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A CALCULUS OF FOURIER INTEGRAL OPERATORS AND THE GLOBAL FUNDAMENTAL SOLUTION FOR A SCHRÖDINGER EQUATION

HITOSHI KITADA

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Introduction

In Kitada and Kumano-go [6] we studied a theory of Fourier integral operators and constructed the fundamental solution $U_h(t, s_0)$ for a pseudo-differential equation of Schrödinger's type:

$$(1) \qquad \begin{cases} L_h u \equiv \left(\frac{1}{i}\frac{\partial}{\partial t}+H_h(t, X, D_x)\right)u = 0\,, \\ u\,|_{t=s_0} = f \in \mathcal{S} \qquad (s_0 \in R^1)\,, \end{cases}$$

in the form of a Fourier integral operator for t near s_0. Here

$$(2) \qquad H_h(t, x, \xi) = h^{\delta-\rho}H(t, h^{-\delta}x, h^{\rho}\xi) \qquad (0<h<1,\ 0\leqq\delta\leqq\rho\leqq 1)$$

covers a rather general class of smooth time-dependent potentials $V(t, x)$ if $H(t, x, \xi)$ is of the form $H(t, x, \xi)=\frac{1}{2}|\xi|^2+V(t, x)$. However, contrary to the generality of $H(t, x, \xi)$ that we can deal with, the time range in which we can represent $U_h(t, s_0)$ as a single Fourier integral operator was very small. The similar situations are also the case in Fujiwara's construction ([2], [3]) of the fundamental solution, except the results in [3, §4].

In this paper we shall make a rather strong restriction on the potential $V(t, x)$ (see Assumption (A) in section 3), and construct the fundamental solution $U_h(t, s_0)$ for (1) with $H(t, x, \xi)=\frac{1}{2}|\xi|^2+V(t, x)$ in the form of a single (conjugate) Fourier integral operator for all $t\geqq s_0$, when s_0 is sufficiently large.

To do so, in sections 1 and 2 we shall introduce a class of (conjugate) Fourier integral operators and investigate their calculus, which is also our purpose in the present paper. The symbol class for our Fourier integral operators is the same as in [6], while the class of phase functions is different from [6] (see Definition 1.1). The characteristic feature of our phase functions $\phi_h(x, \xi)$ is, roughly speaking, that the function $J_h(x, \xi)\equiv\phi_h(x, \xi)-x\cdot\xi$ is "small" in the sense that only the derivatives of $\nabla_x J_h(x, \xi)$ are small, while in [6] we assumed that the derivatives of both $\nabla_x J_h(x, \xi)$ and $\nabla_\xi J_h(x, \xi)$ are small. This relaxation is possible, because, in the present paper, we restrict ourselves to considering only the conjugate Fourier integral operators of the form

$$(3)\qquad P_h(\phi_h^*)f(x) = 0_s\text{-}\iint e^{i(x\cdot\xi-\phi_h(x',\xi))}p_h(\xi, x')f(x')dx'đ\xi\,,$$

while in [6] we considered both Fourier, and conjugate Fourier, integral operators.

Section 2 is devoted to proving a theorem concerning the calculus of conjugate Fourier integral operators, which is different from [6] in the point that we shall treat $\nu+1$ conjugate Fouier integral operators directly, while in [6] the product of two Fourier integral operators and that of Fourier and conjugate Fourier integral operators were basic. This result will allow us in section 4 to make a global calculus in time of the local fundamental solutions represented as conjugate Fourier integral operators.

In section 3, we shall in turn consider the Schrödinger equation (1) with $H(t, x, \xi)=\frac{1}{2}|\xi|^2+V(t, x)$, where $V(t, x)$ is assumed to satisfy

$$(4)\qquad \sup_{x\in R^n}|\partial_x^\alpha V(t, x)|\leqq C_\alpha(1+|t|)^{-|\alpha|-\varepsilon}$$

for $|\alpha|\neq 0$ with $\varepsilon>0$. We shall first give several estimates concerning the classical orbit $(q, p)(t, s; x, \xi)$ defined as the solution of the Hamilton equation

$$(5)\qquad \begin{cases} \dfrac{dq}{dt}(t, s) = p(t, s)\,, \\ \dfrac{dp}{dt}(t, s) = -\nabla_x V(t, q(t, s)) \end{cases}$$

with the initial condition $(q, p)(s, s)=(x, \xi)$. From this $(q, p)(t, s; x, \xi)$, we shall construct the phase function $\phi(s, t; x, \xi)$ as the solution of the eikonal equation

$$(6)\qquad \begin{cases} \partial_s\phi(s, t; x, \xi)+H(s, x, \nabla_x\phi(s, t; x, \xi)) = 0\,, \\ \phi(t, t; x, \xi) = x\cdot\xi\,, \end{cases}$$

which can be solved globally for $t\geqq s$ when s is sufficiently large, as well as locally for $|t-s|\leqq\delta_0(\ll 1)$. Then we shall define the global and local approximate fundamental solutions of order m ($m=0$ or ∞) in the sense of [6] in the form

$$(7)\qquad \begin{aligned} &E_h^m(\phi_h(s, t)^*)f(x) \\ &= 0_s\text{-}\iint e^{i(x\cdot\xi-\phi_h(s,t;x',\xi))}e_h^m(t, s; \xi, x')f(x')dx'đ\xi \end{aligned}$$

for $t\geqq s$ or $|t-s|\leqq\delta_0$, where $\phi_h(s, t; x', \xi)\equiv h^{\delta-\rho}\phi(s, t; h^{-\delta}x', h^\rho\xi)$, and $m=0$ in case $0\leqq\delta\leqq\rho\leqq 1$ and $m=\infty$ in case $0\leqq\delta<\rho\leqq 1$. We shall then summarize the important estimates concerning these approximate fundamental solutions as Theorem 3.11 at the end of section 3.

Using these estimates, in section 4 we shall first construct the local fundamental solution $U_h(t, s)$ for $|t-s|\leqq\delta_0$ as a conjugate Fourier integral operator in quite a similar way to [6]. Then using the global solution $\phi(s, t; x, \xi)$ of (6) and the results of section 2 on the calculus, we shall represent the global fundamental solution $U_h(t, s_0)=U_h(t, t_\nu)U_h(t_\nu, t_{\nu-1})\cdots U_h(t_1, s_0)$ $(0<t_j-t_{j-1}\leqq(t-s_0)/(\nu+1)\leqq\delta_0, s_0<t_1<\cdots<t_\nu\leqq t)$ as a single conjugate Fourier integral operator for sufficiently large s_0. For general s_0, we can therefore represent the global fundamental solution $U_h(t, s_0)$ as a product of a finite number of conjugate Fourier integral operators, the number being independent of t but dependent on s_0. At the same time, we shall also give some estimates for the differences between the fundamental solution $U_h(t, s)$ and the global approximate fundamental solutions $E_h^m(\phi_h(s, t)^*)$ when $t\geqq s$ for sufficiently large s. One of these estimates played a crucial role in the proof of the completeness of modified wave operators in [5].

We note that our assumption on the potential $V(t, x)$, hence on the Hamiltonian $H(t, x, \xi)$, is not symmetric in x and ξ, while the assumption adopted in [6] was symmetric. Moreover, under our present assumption (4), the classical orbit $y=q(s, t; x, \xi)$ in the configuration space is uniquely determined by its initial and final positions x and y for $t\geqq s$ when s is sufficiently large (by (3.11) below), which makes it possible to construct the global phase function (compare this with the situations in §4 of Fujiwara [3]).

Recently, Nishiwada [9] gave an explicit expression, which is written by means of one or two integral transformations, of the global fundamental solution for a Schrödinger equation with a quadratic Hamiltonian. However his assumption and method are different from ours.

1. Fourier integral operators

In this and the next sections, we introduce a class of Fourier integral operators and investigate their properties, especially their calculus. We first explain some basic notations we shall use in the following. For any point $x=(x_1, \cdots, x_n)$ in the n-dimensional Euclidian space R^n, we define its norm $|x|$ by $|x|=(\sum_{j=1}^n x_j^2)^{1/2}$, and for any $n\times n$ real matrix $A=(a_{ij})$ we define $|A|=\sup_{|x|=1, x\in R^n}|Ax|/|x|$. Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be a multi-index whose components α_j are non-negative integers and let $x, y, z\in R^n$. Then we use the following notations:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad \alpha!=\alpha_1!\cdots\alpha_n!,\quad x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n},\quad D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},\quad \partial_{x_j}=\frac{\partial}{\partial x_j},\quad D_{x_j}=\frac{1}{i}\frac{\partial}{\partial x_j},$$

$$\nabla_x = {}^t(\partial_{x_1}, \cdots, \partial_{x_n}),\ \vec{\nabla}_x = {}^t\nabla_x\,,$$
$$\langle x\rangle = \sqrt{1+|x|^2},\ \langle x; y\rangle = \sqrt{1+|x|^2+|y|^2}\,,$$
$$\langle x; y; z\rangle = \sqrt{1+|x|^2+|y|^2+|z|^2}\,.$$

By $\mathcal{S}$ we denote the Schwartz space of rapidly decreasing functions on R^n. For $f\in\mathcal{S}$ we define its Fourier transform $\hat{f}(\xi)=\mathcal{F}f(\xi)$ by

$$\mathcal{F}f(\xi) = \int e^{-ix\cdot\xi}f(x)\,dx,\quad x\cdot\xi = \sum_{j=1}^{n} x_j\xi_j\,.$$

The inverse Fourier transform $\mathcal{F}^{-1}f$ of $f\in\mathcal{S}$ is given by

$$(\mathcal{F}^{-1}f)(x) = \int e^{ix\cdot\xi}f(\xi)\,đ\xi,\quad đ\xi = (2\pi)^{-n}d\xi\,.$$

DEFINITION 1.1. 1° Let $0\leqq\tau<1, 0\leqq\sigma<\infty$ and $0\leqq\delta\leqq\rho\leqq1$. A family $\{\phi_h(x,\xi)\}_{0<h<1}$ of C^∞-functions $\phi_h(x,\xi)$ in $R^n\times R^n$ is said to belong to the class $\{P_{\rho,\delta}^{[x]}(\tau,\sigma;h)\}_{0<h<1}$, if the function $\tilde{J}_h(x,\xi)$ defined by

$$(1.1)\qquad \begin{cases} \tilde{J}_h(x,\xi) = \tilde{\phi}_h(x,\xi)-x\cdot\xi = h^{\rho-\delta}J_h(h^\delta x, h^{-\rho}\xi)\,,\\ \tilde{\phi}_h(x,\xi) = h^{\rho-\delta}\phi_h(h^\delta x, h^{-\rho}\xi)\,,\\ J_h(x,\xi) = \phi_h(x,\xi)-x\cdot\xi \end{cases}$$

satisfies

$$(1.2)\qquad \begin{cases} \text{i)}\ \sup\limits_{h,x,\xi}\{|\nabla_\xi\tilde{J}_h(x,\xi)|/\langle\xi\rangle\}+\sup\limits_{h,x,\xi}|\nabla_x\tilde{J}_h(x,\xi)|<\infty\,,\\ \text{ii)}\ \sup\limits_{h,x,\xi}|\vec{\nabla}_\xi\nabla_\xi\tilde{J}_h(x,\xi)|\leqq\sigma\,,\\ \text{iii)}\ \max\limits_{|\alpha+\beta|=1}\sup\limits_{h,x,\xi}|\partial_\xi^\alpha D_x^\beta\nabla_x\tilde{J}_h(x,\xi)|\leqq\tau \end{cases}$$

and

$$(1.3)\qquad \sup_{h,x,\xi}|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|<\infty \qquad \text{for } |\alpha+\beta|\geqq3\,,$$

where $\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)=\partial_\xi^\alpha D_x^\beta\tilde{J}_h(x,\xi)$. For simplicity we also write this as $\phi_h(x,\xi)\in P_{\rho,\delta}^{[x]}(\tau,\sigma;h)$.

2° For $\phi_h(x,\xi)\in P_{\rho,\delta}^{[x]}(\tau,\sigma;h)$ we define a semi-norm $|J_h|_{l,m}$ for integers $l, m\geqq0$ by

$$(1.4)\qquad |J_h|_{l,m} = \max_{l\leqq|\alpha+\beta|\leqq l+m}\ \sup_{h,x,\xi}|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|\,.$$

REMARK. In section 4 we shall also use the class $\{P_{\rho,\delta}(\tau,l;h)\}_{0<h<1}$ $(0\leqq\tau<1, l=0, 1, 2, \cdots)$ defined in Kitada and Kumano-go [6]. Here for the sake of the later convenience, we state its definition. $\{\phi_h(x,\xi)\}_{0<h<1}\in\{P_{\rho,\delta}(\tau,l;h)\}_{0<h<1}$ or $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,l;h)$ means that J_h and $\tilde{J}_h$ defined by (1.1) satisfy

$$|J_h|_l \equiv \sum_{|\alpha+\beta|\leqq1}\ \sup_{h,x,\xi}\{|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|/\langle x;\xi\rangle^{2-|\alpha+\beta|}\}$$

$$+\sum_{2\leqq|\alpha+\beta|\leqq 2+l}\sup_{h,x,\xi}|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|\leqq\tau$$

and

$$\sup_{h,x,\xi}|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|<\infty \qquad \text{for } |\alpha+\beta|\geqq 3\,.$$

We next define the symbol classes which are the same as those introduced in [6].

DEFINITION 1.2. 1° Let $m\in R^1$ and $0\leqq\delta\leqq\rho\leqq 1$. A family $\{p_h(x,\xi,x')\}_{0<h<1}$ of C^∞-functions $p_h(x,\xi,x')$ is said to belong to the class $\{B^m_{\rho,\delta}(h)\}_{0<h<1}$ if $\{p_h\}_{0<h<1}$ satisfies

$$\begin{aligned}(1.5)\qquad |p_h|_l^{(m)} &= |\{p_h\}_{0<h<1}|_l^{(m)}\\ &\equiv \max_{|\beta+\alpha+\beta'|\geqq l}\sup_{h,x,\xi} h^{-m-\rho|\alpha|+\delta|\beta+\beta'|}|p_{h(\beta,\beta')}^{(\alpha)}(x,\xi,x')|<\infty\end{aligned}$$

for any integer $l\geqq 0$, where $p_{h(\beta,\beta')}^{(\alpha)}=D_x^\beta\partial_\xi^\alpha D_{x'}^{\beta'}p_h(x,\xi,x')$. We write this also as $p_h(x,\xi,x')\in B^m_{\rho,\delta}(h)$.

2° For $m\in R^1$, $r\geqq 0$ and $0\leqq\delta\leqq\rho\leqq 1$, we say that a family $\{p_h(x,\xi,x')\}_{0<h<1}$ of C^∞-functions belongs to the class $\{B^{m;r}_{\rho,\delta}(h)\}_{0<h<1}$ if $\langle h^{-\delta}x;h^\rho\xi;h^{-\delta}x'\rangle^{-r}$ $p_h(x,\xi,x')$ belongs to $B^m_{\rho,\delta}(h)$.

REMARK. 1° $B^{m;0}_{\rho,\delta}(h)=B^m_{\rho,\delta}(h)$.

2° When $p_h(x,\xi,x')=p_h(x,\xi)$ (independent of x') [resp. $p_h(x,\xi,x')=p_h(\xi,x')$ (independent of x)], $p_h(x,\xi,x')\in B^{m;r}_{\rho,\delta}(h)$ is equivalent to $\langle h^{-\delta}x;h^\rho\xi\rangle^{-r}$ $p_h(x,\xi)\in B^m_{\rho,\delta}(h)$ [resp. $\langle h^\rho\xi;h^{-\delta}x'\rangle^{-r}p_h(\xi,x')\in B^m_{\rho,\delta}(h)$]. Such symbols are called single symbols.

Proposition 1.3. *Let $p_{j,h}(x,\xi,x')\in B^{m_j;j}_{\rho,\delta}(h)$ $(j=0,1,2,\cdots)$ such that $m_0\leqq m_1\leqq\cdots\leqq m_j\leqq\cdots\to\infty$ and let χ be a C^∞-function on $[0,\infty]$ such that $0\leqq\chi(\theta)\leqq 1$ on $[0,\infty)$ and $\chi(\theta)=1$ (for $0\leqq\theta\leqq 1/2$), $=0$ (for $\theta\geqq 1$). Then there exists a decreasing sequence $\{\varepsilon_j\}_{j=0}^\infty$ tending to zero as $j\to\infty$ such that*

$$(1.6)\qquad p_h(x,\xi,x')=\sum_{j=0}^\infty\chi(\varepsilon_j^{-1}h)p_{j,h}(x,\xi,x')$$

converges in $B^{m_0;0}_{\rho,\delta}(h)$ and

$$p_h(x,\xi,x')-\sum_{j=0}^{N-1}p_{j,h}(x,\xi,x')\in B^{m_N;N}_{\rho,\delta}(h)$$

for any $N\geqq 1$. Furthermore such $p_h\in B^{m_0;0}_{\rho,\delta}(h)$ is unique modulo $B^\infty(h)\equiv\bigcap_{m\in R^1}B^m_{\rho,\delta}(h)$ (independent of ρ, δ).

For the proof see Theorem 1.3 of [6].

Proposition 1.4. *Let $\phi_h(x,\xi)\in P^{[x]}_{\rho,\delta}(\tau,\sigma;h)$ and $p_h(x,\xi,x')\in B^{m;r}_{\rho,\delta}(h)$ for $0\leqq\tau<1$, $0\leqq\sigma<\infty$, $0\leqq\delta\leqq\rho\leqq 1$, $m\in R^1$ and $r\geqq 0$. Then for any $f\in\mathcal{S}$ and $\chi\in$*

$\mathcal{S}$ with $\chi(0)=1$, the integrals

$$(1.7)\qquad \begin{cases} P_{h,\varepsilon}[f](x)=\iint e^{i(\phi_h(x,\xi)-x'\cdot\xi)}p_h(x,\xi,x')f(x')\chi(\varepsilon\xi)dx'd\!\!{}^-\xi\,, \\ \bar{P}_{h,\varepsilon}[f](x)=\iint e^{i(x\cdot\xi-\phi_h(x',\xi))}p_h(x,\xi,x')f(x')\chi(\varepsilon\xi)dx'd\!\!{}^-\xi \end{cases}$$

have the limits $P_h[f](x)$ and $\bar{P}_h[f](x)$ for $\varepsilon\downarrow 0$, which are independent of χ. Moreover P_h and $\bar{P}_h$ define continuous linear mappings from $\mathcal{S}$ into $\mathcal{S}$ for each $h\in(0,1)$. We write those limits as

$$(1.8)\qquad \begin{cases} P_h[f](x)=0_s\!-\!\iint e^{i(\phi_h(x,\xi)-x'\cdot\xi)}p_h(x,\xi,x')f(x')dx'd\!\!{}^-\xi\,, \\ \bar{P}_h[f](x)=0_s\!-\!\iint e^{i(x\cdot\xi-\phi_h(x',\xi))}p_h(x,\xi,x')f(x')dx'd\!\!{}^-\xi\,. \end{cases}$$

Proof. Putting $\psi_h(x,\xi,x')=x\cdot\xi-\phi_h(x',\xi)=(x-x')\cdot\xi-J_h(x',\xi)$, we see from (1.1)–(1.2) that $\langle\nabla_{x'}\psi_h\rangle\geqq C_h\langle\xi\rangle$ for some constant $C_h>0$. Thus $L\equiv\langle\nabla_{x'}\psi_h\rangle^{-1}(1-i\nabla_{x'}\psi_h\cdot\nabla_{x'})$ is well-defined and we have for any $l\geqq 0$

$$(1.9)\qquad \bar{P}_{h,\varepsilon}[f](x)=\iint e^{i\psi_h}({}^tL)^l[p_h(x,,\xi,x')f(x')\chi(\varepsilon\xi)]dx'd\!\!{}^-\xi\,,$$

where ${}^tL$ is the transposed operator of L. Then taking $l>n+r$, noting $f\in\mathcal{S}$ and letting $\varepsilon\downarrow 0$, we have

$$(1.10)\qquad \bar{P}_h[f](x)=\iint e^{i\psi_h}({}^tL)^l[p_h(x,\xi,x')f(x')]dx'd\!\!{}^-\xi\,,$$

which is independent of χ. Therefore we get

$$\begin{aligned} &x^\alpha D_x^\beta(\bar{P}_h[f])(x) \\ &=\sum_{\beta^1+\beta^2=\beta}\iint e^{ix\cdot\xi}D_\xi^\alpha\{e^{-i\phi_h(x',\xi)}\xi^{\beta^1}({}^tL)^l[D_x^{\beta^2}p_h(x,\xi,x')f(x')]\}dx'd\!\!{}^-\xi\,. \end{aligned}$$

We see from (1.2)–(1.3) that this is uniformly bounded in $x\in R^n$ for each fixed $h\in(0,1)$, if l is taken sufficiently large.

