



Title	Endomorphisms of homogeneous spaces of Lie groups
Author(s)	Laitinen, Erkki
Citation	Osaka Journal of Mathematics. 1995, 32(1), p. 165-170
Version Type	VoR
URL	<a href="https://doi.org/10.18910/9853">https://doi.org/10.18910/9853</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

Laitinen, E.  
Osaka J. Math.  
32 (1995), 165–170

## ENDOMORPHISMS OF HOMOGENEOUS SPACES OF LIE GROUPS

ERKKI LAITINEN

(Received May 6, 1993)

If  $H$  is a closed subgroup of a topological group  $G$  it is well-known that there is a bijection

$$\text{Map}_G(G/H, G/H) \xrightarrow{\sim} (G/H)^H$$

which is actually a homeomorphism when the mapping space is equipped with compact-open topology. Homeomorphisms correspond to the subspaces

$$\text{Homeo}_G(G/H) \xrightarrow{\sim} NH/H.$$

Our main purpose is to prove

**Theorem.** *If  $G$  is a Lie group and  $H$  is a closed subgroup then  $NH/H$  is open in  $(G/H)^H$ .*

In [tD, Ch. IV.1] Tammo tom Dieck defines a universal additive invariant  $U(G)$  of pointed finite  $G$ -CW-complexes for arbitrary topological groups  $G$  and computes it for compact Lie groups. As a corollary we obtain that his result is valid for arbitrary Lie groups, too.

**Corollary.**  *$U(G)$  is a free abelian group on elements  $u(G/H^+)$  where  $H$  runs through a complete set of conjugacy classes of closed subgroups  $H$  in  $G$  for any Lie group  $G$ .*

The condition

$$(O) \quad NH/H \text{ is open in } (G/H)^H$$

was introduced in a study with Wolfgang Lück [LL] in order to define the equivariant Lefschetz class of a  $G$ -endomorphism  $f: X \rightarrow X$  of a finite  $G$ -CW-complex.

The inverse images of the subspaces  $NH/H \subset (G/H)^H \subset G/H$  are

$$NH = \{g \in G \mid g^{-1}Hg = H\} \text{ and } SH = \{g \in G \mid g^{-1}Hg \subset H\}$$

and we claim that  $NH$  is open in  $SH$ , when  $G$  is a Lie group. It is well-known that  $NH = SH$  when  $H$  is compact. As  $H$  is closed in  $G$ , both  $NH$  and  $SH$  are always closed in  $G$ , so that for Lie groups  $G$  the fixed point space splits as a topological sum

$$(G/H)^H = NH/H + (G/H)^{>H}.$$

The Lie theory we need can be found e.g. in the books Helgason [H, Ch.II] or Kawakubo [K, Ch.3].

### Reduction to a discrete subgroup

Let  $G$  be a Lie group and  $H$  be a closed subgroup. We first claim that it suffices to prove the Theorem for all Lie groups  $G$  in the case where  $H$  is discrete. Indeed, let  $H_0$  denote the unit component of  $H$ . Then  $H_0$  is a closed and open subgroup of  $H$  and  $H/H_0 = \pi_0(H)$ . If  $g^{-1}Hg \subset H$  then  $g^{-1}H_0g \subset H$  is a connected set which contains  $e$ , whence  $g^{-1}H_0g \subset H_0$ . Then it holds for the Lie algebras that  $L(g^{-1}H_0g) \subset L(H_0)$ , but as they have the same dimension they must coincide. By connectedness  $g^{-1}H_0g = H_0$  and  $NH \subset SH \subset N(H_0)$ . We can therefore assume that  $G = N(H_0)$ , i.e. that  $H_0$  is normal in  $G$ . Then the normalizer of the discrete subgroup  $\pi_0(H) = H/H_0$  of  $G/H_0$  is  $N\pi_0(H) = NH/H_0$ ,  $S\pi_0(H) = SH/H_0$  and it clearly suffices to prove the claim for the subgroup  $\pi_0(H)$  of  $G/H_0$ .

### Lie algebra of the centralizer

Let  $G$  be a Lie group and let  $H$  be a discrete closed subgroup of  $G$ . The centralizer

$$ZH = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$$

is a closed subgroup of  $G$  and is normal in  $NH$  for any closed  $H$ . When  $H$  is moreover discrete, then it holds  $(NH)_0 = (ZH)_0$ ; each  $g \in (NH)_0$  can be connected to  $e$  by a path  $g$ , in  $NH$ . The corresponding conjugations  $c_g : H \rightarrow H$  give a homotopy from  $c_g$  to  $c_e = id_H$ . As  $H$  is discrete, the homotopy is constant and therefore  $c_g = id_H$ , i.e.  $g \in ZH$ . We conclude that  $LNH = LZH$ .

Recall that the adjoint representation of  $G$  in  $LG$  is defined by attaching to an element  $g \in G$  the differential  $Ad(g) : LG \rightarrow LG$  of the conjugation  $c_g : G \rightarrow G$ .

**Lemma.**  $LZH = \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\}$ .

**Proof.** A closed subgroup  $H$  of a Lie group  $G$  is itself a Lie group with Lie

algebra

$$LH = \{X \in LG \mid \exp(tX) \in H \text{ for } t \in \mathbb{R}\},$$

see [H, Theorem II 2.3] or [K, Theorem 3.36]. Hence

$$\begin{aligned} LZH &= \{X \in LG \mid \exp(tX) \in ZH \text{ for } t \in \mathbb{R}\} \\ &= \{X \in LG \mid c_h(\exp(tX)) = \exp(tX) \text{ for } h \in H, t \in \mathbb{R}\} \end{aligned}$$

as  $Ad(h)$  is the differential of  $c_h$ , the last set equals to

$$\begin{aligned} &= \{X \in LG \mid \exp(tX) = \exp(tAd(h)X) \text{ for } h \in H, t \in \mathbb{R}\} \\ &= \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\} \end{aligned}$$

as  $\exp$  is a diffeomorphism near the origin. This proves the Lemma.

