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CHARACTERIZATIONS OF ALMOST *QF* RINGS

Dedicated to Professor Yukio Tsushima on his sixtieth birthday

SHOJIRO TOZAKI

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Let R be a ring with identity and let M and N be right R-modules. A module M is called almost N-injective if M satisfies either the following (AII) or (AI2) for any monomorphism $\lambda : L \to N$ and any homomorphism $\psi : L \to M$:

(AI1) There exists a homomorphism $\varphi : N \to M$ with $\psi = \varphi\lambda$;

(AI2) There exist a non-zero split epimorphism $\eta : N \to K$ and a homomorphism $\theta : M \to K$ with $\theta\psi = \eta\lambda$, where K is some right R-module (see the diagrams below).

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\lambda"] \arrow[d, "\psi"'] & N \arrow[dl, dashed, "\varphi"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\lambda"] \arrow[d, "\psi"'] & N \arrow[d, "0 \neq \eta \text{ (split epi.)}"] \\ & M \arrow[r, dashed, "\theta"'] & K \end{tikzcd}$$

A module M is called almost injective if M is almost N-injective for every right R-module N. A ring R is called a right almost QF ring if R is a right artinian ring and every indecomposable projective right R-module is almost injective (see [2], [3], and [5]).

In this paper we show anew characterizations of almost QF rings which are obtained in [6].

Throughout this paper we always assume that R is a right artinian ring with identity, J its Jacobson radical, and "a module" means a unitary right R-module. Let M be a module. Then $L \leq M$ (resp. $L < M$) signifies that L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By Top(M) and $E(M)$ we denote the top and an injective hull of M, respectively. We denote the set of primitive idempotents of R by pi(R). We call a module M local if M has a unique maximal submodule. If a module M has a simple socle we call M colocal. A module M is called completely indecomposable if its endomorphism ring End(M) is a local ring. A submodule N of M is called a waist in M if either $N \leq X$ or $N \geq X$ is satisfied for every submodule X of M. A module M is called uniserial if M has a unique composition series. Note that in case R is a right artinian ring, a module M is uniserial if and only if every submodule of M is a waist in M.

Let $M = \prod_{i\in I} M_i$ be a direct product of modules $M_i (i \in I)$. Then note that M is almost injective if and only if M_i is almost injective for every $i \in I$ (see [6, Lemma 1.1]).

The following lemma is fundamental for almost injective modules.

Lemma 1 ([2, Theorem $1^{\#}$] and [5, Lemma 1.2]). *Let R be a right artinian ring, and M a completely indecomposable right R-module. Then the following are equivalent*:

(1) *M is almost injective*;

(2) (i) *M is colocal*;

(ii) *Any proper essential extension N of M is projective.*

In this case M is a waist in $E(M)$ and $E(M)/M$ is uniserial.

We define two subsets $\mathrm{A}(R)$ and $\mathrm{B}(R)$ of $\mathrm{pi}(R)$ as follows:

$\mathrm{A}(R) := \{e \in \mathrm{pi}(R) \mid eR$ is injective$\}$.

$\mathrm{B}(R) := \{f \in \mathrm{pi}(R) \mid fR \cong eJ^i$ for some $e \in \mathrm{A}(R)$ and some integer i such that eJ^j is projective for every j with $0 \leq j \leq i\}$.

Moreover we define an integer $\mathrm{m}(e)$ for every $e \in \mathrm{A}(R)$ as follows:

$\mathrm{m}(e) := \max\{m > 0 \mid eJ^i$ is non-zero and projective for every i with $0 \leq i < m\}$.

Then clearly $\mathrm{A}(R) \subseteq \mathrm{B}(R) \subseteq \mathrm{pi}(R)$.

Right almost QF rings have the following property.

Proposition 2 ([2, Corollary $1^{\#}$ and Proposition 3]). *Let R be a right artinian ring. Then the following are equivalent*:

(1) *R is a right almost QF ring*;

(2) *For each $f \in \mathrm{pi}(R)$ there exist an element e of $\mathrm{A}(R)$ and an integer m with $0 \leq m < \mathrm{m}(e)$ such that $fR \cong eJ^m$.*

This implies that R is a right almost QF ring if and only if $\mathrm{pi}(R) = \mathrm{B}(R)$.

Proof. This is immediate from Lemma 1. □

We call R a right QF-2 ring if every indecomposable projective right R-module has a simple socle, that is, eR is colocal for every $e \in \mathrm{pi}(R)$. We call R a right QF-3 ring if $E(R_R)$ is projective.

The following lemma is essentially shown in the proof of [3, Corollary to Theorem 1] (see also [1, Proposition 3-(B)] and [6, Lemma 2.2]). We give its proof for convenience.

