2\Lambda_2)(1+\Gamma_3+|DDe_i|)(p),$$

where $C^*(s, t)$ stands for a positive function depending on m, Λ_0, and r_0 such that $C^*(s, t)$ remains bounded as long as the variables s, t are kept to be bounded. Now taking the covariant derivative of the hessian of the left-hand side in (A.3), carrying out computations directly and then applying (A.9) to $DDdE_i$, we are able to show the following

Lemma A.3. *Under the conditions* (C.1)~(C.7) *and the same notations as above, it holds on V:*

$$|DDdF - \sum_{\alpha \in A} \xi_\alpha P_\alpha(DDdf_\alpha)| \leqq C^{**}(1+\sum_{\alpha \in A}|DDd\xi_\alpha| + \sum_{i=1}^m |DDe_i| \circ F + \delta |DDdF|),$$

Here $C^{**} = C^{**}(m, \Lambda_0, \delta\Lambda_1, \delta^2\Lambda_2, \lambda, \nu, r_0, \Gamma_1, \Gamma_2, \Gamma_3)$ is a positive constant which depends on the quantities in the parenthesis, and which has the property that it remains bounded if so are the quantities.

References

[1] U. Abresch: *Uber das glatten Riemannisher metriken*, Habilitationsschrift, Reinnischen Friedrisch-Willhelms-Universität Bonn (1988).

[2] T. Adachi and T. Sunada: *Energy spectrum of certain harmonic mappongs*, Compotio Math. **56** (1985), 153–170.

[3] M.T. Anderson and J. Cheeger: *C^α-Compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Differential Geom. **35** (1992), 265–281.

[4] J. Bemelmans, Min-Oo and E.A. Ruh: *Smoothing Riemannian metrics*, Math. Z. **188** (1984), 69–74.

[5] P. Buser and H. Karcher: *Gromov's almost flat manifolds*, Asterique **81** (1981), 1–148.

[6] J. Cheeger, K. Fukaya and M. Gromov: *Nilpotent structures and invariant metrics on collapsed manifolds*, J. of Amer. Math. Soc. **5** (1992), 327–372.

[7] S.Y. Cheng and S.T. Yau: *Differential equations on Riemannian manifolds and their geometric applications*, Commun. Pure Appl. Math. **28** (1975), 333–354.

[8] J. Eells and J. Sampson: *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[9] J. Eells and L. Lemaire: *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.

[10] K. Fukaya: *Collapsing Riemannian manifolds to ones of lower dimension*, J. Differ-

ential Geom. **25** (1987), 139–156.

[11] K. Fukaya: *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator,* Invent. Math. **87** (1987), 517–547.

[12] K. Fukaya: *A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters*, J. Differential Geom. **28** (1988), 1–21.

[13] K. Fukaya: *Collapsing Riemannian manifolds to ones of lower dimension II,* J. Math. Soc. Japan **41** (1989), 333–356.

[14] D. Gilberg and N.S. Trudinger: Elliptic Partial Differential Equations of Second Order (2^{nd} Eidtion), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[15] R.E. Greene and H. Wu: *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math. **131** (1988), 119–141.

[16] M. Gromov (rédige par J. Lafontaine and P. Pansu): Structure métrique pour les variétés riemanniennes, Cedic/Fernand Nathan, Paris, 1981.

[17] M. Grüter and K. -O. Widman: *The Green function for uniformly elliptic equations,* Manuscripta Math. **37** (1982), 303–342.

[18] P. Hartman: *On homotopic harmonic maps*, Can. J. Math. **19** (1967), 673–687.

[19] J. Jost: Harmonic Mappings between Riemannian Manifolds, Proc. Center Math. Analysis. Aust. Nat. Univ. Press, Canberra, 1983.

[20] J. Jost and H. Karcher: *Geometrische Methoden zur Gewinnung von a-priori-Schhranken fur harmonische Abbildungen*, Manuscripta Math. **40** (1982), 27–77.

[21] A. Kasue: *A convergence theorem for Riemannian manifolds and some applications,* Nagoya Math. J. **114** (1989), 21–51.

[22] A. Kasue: *Measured Hausdorff convergence of Riemannian manifolds and Laplace operators II*, "Complex Geometry" ed. by G. Komatsu and Y. Sakane, Marcel Dekker, New York, 1992, 97–111..

[23] A. Kasue: *Harmonic functions of polynomial growth on complete Riemannian manifolds*, "Differential Geomtery" ed. by R.E. Greene and S. -T. Yau, Proc. Symp. Pure Math. **54** (1983), Part I, A.M.S., 281–290; —II, to appear in J. Math. Soc. of Japan.

[24] P. Li and S. -T. Yau: *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[25] A. Lichnerowicz: *Applications harmoniques et variétés kählériennes*, Symp. Mrth. **III**, Bologna (1970), 341–402.

[26] B.O'Neill: *The fundamental equations of a submersion,* Michigan Math. J. **13** (1966), 459–469.

[27] S. Peters: *Convergence of Riemannian manifolds*, Comp. Math. **62** (1987), 3–16.

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