For $P_h[f]$, putting $\varphi_h(x,\xi,x')=\phi_h(x,\xi)-x'\cdot\xi=(x-x')\cdot\xi+J_h(x,\xi)$, we have $\nabla_{x'}\varphi_h=-\xi$. So letting $L\equiv\langle\nabla_{x'}\varphi_h\rangle^{-1}(1-i\nabla_{x'}\varphi_h\cdot\nabla_{x'})$ we have (1.10) for $P_h[f]$ and

$$\begin{aligned} &|x^\alpha D_x^\beta(P_h[f])(x)| \\ &\leqq C_h\iint\langle x';\xi\rangle^{|\alpha|}\langle\xi\rangle^{|\beta|}\sum_{\alpha'\leqq\alpha,\beta'\leqq\beta}|D_\xi^{\alpha'}({}^tL)^l[D_x^{\beta'}p_h(x,\xi,x')f(x')]|dx'd\!\!{}^-\xi\,, \end{aligned}$$

which shows $P_h[f]\in\mathcal{S}$ if l is taken sufficiently large. The continuity of P_h and $\bar{P}_h$ in $\mathcal{S}$ is clear by the above discussions. □

DEFINITION 1.5. 1° For $p_h(x,\xi,x')\in B_{\rho,\delta}^{m,r}(h)$ $(0\leqq\delta\leqq\rho\leqq 1,\ m\in R^1,\ r\geqq 0)$ we define a family of pseudo-differential operators $P_h = p_h(X, D_x, X')$ by

$$P_h f(x) = 0_s - \iint e^{i(x-x')\cdot\xi} p_h(x, \xi, x') f(x') dx' d\xi \tag{1.11}$$

for $f \in \mathscr{S}$, and write this as $\{P_h\}_{0<h<1} \in \{\boldsymbol{B}^{m,r}_{\rho,\delta}(h)\}_{0<h<1}$ or simply as $P_h \in \boldsymbol{B}^{m,r}_{\rho,\delta}(h)$.

2° For $\phi_h(x, \xi) \in P^{[x]}_{\rho,\delta}(\tau, \sigma; h)$ $(0 \leqq \tau < 1, 0 \leqq \sigma < \infty)$ and $p_h(x, \xi, x') \in B^{m,r}_{\rho,\delta}(h)$, we define a family of Fourier, and conjugate Fourier, integral operators $P_h(\phi_h) = p_h(\phi_h; X, D_x, X')$ and $P_h(\phi_h^*) = p_h(\phi_h^*; X, D_x, X')$ by

$$\begin{cases} P_h(\phi_h) f(x) = 0_s - \iint e^{i(\phi_h(x,\xi) - x'\cdot\xi)} p_h(x, \xi, x') f(x') dx' d\xi\,, \\ P_h(\phi_h^*) f(x) = 0_s - \iint e^{i(x\cdot\xi - \phi_h(x',\xi))} p_h(x, \xi, x') f(x') dx' d\xi \end{cases} \tag{1.12}$$

for $f \in \mathscr{S}$. We write this as $\{P_h(\phi_h)\}_{0<h<1} \in \{\boldsymbol{B}^{m,r}_{\rho,\delta}(\phi_h)\}_{0<h<1}$ and $\{P_h(\phi_h^*)\}_{0<h<1} \in \{\boldsymbol{B}^{m,r}_{\rho,\delta}(\phi_h^*)\}_{0<h<1}$, or simply as $P_h(\phi_h) \in \boldsymbol{B}^{m,r}_{\rho,\delta}(\phi_h)$ and $P_h(\phi_h^*) \in \boldsymbol{B}^{m,r}_{\rho,\delta}(\phi_h^*)$.

REMARK. 1° If we define $q_h(x, \xi, x') = \overline{p_h(x', \xi, x)}$ for $p_h(x, \xi, x') \in B^{m,r}_{\rho,\delta}(h)$, then we have $q_h(x, \xi, x') \in B^{m,r}_{\rho,\delta}(h)$ and $(P_h f, g)_{L^2} = (f, Q_h g)_{L^2}$ for $f, g \in \mathscr{S}$.

2° For single symbols $p_h(x, \xi)$ and $q_h(\xi, x') \in B^{m,q}_{\rho,\delta}(h)$, we have from Proposition 1.4 that

$$\begin{cases} P_h(\phi_h) f(x) = \int e^{i\phi_h(x,\xi)} p_h(x, \xi) \hat{f}(\xi) d\xi\,, \\ \widehat{Q_h(\phi_h^*) f}(\xi) = \int e^{-i\phi_h(x',\xi)} q_h(\xi, x') f(x') dx' \end{cases} \tag{1.13}$$

for $f \in \mathscr{S}$ and $\phi_h \in P^{[x]}_{\rho,\delta}(\tau, \sigma; h)$.

Theorem 1.6. *Let $r \geqq 0$, and denote by $\bar{r}$ the minimum integer not less than r. Let $p_h(\xi, x') \in B^{m,r}_{\rho,\delta}(h)$ and $\phi_h(x, \xi) \in P^{[x]}_{\rho,\delta}(\tau, \sigma; h)$ $(0 \leqq \tau < 1, 0 \leqq \sigma < \infty, 0 \leqq \delta \leqq \rho \leqq 1)$ and assume that*

$$p_{h(\beta)}^{(\alpha)}(\xi, x') \in B^{m+|\alpha| - \delta|\beta|, r - |\alpha+\beta|}_{\delta,\delta}(h) \tag{1.14}$$

for $|\alpha+\beta| \leqq \bar{r}$. Let $Q_h(\phi_h^*) = q_h(\phi_h^*; D_x, X') \in \boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h^*)$. Set

$$\begin{cases} s_h(\xi, x', \xi', x'') = p_h(\xi, x' + \tilde{\nabla}_\xi J_h(\xi, x'', \xi')) q_h(\xi', x'')\,, \\ \tilde{\nabla}_\xi J_h(\xi, x'', \xi') = \int_0^1 \nabla_\xi J_h(x'', \xi' + \theta(\xi - \xi')) d\theta\,, \end{cases}$$

and define $r_h(\xi, x'')$ by

$$r_h(\xi, x'') = 0_s - \iint e^{-iy\cdot\eta} s_h(\xi, x'' + y, \xi - \eta, x'') dy d\eta\,. \tag{1.15}$$

Then we have $r_h(\xi, x'') \in B^{m+m',r}_{\rho,\delta}(h)$ and $R_h(\phi_h^*) \equiv r_h(\phi_h^*; D_x, X') = P_h Q_h(\phi_h^*)$. More precisely we have for $N \geqq \bar{r}$

$$r_h(\xi, x'')-\sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\xi'}^{\alpha}\{p_{h(\alpha)}(\xi, \tilde{\nabla}_{\xi}\phi_h(\xi, x'', \xi'))q_h(\xi', x'')\}_{|\xi'=\xi}$$
$$(1.16)\qquad = N\sum_{|\gamma|=N}\frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1(1-\theta)^{N-1}t_{\gamma,h}(\xi, x''; \theta)d\theta \in B_{\rho,\delta}^{m+m'+(\rho-\delta)N}(h),$$

where

$$(1.17)\qquad \begin{aligned} &t_{\gamma,h}(\xi, x''; \theta)\\ &= 0_s-\iint e^{-iy\cdot\eta}\partial_{\xi'}^{\gamma}\{p_{h(\gamma)}(\xi, \theta y+\tilde{\nabla}_{\xi}\phi_h(\xi, x'', \xi'))q_h(\xi', x'')\}_{|\xi'=\xi-\eta}dy\,d\!\!{}^-\eta,\end{aligned}$$

$$\tilde{\nabla}_{\xi}\phi_h(\xi, x'', \xi') = \int_0^1\nabla_{\xi}\phi_h(x'', \xi'+\theta(\xi-\xi'))d\theta.$$

If, in particular $q_h(\xi,x')=1$, *we have for* $\tilde{N}\geqq \bar{r}/2$

$$r_h(\xi, x'')-\sum_{|\alpha|<\tilde{N}}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_{\xi'}^{\alpha}\{p_{h(\alpha)}(\xi, \tilde{\nabla}_{\xi}\phi_h(\xi, x'', \xi'))\}_{|\xi'=\xi}$$
$$(1.16)'\qquad = \tilde{N}\sum_{|\gamma|=\tilde{N}}\frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1(1-\theta)^{\tilde{N}-1}t_{\gamma,h}(\xi, x''; \theta)d\theta \in B_{\rho,\delta}^{m+m'+2(\rho-\delta)N}(h),$$

where

$$(1.17)'\qquad \begin{aligned} &t_{\gamma,h}(\xi, x''; \theta)\\ &= 0_s-\iint e^{-iy\cdot\eta}\partial_{\gamma'}^{\xi}\{p_{h(\gamma)}(\xi, \theta y+\tilde{\nabla}_{\xi}\phi_h(\xi, x'', \xi'))\}_{|\xi'=\xi-\eta}dy\,d\!\!{}^-\eta.\end{aligned}$$

Proof is similar to that of Theorem 3.7 and Proposition 5.6 of [6].

Theorem 1.7. *Let* $\phi_h(x,\xi)\in P_{\rho,\delta}^{[x]}(\tau,\sigma;h)$ *and* $p_h(\xi,x')\in B_{\rho,\delta}^m(h)$ *with* $0\leqq\tau<1$, $0\leqq\sigma<\infty$, $0\leqq\delta\leqq\rho\leqq1$ *and* $m\in R^1$. *Then for* $P_h(\phi_h^*)\equiv p_h(\phi_h^*;D_x,X')\in \boldsymbol{B}_{\rho,\delta}^m(\phi_h^*)$ *we have*

$$(1.18)\qquad \begin{aligned}&\|P_h(\phi_h^*)\|_{L^2\to L^2}\\ &\leqq Ch^m|p_h|_M^{(m)}(1+\max_{1\leqq|\alpha+\beta|\leqq M+1}\sup_{h,x,\xi}|\partial_\xi^\alpha D_x^\beta\nabla_x\tilde{J}_h(x,\xi)|)^{(M+1)/2},\end{aligned}$$

where $M=2([n/2]+[5n/4]+2)$; $\tilde{J}_h$ *is defined by* (1.1); *and* C *is a positive constant independent of* $h\in(0,1)$, $\{\phi_h\}_{0<h<1}$ *and* $\{p_h\}_{0<h<1}$.

Proof. For $f\in\mathcal{S}$ we have from (1.13)

$$\widehat{P_h(\phi_h^*)f}(\xi) = \int e^{-i\phi_h(x',\xi)}p_h(\xi, x')f(x')dx'.$$

Thus we have

$$\|P_h(\phi_h^*)f\|_{L^2}^2 = (K_hf,f)_{L^2},$$

where

$$K_h f(x) = 0_s\text{-}\iint e^{i(\phi_h(x,\xi)-\phi_h(y,\xi))} p_h(\xi, y)\overline{p_h(\xi, x)} f(y) dy d\xi .$$

Noting that $\phi_h(x,\xi)-\phi_h(y,\xi)=(x-y)\cdot\tilde{\nabla}_x\phi_h(x,\xi,y)\equiv(x-y)\cdot\int_0^1 \nabla_x\phi_h(y+\theta(x-y), \xi)d\theta$ and that the mapping $\xi\mapsto\eta=\tilde{\nabla}_x\phi_h(x,\xi,y)$ has the inverse C^∞-diffeomorphism $\eta\mapsto\tilde{\nabla}_x\phi_h^{-1}(x,\eta,y)$, since $|\vec{\nabla}_\xi\tilde{\nabla}_x\phi_h(x,\xi,y)-I|=|\vec{\nabla}_\xi\tilde{\nabla}_x\tilde{J}_h(h^{-\delta}x,h^\rho\xi,h^{-\delta}y)|\leqq\sigma<1$ by (1.2)-iii), we make a change of variable: $\eta=\tilde{\nabla}_x\phi_h(x,\xi,y)$. Then we obtain

$$\begin{aligned} K_h f(x) = 0_s\text{-}\iint e^{i\eta\cdot(x-y)} & p_h(\tilde{\nabla}_x\phi_h^{-1}(x, \eta, y), y)\times \\ & \times\overline{p_h(\tilde{\nabla}_x\phi_h^{-1}(x, \eta, y), x)}\left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\eta)}(x, \eta, y)\right| f(y)dy d\eta .\end{aligned}$$

Putting

$$\begin{aligned} r_h(\tilde{x}, \tilde{\eta}, \tilde{y}) = h^{-2m} & \tilde{p}_h(\tilde{\nabla}_x\tilde{\phi}_h^{-1}(\tilde{x}, h^{\rho-\delta}\tilde{\eta}, \tilde{y}), \tilde{y})\times \\ & \times\overline{\tilde{p}_h(\tilde{\nabla}_x\tilde{\phi}_h^{-1}(\tilde{x}, h^{\rho-\delta}\tilde{\eta}, \tilde{y}), \tilde{x})}\left|\frac{D(\tilde{\nabla}_x\tilde{\phi}_h^{-1})}{D(\eta)}(\tilde{x}, h^{\rho-\delta}\tilde{\eta}, \tilde{y})\right| ,\end{aligned}$$

where $\tilde{p}_h(\xi,x)=p_h(h^{-\rho}\xi,h^\delta x)$, and making again a change of variables $x=h^\delta\tilde{x}$, $\eta=h^{-\delta}\tilde{\eta}$, $y=h^\delta\tilde{y}$, we obtain

$$K_h f(h^\delta\tilde{x}) = h^{2m}\, 0_s\text{-}\iint e^{i\tilde{\eta}\cdot(\tilde{x}-\tilde{y})} r_h(\tilde{x}, \tilde{\eta}, \tilde{y}) f(h^\delta\tilde{y}) d\tilde{y} d\tilde{\eta} .$$

Thus by the Calderón-Vaillancourt theorem ([1]) we have

$$\begin{aligned} & ||K_h f(h^\delta\tilde{x})||_{L^2(R^n_{\tilde{x}})} \\ & \leqq Ch^{2m} \max_{|\beta+\alpha+\beta'|\leqq M} \sup_{\tilde{x},\tilde{\eta},\tilde{y}} |\partial_{\tilde{x}}^\beta\partial_{\tilde{\eta}}^\alpha\partial_{\tilde{y}}^{\beta'} r_h(\tilde{x}, \tilde{\eta}, \tilde{y})|\, ||f(h^\delta\tilde{y})||_{L^2(R^n_{\tilde{y}})}\end{aligned}$$

for some constant $C>0$ independent of $h\in(0, 1)$. From $\rho-\delta\geqq0$ and the definition of r_h we get

$$\begin{aligned} & |\partial_{\tilde{x}}^\beta\partial_{\tilde{\eta}}^\alpha\partial_{\tilde{y}}^{\beta'} r_h(\tilde{x}, \tilde{\eta}, \tilde{y})| \\ & \leqq C(1+\max_{2\leqq|\alpha+\beta|\leqq M+1} \sup_{h,x,\xi} |\partial_\xi^\alpha D_x^\beta\nabla_x\tilde{J}_h(x, \xi)|)^{M+1}(|p_h|_M^{(m)})^2\end{aligned}$$

for $|\beta+\alpha+\beta'|\leqq M$, where C is independent of $h\in(0, 1)$. Thus we have (1.18). □

2. Multi-products of conjugate Fourier integral operators

Now we turn to the study of the multi-products of conjugate Fourier integral operators. We first introduce the following condition (♯) for $(\nu+1)$-tuple $(\phi_{1,h}, \cdots, \phi_{\nu+1,h})$ ($\nu\geqq1$, integer) of phase functions $\phi_{j,h}\in P_{\rho,\delta}^{[x]}(\tau_j,\sigma_j;h)$ $(j=1, \cdots, \nu+1)$:

(♯) For each fixed $h\in(0, 1)$, there exists a unique C^∞ solution $\{X_{\nu,h}^j,$

$\Xi^j_{\nu,h}\}^{\nu}_{j=1}(x,\xi)$ of the equation

$$
(2.1)\qquad \begin{cases} X^j_{\nu,h} = \nabla_\xi \phi_{j,h}(X^{j-1}_{\nu,h}, \Xi^j_{\nu,h}), \\ \Xi^j_{\nu,h} = \nabla_x \phi_{j+1,h}(X^j_{\nu,h}, \Xi^{j+1}_{\nu,h}) \end{cases} \quad (j=1,\cdots,\nu)
$$

where $X^0_{\nu,h}=x$ and $\Xi^{\nu+1}_{\nu,h}=\xi$.

DEFINITION 2.1. For $(\nu+1)$-tuple $(\phi_{1,h},\cdots,\phi_{\nu+1,h})$ of phase functions $\phi_{j,h}\in P^{[x]}_{\rho,\delta}(\tau_j,\sigma_j;h)$ satisfying (#), we define its #-$(\nu+1)$ product $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$ by

$$
(2.2)\qquad \begin{aligned} &\Phi_{\nu+1,h}(x,\xi) \\ &= \sum_{j=1}^{\nu} (\phi_{j,h}(X^{j-1}_{\nu,h}, \Xi^j_{\nu,h}) - X^j_{\nu,h}\cdot\Xi^j_{\nu,h}) + \phi_{\nu+1,h}(X^\nu_{\nu,h}, \xi), \end{aligned}
$$

where $X^0_{\nu,h}=x$ and $\{X^j_{\nu,h}, \Xi^j_{\nu,h}\}^{\nu}_{j=1}(x,\xi)$ is the assumed solution of (2.1).

REMARK. 1° The condition (#) is satisfied by the phase functions defined as the solution of some Hamilton-Jacobi equations (see Proposition 4.3 of section 4).

2° Let $\{\tilde{X}^j_{\nu,h}, \tilde{\Xi}^j_{\nu,h}\}^{\nu}_{j=1}(x,\xi)\equiv\{h^{-\delta}X^j_{\nu,h}, h^{\rho}\Xi^j_{\nu,h}\}^{\nu}_{j=1}(h^\delta x, h^{-\rho}\xi)$. Then $\{\tilde{X}^j_{\nu,h}, \tilde{\Xi}^j_{\nu,h}\}^{\nu}_{j=1}(x,\xi)$ is the solution of (2.1) with $\phi_{j,h}(x,\xi)$ replaced by $\tilde{\phi}_{j,h}(x,\xi)\equiv h^{\rho-\delta}\phi_{j,h}(h^\delta x, h^{-\rho}\xi)$. Thus $\tilde{\phi}_{j,h}\in P^{[x]}_{0,0}(\tau_j,\sigma_j;h)$ $(j=1,2,\cdots)$ satisfy the condition (#) and we can define #-$(\nu+1)$ product $\tilde{\Phi}_{\nu+1,h}=\tilde{\phi}_{1,h}\#\cdots\#\tilde{\phi}_{\nu+1,h}$ by (2.2) with $\phi_{j,h}$ and $\{X^j_{\nu,h}, \Xi^j_{\nu,h}\}^{\nu}_{j=1}$ replaced by $\tilde{\phi}_{j,h}$ and $\{\tilde{X}^j_{\nu,h}, \tilde{\Xi}^j_{\nu,h}\}^{\nu}_{j=1}$. In this case we have the relation $\tilde{\Phi}_{\nu+j,h}(x,\xi)=h^{\rho-\delta}\Phi_{\nu+1,h}(h^\delta x, h^{-\rho}\xi)$.

We next prepare a technical key lemma.