Lemma 3. *Let R be a right artinian right QF-2 ring, and let L and M be R-modules with $L \leq M$. If L is a non-zero projective and M is local, then M is also* *projective.*

Proof. Since M is local and $L \leq M$, there exists an indecomposable projective module P such that $M \cong P/K$ and $L \cong Q/K$, where $K \leq Q \leq P$. Then since $Q/K(\cong L)$ is projective by assumption, a canonical epimorphism $Q \to Q/K$ splits. But since P is colocal, so is Q. Hence $K = 0$, so that $M \cong P$. Therefore M is projective. $\square$

Lemma 4. *Let R be a right artinian ring. The following are equivalent:*
(1) *R is a right QF-2 and right QF-3 ring;*
(2) *$E(eR)$ is local for every $e \in \mathrm{pi}(R)$.*

Proof. (1) $\Rightarrow$ (2). This is clear.
(2) $\Rightarrow$ (1). For every $e \in \mathrm{pi}(R)$ since $E(eR)$ is indecomposable by assumption, $E(eR)$ is colocal and hence eR is also colocal. Thus R is a right QF-2 ring. On the other hand since $E(eR)$ is local, $E(eR)$ is projective by Lemma 3. Hence $E(R_R)$ is projective, that is, R is a right QF-3 ring. $\square$

Note that the assertion (1) in Lemma 4 holds if and only if for every $e \in \mathrm{pi}(R)$ there exists an $f \in \mathrm{A}(R)$ with $E(eR) \cong fR$.

Theorem 5. *Let R be a right artinian ring. The following are equivalent:*
(1) *R is a right almost QF ring;*
(2) *Every essential extension of eR is local for every $e \in \mathrm{pi}(R)$;*
(3) *eR is a waist in $E(eR)$ and $E(eR)/eR$ is uniserial for every $e \in \mathrm{pi}(R)$.*

Proof. (1) $\Rightarrow$ (3). This follows from the last statement of Lemma 1.
(3) $\Rightarrow$ (2). Let M be an essential extension of eR such that $eR < M \leq E(eR)$. Since M/eR is local, we have $M = xR + eR$ for some $x = xf \in M$, where $f \in \mathrm{pi}(R)$. But eR is a waist in $E(eR)$, which implies $eR < xR$. Hence $M = xR$ is local.
(2) $\Rightarrow$ (1). For every $e \in \mathrm{pi}(R)$, $E(eR)$ is local by assumption. Hence R is a right QF-2 ring by Lemma 4. Therefore every essential extension of eR is projective by Lemma 3. Thus R is a right almost QF ring by Lemma 1. $\square$

Lemma 6 (Harada [4, Proposition 1]). *Let R be a right almost QF ring and $e \in \mathrm{A}(R)$. Suppose that $\mathrm{Top}(eJ^m)g \neq 0$ for an integer m with $0 \leq m \leq \mathrm{m}(e)$, where $g \in \mathrm{pi}(R)$. If gR is not injective, then $eJ^m \cong gR$ and hence $|\mathrm{Top}(eJ^m)| = 1$.*
This implies that gR is injective if $\mathrm{Top}(eJ^{m(e)})g \neq 0$, where $g \in \mathrm{pi}(R)$.

Let e be an element of $\mathrm{A}(R)$. We define an integer $\mathrm{k}(e)$ for e as follows:

$\mathrm{k}(e) := \max\{k > 0 | eJ^j$ is local for every j with $0 \leq j < k\}$.

This implies that $eR = eJ^0$, eJ, eJ^2, $\cdots$, and $eJ^{\mathrm{k}(e)-1}$ are all local but $eJ^{\mathrm{k}(e)}$ is not local.

Now we consider the following conditions (C1) and (C2) on a ring R:

(C1) $E(eR)$ is local for each $e \in \mathrm{pi}(R)$.

(C2) Let e be an element of $\mathrm{A}(R)$ and put $k := \mathrm{k}(e)$. If an epimorphism $P \to eJ^k$ is a projective cover of eJ^k, then P is injective.

Note that the condition (C1) is just the assertion (2) in Lemma 4.

The following theorem is the main theorem in [6] (see also [1]).

Theorem 7. *Let R be a right artinian ring. The following are equivalent:*

(1) *R is a right almost QF ring;*

(2) *R satisfies the conditions* (C1) *and* (C2).

Proof. (1) $\Rightarrow$ (2). Suppose the assertion (1). First it is clear that R satisfies the condition (C1) by Lemma 1.