Lemma 2.2. *Let x^0, x^j, ξ^j, u^j, $v^j\in R^n$ and let r_j, s_j, $\tilde{s}_j$ and t_j be $n\times n$ real matrices for $j=1, 2, \cdots$ such that*

$$
(2.3)\qquad |r_j|\leqq\sigma,\quad |t_j|,\ |s_j|,\ |\tilde{s}_j|\leqq\tau_j
$$

for some $0\leqq\sigma<\infty$ and $0\leqq\tau_j<\infty$ $(j=1, 2, \cdots)$. Then we have for any integer $\nu\geqq1$

$$
(2.4)\qquad \begin{aligned} &|x^1-(I+\tilde{s}_1)x^0-r_1\xi^1-u^1| \\ &+\sum_{j=2}^{\nu}\{|x^j-(I+\tilde{s}_j)x^{j-1}-r_j\xi^j-u^j|+|\xi^{j-1}-(I+s_j)\xi^j-t_jx^{j-1}-v^j|\} \\ &+|\xi^\nu-(I+s_{\nu+1})\xi^{\nu+1}-t_{\nu+1}x^\nu-v^{\nu+1}| \\ \geqq\ &(1-2\sigma\bar{\tau}_{\nu+1}-\bar{\tau}_{\nu+1})\sum_{j=1}^{\nu}|\xi^j-\xi^{j+1}|-(2\sigma\bar{\tau}_{\nu+1}+\bar{\tau}_{\nu+1})|\xi^{\nu+1}| \\ &+(1-2\bar{\tau}_{\nu+1})\sum_{j=1}^{\nu}|x^j-x^{j-1}-r_j\xi^j|-2\bar{\tau}_{\nu+1}|x^0| \\ &-\sum_{j=1}^{\nu}(|u^j|+|v^{j+1}|), \end{aligned}
$$

provided that $\overline{\overline{\tau}}_{\nu+1}=\sum_{k=1}^{\nu+1} k\tau_k$ *and* $\overline{\tau}_{\nu+1}=\sum_{k=1}^{\nu+1}\tau_k$ *satisfy* $0\leqq 2(\sigma\overline{\overline{\tau}}_{\nu+1}+\overline{\tau}_{\nu+1})<1$. *The inequality* (2.4) *also holds for* $n\times n$ *real matrices* x^0, x^j, ξ^j, u^j, *and* v^j.

Proof. We first observe that the left hand side of (2.4) is bounded from below by $\Xi+X+U$, where

$$(2.5)\qquad \begin{cases} \Xi=\sum_{j=2}^{\nu+1}|\xi^{j-1}-\xi^j|-\sum_{j=2}^{\nu+1}\tau_j|\xi^j|\,, \\ X=\sum_{j=1}^{\nu}|x^j-x^{j-1}-r_j\xi^j|-2\sum_{j=1}^{\nu}\tau_j|x^{j-1}|-\tau_{\nu+1}|x^\nu|\,, \\ U=-\sum_{j=1}^{\nu}(|u^j|+|v^{j+1}|)\,. \end{cases}$$

Put $\tau_{\nu,j}=\tau_\nu+\cdots+\tau_j$. Then X is estimated as

$$\begin{aligned} X&=\sum_{j=1}^{\nu}(1-(\tau_{\nu+1}+2\tau_{\nu,j+1}))|x^j-x^{j-1}-r_j\xi^j| \\ &\quad+\sum_{j=1}^{\nu}(\tau_{\nu+1}+2\tau_{\nu,j+1})|x^j-x^{j-1}-r_j\xi^j|-2\sum_{j=1}^{\nu}\tau_j|x^{j-1}|-\tau_{\nu+1}|x^\nu| \\ &\geqq(1-2\overline{\tau}_{\nu+1})\sum_{j=1}^{\nu}|x^j-x^{j-1}-r_j\xi^j|+\sum_{j=1}^{\nu}(\tau_{\nu+1}+2\tau_{\nu,j+1})|x^j| \\ &\quad-\sum_{j=1}^{\nu}(\tau_{\nu+1}+2\tau_{\nu,j+1})(|x^{j-1}|+\sigma|\xi^j|) \\ &\quad-2\sum_{j=1}^{\nu}\tau_j|x^{j-1}|-\tau_{\nu+1}|x^\nu| \\ &\geqq(1-2\overline{\tau}_{\nu+1})\sum_{j=1}^{\nu}|x^j-x^{j-1}-r_j\xi^j|-2\sigma\sum_{j=1}^{\nu}\tau_{\nu+1,j+1}|\xi^j|-2\overline{\tau}_{\nu+1}|x^0|\,. \end{aligned}$$

Thus we have

$$(2.6)\qquad X+\Xi\geqq A+B\,,$$

where

$$(2.7)\qquad \begin{cases} A=(1-2\overline{\tau}_{\nu+1})\sum_{j=1}^{\nu}|x^j-x^{j-1}-r_j\xi^j|-2\overline{\tau}_{\nu+1}|x^0|\,, \\ B=\sum_{j=2}^{\nu+1}|\xi^{j-1}-\xi^j|-\sum_{j=2}^{\nu+1}\tau_j|\xi^j|-2\sigma\sum_{j=1}^{\nu}\tau_{\nu+1,j+1}|\xi^j|\,. \end{cases}$$

Here noting $\sum_{l=1}^{j+1}\tau_{\nu+1,l}\leqq\overline{\overline{\tau}}_{\nu+1}$ for $1\leqq j\leqq\nu$, we have

$$\begin{aligned} B&\geqq\sum_{j=1}^{\nu}|\xi^j-\xi^{j+1}|-\sum_{j=1}^{\nu}(2\sigma\tau_{\nu+1,j+1}+\tau_j)|\xi^j|-\tau_{\nu+1}|\xi^{\nu+1}| \\ &\geqq\sum_{j=1}^{\nu}(1-(2\sigma\sum_{l=1}^{j+1}\tau_{\nu+1,l}+\overline{\tau}_j))|\xi^j-\xi^{j+1}| \\ &\quad+\sum_{j=1}^{\nu}(2\sigma\sum_{l=1}^{j+1}\tau_{\nu+1,l}+\overline{\tau}_j)(|\xi^j|-|\xi^{j+1}|) \end{aligned}$$

$$-\sum_{j=1}^{\nu}(2\sigma\tau_{\nu+1,j+1}+\tau_j)|\xi^j|-\tau_{\nu+1}|\xi^{\nu+1}|$$
$$\geqq\sum_{j=1}^{\nu}(1-(2\sigma\bar{\bar{\tau}}_{\nu+1}+\bar{\tau}_{\nu+1}))|\xi^j-\xi^{j+1}|-(2\sigma\bar{\bar{\tau}}_{\nu+1}+\bar{\tau}_{\nu+1})|\xi^{\nu+1}|\ .$$

Combining this with (2.5)–(2.7) proves the lemma. □

Proposition 2.3. *Let $\phi_{j,h}\in P_{\rho,\delta}^{[x]}(\tau_j,\sigma_j;h)$ $(j=1,2,\cdots)$ satisfy the condition (♯) for any integer $\nu\geqq 1$. Let $\{\tau_j\}_{j=1}^{\infty}$ and $\{\sigma_j\}_{j=1}^{\infty}$ satisfy*

$$\begin{cases}0\leqq\sigma_j\leqq\sigma_0\\ 0\leqq 2(\sigma_0\bar{\bar{\tau}}_\infty+\bar{\tau}_\infty)\leqq\tau_0\end{cases}\tag{2.8}$$

for some $0\leqq\sigma_0<\infty$ and $0\leqq\tau_0<1$, where $\bar{\tau}_\infty=\sum_{k=1}^{\infty}\tau_k$ and $\bar{\bar{\tau}}_\infty=\sum_{k=1}^{\infty}k\tau_k$. Let $\{\tilde{X}_{\nu,h}^j,\tilde{\Xi}_{\nu,h}^j\}_{j=1}^{\nu}(x,\xi)=\{h^{-\delta}X_{\nu,h}^j,h^{\sigma}\Xi_{\nu,h}^j\}_{j=1}^{\nu}(h^{\delta}x,h^{-\rho}\xi)$, where $\{X_{\nu,h}^j,\Xi_{\nu,h}^j\}_{j=1}^{\nu}$ is the assumed solution of (2.1). Then the following estimates hold.

i) *For any $\nu\geqq 1$ there exists a constant $C_\nu>0$ such that for $j=1,\cdots,\nu$, $h\in(0,1)$ and $(x,\xi)\in R^{2n}$*

$$\begin{cases}|\tilde{X}_{\nu,h}^j-\tilde{X}_{\nu,h}^{j-1}|\leqq C_\nu\langle\xi\rangle\,,\\ |\tilde{\Xi}_{\nu,h}^j-\tilde{\Xi}_{\nu,h}^{j+1}|\leqq C_\nu\,.\end{cases}\tag{2.9}$$

ii) *For any $\nu\geqq 1$, $1\leqq k\leqq\nu$, $h\in(0,1)$ and $(x,\xi)\in R^{2n}$, one has*

$$\begin{cases}\text{i)}\ \sum_{j=1}^{k}|\nabla_x(\tilde{X}_{\nu,h}^j-\tilde{X}_{\nu,h}^{j-1})|\leqq\dfrac{\tau_0}{1-\tau_0}(1+\bar{\sigma}_k),\ |\nabla_x\tilde{X}_{\nu,h}^k-I|\leqq\dfrac{\tau_0}{1-\tau_0}(1+\bar{\sigma}_k)\,,\\ \text{ii)}\ \sum_{j=1}^{k}|\nabla_x(\tilde{\Xi}_{\nu,h}^j-\tilde{\Xi}_{\nu,h}^{j+1})|\leqq\dfrac{\tau_0}{1-\tau_0}\,,\quad|\nabla_x\tilde{\Xi}_{\nu,k}^h|\leqq\dfrac{\tau_0}{1-\tau_0}\,,\end{cases}\tag{2.10}$$

and

$$\begin{cases}\text{i)}\ \sum_{j=1}^{k}|\nabla_\xi(\tilde{X}_{\nu,h}^j-\tilde{X}_{\nu,h}^{j-1})|\leqq\dfrac{1+\bar{\sigma}_k}{1-\tau_0},\quad|\nabla_\xi\tilde{X}_{\nu,h}^k|\leqq\dfrac{1+\bar{\sigma}_k}{1-\tau_0}\,,\\ \text{ii)}\ \sum_{j=1}^{k}|\nabla_\xi(\tilde{\Xi}_{\nu,h}^j-\tilde{\Xi}_{\nu,h}^{j+1})|\leqq\dfrac{\tau_0}{1-\tau_0},\quad|\nabla_\xi\tilde{\Xi}_{\nu,h}^k-I|\leqq\dfrac{\tau_0}{1-\tau_0}\,,\end{cases}\tag{2.11}$$

where $\bar{\sigma}_k=\sigma_1+\cdots+\sigma_k$.

iii) *For any α, β satisfying $|\alpha+\beta|\geqq 1$, one has*

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta(\tilde{X}_{\nu,h}^j-x,\ \tilde{\Xi}_{\nu,h}^j-\xi)|\\ &\leqq C_{\alpha,\beta}\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha+\beta|-1}\Big(\sum_{j=1}^{\nu+1}|J_{j,h}|_{3,|\alpha+\beta|-1}\Big)^{|\alpha+\beta|-1}\,,\end{aligned}\tag{2.12}$$

where the constant $C_{\alpha,\beta}$ is independent of $\nu\geqq j\geqq 1$, $h\in(0,1)$, $(x,\xi)\in R^{2n}$, $\{\sigma_j\}_{j=1}^{\infty}$, $\{\tau_j\}_{j=1}^{\infty}$, $0\leqq\sigma_0\leqq 1$ and $0\leqq\tau_0\leqq 1$.

Proof. i) Since $\{\tilde{X}_{\nu,h}^j,\tilde{\Xi}_{\nu,h}^j\}_{j=1}^{\nu}$ is the solution of (2.1) with $\phi_{j,h}$ replaced by

$\tilde{\phi}_{j,h}$, we have

$$\begin{cases} \tilde{X}^j_{\nu,h}-\tilde{X}^{j-1}_{\nu,h} = \nabla_\xi \tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h}) , \\ \tilde{\Xi}^j_{\nu,h}-\tilde{\Xi}^{j+1}_{\nu,h} = \nabla_x \tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{\nu,h}) , \end{cases} \quad (j=1, \cdots, \nu) \tag{2.13}$$

where $\tilde{X}^0_{\nu,h}=x$ and $\tilde{\Xi}^{\nu+1}_{\nu,h}=\xi$. From this and (1.2) we have (2.9).

ii) Differentiating (2.13) we have for $j=1,\cdots,\nu$

$$\begin{cases} \nabla_x \tilde{X}^j_{\nu,h} = (I+\vec{\nabla}_x \nabla_\xi \tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h}))\cdot \nabla_x \tilde{X}^{j-1}_{\nu,h} \\ \qquad +\vec{\nabla}_\xi \nabla_\xi \tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h})\cdot \nabla_x \tilde{\Xi}^j_{\nu,h} , \\ \nabla_x \tilde{\Xi}^j_{\nu,h} = (I+(\vec{\nabla}_\xi \nabla_x \tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{\nu,h}))\cdot \nabla_x \tilde{\Xi}^{j+1}_{\nu,h} \\ \qquad +\vec{\nabla}_x \nabla_x \tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{\nu,h})\cdot \nabla_x \tilde{X}^j_{\nu,h} \end{cases} \tag{2.14}$$

and

$$\begin{cases} \nabla_\xi \tilde{X}^j_{\nu,h} = (I+\vec{\nabla}_x \nabla_\xi \tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h}))\cdot \nabla_\xi \tilde{X}^{j-1}_{\nu,h} \\ \qquad +\vec{\nabla}_\xi \nabla_\xi \tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h})\cdot \nabla_\xi \tilde{\Xi}^j_{\nu,h} , \\ \nabla_\xi \tilde{\Xi}^j_{\nu,h} = (I+\vec{\nabla}_\xi \nabla_x \tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{n,h}))\cdot \nabla_\xi \tilde{\Xi}^{j+1}_{\nu,h} \\ \qquad +\vec{\nabla}_x \nabla_x \tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{\nu,h})\cdot \nabla_\xi \tilde{X}^j_{\nu,h} . \end{cases} \tag{2.15}$$

Writing $y^j=\nabla_x \tilde{X}^j_{\nu,h}$, $\eta^j=\nabla_x \tilde{\Xi}^j_{\nu,h}$ and putting $\tilde{s}_j=\vec{\nabla}_x \nabla_\xi \tilde{J}_{j,k}$, $s_{j+1}=\vec{\nabla}_\xi \nabla_x \tilde{J}_{j+1,k}$, $t_{j+1}=\vec{\nabla}_x \nabla_x \tilde{J}_{j+1,h}$ and $r_j=\vec{\nabla}_\xi \nabla_\xi \tilde{J}_{j,h}$, we can rewrite (2.14) as

$$\begin{cases} y^j = (I+\tilde{s}_j)y^{j-1}+r_j\eta^j , \\ \eta^j = (I+s_{j+1})\eta^{j+1}+t_{j+1}y^j , \end{cases} \quad (j=1, \cdots, \nu) \tag{2.16}$$

where $y^0=I$ and $\eta^{\nu+1}=0$. Since $\tilde{s}_j$, s_j, t_j and r_j satisfy $|\tilde{s}_j|$, $|s_j|$, $|t_j|\leqq\tau_j$ and $|r_j|\leqq\sigma_j\leqq\sigma_0$, we can apply Lemma 2.2 and we have

$$0\geqq(1-\tau_0)\sum_{j=1}^{\nu}(|\eta^j-\eta^{j+1}|+|y^j-y^{j-1}-r_j\eta^j|)-\tau_0|\eta^{\nu+1}|-\tau_0|y^0| , \tag{2.17}$$

hence

$$\sum_{j=}^{\nu}(|\eta^j-\eta^{j+1}|+|y^j-y^{j-1}-r_j\eta^j|)\leqq\frac{\tau_0}{1-\tau_0} . \tag{2.18}$$

Thus we obtain (2.10)-ii). So using (2.18) again we obtain for $k=1,\cdots,\nu$,

$$\begin{aligned} \sum_{j=1}^{k}|y^j-y^{j-1}| &\leqq \sum_{j=1}^{k}(|y^j-y^{j-1}-r_j\eta^j|+\sigma_j|\eta^j|) \\ &\leqq\frac{\tau_0}{1-\tau_0}+\sum_{j=1}^{k}\sigma_j\sum_{l=j}^{\nu}|\eta^l-\eta^{l+1}| \\ &\leqq\frac{\tau_0}{1-\tau_0}(1+\bar{\sigma}_k) , \end{aligned} \tag{2.19}$$

which proves (2.10).

We next prove (2.11). Under obvious notations, (2.15) is rewritten as (2.16) with $y^0=0$ and $\eta^{\nu+1}=I$. Applying Lemma 2.2 we obtain (2.17) hence (2.18), from which we get (2.11)-ii). So using (2.18) again we have for $k=1,\cdots,\nu$

$$
\begin{aligned}
\sum_{j=1}^{k}|y^j-y^{j-1}| &\leqq \frac{\tau_0}{1-\tau_0}+\sum_{j=1}^{k}\sigma_j\Big(\sum_{l=j}^{\nu}|\eta^l-\eta^{l+1}|+|\eta^{\nu+1}|\Big)\\
&\leqq \frac{\tau_0+\bar{\sigma}_k}{1-\tau_0}\leqq\frac{1+\bar{\sigma}_k}{1-\tau_0},
\end{aligned}
\tag{2.20}
$$

which proves (2.11).

iii) For any multi-indices α, β with $|\alpha+\beta|\geqq 1$, differentiating (2.15) we have for $j=1,\cdots,\nu$

$$
\begin{cases}
\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{X}^j_{\nu,h} = (I+\vec{\nabla}_x\nabla_\xi\tilde{J}_{j,h})\cdot\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{X}^{j-1}_{\nu,h}\\
\qquad\qquad +\vec{\nabla}_\xi\nabla_\xi\tilde{J}_{j,h}\cdot\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{\Xi}^j_{\nu,h}+U_j\,,\\
\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{\Xi}^j_{\nu,h} = (I+\vec{\nabla}_\xi\nabla_x\tilde{J}_{j+1,h})\cdot\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{\Xi}^{j+1}_{\nu,h}\\
\qquad\qquad +\vec{\nabla}_x\nabla_x\tilde{J}_{j+1,h}\cdot\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{X}^j_{\nu,h}+V_{j+1}\,,
\end{cases}
\tag{2.21}
$$

where U_j and V_j are the polynomial of $\partial_\xi^\gamma\partial_x^\delta\tilde{J}_{j,h}(3\leqq|\gamma+\delta|\leqq|\alpha+\beta|+2)$ and $\partial_\xi^{\gamma'}\partial_x^{\delta'}\tilde{X}^{j-1}_{\nu,h}$, $\partial_\xi^{\gamma'}\partial_x^{\delta'}\tilde{\Xi}^j_{\nu,h}(1\leqq|\gamma'+\delta'|\leqq|\alpha+\beta|)$ of order $|\alpha+\beta|+2$; especially the orders of $\partial_x\tilde{X}^{j-1}_{\nu,h}$, $\partial_\xi\tilde{X}^{j-1}_{\nu,h}$, $\partial_x\tilde{\Xi}^j_{\nu,h}$ and $\partial_\xi\tilde{\Xi}^j_{\nu,h}$ are at most $|\alpha+\beta|+1$ and the order of $\partial_\xi^\gamma\partial_x^\delta\tilde{J}_{j,h}$ is at most 1. Moreover the sum of $|\gamma+\delta|$ of $\partial_\xi^\gamma\partial_x^\delta\tilde{X}^{j-1}_{\nu,h}$ and $\partial_\xi^\gamma\partial_x^\delta\tilde{\Xi}^j_{\nu,h}$ $(1\leqq|\gamma+\delta|\leqq|\alpha+\beta|)$ in every term of U_j and V_j does not exceed $|\alpha+\beta|+1$. Similar results hold for the differentials of (2.14). Thus using Lemma 2.2 and ii) we obtain by induction

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{X}^j_{\nu,h}|,\ |\partial_\xi^\alpha\partial_x^\beta\nabla_\xi\tilde{\Xi}^j_{\nu,h}|,\ |\partial_\xi^\alpha\partial_x^\beta\nabla_x\tilde{X}^j_{\nu,h}|,\ |\partial_\xi^\alpha\partial_x^\beta\nabla_x\tilde{\Xi}^j_{\nu,h}|\\
&\leqq C_{\alpha,\beta}\Big(\sum_{j=1}^{\nu+1}|J_{j,h}|_{3,|\alpha+\beta|}\Big)^{|\alpha+\beta|}\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha+\beta|+1}. \quad\square
\end{aligned}
\tag{2.22}
$$

Proposition 2.4. *Let $\phi_{j,h}\in P^{[x]}_{\rho,\delta}(\tau_j,\sigma_j;h)$ $(j=1,2,\cdots)$ satisfy the condition (♯) for any integer $\nu\geqq 1$. Let $\{\tau_j\}_{j=1}^\infty$ and $\{\sigma_j\}_{j=1}^\infty$ satisfy*

$$
\begin{cases}
0\leqq\sigma_j\leqq\sigma_0\,,\\
0\leqq 2(\sigma_0\bar{\bar{\tau}}_\infty+\bar{\tau}_\infty)\leqq\tau_0
\end{cases}
\tag{2.23}
$$

for some $0\leqq\sigma_0<\infty$ and $0\leqq\tau_0\leqq 1/4$. Let $\Phi_{\nu+1,h}=\phi_{1,h}\sharp\cdots\sharp\phi_{\nu+1,h}$ be defined by (2.2). *Then $\Phi_{\nu+1,h}$ satisfies the following properties*:

i) *We have*

$$
\Phi_{\nu+1,h}\in P^{[x]}_{\rho,\delta}\Big(3\tau_0,\ \frac{5}{3}(1+\bar{\sigma}_{\nu+1});h\Big)
\tag{2.24}
$$

for any $\nu\geqq 1$.

ii) *We have for any $\nu\geqq 1$*

$$(2.25) \qquad \begin{cases} \nabla_x \Phi_{\nu+1,h}(x, \xi) = \nabla_x \phi_{1,h}(x, \Xi^1_{\nu,h}), \\ \nabla_k \Phi_{\nu+1,h}(x, \xi) = \nabla_\xi \phi_{\nu+1,h}(X^\nu_{\nu,h}, \xi) \end{cases}$$

and

$$(2.26) \qquad \begin{cases} \Delta_x \boldsymbol{J}_{\nu+1,h}(x, \xi) = \nabla_x J_{1,h}(x, \Xi^1_{\nu,h}) + \Xi^1_{\nu,h} - \xi, \\ \nabla_\xi \boldsymbol{J}_{\nu+1,h}(x, \xi) = \nabla_\xi J_{\nu+1,h}(X^\nu_{\nu,h}, \xi) + X^\nu_{\nu,h} - x, \end{cases}$$

where $\boldsymbol{J}_{\nu+1,h}(x, \xi) \equiv \Phi_{\nu+1,h}(x, \xi) - x \cdot \xi$.