Let e be an element of $\mathrm{A}(R)$, and put $k := \mathrm{k}(e)$. Suppose $\mathrm{Top}(eJ^k)g \neq 0$ for some $g \in \mathrm{pi}(R)$. Since eJ^{k-1} is local by the definition of $\mathrm{k}(e)$, there exists an $f \in \mathrm{pi}(R)$ with $eJ^{k-1} \cong fR/K$, where $K \leq fR$. Then by Proposition 2 there exist an $h \in \mathrm{A}(R)$ and an integer m with $0 \leq m < \mathrm{m}(h)$ such that $fR \cong hJ^m$. So $fJ \cong hJ^{m+1}$. Now since there exists an epimorphism $fR \to eJ^{k-1}$, we have also an epimorphism $hJ^{m+1}(\cong fJ) \to eJ^k$. Hence $\mathrm{Top}(hJ^{m+1})g \neq 0$ since $\mathrm{Top}(eJ^k)g \neq 0$. From the definition of $\mathrm{k}(e)$ eJ^k is not local, and hence hJ^{m+1} is not local. Hence $|\mathrm{Top}(hJ^{m+1})| > 1$. This implies exactly $m + 1 = \mathrm{m}(h)$ from the definition of $\mathrm{m}(h)$. So applying Lemma 6, we have that gR is injective. Consequently R satisfies the condition (C2).

(2) $\Rightarrow$ (1). Suppose the assertion (2). Then by Proposition 2, it suffices to show that $\mathrm{pi}(R) = \mathrm{B}(R)$. Now we divide the proof into two steps.

STEP 1. Let f be an element of $\mathrm{B}(R)$ and let g be an element of $\mathrm{pi}(R)$. If $\mathrm{Top}(fJ)g \neq 0$, then g belongs to $\mathrm{B}(R)$.

Proof of Step 1. Suppose $\mathrm{Top}(fJ)g \neq 0$ for f and g above. Since $f \in \mathrm{B}(R)$, there exist an $e \in \mathrm{A}(R)$ and an integer m with $0 \leq m < \mathrm{m}(e)(\leq \mathrm{k}(e))$ such that $fR \cong eJ^m$. So $fJ \cong eJ^{m+1}$. Hence we have $\mathrm{Top}(eJ^{m+1})g \neq 0$ since $\mathrm{Top}(fJ)g \neq 0$.

(i) Assume $|\mathrm{Top}(eJ^{m+1})| > 1$. Then eJ^{m+1} is not local. This implies $m + 1 = \mathrm{k}(e)$. Hence gR is injective by the condition (C2). Thus $g \in \mathrm{A}(R) \subseteq \mathrm{B}(R)$.

(ii) Assume $|\mathrm{Top}(eJ^{m+1})| = 1$. Then eJ^{m+1} is local (i.e., $m+1 < \mathrm{k}(e)$). Hence we have an epimorphism $gR \to eJ^{m+1}$. Here suppose $g \notin \mathrm{A}(R)$. Then we have an $h \in \mathrm{A}(R)$ with $gR < E(gR) \cong hR$ by Lemma 4. Hence we have a right ideal I of

R such that $gR \cong hI \leq hJ$. We consider an epimorphism $\varphi : hI(\cong gR) \to eJ^{m+1}$. Since eR is injective, for a composition map $\varepsilon\varphi : hI \to eR$ there exist a homomorphism $\theta : hR \to eR$ with $\theta|_{hI} = \varepsilon\varphi$, where ε is an inclusion map from eJ^{m+1} into eR (see the diagram below).

$$\begin{tikzcd}
0 \arrow[r] & hI \arrow[r] \arrow[d, "\text{epi. } \varphi"'] & hR \arrow[ddl, dashed, "\theta"] \\
& eJ^{m+1} \arrow[d, "\varepsilon"'] & \\
& eR &
\end{tikzcd}$$

Then eJ^{m+1} is a waist in eR since $m + 1 < k(e)$. So $\theta(hR) \leq eJ^{m+1}$ or $\theta(hR) > eJ^{m+1}$. Suppose $\theta(hR) \leq eJ^{m+1}$. Then $0 \neq eJ^{m+1} = \varphi(hI) = \varepsilon\varphi(hI) = \theta(hI) \leq \theta(hJ) \leq eJ^{m+2}$. This is a contradiction. Hence $\theta(hR) > eJ^{m+1}$. Since eR/eJ^{m+1} is uniserial, we have $\theta(hR) \cong eJ^j$ for some j with $0 \leq j \leq m$. On the other hand eJ^j is projective from $j < \mathrm{m}(e)$, so that $\theta(hR)$ is also projective. Hence we have Ker $\theta = 0$ by the indecomposability of hR. Hence $\varphi : hI \to eJ^{m+1}$ is an isomorphism, that is, $gR \cong hI \cong eJ^{m+1}$. Thus $m + 1 < m(e)$, and consequently g belongs to $\mathrm{B}(R)$.