Proof. Since the properties except (2.24) can be proved without using (2.24) in quite the same way as in the proof of Theorem 2.7 of [6], we only prove (2.24). Since

$$(2.27) \qquad \begin{cases} \nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi) = \nabla_x \tilde{J}_{1,h}(x, \tilde{\Xi}^1_{\nu,h}) + \tilde{\Xi}^1_{\nu,h} - \xi, \\ \nabla_\xi \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi) = \nabla_\xi \tilde{J}_{\nu+1,h}(\tilde{X}^\nu_{\nu,h}, \xi) + \tilde{X}^\nu_{\nu,h} - x, \end{cases}$$

we easily see from (2.9) that

$$(2.28) \qquad \sup_{h,x,\xi} \{ |\nabla_\xi \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi)| / \langle \xi \rangle + |\nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi)| \} < \infty.$$

From (2.27) we have

$$(2.29) \qquad \begin{cases} \nabla_x \nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi) = \nabla_x \nabla_x \tilde{J}_{1,h}(x, \tilde{\Xi}^1_{\nu,h}) \\ \qquad\qquad + (\nabla_\xi \nabla_x \tilde{J}_{1,h}(x, \tilde{\Xi}^1_{\nu,h}) + I) \cdot \nabla_x \tilde{\Xi}^1_{\nu,h}, \\ \nabla_\xi \nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi) = (\nabla_\xi \nabla_x \tilde{J}_{1,h}(x, \tilde{\Xi}^1_{\nu,h}) + I) \cdot \nabla_\xi \tilde{\Xi}^1_{\nu,h} - I, \\ \nabla_\xi \nabla_\xi \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi) = (\nabla_x \nabla_\xi \tilde{J}_{\nu+1,h}(\tilde{X}^\nu_{\nu,h}, \xi) + I) \cdot \nabla_\xi \tilde{X}^\nu_{\nu,h} \\ \qquad\qquad + \nabla_\xi \nabla_\xi \tilde{J}_{\nu+1,h}(\tilde{X}^\nu_{\nu,h}, \xi). \end{cases}$$

Thus from (2.10) and (2.11) we get

$$(2.30) \qquad \begin{cases} |\nabla_x \nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi)| \leqq \tau_1 + (\tau_1 + 1) \dfrac{\tau_0}{1 - \tau_0} \leqq 3\tau_0, \\ |\nabla_\xi \nabla_x \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi)| \leqq (\tau_1 + 1) \dfrac{\tau_0}{1 - \tau_0} + \tau_1 \leqq 3\tau_0, \\ |\nabla_\xi \nabla_\xi \tilde{\boldsymbol{J}}_{\nu+1,h}(x, \xi)| \leqq (\tau_{\nu+1} + 1) \dfrac{1 + \bar{\sigma}_\nu}{1 - \tau_0} + \sigma_{\nu+1} \\ \qquad \leqq \dfrac{1 + \tau_0}{1 - \tau_0}(1 + \bar{\sigma}_{\nu+1}) \leqq \dfrac{5}{3}(1 + \bar{\sigma}_{\nu+1}). \end{cases}$$

Differentiating (2.29) further and using (2.10)–(2.12) we can easily see that (1.3) holds for our $\boldsymbol{J}_{\nu+1,h}$. Thus we have proved (2.24). □

Before stating a theorem concerning the calculus of conjugate Fourier integral operators we prepare a lemma.

Lemma 2.5. *Let $\phi_{j,h}\in P^{[x]}_{\rho,\delta}(\tau_j,\sigma_j;h)(j=1,2,\cdots)$ satisfy the condition (♯) for any $\nu\geqq 1$, and assume that $0\leqq\sigma_j\leqq\sigma_0(j=1,2,\cdots)$ and $0\leqq 2(\sigma_0\bar{\tau}_\infty+\bar{\tau}_\infty)\leqq\tau_0$ for some $0\leqq\sigma_0<\infty$ and $0\leqq\tau_0\leqq 1/4$. Let $\nu\geqq 1$, and let $\{\tilde{X}^j_{\nu,h},\ \tilde{\Xi}^j_{\nu,h}\}^{\nu}_{j=1}(x,\ \xi)$ and $\tilde{\Phi}_{\nu+1,h}$ be defined as in 2° of the remark after Definition 2.1. Define $\tilde{\varphi}_h$ by*

$$
\begin{aligned}
&\tilde{\varphi}_h(y^1,\ \cdots,\ y^\nu,\ \eta^1,\ \cdots,\ \eta^\nu;\ x,\ \xi)\\
(2.31)\qquad &=\sum_{j=1}^{\nu}(\tilde{\phi}_{j,h}(\tilde{X}^{j-1}_{\nu,h}(x,\ \xi)+y^{j-1},\ \tilde{\Xi}^j_{\nu,h}+\eta^j)-\tilde{X}^j_{\nu,h}(x,\ \xi)\cdot\tilde{\Xi}^j_{\nu,h}(x,\ \xi))\\
&\quad+\tilde{\phi}_{\nu+1,h}(\tilde{X}^{\nu}_{\nu,h}(x,\ \xi)+y^\nu,\ \xi)-\tilde{\Phi}_{\nu+1,h}(x,\ \xi)
\end{aligned}
$$

with $y^0=0$ and $\tilde{X}^0_{\nu,h}=x$. Then the following estimates hold:

i) *For any $y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu,x,\xi\in R^n$, $\sigma\in R^1$, $h\in(0,1)$, and $\nu\geqq 1$*

$$
\begin{aligned}
&2\nu+h^{-2\sigma}\sum_{j=1}^{\nu}(|\nabla_{y^j}\tilde{\varphi}_h|^2+|\nabla_{\eta^j}\tilde{\varphi}_h|^2)\\
(2.32)\qquad &\geqq\frac{(1-\tau_0)^2}{4\nu}\{\sum_{j=1}^{\nu}(2+h^{-\sigma}|\eta^j-\eta^{j+1}|+h^{-\sigma}|y^j-y^{j-1}-r_j\eta^j|)\}^2\\
&\geqq\nu(1-\tau_0)^2\prod_{j=1}^{\nu}\langle h^{-\sigma}(\eta^j-\eta^{j+1})\rangle^{1/\nu}\langle h^{-\sigma}(y^j-y^{j-1}-r_j\eta^j)\rangle^{1/\nu},
\end{aligned}
$$

where $r_j=\vec{\nabla}_\xi\nabla_\xi\tilde{J}_{j,h}(\tilde{\Xi}^j_{\nu,h},\ \tilde{X}^{j-1}_{\nu,h}+y^{j-1},\ \tilde{\Xi}^j_{\nu,h}+\eta^j)$.

ii) *For any $y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu,x,\xi\in R^n$, $h\in(0,1)$, and for any multi-indices $\alpha,\beta,\alpha^1,\beta^1,\cdots,\alpha^\nu,\beta^\nu$ and any integer j with $1\leqq j\leqq\nu$, we have*

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\partial_{\eta^1}^{\alpha^1}\partial_{y^1}^{\beta^1}\cdots\partial_{\eta^\nu}^{\alpha^\nu}\partial_{y^\nu}^{\beta^\nu}\nabla_{y^j}\tilde{\varphi}_h|\\
&\leqq C_{\alpha,\beta,\alpha^{j+1},\beta^j}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{(2|\alpha+\beta|-1)\vee 0}\times\\
(2.33)\qquad &\times(1+\sum_{k=1}^{\nu+1}|J_{k,h}|_{3,(|\alpha+\beta|-1)\vee 0})^{|\alpha+\beta|}\times\\
&\times(1+|J_{j+1,h}|_{2,|\alpha+\beta+\alpha^{j+1}+\beta^j|})\langle y^j;\ \eta^j;\ \eta^{j+1}\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\partial_{\eta^1}^{\alpha^1}\partial_{y^1}^{\beta^1}\cdots\partial_{\eta^\nu}^{\alpha^\nu}\partial_{y^\nu}^{\beta^\nu}\nabla_{\eta^j}\tilde{\varphi}_h|\\
&\leqq C_{\alpha,\beta,\alpha^j,\beta^{j-1}}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{(|\alpha+\beta|-1)\vee 0}\times\\
(2.34)\qquad &\times(1+\sum_{k=1}^{\nu+1}|J_{k,h}|_{3,(|\alpha+\beta|-1)\vee 0})^{|\alpha+\beta|}\times\\
&\times(1+|J_{j,h}|_{2,|\alpha+\alpha+\alpha^j+\beta^{j-1}|})\langle y^{j-1};\ y^j;\ \eta^j\rangle,
\end{aligned}
$$

where $a\vee b=\max(a,\ b)$ for $a,\ b\in R^1$; $y^0=\eta^{\nu+1}=0$; and $\alpha^{\nu+1}=\beta^0=0$.

iii) *For any $y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu,x,\xi\in R^n$, $h\in(0,1)$ and any multi-indices α,β with $|\alpha+\beta|\geqq 1$, we have*

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\tilde{\varphi}_h|\\
(2.35)\qquad &\leqq C_{\alpha,\beta}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{2|\alpha+\beta|-1}(1+\sum_{k=1}^{\nu+1}|J_{k,h}|_{3,|\alpha+\beta|-1})^{|\alpha+\beta|}\times
\end{aligned}
$$

$$\times \{\sum_{j=1}^{\nu}(|y^j|+|\eta^j|)\}^2 .$$

Proof. i) We can rewrite (2.31) using (2.2) as

$$\begin{aligned}&\tilde{\Phi}_h(y_1 \cdots, y^\nu, \eta^1, \cdots, \eta^\nu; x, \xi)\\ &=\sum_{j=1}^{\nu}\{(\tilde{\phi}_{j,h}(\tilde{X}^{j-1}_{\nu,h}+y^{j-1}, \tilde{\Xi}^j_{\nu,h}+\eta^j)-\tilde{\phi}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h}))\\ &\quad-((\tilde{X}^j_{\nu,h}+y^j)\cdot(\tilde{\Xi}^j_{\nu,h}+\eta^j)-\tilde{X}^j_{\nu,h}\cdot\tilde{\Xi}^j_{\nu,h})\}\\ &\quad+(\tilde{\phi}_{\nu+1,h}(\tilde{X}^\nu_{\nu,h}+y^\nu, \xi)-\tilde{\phi}_{\nu+1,h}(\tilde{X}^\nu_{\nu,h}, \xi)) .\end{aligned} \tag{2.36}$$

From this we have for $j=1, \cdots, \nu$

$$\begin{cases}\nabla_y\tilde{\Phi}_h = -\eta^j+(I+\vec{\tilde{\nabla}}_\xi\nabla_x\tilde{J}_{j+1,h}(\tilde{\Xi}^{j+1}_{\nu,h}, \tilde{X}^j_{\nu,h}+y^j, \tilde{\Xi}^{j+1}_{\nu,h}+\eta^{j+1}))\cdot\eta^{j+1}\\ \qquad\qquad +\vec{\tilde{\nabla}}_x\nabla_x\tilde{J}_{j+1,h}(\tilde{X}^j_{\nu,h}, \tilde{\Xi}^{j+1}_{\nu,h}, \tilde{X}^j_{\nu,h}+y^j)\cdot y^j ,\\ \nabla_{\eta^j}\tilde{\Phi}_h= -y^j+(I+\vec{\tilde{\nabla}}_x\nabla_\xi\tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h}, \tilde{\Xi}^j_{\nu,h}, \tilde{X}^{j-1}_{\nu,h}+y^{j-1}))\cdot y^{j-1}\\ \qquad\qquad +\vec{\tilde{\nabla}}_\xi\nabla_\xi\tilde{J}_{i,h}(\tilde{\Xi}^j_{\nu,h}, \tilde{X}^{j-1}_{\nu,h}+y^{j-1}, \tilde{\Xi}^j_{\nu,h}+\eta^j)\cdot\eta^j ,\end{cases} \tag{2.37}$$

where $y^0=\eta^{\nu+1}=0$, $\tilde{X}^0_{\nu,h}=x$ and $\tilde{\Xi}^{\nu+1}_{\nu,h}=\xi$. Putting $s_j=\vec{\tilde{\nabla}}_\xi\nabla_x\tilde{J}_{j,h}$, $\tilde{s}_j=\vec{\tilde{\nabla}}_x\nabla_\xi\tilde{J}_{j,h}$, $t_j=\vec{\tilde{\nabla}}_x\nabla_x\tilde{J}_{j,h}$ and $r_j=\vec{\tilde{\nabla}}_\xi\nabla_\xi\tilde{J}_{j,h}$, we have

$$\begin{cases}\nabla_{y^j}\tilde{\Phi}_h = -\eta^j+(I+s_{j+1})\eta^{j+1}+t_{j+1}y^j ,\\ \nabla_{\eta^j}\tilde{\Phi}_h = -y^j+(I+\tilde{s}_j)y^{j-1}+r_j\eta^j ,\end{cases} \quad (j=1, \cdots, \nu,\ y^0=\eta^{\nu+1}=0) \tag{2.38}$$

and

$$|s_j|,\ |\tilde{s}_j|,\ |t_j|\leqq\tau_j , \qquad |r_j|\leqq\sigma_j . \tag{2.39}$$

Then applying Lemma 2.2 we have

$$\begin{aligned}&4\nu[2\nu+h^{-2\sigma}\sum_{j=1}^{\nu}(|\nabla_{y^j}\tilde{\Phi}_h|^2+|\nabla_{\eta^j}\tilde{\Phi}_h|^2)]\\ &\geqq 2[(2\nu)^2+\{\sum_{j=1}^{\nu}(|h^{-\sigma}\nabla_{y^j}\tilde{\Phi}_h|+|h^{-\sigma}\nabla_{\eta^j}\tilde{\Phi}_h|)\}^2]\\ &\geqq \{2\nu+\sum_{j=1}^{\nu}(|h^{-\sigma}\nabla_{y^j}\tilde{\Phi}_h|+|h^{-\sigma}\nabla_{\eta^j}\tilde{\Phi}_h|)\}^2\\ &\geqq \{(1-\tau_0)\sum_{j=1}^{\nu}(2+h^{-\sigma}|\eta^j-\eta^{j+1}|+h^{-\sigma}|y^j-y^{j-1}-r_j\eta^j|)\}^2\\ &\geqq \{2\nu(1-\tau_0)\prod_{j=1}^{\nu}\langle h^{-\sigma}(\eta^j-\eta^{j+1})\rangle^{1/(2\nu)}\langle h^{-\sigma}(y^j-y^{j-1}-r_j\eta^j)\rangle^{1/(2\nu)}\}^2 ,\end{aligned} \tag{2.40}$$

which proves (2.32).

ii) is easily seen from (2.37) and Proposition 2.3-iii).

iii) Using Taylor's expansion formula of order two we see from (2.36) and (2.1) that

$$\tilde{\Phi}_h(y^1, \cdots, y^\nu, \eta^1, \cdots, \eta^\nu; x, \xi)$$

$$(2.41)\qquad =\sum_{j=1}^{\nu}\{\tilde{\nabla}_x^2\tilde{\phi}_{j+1,h}(\tilde{X}_{\nu,h}^j,\tilde{\Xi}_{\nu,h}^{j+1},\tilde{X}_{\nu,h}^j+y^j)y^j\cdot y^j + \tilde{\nabla}_\xi^2\tilde{\phi}_{j,h}(\tilde{\Xi}_{\nu,h}^j,\tilde{X}_{\nu,h}^{j-1}+y^{j-1},\tilde{\Xi}_{\nu,h}^j+\eta^j)\eta^j\cdot\eta^j + \vec{\tilde{\nabla}}_x\nabla_\xi\tilde{\phi}_{j,h}(\tilde{X}_{\nu,h}^{j-1},\tilde{\Xi}_{\nu,h}^j,\tilde{X}_{\nu,h}^{j-1}+y^{j-1})y^{j-1}\cdot\eta^j-y^j\cdot\eta^j\},$$

where $\tilde{\Xi}_{\nu,h}^{\nu+1}=\xi$, $\tilde{X}_{\nu,h}^0=x$, $y^0=0$, and

$$(2.42)\qquad \begin{cases}\tilde{\nabla}_x^2 f(x,\xi,y)=\int_0^1(1-\theta)\vec{\tilde{\nabla}}_x\nabla_x f(x+\theta(y-x),\xi)d\theta\,,\\ \tilde{\nabla}_\xi^2 f(\xi,x,\eta)=\int_0^1(1-\theta)\vec{\tilde{\nabla}}_\xi\nabla_\xi f(x,\xi+\theta(\eta-\xi))d\theta\end{cases}$$

for any C^2-function $f(x,\xi)$. If we use $\tilde{J}_{j,h}$ we get another expression of $\tilde{\varphi}_h$:

$$(2.43)\qquad \begin{aligned}&\tilde{\varphi}_h(y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu;x,\xi)\\ &=\sum_{j=1}^{\nu}\{\tilde{\nabla}_x^2\tilde{J}_{j+1,h}(\tilde{X}_{\nu,h}^j,\tilde{\Xi}_{\nu,h}^{j+1},\tilde{X}_{\nu,h}^j+y^j)y^j\cdot y^j\\ &\quad+\tilde{\nabla}_\xi^2\tilde{J}_{j,h}(\tilde{\Xi}_{\nu,h}^j,\tilde{X}_{\nu,h}^{j-1}+y^{j-1},\tilde{\Xi}_{\nu,h}^j+\eta^j)\eta^j\cdot\eta^j\\ &\quad+\vec{\tilde{\nabla}}_x\nabla_\xi\tilde{J}_{j,h}(\tilde{X}_{\nu,h}^{j-1},\tilde{\Xi}_{\nu,h}^j,\tilde{X}_{\nu,h}^{j-1}+y^{j-1})y^{j-1}\cdot\eta^j\\ &\quad+(y^{j-1}-y^j)\cdot\eta^j\}\,.\end{aligned}$$

From this and Proposition 2.3-iii) we obtain for $|\alpha+\beta|\geqq 1$

$$(2.44)\qquad \begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta\tilde{\varphi}_h|\\ &\leqq C_{\alpha,\beta}\Big(\sum_{j=1}^{\nu+1}|\tilde{J}_{j,h}|_{3,|\alpha+\beta|-1}\Big)\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha+\beta|-1}\times\\ &\quad\times\Big(1+\sum_{k=1}^{\nu+1}|J_{k,h}|_{3,|\alpha+\beta|-1}\Big)^{|\alpha+\beta|-1}(|y^j|^2+|\eta^j|^2+|y^{j-1}||\eta^j|)\\ &\leqq C_{\alpha,\beta}\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha+\beta|-1}\Big(1+\sum_{k=1}^{\nu+1}|J_{k,h}|_{3,|\alpha+\beta|-1}\Big)^{|\alpha+\beta|}\times\\ &\qquad\times\sum_{j=1}^{\nu}(|y^j|^2+|\eta^j|^2)\,,\end{aligned}$$

which proves iii). □

Now we can state and prove the main result of this section.

Theorem 2.6. *Let* $\phi_{j,h}\in P_{\rho,\delta}^{[x]}(\tau_j,\sigma_j;h)$ $(j=1,2,\cdots)$ *satisfy the condition* (#) *for any* $\nu\geqq 1$, *and assume that* $0\leqq\sigma_j\leqq\sigma_0$ $(j=1,2,\cdots)$ *and* $0\leqq 2(\sigma_0\bar{\tau}_\infty+\bar{\tau}_\infty)\leqq\tau_0$ *for some* $0\leqq\sigma_0<\infty$ *and* $0\leqq\tau_0\leqq 1/4$. *Let* $\nu\geqq 1$ *and put* $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$. *Let* $p_{j,h}(\xi,x')\in B_{\rho,\delta}^{m_j}(h)$ *for* $j=1,\cdots,\nu+1$. *Then there exists a symbol* $r_{\nu+1,h}(\xi,x')\in B_{\rho,\delta}^{\bar{m}_{\nu+1}}(h)$ $(\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1})$ *such that*

$$(2.45)\qquad P_{\nu+1,h}(\phi_{\nu+1,h}^*)\cdots P_{1,h}(\phi_{1,h}^*)=R_{\nu+1,h}(\Phi_{\nu+1,h}^*)$$

and

$$
(2.46)\quad \begin{aligned}
&|r_{\nu+1,h}|_l^{(\bar{m}_{\nu+1})}\\
&\leqq C_l c_0^{6\nu n+8l+3} \prod_{m=1}^{l} (4\nu n+4l+1+m)\cdot \nu^{14\nu n+2\nu+21l+7}\times\\
&\times(1+\bar{\sigma}_\nu)^{6\nu n+16l+3}(1-\tau_0)^{-8\nu n-16l-4}(1+\sum_{s=1}^{\nu+1} |J_{s,h}|_{3,l})^{3l}\times\\
&\times(1+\max_{1\leqq j\leqq \nu+1} |J_{j,h}|_{2,2\nu n+3l+1})^{6\nu n+8l+3} \prod_{j=1}^{\nu+1} |p_{j,h}|_{2\nu n+3l+1}^{(m_j)}
\end{aligned}
$$

for any integer $l\geqq 0$, *where* $P_{j,h}(\phi_{j,h}^*)=p_{j,h}(\phi_{j,h}^*;D_x,X')$; $R_{\nu-1,h}(\Phi_{\nu+1,h}^*)=r_{\nu+1,h}(\Phi_{\nu+1,h}^*;D_x,X')$; *and* $c_0>1$ *is a constant.*

Proof. We can write formally for $f\in\mathcal{S}$

$$
\begin{aligned}
&P_{\nu+1,h}(\phi_{\nu+1,h}^*)\cdots P_{1,h}(\phi_{1,h}^*)f(x^{\nu+1})\\
&=0_s-\iint e^{i(x^{\nu+1}\cdot\xi^{\nu+1}-\Phi_{\nu+1,h}(x^0,\xi^{\nu+1}))}r_{\nu+1,h}(\xi^{\nu+1},x^0)\times\\
&\qquad\times f(x^0)dx^0 d\xi^{\nu+1},
\end{aligned}
$$

where

$$
\begin{aligned}
&r_{\nu+1,h}(\xi^{\nu+1},x^0)\\
&=0_s-\overbrace{\int\cdots\int}^{2\nu} \exp\frac{1}{i}\Big[\sum_{j=1}^{\nu}(\phi_{j,h}(x^{j-1},\xi^j)-x^j\cdot\xi^j)\\
&\qquad+\phi_{\nu+1,h}(x^\nu,\xi^{\nu+1})-\Phi_{\nu+1,h}(x^0,\xi^{\nu+1})\Big]\times\\
&\qquad\times p_{\nu+1,h}(\xi^{\nu+1},x)\cdots p_{1,h}(\xi^1,x^0)dx^1\cdots dx^\nu d\xi^1\cdots d\xi^\nu .
\end{aligned}
$$

So we get (2.45) by limit process as in the proof of Proposition 1.4 if we show that $r_{\nu+1,h}(\xi^{\nu+1},x^0)$ is well-defined as an oscillatory integral and satisfies (2.46). Set

$$
(2.47)\quad \begin{aligned}
&\tilde{r}_{\nu+1,h}(\xi^{\nu+1},x^0)\equiv r_{\nu+1,h}(h^{-\rho}\xi^{\nu+1},h^\delta x^0)\\
&=h^{-2\nu n\sigma}0_s-\overbrace{\int\cdots\int}^{2\nu} \exp\frac{h^{-2\sigma}}{i}\Big[\sum_{j=1}^{\nu}(\tilde{\phi}_{j,h}(x^{j-1},\xi^j)-x^j\cdot\xi^j)\\
&\qquad+\tilde{\phi}_{\nu+1,h}(x^\nu,\xi^{\nu+1})-\tilde{\Phi}_{\nu+1,h}(x^0,\xi^{\nu+1})\Big]\times\\
&\qquad\times \tilde{p}_{\nu+1,h}(\xi^{\nu+1},x^\nu)\cdots \tilde{p}_{1,h}(\xi^1,x^0)dx^1\cdots dx^\nu d\xi^1\cdots d\xi^\nu ,
\end{aligned}
$$

where $\tilde{\phi}_{j,h}(x,\xi)=h^{\rho-\delta}\phi_{j,h}(h^\delta x,h^{-\rho}\xi)$, $\tilde{\Phi}_{\nu+1,h}(x,\xi)=h^{\rho-\delta}\Phi_{\nu+1,h}(h^\delta x,h^{-\rho}\xi)$, $\tilde{p}_{j,h}(\xi,x)=p_{j,h}(h^{-\rho}\xi,h^\delta x)\in B_{0,0}^{m_j}(h)$, and $\sigma=(\rho-\delta)/2$. So since $|\tilde{r}_{\nu+1,h}|^{(\bar{m}_{\nu+1})}$ (in $B_{0,0}^{\bar{m}_{\nu+1}}(h)$) $=|r_{\nu+1,h}|^{(\bar{m}_{\nu+1})}$ (in $B_{\rho,\delta}^{\bar{m}_{\nu+1}}(h)$), we have to prove $\tilde{r}_{\nu+1,h}(\xi,x)\in B_{0,0}^{\bar{m}_{\nu+1}}(h)$ and to estimate $|\tilde{r}_{\nu+1,h}|^{(\bar{m}_{\nu+1})}$ in $B_{0,0}^{\bar{m}_{\nu+1}}(h)$.

In (2.47) making a change of variables:

$$
\begin{cases}
x^j=\tilde{X}_{\nu,h}^j(x^0,\xi^{\nu+1})+y^j\equiv h^{-\delta}X_{\nu,h}^j(h^\delta x^0,h^{-\rho}\xi^{\nu+1})+y^j ,\\
\xi^j=\tilde{\Xi}_{\nu,h}^j(x^0,\xi^{\nu+1})+\eta^j\equiv h^\rho\Xi_{\nu,h}^j(h^\delta x^0,h^{-\rho}\xi^{\nu+1})+\eta^j
\end{cases}
$$

for $j=1,\cdots,\nu$, we get

$$\tilde{r}_{\nu+1,h}(\xi,x) = h^{-2\nu n\sigma} 0_s - \overbrace{\int\cdots\int}^{2\nu} e^{-ih^{-2\sigma}\tilde{\varphi}_h(y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu;x,\xi)} \times \times \prod_{j=1}^{\nu+1} q_{j,h}(\xi,x;\eta^j,y^{j-1}) dy^1\cdots dy^\nu đ\eta^1 \cdots đ\eta^\nu, \tag{2.48}$$

where $\tilde{\varphi}_h$ is defined by (2.31) in Lemma 2.5 and

$$q_{j,h}(\xi,x;\eta^j,y^{j-1}) = \tilde{p}_{j,h}(\tilde{\Xi}^j_{\nu,h}(x,\xi)+\eta^j, \tilde{X}^{j-1}_{\nu,h}(x,\xi)+y^{j-1}) \tag{2.49}$$

for $j=1,\cdots,\nu+1$ with $y^0=\eta^{\nu+1}=0$, $\tilde{\Xi}^{\nu+1}_{\nu,h}=\xi$ and $\tilde{X}^0_{\nu,h}=x$. Now setting

$$\begin{cases} \Gamma_h = 2\nu + h^{-2\sigma}\sum_{j=1}^{\nu}(|\nabla_{y^j}\tilde{\varphi}_h|^2+|\nabla_{\eta^j}\tilde{\varphi}_h|^2), \\ L_h = \Gamma_h^{-1}\{2\nu - i\sum_{j=1}^{\nu}(\nabla_{y^j}\tilde{\varphi}_h\cdot\nabla_{y^j}+\nabla_{\eta^j}\tilde{\varphi}_h\cdot\nabla_{\eta^j})\}, \end{cases} \tag{2.50}$$

we can write

$$\tilde{r}_{\nu+1,h}(\xi,x) = h^{-2\nu n\sigma} \overbrace{\int\cdots\int}^{2\nu} e^{-ih^{-2\sigma}\tilde{\varphi}_h}({}^tL_h)^l\Big(\prod_{j=1}^{\nu+1} q_{j,h}(\xi,x;\eta^j,y^{j-1})\Big)\times \times dy^1\cdots dy^\nu đ\eta^1\cdots đ\eta^\nu \tag{2.51}$$

for $l\geqq 0$ at least formally, where ${}^tL_h$ is the transposed operator of L_h. We shall show that the right hand side of (2.51) converges absolutely for $l\geqq 2\nu n$.

Noting

$$\begin{aligned} {}^tL_h = & -\frac{1}{\Gamma_h}\sum_{j=1}^{\nu}(\nabla_{y^j}\tilde{\varphi}_h\cdot\nabla_{y^j}+\nabla_{\eta^j}\tilde{\varphi}_h\cdot\nabla_{\eta^j}) \\ & +\frac{1}{\Gamma_h^2}\{2\nu\Gamma_h + i\sum_{j=1}^{\nu}(\nabla_{y^j}\tilde{\varphi}_h\cdot\nabla_{y^j}\Gamma_h+\nabla_{\eta^j}\tilde{\varphi}_h\cdot\nabla_{\eta^j}\Gamma_h)\} \end{aligned} \tag{2.52}$$

and

$$\begin{cases} \nabla_{y^j}\Gamma_h = 2h^{-2\sigma}\{\nabla_{y^j}\tilde{\varphi}_h\cdot\tilde{\nabla}_x\nabla_x\tilde{J}_{j+1,h}+\nabla_{\eta^{j+1}}\tilde{\varphi}_h\cdot(I+\tilde{\nabla}_x\nabla_\xi\tilde{J}_{j+1,h}) \\ \quad -\nabla_{\eta^j}\tilde{\varphi}_h+[\nabla_{y^j}\tilde{\varphi}_h\cdot\nabla_{y^j}\tilde{\nabla}_x\nabla_x\tilde{J}_{j+1,h}+\nabla_{\eta^{j+1}}\tilde{\varphi}_h\cdot\nabla_{y^j}\tilde{\nabla}_x\nabla_\xi\tilde{J}_{j+1,h}]y^j\}, \\ \nabla_{\eta^j}\Gamma_h = 2h^{-2\sigma}\{\nabla_{y^{j-1}}\tilde{\varphi}_h\cdot(I+\tilde{\nabla}_\xi\nabla_x\tilde{J}_{j+1,h})+\nabla_{\eta^j}\tilde{\varphi}_h\cdot\tilde{\nabla}_\xi\nabla_\xi\tilde{J}_{j,h} \\ \quad -\nabla_{y^j}\tilde{\varphi}_h+[\nabla_{y^{j-1}}\tilde{\varphi}_h\cdot\nabla_{\eta^j}\tilde{\nabla}_\xi\nabla_x\tilde{J}_{j,h}+\nabla_{\eta^j}\tilde{\varphi}_h\cdot\nabla_{\eta^j}\tilde{\nabla}_\xi\nabla_\xi\tilde{J}_{j,h}]\eta^j\}, \end{cases} \tag{2.53}$$

and using (2.37) we see by induction that $({}^tL_h)^l$ has the form

$$({}^tL_h)^l = \frac{1}{\Gamma_h^{2l}}\sum_{\substack{|\mu|\leqq 3l \\ |\rho|\leqq l}} a^{(l)}_{\mu,\rho,h}\,(h^{-\sigma}(\boldsymbol{y},\boldsymbol{\eta}))^\mu\partial^\rho_{(\boldsymbol{y},\boldsymbol{\eta})}, \tag{2.54}$$

where $(\boldsymbol{y},\boldsymbol{\eta})=(y^1,\cdots,y^\nu,\eta^1,\cdots,\eta^\nu)$ and $a^{(l)}_{\mu,\rho,h}$ is a polynomial of $\partial^\alpha_{(\boldsymbol{y},\boldsymbol{\eta})}\nabla^2_{(x,\xi)}\tilde{J}_{j,h}$

$(|\alpha|\leqq l-|\rho|,\ 1\leqq j\leqq \nu+1)$ of order $3l$, $\nabla^2_{(x,\xi)}\tilde{J}_{j,h}$ denoting one of the terms $\tilde{\nabla}_x\nabla_\xi\tilde{J}_{j,h}(\tilde{X}^{j-1}_{\nu,h},\tilde{\Xi}^j_{\nu,h},\tilde{X}^{j-1}_{\nu,h}+y^{j-1})$, etc. in (2.37), and satisfies

$$|\partial_\xi^\alpha\partial_x^\beta a^{(l)}_{\mu,\rho,h}| \tag{2.55}$$
$$\leqq C_{\alpha,\beta}c^l\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha+\beta|}(1+\max_{1\leqq j\leqq \nu+1}|J_{j,h}|_{2,l+|\alpha+\beta|})^{3l}\times$$
$$\times(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,(|\alpha+\beta|-1)\vee 0})^{(|\alpha+\beta|-1)\vee 0}$$

for some constant $c<1$, where we have used Proposition 2.3-iii). Thus taking $l>2\nu n$, and using (2.32) and the inequalities

$$\begin{cases}|\eta^j|\leqq\sum_{k=j}^{\nu}|\eta^k-\eta^{k+1}|\leqq\sum_{k=1}^{\nu}|\eta^k-\eta^{k+1}|\,, \\ |y^j|\leqq\sum_{k=1}^{j}|y^k-y^{k-1}|\leqq\sum_{k=1}^{j}|y^k-y^{k-1}-r_k\eta^k|+\sum_{k=1}^{j}\sigma_j|\eta^k| \\ \qquad\leqq(1+\bar{\sigma}_j)\sum_{k=1}^{\nu}(|y^k-y^{k-1}-r_k\eta^k|+|\eta^k-\eta^{k+1}|)\,,\end{cases} \tag{2.56}$$

where $y^0=\eta^{\nu+1}=0$ and $r_k=\vec{\tilde{\nabla}}_\xi\nabla_\xi\tilde{J}_{k,h}(\tilde{\Xi}^k_{\nu,h},\tilde{X}^{k-1}_{\nu,h}+y^{k-1},\tilde{\Xi}^k_{\nu,h}+\eta^k)$, we see that the integral (2.51) is well-defined.

Thus we have again formally for any α, β

$$\partial_\xi^\alpha\partial_x^\beta\tilde{r}_{\nu+1,h}(\xi,x)$$
$$=h^{-2\nu n\sigma}\sum_{\substack{\alpha^1+\alpha^2=\alpha\\ \beta^1+\beta^2=\beta}}\overbrace{\int\cdots\int}^{2\nu}\partial_\xi^{\alpha^1}\partial_x^{\beta^1}(e^{-ih^{-2\sigma}\tilde{\varphi}_h})\times \tag{2.57}$$
$$\times\partial_\xi^{\alpha^2}\partial_x^{\beta^2}[({}^tL_h)^l(\prod_{j=1}^{\nu+1}q_{j,h}(\xi,x;\eta^j,y^{j-1}))]dy^1\cdots dy^\nu d\eta^1\cdots d\eta^\nu\,.$$

From Lemma 2.5-iii) we get for $|\alpha^1+\beta^1|\geqq 1$

$$|\partial_\xi^{\alpha^1}\partial_x^{\beta^1}(e^{-ih^{-2\sigma}\tilde{\varphi}_h})|$$
$$\leqq\sum_{l=1}^{|\alpha^1+\beta^2|}\sum_{\substack{k_1+\cdots+k_l=|\alpha^1+\beta^1|\\ |k_j|\geqq 1}}\prod_{j=1}^{l}\{C_{k_j}\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2k_j-1}\times$$
$$\times(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,k_j-1})^{k_j}(h^{-\sigma}\sum_{s=1}^{\nu}(|y^s|+|\eta^s|))^2\} \tag{2.58}$$
$$\leqq C_{\alpha^1,\beta^1}\Big(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\Big)^{2|\alpha^1+\beta^1|-1}(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\alpha^1+\beta^1|-1}))^{|\alpha^1+\beta^1|}\times$$
$$\times(1+h^{-\sigma}\sum_{s=1}^{\nu}(|y^s|+|\eta^s|))^{2|\alpha^1+\beta^1|}\,.$$

Using (2.56) and i) and ii) of Lemma 2.5 we have for $|\alpha+\beta|\geqq 1$

$$|\partial_\xi^\alpha\partial_x^\beta\Big(\frac{1}{\Gamma_h^{2l}}\Big)|$$

$$
\begin{aligned}
&\leqq \sum_{m=1}^{|\alpha+\beta|} 2l(2l+1)\cdots(2l+m-1)\Gamma_h^{-2l-m} \sum_{\substack{k_1+\cdots+k_m=|\alpha+\beta| \\ k_i\geqq 1}} \prod_{i=1}^{m}\sum_{j=1}^{\nu} \times \\
&\times \sum_{a^1+a^2=k_i} h^{-2\sigma}\{|\partial_{(\xi,x)}^{a^1}\nabla_{y^j}\tilde{\varphi}_h|\,|\partial_{(\xi,x)}^{a^2}\nabla_{y^j}\tilde{\varphi}_h| \\
&\qquad + |\partial_{(\xi,x)}^{a^1}\nabla_{\eta^j}\tilde{\varphi}_h|\,|\partial_{(\xi,x)}^{a^2}\nabla_{\eta^j}\tilde{\varphi}_h|\} \\
&\leqq C_{\alpha,\beta} 2l(2l+1)\cdots(2l+|\alpha+\beta|-1)\nu^{3|\alpha+\beta|}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{4|\alpha+\beta|}\times \\
&\times(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\alpha+\beta|})^{|\alpha+\beta|}(1+\max_{1\leqq j\leqq\nu+1}|J_{j,h}|_{2,|\alpha+\beta|})^{2|\alpha+\beta|}\frac{1}{\Gamma_h^{2l}}\,.
\end{aligned}
\tag{2.59}
$$

On the other hand, from Proposition 2.3-iii) we get

$$
\begin{aligned}
&|\partial_\xi^\gamma\partial_x^\delta\partial_{\eta^j}^a\partial_{y^{j-1}}^b q_{j,h}(\xi,x;\eta^j,y^{j-1})| \\
&\leqq C_{\gamma,\delta}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{2|\gamma+\delta|}(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\gamma+\delta|})^{|\gamma+\delta|}h^{m_j}|p_{j,h}|_{|\gamma+\delta+a+b|}^{(m_j)}\,.
\end{aligned}
\tag{2.60}
$$