STEP 2. Let e be an element of $\mathrm{A}(R)$. If $\mathrm{Top}(eJ^n)g \neq 0$ for an element $g \in \mathrm{pi}(R)$ and an integer $n \geq 0$, then g belongs to $\mathrm{B}(R)$.

Proof of Step 2. We shall use induction on n.

In the case $n = 0$, $eR \cong gR$. So $g \in \mathrm{A}(R) \subseteq \mathrm{B}(R)$.

Suppose that the claim above holds for $n \geq 0$. Let $\mathrm{Top}(eJ^{n+1})g \neq 0$ and $\mathrm{Top}(eJ^n) \cong \mathrm{Top}(f_1R) \bigoplus \mathrm{Top}(f_2R) \bigoplus \cdots \bigoplus \mathrm{Top}(f_sR)$, where $f_i \in \mathrm{pi}(R)$, $i = 1, 2, ..., s$. By the inductive assumption, $f_t \in \mathrm{B}(R)$ for every t with $1 \leq t \leq s$. Now from the the isomorphism $\mathrm{Top}(eJ^n) \cong \mathrm{Top}(f_1R) \bigoplus \mathrm{Top}(f_2R) \bigoplus \cdots \bigoplus \mathrm{Top}(f_sR)$ we have an epimorphism $(f_1R \bigoplus f_2R \bigoplus \cdots \bigoplus f_sR) \to eJ^n$. So there exists an epimorphism $(f_1J \bigoplus f_2J \bigoplus \cdots \bigoplus f_sJ) \to eJ^{n+1}$. Since $\mathrm{Top}(eJ^{n+1})g \neq 0$, we have an $f_u \in \mathrm{B}(R)$ such that $\mathrm{Top}(f_uJ)g \neq 0$ for some u with $1 \leq u \leq s$. Therefore we have $g \in \mathrm{B}(R)$ by Step 1. Thus Step2 is proved.

Now we can easily show that $\mathrm{pi}(R) = \mathrm{B}(R)$. Let g be an arbitrary element in $\mathrm{pi}(R)$. Since R satisfies the condition (C1), there exists an $e \in \mathrm{A}(R)$ such that $gR \leq E(gR) \cong eR$ by Lemma 4. Then we have an integer k (≥ 0) such that $\mathrm{Top}(eJ^k)g \neq 0$. Hence by Step 2 we have $g \in \mathrm{B}(R)$, that is, $\mathrm{pi}(R) \subseteq \mathrm{B}(R)$. The theorem is proved. $\square$

Finally we consider another condition (C3) on a ring R as follows:

(C3) Let e be an element of $\mathrm{A}(R)$ and put $m := \mathrm{m}(e)$. If an epimorphism $P \to eJ^m$ is a projective cover of eJ^m, then P is injective.

Remark that, according to Lemma 6, if R is a right almost QF ring then R satisfies the condition (C3).

In Theorem 7 if we replace the condition (C2) in the assertion (2) with the condition (C3), then we obtain the following proposition as another characterization of almost QF rings, which is one of the main results in [6] (see also [1]).

Proposition 8. *Let R be a right artinian ring. The following are equivalent*:

(1) *R is a right almost QF ring*;

(2) *R satisfies the conditions* (C1) *and* (C3).

Proof. (1) $\Rightarrow$ (2). Suppose the assertion (1). Then the assertion (2) follows from Theorem 7 and the preceding remark.

(2) $\Rightarrow$ (1). Since we can prove this in the same way as Theorem 7 except the claim of Step 1, it suffices to prove the following claim corresponding to Step 1 in the proof of Theorem 7.

CLAIM. Suppose the assertion (2). Let f be an element of $\mathrm{B}(R)$ and let g be an element of $\mathrm{pi}(R)$. If $\mathrm{Top}(fJ)g \neq 0$, then g belongs to $\mathrm{B}(R)$.

Proof of the Claim. Suppose $\mathrm{Top}(fJ)g \neq 0$ for f and g above. Since f is an element of $\mathrm{B}(R)$, we have an $e \in \mathrm{A}(R)$ and an integer m with $0 \leq m < \mathrm{m}(e)$ such that $fR \cong eJ^m$. So $fJ \cong eJ^{m+1}$. Hence we have $\mathrm{Top}(eJ^{m+1})g \neq 0$ since $\mathrm{Top}(fJ)g \neq 0$.

Then in the case $m + 1 = \mathrm{m}(e)$, gR is injective by the condition (C3). Hence $g \in \mathrm{A}(R) \subseteq \mathrm{B}(R)$.

In the case $m + 1 < \mathrm{m}(e)$, eJ^{m+1} is projective by the definition of $\mathrm{m}(e)$. Hence $gR \cong eJ^{m+1}$. This implies g belongs to $\mathrm{B}(R)$. The proposition is proved. $\square$

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558–0022, Japan