Then using (2.54), (2.55), (2.59) and (2.60) we obtain

$$
\begin{aligned}
&|\partial_\xi^{\alpha^2}\partial_x^{\beta^2}[({}^tL_h)^l(\prod_{j=1}^{\nu+1}q_{j,h}(\xi,x;\eta^j,y^{j-1}))]| \\
&\leqq C_{\alpha^2,\beta^2}c^l\nu^{3l}2l(2l+1)\cdots(2l+|\alpha^2+\beta^2|-1)\nu^{3|\alpha^2+\beta^2|}\times \\
&\times\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{8|\alpha^2+\beta^2|}(1+\max_{1\leqq j\leqq\nu+1}|J_{j,h}|_{2,l+|\alpha^2+\beta^2|})^{3l+2|\alpha^2+\beta^2|}\times \\
&\times(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\alpha^2+\beta^2|})^{3|\alpha^2+\beta^2|}\frac{|h^{-\sigma}(\boldsymbol{y},\boldsymbol{\eta})|^{3l}}{\Gamma_h^{2l}}h^{\bar{m}_{\nu+1}}\times \\
&\times\prod_{j=1}^{\nu+1}|p_{j,h}|_{|\alpha^2+\beta^2|+l}^{(m_j)}\,.
\end{aligned}
\tag{2.61}
$$

Thus from (2.58) and (2.61) we have

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\tilde{r}_{\nu+1,h}(\xi,x)| \\
&\leqq C_{\alpha,\beta}c^l 2l(2l+1)\cdots(2l+|\alpha+\beta|-1)\nu^{3l+3|\alpha+\beta|}\left(\frac{1+\bar{\sigma}_\nu}{1-\tau_0}\right)^{8|\alpha+\beta|}\times \\
&\times(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\alpha+\beta|})^{3|\alpha+\beta|}(1+\max_{1\leqq j\leqq\nu+1}|J_{j,h}|_{2,l+|\alpha+\beta|})^{3l+2|\alpha+\beta|}\times \\
&\times h^{\bar{m}_{\nu+1}}\prod_{j=1}^{\nu+1}|p_{j,h}|_{|\alpha+\beta|+l}^{(m_j)}\overbrace{\int\cdots\int}^{2\nu}h^{-2\nu n\sigma}\Gamma_h^{-2l}\times \\
&\times\{1+h^{-\sigma}\sum_{s=1}^{\nu}(|y^s|+|\eta^s|)\}^{2|\alpha+\beta|}|h^{-\sigma}(\boldsymbol{y},\boldsymbol{\eta})|^{3l}d\boldsymbol{y}d\boldsymbol{\eta}\,.
\end{aligned}
\tag{2.62}
$$

Since (2.56) shows

$$
\begin{aligned}
&\{1+h^{-\sigma}\sum_{s=1}^{\nu}(|y^s|+|\eta^s|)\}^{2|\alpha+\beta|}|h^{-\sigma}(\boldsymbol{y},\boldsymbol{\eta})|^{3l} \\
&\leqq\{1+\sum_{s=1}^{\nu}h^{-\sigma}(|y^s|+|\eta^s|)\}^{2|\alpha+\beta|+3l}
\end{aligned}
$$

$$\leqq \{2\nu(1+\bar{\sigma}_\nu)[1+h^{-\sigma}\sum_{k=1}^{\nu}(|\eta^k-\eta^{k+1}|+|y^k-y^{k-1}-r_k\eta^k|)]\}^{2|\alpha+\beta|+3l},$$

we see from (2.32) that the integrand of the right hand side of (2.62) is bounded by

$$\begin{aligned}(2.63)\qquad & 2^{6l+4|\alpha+\beta|}\nu^{4l+4|\alpha+\beta|}(1+\bar{\sigma}_\nu)^{3l+2|\alpha+\beta|}(1-\tau_0)^{-4l}\times\\ &\times h^{-2\nu n\sigma}\prod_{j=1}^{\nu}(\langle h^{-\sigma}(\eta^j-\eta^{j+1})\rangle\langle h^{-\sigma}(y^j-y^{j-1}-r_j\eta^j)\rangle)^{-(l-2|\alpha+\beta|)/2\nu}.\end{aligned}$$

This is integrable in $(\boldsymbol{y},\boldsymbol{\eta})$ uniformly in $h\in(0,1)$ whenever $l>2\nu n+2|\alpha+\beta|$, which shows (2.57) is actually valid. Thus from (2.62) and (2.63) we have for any $l>2\nu n+2|\alpha+\beta|$ and some constant $c_0>1$

$$\begin{aligned}(2.64)\qquad & |\partial_\xi^\alpha\partial_x^\beta\tilde{r}_{\nu+1,h}(\xi,x)|\\ &\leqq C_{\alpha,\beta}c_0^{3l+2|\alpha+\beta|}2l(2l+1)\cdots(2l+|\alpha+\beta|-1)\nu^{7l+7|\alpha+\beta|}\times\\ &\times(1+\bar{\sigma}_\nu)^{3l+10|\alpha+\beta|}(1-\tau_0)^{-4l-8|\alpha+\beta|}(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{3,|\alpha+\beta|})^{3|\alpha+\beta|}\times\\ &\times(1+\max_{1\leqq j\leqq\nu+1}|J_{j,h}|_{2,l+|\alpha+\beta|})^{3l+2|\alpha+\beta|}h^{\bar{m}_{\nu+1}}\prod_{j=1}^{\nu+1}|p_{j,h}|_{l+|\alpha+\beta|}^{(m_j)}\times\\ &\times\overbrace{\int\cdots\int}^{2\nu}\prod_{j=1}^{\nu}(\langle\eta^j-\eta^{j+1}\rangle\langle y^j-y^{j-1}-r_j\eta^j\rangle)^{-(l-2|\alpha+\beta|)/(2\nu)}d\boldsymbol{y}d\boldsymbol{\eta}.\end{aligned}$$

Taking $l=2\nu n+2|\alpha+\beta|+1$ and noting that the integral is bounded by $c_1^\nu\nu^{2\nu}$ for some constant $c_1>1$ prove the theorem. □

3. Approximate fundamental solutions

We consider the Hamiltonian $H(t,x,\xi)$ of the form

$$(3.1)\qquad H(t,x,\xi)=\frac{1}{2}|\xi|^2+V(t,x),$$

where the time-dependent potential $V(t,x)$ satisfies the following assumption (A):

ASSUMPTION (A)

i) For each $t\in R^1$, $V(t,x)$ is a real-valued C^∞-function of $x\in R^n$.

ii) For any multi-index α, $\partial_x^\alpha V(t,x)$ is continuous in $(t,x)\in R^1\times R^n$.

iii) There exists a constant $\varepsilon>0$ such that for any multi-index α with $|\alpha|\neq 0$

$$|\partial_x^\alpha V(t,x)|\leqq C_\alpha\langle t\rangle^{-|\alpha|-\varepsilon},$$

where the constant $C_\alpha>0$ is independent of t, x.

This assumption is the same as in [5], where we have studied the scattering problem by $V(t, x)$. For the examples covered by this assumption, see [5, §1].

Let $0 \leqq \delta \leqq \rho \leqq 1$ and set

$$H_h(t, x, \xi) = h^{\delta-\rho} H(t, h^{-\delta} x, h^{\rho} \xi) . \tag{3.2}$$

We consider the Schrödinger equation

$$\begin{cases} L_h u \equiv (D_t + H_h(t, X, D_x))u = 0 & \text{on } R^1 , \\ u|_{t=s_0} = f(\in \mathcal{S}) & (s_0 \in R^1) . \end{cases} \tag{3.3}$$

In this and the next sections we shall construct the fundamental solution of (3.3) globally in time in the form of a product of a certain finite number of (conjugate) Fourier integral operators, the number depending on s_0 but not on t. Here by the fundamental solution of (3.3) we mean an operator $U_h(t, s_0)$ such that

$$\begin{cases} U_h(s_0, s_0) = I , \\ L_h U_h(t, s_0) = 0 & (t, s_0 \in R^1) . \end{cases} \tag{3.4}$$

It is easily seen from (3.1) that $H(t, X, D_x)$ is symmetric in $L^2(R^n)$. Thus we have by 1° of the remark after Definition 1.5

$$\begin{aligned} H_h(t, X, D_x) f(x) &= H_h(t, X', D_x) f(x) \\ &= 0_s - \iint e^{i(x-x')\cdot\xi} H_h(t, x', \xi) f(x') dx' d\xi \end{aligned} \tag{3.5}$$

for $f \in \mathcal{S}$. So we consider the Cauchy problem

$$\begin{cases} L_h' u \equiv (D_t + H_h(t, X', D_x))u = 0 & \text{on } R^1 , \\ u|_{t=s_0} = f(\in \mathcal{S}) & (s_0 \in R^1) \end{cases} \tag{3.3$'$}$$

instead of (3.3). In the following, for the sake of simplicity we restrict ourselves to considering only the case $t \geqq s_0$, since the other case can be dealt with similarly.

Let $(q(t, s; x, \xi), p(t, s; x, \xi))$ be the solution of the Hamilton equation

$$\begin{cases} \dfrac{dq}{dt}(t, s) = \nabla_\xi H(t, q(t, s), p(t, s)) , \\ \dfrac{dp}{dt}(t, s) = -\nabla_x H(t, q(t, s), p(t, s)) \end{cases} \tag{3.6}$$

on R^1 with the initial condition

$$q(s, s) = x, \ \ p(s, s) = \xi \qquad (s \in R^1) . \tag{3.7}$$

The equation (3.6)–(3.7) is equivalent to the integral equation

$$(3.8)\qquad \begin{cases} q(t, s) = x+\int_s^t p(\tau, s)d\tau\,, \\ p(t, s) = \xi-\int_s^t \nabla_x V(\tau, q(\tau, s))d\tau\,. \end{cases}$$

Then we can easily prove the following proposition by successive approximation. Let $\mathcal{B}^{k,\infty}(R^m)$ denote the Fréchet space of C^∞-functions $f(y)$ on R^m such that $\partial_y^\alpha f(y)$ ($|\alpha|\geqq k$) are all bounded on R^m with semi-norms $|f|_{k,l}$ ($l=0, 1, 2, \cdots$) defined by

$$|f|_{k,l} = \sum_{|\alpha|\leqq k-1} \sup_y \{|\partial_y^\alpha f(y)|/\langle y\rangle^{k-|\alpha|}\} + \sum_{k\leqq|\alpha|\leqq k+l} \sup_y |\partial_y^\alpha f(y)|\,.$$

We often write $\mathcal{B}^\infty(R^m)=\mathcal{B}^{0,\infty}(R^m)$. We also use the class $C^l(\Omega|\mathcal{B}^{k,\infty}(R_y^m))$ for a domain $\Omega\subset R^p$ which consists of the functions $f(\omega, y)$ on $\Omega\times R^m$ such that for each $\omega\in\Omega$ $f(\omega, y)$ is in $\mathcal{B}^{k,\infty}(R_y^m)$ and any derivative $\partial_y^\alpha f(\omega, y)$ is in $C^l(\Omega\times R^m)$. Then:

Proposition 3.1. *There exists a unique solution of* (3.8). *The solution* (q, p) $(t, s; x, \xi)$ *belongs to* $C^1(R_t^1\times R_s^1|\mathcal{B}^{1,\infty}(R_x^n\times R_\xi^n))$. *Furthermore there exist positive constants* T_0 *and* C_0 *such that the following estimates hold*:

i) *For any* $t\geqq s\geqq T_0$ *and* $x, \xi\in R^n$

$$(3.9)\qquad \begin{cases} |q(s, t; x, \xi)-x|+|q(t, s; x, \xi)-x|\leqq C_0(t-s)\,(\langle s\rangle^{-\varepsilon}+|\xi|)\,, \\ |p(s, t; x, \xi)-\xi|+|p(t, s; x, \xi)-\xi|\leqq C_0\langle s\rangle^{-\varepsilon}; \end{cases}$$

$$(3.10)\qquad \begin{cases} |\nabla_x q(s, t; x, \xi)-I|\leqq C_0\langle s\rangle^{-\varepsilon},\ |\nabla_x q(t, s; x, \xi)-I|\leqq C_0(t-s)\langle s\rangle^{-1-\varepsilon}\,, \\ |\nabla_x p(s, t; x, \xi)|+|\nabla_x p(t, s; x, \xi)|\leqq C_0\langle s\rangle^{-1-\varepsilon}; \end{cases}$$

$$(3.11)\qquad \begin{cases} |\nabla_\xi q(s, t; x, \xi)-(s-t)I|\leqq C_0(t-s)\langle s\rangle^{-\varepsilon}\,, \\ |\nabla_\xi p(s, t; x, \xi)-I|\leqq C_0(t-s)\langle s\rangle^{-1-\varepsilon}; \end{cases}$$

and

$$(3.12)\qquad \begin{cases} |\nabla_\xi q(t, s; x, \xi)-(t-s)I|\leqq C_0(t-s)\langle s\rangle^{-\varepsilon}\,, \\ |\nabla_\xi p(t, s; x, \xi)-I|\leqq C_0\langle s\rangle^{-\varepsilon}\,. \end{cases}$$

In particular, when $0\leqq t-s\leqq 1$ *and* $s\geqq T_0$, *we have*

$$(3.13)\qquad \begin{cases} |\nabla_x q(s, t; x, \xi)-I\leqq C_0(t-s)^2\langle s\rangle^{-2-\varepsilon}\,, \\ |\nabla_x p(t, s; x, \xi)|\leqq C_0(t-s)\langle s\rangle^{-2-\varepsilon}\,, \\ |\nabla_\xi p(t, s; x, \xi)-I|\leqq C_0(t-s)^2\langle s\rangle^{-2-\varepsilon}\,. \end{cases}$$

ii) *For any* α, β *with* $|\alpha+\beta|\geqq 2$, *there is a constant* $C_{\alpha,\beta}$ *independent of* $t\geqq s(\geqq T_0)$ *and* x, ξ *such that*

$$(3.14)\qquad \begin{cases} |\partial_\xi^\alpha\partial_x^\beta q(s, t; x, \xi)| \leqq C_{\alpha,\beta}(t-s)^{|\alpha|}\langle s\rangle^{-1-\varepsilon}, \\ |\partial_\xi^\alpha\partial_x^\beta p(s, t; x, \xi)| \leqq C_{\alpha,\beta}(t-s)^{|\alpha|}\langle s\rangle^{-2-\varepsilon}, \end{cases}$$

and

$$(3.15)\qquad \begin{cases} |\partial_\xi^\alpha\partial_x^\beta q(t, s; x, \xi)| \leqq C_{\alpha,\beta}(t-s)\langle s\rangle^{-\varepsilon}, \\ |\partial_\xi^\alpha\partial_x^\beta p(t, s; x, \xi)| \leqq C_{\alpha,\beta}\langle s\rangle^{-\varepsilon}. \end{cases}$$

Proof. (3.13) follows from (3.10)–(3.12) and the equalities obtained by differentiating (3.8) with respect to x or ξ. For the proof of the other results, see the proof of Proposition 2.1 of [5]. □

From this proposition, we can easily get the following important proposition.

Proposition 3.2. *Take $T>T_0$ so large that $C_0\langle T\rangle^{-\varepsilon}<1/2$ for the constant C_0 in Proposition 3.1. Then for $t\geqq s\geqq T$ there exist the inverse C^∞ diffeomorphisms $x\mapsto y(s, t; x, \xi)$ and $\xi\mapsto\eta(t, s; x, \xi)$ of the mappings $y\mapsto x=q(s, t; y, \xi)$ and $\eta\mapsto\xi=p(t, s; x, \eta)$, respectively. These mappings y and η belong to $C^1(A_T|\mathcal{B}^{1,\infty}(R_x^n\times R_\xi^n))$, where $A_T\equiv\{(t, s)\,|\,t\geqq s\geqq T\}$, and they satisfy the following properties:*

i) $q(s, t; y(s, t; x, \xi), \xi) = x$, $\quad p(t, s; x, \eta(t, s; x, \xi)) = \xi$.

ii) $\begin{cases} q(t, s; x, \eta(t, s; x, \xi)) = y(s, t; x, \xi), \\ p(s, t; y(s, t; x, \xi), \xi) = \eta(t, s; x, \xi). \end{cases}$

iii) *There is a constant $C_1>1$ such that for any $(t, s)\in A_T$ and $x, \xi\in R^n$*

$$(3.16)\qquad \begin{cases} |\eta(t, s; x, \xi)-\xi| \leqq C_1\langle s\rangle^{-\varepsilon}, \\ |y(s, t; x, \xi)-x| \leqq C_1(t-s)\,(\langle s\rangle^{-\varepsilon}+|\xi|); \end{cases}$$

$$(3.17)\qquad \begin{cases} |\nabla_x y(s, t; x, \xi)-I| \leqq C_1\langle s\rangle^{-\varepsilon}, \\ |\nabla_\xi y(s, t; x, \xi)-(t-s)I| \leqq (C_1-1)\,(t-s)\langle s\rangle^{-\varepsilon}; \end{cases}$$

and

$$(3.18)\qquad \begin{cases} |\nabla_x \eta(t, s; x, \xi)| \leqq C_1\langle s\rangle^{-1-\varepsilon}, \\ |\nabla_\xi \eta(t, s; x, \xi)-I| \leqq C_1\langle s\rangle^{-\varepsilon}. \end{cases}$$

If, in particular $0\leqq t-s\leqq 1$, we have

$$(3.19)\qquad \begin{cases} |\nabla_x y(s, t; x, \xi)-I| \leqq C_1(t-s)^2\langle s\rangle^{-2-\varepsilon}, \\ |\nabla_x \eta(t, s; x, \xi)| \leqq C_1(t-s)\langle s\rangle^{-2-\varepsilon}, \\ |\nabla_\xi \eta(t, s; x, \xi)-I| \leqq C_1(t-s)^2\langle s\rangle^{-2-\varepsilon}. \end{cases}$$

iv) *For any α, β with $|\alpha+\beta|\geqq 2$, there is a constant $C_{\alpha,\beta}$ such that for any $(t, s)\in A_T$ and $x, \xi\in R^n$*

$$(3.20)\qquad \begin{cases} |\partial_\xi^\alpha\partial_x^\beta \eta(t, s; x, \xi)| \leqq C_{\alpha,\beta}\langle s\rangle^{-\varepsilon}, \\ |\partial_\xi^\alpha\partial_x^\beta y(s, t; x, \xi)| \leqq C_{\alpha,\beta}(t-s+1)\langle s\rangle^{-\varepsilon}. \end{cases}$$

Proof. The proof except for (3.19) is similar to that of Proposition 2.2 of [5]. (3.19) follows from (3.13), (3.10) and (3.12) of Proposition 3.1 by virtue of the relations in i). □

When $|t-s|$ is small, we have the following estimates for (q, p) and (y, η) (see [4, §3]).

Proposition 3.3. *There exists a small constant* $0<\tilde{\delta}<1$ *such that the following assertions hold*:

i) *We have*

$$(q, p)(t, s; x, \xi)\in C^1(B_{\tilde{\delta}}\,|\,\mathcal{B}^{1,\infty}(R^{2n})), \tag{3.21}$$

where $B_{\tilde{\delta}}\equiv\{(t, s)\,|\,t, s\in R^1, \ |t-s|\leqq\tilde{\delta}\}$, *and*

$$\text{“}\{[(q, p)(t, s; x, \xi)-(x, \xi)]/(t-s)\}_{(t,s)\in B_{\tilde{\delta}}} \textit{ is bounded in } \mathcal{B}^{1,\infty}(R^{2n}).\text{”} \tag{3.22}$$

ii) *For* $(t, s)\in B_{\tilde{\delta}}$, *there exist the inverse* C^∞ *diffeomorphisms* $x\mapsto y(t, s; x, \xi)$ *and* $\xi\mapsto\eta(t, s; x, \xi)$ *of the mappings* $y\mapsto x=q(t, s; y, \xi)$ *and* $\eta\mapsto\xi=p(t, s; x, \eta)$, *respectively, and they satisfy*

$$(y, \eta)(t, s; x, \xi)\in C^1(B_{\tilde{\delta}}\,|\,\mathcal{B}^{1,\infty}(R^{2n})), \tag{3.23}$$

and

$$\text{“}\{[(y, \eta)(t, s; x, \xi)-(x, \xi)]/(t-s)\}_{(t,s)\in B_{\tilde{\delta}}} \textit{ is bounded in } \mathcal{B}^{1,\infty}(R^{2n}).\text{”} \tag{3.24}$$

DEFINITION 3.4. For $(t, s)\in A_T\cup B_{\tilde{\delta}}$, define

$$\phi(s, t; x, \xi) = u(s, t; y(s, t; x, \xi), \xi), \tag{3.25}$$

where

$$u(s, t; y, \eta) = y\cdot\eta+\int_t^s(\xi\cdot\nabla_\xi H-H)(\tau, q(\tau, t; y, \eta), p(\tau, t; y, \eta))d\tau\,. \tag{3.26}$$

Proposition 3.5. *Let* $(t, s)\in A_T$(or $B_{\tilde{\delta}}$). *Then* $\phi(s, t; x, \xi)$ *defined above satisfies*

$$\begin{cases}\partial_s\phi(s, t; x, \xi)+H(s, x, \nabla_x\phi(s, t; x, \xi)) = 0\,,\\ \phi(t, t; x, \xi) = x\cdot\xi;\end{cases} \tag{3.27}$$

$$\partial_t\phi(s, t; x, \xi)-H(t, \nabla_\xi\phi(s, t; x, \xi), \xi) = 0; \tag{3.28}$$

and

$$\begin{cases}\nabla_x\phi(s, t; x, \xi) = \eta(t, s; x, \xi)\,,\\ \nabla_\xi\phi(s, t; x, \xi) = y(s, t; x, \xi)\,.\end{cases} \tag{3.29}$$

Furthermore we have

(3.30) $\phi(s, t; x, \xi) \in C^1(A_T \mid \mathscr{B}^{2,\infty}(R^{2n}))$ (or $\in C^1(B_{\tilde{\delta}} \mid \mathscr{B}^{2,\infty}(R^{2n}))$).

Proof. (3.27)–(3.29) can be shown by direct clculations (or see Kumano-go [7] and Kumano-go, Taniguchi, Tozaki [8]). (3.30) follows from Proposition 3.2 (or Proposition 3.3). □

DEFINITION 3.6. Let $\phi_h(s, t; x, \xi)$ be defined by

$$\phi_h(s, t; x, \xi) = h^{\delta-\rho}\phi(s, t; h^{-\delta}x, h^\rho \xi) \qquad (0 \leqq \delta \leqq \rho \leqq 1) \tag{3.31}$$

for $(t, s) \in A_T \cup B_{\tilde{\delta}}$.

In the following, we switch to another large $T > T_0$ such that $C_1\langle T\rangle^{-\varepsilon} < 1$, if necessary.

Proposition 3.7. i) *For $(t, s) \in A_T$, we have*

$$\phi_h(s, t; x, \xi) \in P_{\rho,\delta}^{[x]}(C_1\langle s\rangle^{-\varepsilon}, C_1(t-s); h). \tag{3.32}$$

When $(t, s) \in A_T$ and $|t-s| \leqq 1$, we have

$$\phi_h(s, t; x, \xi) \in P_{\rho,\delta}^{[x]}(C_1(t-s)\langle s\rangle^{-2-\varepsilon}, C_1(t-s); h). \tag{3.33}$$

ii) *For any $l \geqq 0$, there exist constants $0 < \tilde{\delta}_l \leqq \tilde{\delta}$ and $c_l \geqq 1$ such that $c_l \tilde{\delta}_l < 1$ and*

$$\phi_h(s, t; x, \xi) \in P_{\rho,\delta}(c_l |t-s|, l; h) \tag{3.34}$$

for any $(t, s) \in B_{\tilde{\delta}_l}$.

Proof. i) is clear from Propositions 3.5 and 3.2. ii) is also clear from Propositions 3.5 and 3.3. □

In the sequel we switch to another small $\tilde{\delta} > 0$ such that $\tilde{\delta} \leqq \tilde{\delta}_0$, if necessary. We next solve the transport qeuations.

DEFINITION 3.8. For $(t, s) \in A_T \cup B_{\tilde{\delta}}$, we define a sequence of functions $a_j(t, s; \xi, x')$ $(j=0, 1, \cdots)$ inductively as follows:

$$\begin{aligned} &a_0(t, s; \xi, x') \\ &= \exp\{-\frac{1}{2}\sum_{l,k=1}^{n}\int_t^s (\partial_{x_k}\partial_{x_l}H)(\tau, y(s, \tau; x', \Xi(\tau)), \Xi(\tau)) \times \\ &\qquad \times (\partial_{\xi_k}\partial_{\xi_l}\phi)(s, \tau; x', \Xi(\tau))d\tau\} \end{aligned} \tag{3.35}$$

and for $j \geqq 1$

$$a_j(t, s; \xi, x') = -a_0(t, s; \xi, x')\int_t^s \frac{B_j(\tau, s; \Xi(\tau), x')}{a_0(\tau, s; \Xi(\tau), x')} d\tau, \tag{3.36}$$

where

$$\Xi(\tau) = p(\tau, s; x', \eta(t, s; x', \xi)) \tag{3.37}$$

and

$$\begin{aligned} &B_j(t, s; \xi, x') \\ &= \sum_{2\leq|\alpha|\leq j+1} \frac{1}{\alpha!} \partial_{\xi'}^{\alpha} \{(\partial_x^{\alpha} H)(t, \tilde{\nabla}_{\xi}\phi(s, t; \xi, x', \xi'), \xi) \times \\ &\qquad \times a_{j+1-|\alpha|}(t, s; \xi', x')\}_{|\xi'=\xi} . \end{aligned} \tag{3.38}$$

Moreover, in case $0\leqq\delta<\rho\leqq1$, we define $e_h(t, s; \xi, x')$ by

$$e_h(t, s; \xi, x') = \sum_{j=0}^{\infty} \chi(\varepsilon_j^{-1}h)(ih^{\rho-\delta})^j a_j(t, s; h^{\rho}\xi, h^{-\delta}x') , \tag{3.39}$$

where χ and $\{\varepsilon_j\}_{j=0}^{\infty}$ are taken as in Proposition 1.3.

Let $C^l(\Omega\,|\,B_{\rho,\delta}^m(h))$ ($\Omega\subset R^p$ domain) denote the set of families $\{f_h(\omega, y)\}_{0<h<1}$ of functions $f_h(\omega, y)$ such that for each $\omega\in\Omega$ $\{f_h(\omega, y)\}_{0<h<1}\in\{B_{\rho,\delta}^m(h)\}_{0<h<1}$ and the derivative $\partial_y^{\alpha}f_h(\omega, y)$ belongs to $C^l(\Omega\times R_y^k)$ for each $h\in(0, 1)$.

Proposition 3.9. i) *The function* $a_j(t, s; \xi, x')$ $(j=0, 1, 2, \cdots)$ *belongs to* $C^1(A_T\cup B_{\tilde{\delta}}\,|\,\mathcal{B}^{\infty}(R^{2n}))$ *and is the solution of the transport equation*

$$\begin{aligned} &-\partial_t a_j(t, s; \xi, x') + \sum_{k=1}^{n} (\partial_{x_k}H)(t, y(s, t; x', \xi), \xi)(\partial_{\xi_k}a_j)(rt, s; \xi, x') \\ &+\frac{1}{2}\sum_{l,k=1}^{n} (\partial_{x_k}\partial_{x_l}H)(t, y(s, t; x', \xi), \xi)(\partial_{\xi_k}\partial_{\xi_l}\phi)(s, t; x', \xi)\times \\ &\times a_j(t, s; \xi, x') + B_j(t, s; \xi, x') = 0 \end{aligned} \tag{3.40}$$

for $(t, s)\in A_T\cup B_{\tilde{\delta}}$ *with the initial condition*

$$a_0(s, s; \xi, x') = 1 , \quad a_j(s, s; \xi, x') = 0 \qquad (j = 1, 2, \cdots) , \tag{3.41}$$

where we put $B_0(t, s; \xi, x')=0$. *More precisely* $a_j(t, s; \xi, x')$ *satisfies the estimates*

$$\begin{cases} |\partial_{\xi}^{\alpha}\partial_{x'}^{\beta}(a_0(t, s; \xi, x')-1)| \leqq C_{\alpha,\beta}\langle s\rangle^{-\varepsilon} \text{ (or } \leqq C_{\alpha,\beta}|t-s|^2) , \\ |\partial_{\xi}^{\alpha}\partial_{x'}^{\beta}a_j(t, s; \xi, x')| \leqq C_{\alpha,\beta}\langle s\rangle^{-1-\varepsilon} \text{ (or } \leqq C_{\alpha,\beta}|t-s|^2) \ (j\geqq1) \end{cases} \tag{3.42}$$

for $(t, s)\in A_T$(or $\in B_{\tilde{\delta}}$), *where the constant* $C_{\alpha,\beta}>0$ *is independent of* ξ, x' *and* $(t, s)\in A_T$ (or $\in B_{\tilde{\delta}}$).

ii) *When* $0\leqq\delta<\rho\leqq1$, $e_h(t, s; \xi, x')$ *of* (3.39) *is well-defined and belongs to* $C^1(A_T\cup B_{\tilde{\delta}}\,|\,B_{\rho,\delta}^0(h))$. *Moreover the following estimate holds: For any* $(t, s)\in A_T$ (or $\in B_{\tilde{\delta}}$)

$$|e_h(t, s)-1|_l^{(0)} \leqq C_l\langle s\rangle^{-\varepsilon} \text{ (or } \leqq C_l|t-s|^2) \tag{3.43}$$

in $B_{\rho,\delta}^0(h)$, *where the constant* $C_l>0$ *is independent of t and s.*

Proof. i) We have only to prove (3.42), since the others are obvious from the theory of the first order differential equations. We first note by (3.37), (3.10), (3.12), (3.15), and (3.20) that

$$|\partial_\xi^\alpha \partial_{x'}^\beta \Xi(\tau)| \leqq C_{\alpha,\beta} \tag{3.44}$$

for some constant $C_{\alpha,\beta}>0$ independent of t, τ, s, ξ and x'. Thus from (3.35) we have the first estimate of (3.42). Then by induction we get the second estimate of (3.42) and

$$|\partial_\xi^\alpha \partial_{x'}^\beta B_j(t, s; \xi, x')| \leqq C_{\alpha,\beta}\langle t\rangle^{-2-\varepsilon} \text{ (or } \leqq C_{\alpha,\beta}|t-s|) \tag{3.45}$$

for $(t, s)\in A_T$ (or $\in B_{\tilde{\delta}}$), using (3.36), (3.38), (3.29), (3.17) and (3.20) (or (3.24)).
ii) is then easily seen from (3.39) and (3.42). □

DEFINITION 3.10. For $(t, s)\in A_T\cup B_{\tilde{\delta}}$ and $f\in\mathcal{S}$, we define

$$E_h^0(\phi_h(s, t)^*)f(x) = 0_s-\iint e^{i(x\cdot\xi-\phi_h(s,t\,;\,x',\xi))}f(x')dx'd\!\!{}^-\xi \tag{3.46}$$

in case $0\leqq\delta\leqq\rho\leqq1$, and

$$\begin{aligned} &E_h^\infty(\phi_h(s, t)^*)f(x) \\ &= 0_s-\iint e^{i(x\cdot\xi-\phi_h(s,t\,;\,x',\xi))}e_h(t, s; \xi, x')f(x')dx'd\!\!{}^-\xi \end{aligned} \tag{3.47}$$

in case $0\leqq\delta<\rho\leqq1$.

Theorem 3.11. *Let Assumption (A) be satisfied. Let $(t, s)\in A_T$ (or $\in B_{\tilde{\delta}}$) and $f\in\mathcal{S}$, and define*

$$G_h^m(\phi_h(s, t)^*)f(x) = \frac{1}{i}((D_t+H_h(t, X', D_x))E_h^m(\phi_h(s, t)^*)f(x)\,, \tag{3.48}$$

where $m=0$ in case $0\leqq\delta\leqq\rho\leqq1$ and $m=\infty$ in case $0\leqq\delta<\rho\leqq1$. Then:
i) *We have*

$$E_h^m(\phi_h(s, s)^*) = I \tag{3.49}$$

and

$$g_h^m(t, s; \xi, x')\in C^0(A_T|B_{\rho,\delta}^m(h))(\text{or } \in C^0(B_{\tilde{\delta}}|B_{\rho,\delta}^m(h))) \tag{3.50}$$

for $m=0$ or ∞. Here $g_h^m(t, s; \xi, x')=\sigma(G_h^m(\phi_h(s, t)^*))(\xi, x')$ is the symbol function of $G_h^m(\phi_h(s, t)^*)$:

$$\begin{aligned} &G_h^m(\phi_h(s, t)^*)f(x) \\ &= 0_s-\iint e^{i(x\cdot\xi-\phi_h(s,t\,;\,x',\xi))}g_h^m(t, s; \xi, x')f(x')dx'd\!\!{}^-\xi\,. \end{aligned} \tag{3.51}$$

ii) *More precisely, we have for $(t, s)\in A_T$ (or $\in B_{\tilde{\delta}}$) and any $N\geqq1$*

$$|g_h^0(t, s)|_l^{(0)}\leqq C_l\langle t\rangle^{-1-\varepsilon} \text{ (or } \leqq C_l|t-s|) \tag{3.52}$$

and

$$|g_h^\infty(t, s)|_l^{(0)} \leqq C_{l,N} h^N \langle t\rangle^{-2-\varepsilon} \text{ (or } \leqq C_{l,N} h^N |t-s|) . \tag{3.53}$$

Here $|\ |_l^{(0)}$ *denotes the semi-norm of* $B^0_{\rho,\delta}(h)$, *and the constants* C_l *and* $C_{l,N}$ *are independent of* t, s *and* h. *Hence we have for* $(t, s) \in A_T$ (or $\in B_{\tilde{\delta}}$) *and any* $N \geqq 1$

$$||G_h^0(\phi_h(s, t)^*)||_{L^2\to L^2} \leqq C\langle t\rangle^{-1-\varepsilon} \text{ (or } \leqq C|t-s|) \tag{3.54}$$

and

$$||G_h^\infty(\phi_h(s, t)^*)||_{L^2\to L^2} \leqq C_N h^N \langle t\rangle^{-2-\varepsilon} \text{ (or } \leqq C_N h^N |t-s|) , \tag{3.55}$$

where C *and* C_N *are independent of* t, s *and* h.

REMARK. Theorem 3.11-i) says that $E_h^m(\phi_h(s, t)^*)$ is the approximate fundamental solution of order $m(m=0$ or $\infty)$ in the sense of Kitada and Kumano-go (see [6, §5]), though the condition (3.50) is weaker than (5.29)-i) and (5.30)-i) of [6].

Proof. (3.49) is obvious by definition. So we have only to prove (3.52) and (3.53), since (3.50), (3.54) and (3.55) follows from them by Theorem 1.7. Using Theorem 1.6 and (3.28) we have for $(t, s) \in A_T \cup B_{\tilde{\delta}}$

$$\begin{aligned} & g_h^0(t, s; \xi, x') \\ & = \sum_{l,k=1}^n \int_0^1 O_s\text{-}\iint e^{-iy\cdot\eta}(\partial_{x_k}\partial_{x_l} H_h)(t, \theta y + \tilde{\nabla}_\xi \phi_h(s, t; \xi, x', \xi-\eta), \xi) \times \\ & \qquad \times \Big(\int_0^1 r(\partial_{\xi_k}\partial_{\xi_l}\phi_h)(s, t; x', \xi - r\eta)dr\Big) dy\, đ\eta\, d\theta , \end{aligned} \tag{3.56}$$

from which follows (3.52) by virtue of the estimates in Propositions 3.2 and 3.3 and our Assumption (A)-iii).

Similarly using Theorem 1.6, (3.28) and (3.40) we get for $N \geqq 2$

$$\begin{aligned} & g_h^\infty(t, s; \xi, x') \\ & = -i\sum_{j=0}^\infty \chi(\varepsilon_j^{-1}h)(ih^{\rho-\delta})^j B_j(t, s; h^\rho\xi, h^{-\delta}x') \\ & \quad + i\sum_{m=1}^\infty (ih^{\rho-\delta})^m \sum_{2\leqq|\alpha|\leqq\min(N-1,m+1)} \frac{1}{\alpha!}\chi(\varepsilon_{m+1-|\alpha|}^{-1}h) \times \\ & \quad \times \partial_{\xi'}^\alpha \{(\partial_x^\alpha H)(t, \tilde{\nabla}_\xi \phi(s, t; h^\rho\xi, h^{-\delta}x', \xi'), h^\rho\xi) \times \\ & \qquad\qquad \times a_{m+1-|\alpha|}(t, s; \xi', h^{-\delta}x')\}|_{\xi'=h^\rho\xi} \\ & \quad + h^{(N-1)(\rho-\delta)} N \sum_{|\gamma|=N} \frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1 (1-\theta)^{N-1} t_{\gamma,h}(\xi, x'; \theta)d\theta , \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} & t_{\gamma,h}(\xi, x'; \theta) \\ & = O_s\text{-}\iint e^{-iy\cdot\eta}\partial_{\xi'}^\gamma\{(D_x^\gamma H)(t, h^{-\delta}\theta y + \tilde{\nabla}_\xi\phi(s, t; h^\rho\xi, h^{-\delta}x', \xi'), \xi) \times \end{aligned} \tag{3.58}$$

$$\times e_h(t, s; h^{-\rho}\xi', x')\}|_{\xi'=h^\rho(\xi-\eta)}dyd\eta .$$

Thus by (3.42), (3.45), (3.43) and Proposition 3.2 (or 3.3), we obtain (3.53). □

4. Fundamental solution global in time

We first construct the fundamental solution locally in time. For this purpose we record a theorem concerning the multi-products of conjugate Fourier integral operators which is a version of Theorem 4.3 of [6].

Theorem 4.1. *Let $n_0>n$ be an even integer and put $\tilde{l}=21n_0+1$. Let $\tilde{\tau}>0$ be sufficiently small as in Theorem 3.8 of [6]. Let $\phi_{j,h}(x, \xi)\in P_{\rho,\delta}(\tau_j, \tilde{l}: h)$ for $j=1, 2, \cdots$, and let $\tilde{\tau}_\infty\equiv\sum_{j=1}^{\infty}\tau_j\leqq\tilde{\tau}$. Let $\nu\geqq 1$ be an integer and put $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$. Let $p_{j,h}(\xi, x')\in B^{m_j}_{\rho,\delta}(h)$ for $j=1, \cdots, \nu+1$. Then there exists a symbol $r_{\nu+1,h}(\xi, x')\in B^{\bar{m}_{\nu+1}}_{\rho,\delta}(h)$ $(\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1})$ such that*

$$P_{\nu+1,h}(\phi_{\nu+1,h}^*)\cdots P_{1,h}(\phi_{1,h}^*) = R_{\nu+1,h}(\Phi_{\nu+1,h}^*) \tag{4.1}$$

and

$$\begin{aligned} &|r_{\eta+1,h}|_l^{(\bar{m}_{\nu+1})} \\ &\leqq \tilde{C}_l^{l+2}\exp(\tilde{c}_l(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{2,k_1})^{k_1+2})\times \\ &\times\sum_{l_1+\cdots+l_{\nu+1}\leqq l+2n_0}\prod_{j=1}^{\nu+1}|p_{j,h}|_{9n_0+l_j}^{(m_j)}, \end{aligned} \tag{4.2}$$

where $P_{j,h}(\phi_{j,h}^*)=p_{j,h}(\phi_{xj,h}^*; D_x, X')$; $R_{\nu+1,h}(\Phi_{\nu+1,h}^*)=r_{\nu+1,h}(\Phi_{\nu+1,h}^*; D_x, X')$; $J_{j,h}=\phi_{j,h}-x\cdot\xi$; $k_1=2l+25n_0+1$; and $\tilde{C}_l$ are $\tilde{c}_l$ positive constants.

Proof. This theorem follows from Theorem 4.3 of [6], if we note the following fact: For $P_h(\phi_h^*)=p_h(\phi_h^*; D_x, X')\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$, $\phi_h\in P_{\rho,\delta}(\tau, 0; h)$, we have for $f\in\mathcal{S}$

$$P_h(\phi_h^*)^*f(x) = q_h(\phi_h; X, D_x)f(x)\,, \tag{4.3}$$

where $q_h(x, \xi)=\overline{p_h(\xi, x)}$. □

Now we construct the local fundamental solution.

Theorem 4.2. *Let Assumption (A) be satisfied. Then for a sufficiently small $0<\delta_0\leqq\delta(<1)$, there exists uniquely the fundamental solution $U_h(t,s)$ of the equation*

$$\begin{cases} U_h(s, s) = I & (s\in R^1)\,, \\ L_hU_h(t, s) = 0 & (|t-s|\leqq\delta_0)\,. \end{cases} \tag{4.4}$$

Moreover the fundamental solution $U_h(t, s)$ is uniquely represented as a conjugate Fourier integral operator with phase function $\phi_h(s, t)$ and a symbol of class

$C^1(B_{\delta_0}|B^0_{\rho,\delta}(h))$. *More precisely there exist symbols* $d^m_h(t, s; \xi, x')\in C^1(B_{\delta_0}|B^m_{\rho,\delta}(h))$ ($m=0$ *in case* $0\leqq\delta\leqq\rho\leqq 1$ *and* $m=\infty$ *in case* $0\leqq\delta<\rho\leqq 1$) *such that for* $D^m_h(\phi_h(s, t)^*)$ *defined by*

$$D^m_h(\phi_h(s, t)^*)f(x) \tag{4.5}$$
$$= 0_s-\iint e^{i(x\cdot\xi-\phi_h(s,t,x',\xi))}d^m_h(t, s; \xi, x')f(x')dx'd\xi \quad (f\in\mathcal{S}),$$

we can write

$$U_h(t, s) = E^m_h(\phi_h(s, t)^*)+D^m_h(\phi_h(s, t)^*) \tag{4.6}$$

for $(t, s)\in B_{\delta_0}$. *The operator* $U_h(t, s)$ *is extended to a unitary operator in* $L^2(R^n)$, *and the following relations hold*:

$$U_h(t, \theta)U_h(\theta, r) = U_h(t, r), \qquad t, \theta, r\in[s-\delta_0/2, s+\delta_0/2], \tag{4.7}$$

$$D_sU_h(t, s)-U_h(t, s)H_h(s, X, D_x) = 0, \qquad |t-s|\leqq\delta_0. \tag{4.8}$$

Proof. We proceed quite similarly as in the proof of Theorem 6.1 of [6]. Let $n_0>n$ be even and let $\tilde{l}=21n_0+1$. Let $\tilde{\tau}>0$ be sufficiently small so that Theorem 3.8 of [6] holds for our case. For $c_{\tilde{l}}$ in Proposition 3.7-ii), we take $\delta_0>0$ as $c_{\tilde{l}}\delta_0\leqq\tilde{\tau}$. Then for any subdivision Δ: $t>t_\nu>t_{\nu-1}>\cdots>t_1>s$ of $[s, t]$, we can easily see that

$$\phi_h(s, t_1)\#\phi_h(t_1, t_2)\#\cdots\#\phi_h(t_\nu, t) = \phi_h(s, t) \tag{4.9}$$

holds (cf. Kumano-go, Taniguchi and Tozaki [8]). Now using Theorem 4.1, we define $W^m_{\nu,h}(\phi_h(s, t)^*)$ by

$$W^m_{1,h}(\phi_h(s, t)^*) = G^m_h(\phi_h(s, t)^*) = \frac{1}{i} L_hE^m_h(\phi_h(s, t)^*) \tag{4.10}$$

and

$$\begin{aligned} W^m_{\nu+1,h}(\phi_h(s, t)^*) &= \int_s^t W^m_{1,h}(\phi_h(\theta, t)^*)W^m_{\nu,h}(\phi_h(s, \theta)^*)d\theta \\ &= \int_s^t\int_s^{t_\nu}\cdots\int_s^{t_2} W^m_{1,h}(\phi_h(t_\nu, t)^*)W^m_{1,h}(\phi_h(t_{\nu-1}, t_\nu)^*)\cdots \\ &\qquad \cdots W^m_{1,h}(\phi_h(s, t_1)^*)dt_1\cdots dt_\nu, \end{aligned} \tag{4.11}$$

where $m=0$ in case $0\leqq\delta\leqq\rho\leqq 1$ and $m=\infty$ in case $0\leqq\delta<\rho\leqq 1$. Then, in quite the same way as in [6], we see from Theorem 3.11-i) that the following series converges in $C^0(B_{\delta_0}|B^m_{\rho,\delta}(h))$:

$$\tilde{d}^m_h(t, s; \xi, x') = \sum_{\nu=1}^{\infty} w^m_{\nu,h}(t, s; \xi, x'), \tag{4.12}$$

where $w^m_{\nu,h}(t, s; \xi, x')\equiv\sigma(W^m_{\nu,h}(\phi_h(s, t)^*))(\xi, x')\in C^0(B_{\delta_0}|B^m_{\rho,\delta}(h))$. Hence setting $\tilde{D}^m_h(\phi_h(s, t)^*)=\tilde{d}^m_h(\phi_h(s, t)^*; t, s; D_x, X')$, we define

$$(4.13) \qquad D_h^m(\phi_h(s, t)^*) = \int_s^t E_h^m(\phi_h(\theta, t)^*)\tilde{D}_h^m(\phi_h(s, \theta)^*)d\theta,$$

where we again use Theorem 4.1. Then $U_h(t, s)$ defined by (4.6) satisfies (4.4). The uniqueness, the unitarity, and the relations (4.7) and (4.8) are proved in a way quite similar to [6]. □

REMARK. As can be easily seen from the proof, this theorem also holds under the same assumption on the Hamiltonian as in [6].

From this theorem we can construct uniquely the global fundamental solution $U_h(t,s_0)$ of the equation (3.4) for $t\in R^1$ by $U_h(t,s_0)=U_h(t,s_N)U_h(s_N,s_{N-1})\cdots U_h(s_1, s_0)$, where $s_k=s_0+k(t-s_0)/(N+1)$ $(k=1, \cdots, N)$ with N being an integer such that $N\geqq|t-s_0|/\delta_0-1$. The operator $U_h(t, s_0)$ thus constructed is unitary in $L^2(R^n)$ and obviously satisfies the relations (4.7) and (4.8) for any $t, \theta, r, s\in R^1$. Especially we have $U_h(t, s)^{-1}=U_h(s, t)$ for $t, s\in R^1$.

We now study a simple expression of $U_h(t, s_0)$, restricting ourselves to considering only the case $t\geqq s_0$. (The other case can be dealt with similarly.) Then we have

$$(4.14) \qquad \begin{aligned} U_h(t, s_0) = {} & U_h(t, t_\nu)U_h(t_\nu, t_{\nu-1})\cdots U_h(t_1, \tilde{T})\cdot \\ & \cdot U_h(\tilde{T}, s_L)U_h(s_L, s_{L-1})\cdots U_h(s_1, s_0), \end{aligned}$$

where $t_{j+1}=t_j+\delta_0$ $(j=0, 1, \cdots)$, $t_0=\tilde{T}$, $t_{\nu+1}\geqq t>t_\nu$, $s_{l-1}=s-\delta_0$ $(l=1, \cdots, L+1)$, $s_{L+1}=\tilde{T}$, and $s_1>s_0\geqq s_1-\delta_0$, $\tilde{T}$ being a large number. Then L is determined only by s_0 and $\tilde{T}$. So if we can represent $U_h(t, t_\nu)\cdots U_h(t_1, \tilde{T})$ as a single conjugate Fourier integral operator, then the fundamental solution $U_h(t, s_0)$ is represented as a product of a finite number of conjugate Fourier integral operators independently of t. Before proving this we prepare a proposition.

Proposition 4.3. *Let $T(>T_0)$ be as in section 3 and let $0<\delta_0<1$ be as in Theorem 4.2. Then:*

i) For any (t,s) satisfying $T\leqq s\leqq t\leqq s+\delta_0$ we have $\phi_h(s,t)\in P_{\rho,\delta}^{[x]}(C_1\delta_0\langle s\rangle^{-2-\varepsilon}, C_1\delta_0; h)$, where C_1 is the constant in Proposition 3.2.

ii) For $s\geqq T$ and $t>s+\delta_0$, let $\nu\geqq 1$ be an integer such that $s+(\nu+1)\delta_0\geqq t> s+\nu\delta_0$, and put $t_j=s+j\delta_0$ for $j=0, 1, \cdots, \nu$ and $t_{\nu+1}=t$. Then the $(\nu+1)$-tuple $(\phi_h(s, t_1), \phi_h(t_1, t_2), \cdots, \phi_h(t_\nu, t))$ of phase functions satisfies the condition (#) of section 2. Moreover we have $\phi_h(s, t_1)\#\phi_h(t_1, t_2)\#\cdots\#\phi_h(t_\nu, t)=\phi_h(s, t)$.

Proof. i) is clear from Proposition 3.7-i).

ii) Put $\phi_{j,h}=\phi_h(t_{j-1}, t_j)$ for $j=1, \cdots, \nu+1$. Then by Definition 3.6 and Proposition 3.5-(3.29) the equation (2.1) is equivalent to

$$(4.15) \qquad \begin{cases} \text{i)} & X_\nu^j = y(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j) \\ \text{ii)} & \Xi_\nu^j = \eta(t_{j+1}, t_j; X_\nu^j, \Xi_\nu^{j+1}) \end{cases} \qquad (j=1, \cdots, \nu)$$

with $X_\nu^0=x$; $\Xi_\nu^{\nu+1}=\xi$; and

$$\text{(4.16)}\qquad \begin{cases} X_{\nu,h}^j(x,\xi)=h^\delta X_\nu^j(h^{-\delta}x,h^\rho\xi) \\ \Xi_{\nu,h}^j(x,\xi)=h^{-\rho}\Xi_\nu^j(h^{-\delta}x,h^\rho\xi)\,. \end{cases} \qquad (j=1,\cdots,\nu)$$

Assume that $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^\nu(x,\xi)$ is the solution of (4.15). Then from (4.15) and Proposition 3.2 we have

$$\text{(4.17)}\qquad \begin{cases} \text{i)}\ \ X_\nu^{j-1}=q(t_{j-1},t_j;X_\nu^j,\Xi_\nu^j) \\ \text{ii)}\ \ \Xi_\nu^{j+1}=p(t_{j+1},t_j;X_\nu^j,\Xi_\nu^j)\,. \end{cases} \qquad (j=1,\cdots,\nu)$$

On the other hand using Proposition 3.2-ii) we have from (4.15)

$$\text{(4.18)}\qquad \begin{cases} \text{i)}\ \ X_\nu^j=q(t_j,t_{j-1};X_\nu^{j-1},\Xi_\nu^{j-1}) \\ \text{ii)}\ \ \Xi_\nu^j=p(t_j,t_{j+1};X_\nu^{j+1},\Xi_\nu^{j+1})\,, \end{cases} \qquad (j=1,\cdots,\nu)$$

where we put $\Xi_\nu^0=\eta(t_1,t_0;x,\Xi_\nu^1)$ and $X_\nu^{\nu+1}=y(t_\nu,t_{\nu+1};X_\nu^\nu,\xi)$. Thus from (4.17)-i) and (4.18)-ii) we get

$$\text{(4.19)}\qquad \begin{aligned} (X_\nu^j,\Xi_\nu^j)&=(q,p)(t_j,t_{j+1};X_\nu^{j+1},\Xi_\nu^{j+1}) \\ &=(q,p)(t_j,t_{\nu+1};X_\nu^{\nu+1},\xi) \qquad (j=0,1,\cdots,\nu)\,. \end{aligned}$$

On the other hand from (4.17)-ii) and (4.18)-i) we have

$$\text{(4.20)}\qquad \begin{aligned} (X_\nu^j,\Xi_\nu^j)&=(q,p)(t_j,t_{j-1};X_\nu^{j-1},\Xi_\nu^{j-1}) \\ &=(q,p)(t_j,t_0;x,\Xi_\nu^0) \qquad (j=1,\cdots,\nu+1)\,. \end{aligned}$$

Hence we have from (4.19) and (4.20)

$$\text{(4.21)}\qquad \begin{cases} x=X_\nu^0=q(t_0,t_{\nu+1};X_\nu^{\nu+1},\xi)\,, \\ \xi=\Xi_\nu^{\nu+1}=p(t_{\nu+1},t_0;x,\Xi_\nu^0)\,, \end{cases}$$

from which we get

$$\text{(4.22)}\qquad \begin{cases} X_\nu^{\nu+1}=y(t_0,t_{\nu+1};x,\xi)\,, \\ \Xi_\nu^0=\eta(t_{\nu+1},t_0;x,\xi)\,. \end{cases}$$

Combining this with (4.18) and (4.17) gives

$$\text{(4.23)}\qquad \begin{cases} X_\nu^j=q(t_j,t_0;x,\eta(t_{\nu+1},t_0;x,\xi))=q(t_j,t_{\nu+1};y(t_0,t_{\nu+1};x,\xi),\xi)\,, \\ \Xi_\nu^j=p(t_j,t_0;x,\eta(t_{\nu+1},t_0;x,\xi))=p(t_j,t_{\nu+1};y(t_0,t_{\nu+1};x,\xi),\xi)\,. \end{cases}$$

Obviously this is C^∞ in (x,ξ) and satisfies (4.15). Thus we have proved ii). □

Now we can prove the main result of this paper.

Theorem 4.4. *Let Assumption (A) be satisfied. Then the following as-*

sertions hold.

i) *There exists uniquely the fundamental solution* $U_h(t, s_0)$ $(t, s_0 \in R^1)$ *of the equation* (3.4). *This operator* $U_h(t, s_0)$ *satisfies the relation*

$$\begin{cases} U_h(t, \theta)U_h(\theta, s) = U_h(t, s), & t, \theta, s \in R^1, \\ D_s U_h(t, s) - U_h(t, s)H_h(s, X, D_x) = 0, & t, s \in R^1, \end{cases} \tag{4.24}$$

and is extended to a unitary operator in $L^2(R^n)$.

ii) *Let* $\tilde{T}(>T)$ *be sufficiently large. Then the fundamental solution* $U_h(t, s)$ *for* $t \geqq s \geqq \tilde{T}$ *is uniquely represented as a single conjugate Fourier integral operator with phase function* $\phi_h(s, t)$ $(\in P^{[x]}_{\rho,\delta}(C_1\langle s\rangle^{-\varepsilon}, C_1(t-s); h))$ *and a symbol of class* $C^1(A_{\tilde{T}} \mid B^0_{\rho,\delta}(h))$. *More precisely, let* $0<\delta_0<1$ *be sufficiently small as in Theorem* 4.2. *For* $t \geqq s \geqq \tilde{T}$ *and* $t > s+\delta_0$, *let* $\nu \geqq 1$ *be an integer such that* $s+(\nu+1)\delta_0 \geqq t > s+\nu\delta_0$, *and put* $t_j = s+j\delta_0$ *for* $j=0, 1, \cdots, \nu$ *and* $t_{\nu+1}=t$. *Then we have*

$$\begin{aligned} U_h(t, s)f(x) &= U_h(t, t_\nu)U_h(t_\nu, t_{\nu-1}) \cdots U_h(t_1, s)f(x) \\ &= 0_s - \iint e^{i(x\cdot\xi - \phi_h(s,t\,;\,x',\xi))} u_h(t, s; \xi, x')f(x')dx'd\xi \end{aligned} \tag{4.25}$$

for $f \in \mathcal{S}$, *where* $u_h(t, s; \xi, x') \in C^1(A_{\tilde{T}} \mid B^0_{\rho,\delta}(h))$ *is uniquely determined. Thus the fundamental solution* $U_h(t, s_0)$ *of* (3.4) *for* $t \geqq s_0$ *is represented as a product of a finite number of conjugate Fourier integral operators, the number depending on* s_0 *but not on* t.

iii) *For* $t \geqq s \geqq \tilde{T}$ *define*

$$F^m_h(\phi_h(s, t)^*) = U_h(t, s) - E^m_h(\phi_h(s, t)^*), \tag{4.26}$$

where $m=0$ *in case* $0 \leqq \delta \leqq \rho \leqq 1$ *and* $m=\infty$ *in case* $0 \leqq \delta < \rho \leqq 1$. *Then we have for* $t \geqq s \geqq \tilde{T}$

$$f^m_h(t, s) \equiv \sigma(F^m_h(\phi_h(s, t)^*)) \in C^1(A_{\tilde{T}} \mid B^m_{\rho,\delta}(h)) \tag{4.27}$$

and

$$\begin{cases} \|F^0_h(\phi_h(s, t)^*)\|_{L^2\to L^2} \leqq C\langle s\rangle^{-\varepsilon}, \\ \|F^\infty_h(\phi_h(s, t)^*)\|_{L^2\to L^2} \leqq C_N h^N \langle s\rangle^{-1-\varepsilon} \end{cases} \tag{4.28}$$

for any $N \geqq 1$, *where the constants* C *and* C_N *are independent of* t, s *and* h.

Proof. i) is already proved.

ii) We have only to prove the second equality of (4.25). Since $0 \leqq t_j - t_{j-1} \leqq \delta_0$, from Proposition 4.3-i) we have $\phi_{j,h} \equiv \phi_h(t_{j-1}, t_j) \in P^{[x]}_{\rho,\delta}(\tau_j, \sigma_0; h)$ for $\sigma_0 = C_1\delta_0$ and $\tau_j = C_1\delta_0\langle t_{j-1}\rangle^{-2-\varepsilon}$ $(j=1, \cdots, \nu+1)$. So putting $\phi_{k,h} = x\cdot\xi$ and $\tau_k=0$ for $k \geqq \nu+2$, we have for the sequence $\{\phi_{j,h}\}_{j=1}^\infty$ of phase functions $\phi_{j,h} \in P^{[x]}_{\rho,\delta}(\tau_j, \sigma_0; h)$

$$\begin{aligned}\bar{\tau}_\infty &\equiv \sum_{j=1}^{\infty} \tau_j \leqq \sum_{j=1}^{\infty} C_1\delta_0\langle t_{j-1}\rangle^{-2-\varepsilon} \\ &\leqq 2C_1 \sum_{j=1}^{\infty} \delta_0\langle \tilde{T}+(j-1)\delta_0\rangle^{-2-\varepsilon} \\ &\leqq 2C_1\int_0^\infty \langle \tilde{T}+\tau\rangle^{-2-\varepsilon} d\tau \leqq \frac{2C_1}{1+\varepsilon}\langle T\rangle^{-1-\varepsilon}\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}\sigma_0\bar{\bar{\tau}}_\infty &\equiv \sigma_0 \sum_{j=1}^{\infty} j\tau_j \leqq 2C_1^2 \sum_{j=1}^{\infty} j\delta_0^2\langle \tilde{T}+(j-1)\delta_0\rangle^{-2-\varepsilon} \\ &\leqq 2C_1^2 \sum_{j=1}^{\infty} (\delta_0\langle \tilde{T}+(j-1)\delta_0\rangle^{-1-\varepsilon}+\delta_0^2\langle \tilde{T}+(j-1)\delta_0\rangle^{-2-\varepsilon}) \\ &\leqq 2C_1^2\int_0^\infty (\langle \tilde{T}+\tau\rangle^{-1-\varepsilon}+\delta_0\langle \tilde{T}+\tau\rangle^{-2-\varepsilon}) d\tau \\ &\leqq 2C_1^2\varepsilon^{-1}\langle \tilde{T}\rangle^{-\varepsilon}+2C_1^2(1+\varepsilon)^{-1}\langle \tilde{T}\rangle^{-1-\varepsilon}.\end{aligned} \tag{4.30}$$

Thus if we take $\tilde{T}(\geqq T)$ sufficiently large, then we have $2(\sigma_0\bar{\bar{\tau}}_\infty+\bar{\tau}_\infty)\leqq\tau_0$ for some $0\leqq\tau_0\leqq 1/4$ independently of ν. This, together with Proposition 4.3-ii), shows that Theorem 2.6 is applicable to the product $U_h(t, t_\nu)\cdots U_h(t_1, T)$. Thus by Theorem 2.6 we have (4.25). The smoothness of $u_h(t, s)$ in (t, s) at $t=t_j$ follows from the uniqueness by taking another small $\delta_0>0$.

iii) From (4.26) and (3.48) we have for $t\geqq s\geqq T$ and $f\in\mathcal{S}$

$$\begin{aligned}F_h^m(\phi_h(s, t)^*)f &= U_h(t, s)\,(I-U_h(t, s)^{-1}E_h^m(\phi_h(s, t)^*))f \\ &= U_h(t, s)\int_t^s \frac{d}{d\theta}\,[U_h(\theta, s)^{-1}E_h^m(\phi_h(s, \theta)^*)f]d\theta \\ &= U_h(t, s)\int_s^t U_h(\theta, s)^{-1}G_h^m(\phi_h(s, \theta)^*)f d\theta \\ &= \int_s^t U_h(t, \theta)G_h^m(\phi_h(s, \theta)^*)f d\theta .\end{aligned} \tag{4.31}$$

From this and Theorem 3.11–(3.50) follows (4.27), if we use the expression (4.25) and Theorem 2.6. The estimate (4.28) also follows from (4.31) and (3.54)–(3.55) of Theorem 3.11. □

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Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku, Tokyo
153 Japan