### Proof of the Theorem

Let  $G$  be a Lie group and let  $H$  be a closed discrete subgroup. The quotient space  $G/H$  is then a smooth manifold and the projection  $\pi: G \rightarrow G/H$  is a smooth covering projection. Then the diagrams

$$\begin{array}{ccc} G & \xrightarrow{c_h} & G \\ \pi \downarrow & & \downarrow \pi \\ G/H & \xrightarrow{l_h} & G/H \end{array} \quad \begin{array}{ccc} LG & \xrightarrow{Ad(h)} & LG \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{c_h} & G \end{array}$$

commute for each  $h \in H$ , where  $l_g: G/H \rightarrow G/H$  is left translation by  $g$  and  $c_g: G \rightarrow G$  is conjugation by  $g$ . As the exponential map  $\exp$  is a local diffeomorphism at the origin 0,  $\exp(0) = e$  and similarly  $\pi$  is a local diffeomorphism at the unit element  $e$  and  $\pi(e) = eH$ , the composite  $\pi \exp$  is a diffeomorphism of a small enough open disk  $U_h \subset LG$  onto its image  $V_h \subset G/H$ .  $V_h$  is an open neighborhood of  $eH$  and the diagram

$$\begin{array}{ccc} U_h & \xrightarrow{Ad(h)} & Ad(h)U_h \\ \pi \exp \downarrow & & \downarrow \pi \exp \\ V_h & \xrightarrow{l_h} & hV_h \end{array}$$

commutes. In particular  $U_h^{Ad(h)}$  is diffeomorphic to  $V_h^h$ .

By the Lemma  $LZH = \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\}$ . As the spaces in question are finite-dimensional vector spaces, we can choose a finite set  $h_1, h_2, \dots, h_n \in H$

such that  $LZH = \{X \in LG \mid X = Ad(h_i)X \text{ for } i = 1, \dots, n\}$ . Let  $U = \bigcap_{i=1}^n U_{h_i}$  and  $V = \bigcap_{i=1}^n V_{h_i}$ . The map  $\pi \exp$  restricts to a diffeomorphism  $U \rightarrow V$ , which induces a diffeomorphism

$$U \cap LZH = \bigcap_{i=1}^n U_{h_i}^{Ad(h_i)} \cong \bigcap_{i=1}^n V_{h_i}^{h_i} = V^{(h_1, \dots, h_n)}.$$

Choose  $U$  and consequently  $V$  is so small that  $\pi \exp(U \cap LZH) = V \cap (ZH/H)$  holds. Then the neighborhood  $V$  of the point  $eH = G/H$  satisfies

$$V \cap (ZH/H) = V \cap (G/H)^{(h_1, \dots, h_n)}.$$

But clearly we have  $ZH/H \subset NH/H \subset (G/H)^H \subset (G/H)^{(h_1, \dots, h_n)}$  so in fact equality

$$V \cap (NH/H) = V \cap (G/H)^H$$

holds. Hence  $NH/H \subset (G/H)^H$  is open at the point  $eH \in NH/H$ . Using the left action of  $NH/H$  we see that  $NH/H$  is open in  $(G/H)^H$ . This proves the Theorem.

### Proof of the Corollary

Recall tom Dieck's definition of the universal additive invariant of a topological group  $G$ . An additive invariant consists of a pair  $(B, b)$ ,  $B$  an abelian group and  $b$  an assignment which associates to each pointed finite  $G$ -CW-complex  $X$  an element  $b(X) \in B$  such that  $b(X) = b(Y)$  if  $X$  and  $Y$  are pointed  $G$ -homotopy equivalent and that the condition

$$b(X) = b(A) + b(X/A)$$

holds when  $A$  is a pointed subcomplex of  $X$ . An additive invariant  $(U, u)$  is universal if every other additive invariant factors through it uniquely. A universal additive invariant is uniquely determined by the usual Grothendieck construction, and it is denoted by  $(U(G), u)$ .

It follows by an easy argument that  $U(G)$  is always generated by the classes  $u(G/H^+)$  [tD, Proposition IV.1.8]. Although not explicitly stated, the proof that there are no relations between the classes  $u(G/H^+)$  uses implicitly the fact that  $G$  is a compact Lie group since it is based on the Euler characteristics  $\chi(X^H/NH)$ , which are guaranteed to exist if  $G$  is a compact Lie group since then  $X^H/NH$  is a compact ENR but not otherwise (cf. the example given below.)

Let  $G$  be a Lie group. Using our Theorem we can alternatively proceed as follows. As noted in [LL, p. 495], the condition  $(O)$  implies that a  $G$ -CW-complex structure on  $X$  induces a relative  $NH/H$ -CW-complex structure on the pair  $(X^H, X^{>H})$ . The quotient  $(X^H/NH, X^{>H}/NH)$  is then an ordinary relative CW-complex (possibly non-Hausdorff) whose cells correspond to the  $G$ -cells of  $X$  of type  $G/H$ . If  $n(X, H, i)$  is the number of such  $i$ -cells, it follows that the numbers

$n(X, H) = \sum_{i \geq 0} n(X, H, i)$  are  $G$ -homotopy invariants of  $X$  as

$$n(X, H) = \chi(X^H/NH, X^{>H}/NH).$$

Then  $(\mathbf{Z}, n(X, H))$  is an additive invariant such that

$$n(G/H^+, H) = 1, n(G/K^+, H) = 0 \text{ for } K \text{ not conjugate to } H.$$

This proves the Corollary.

### Example

We conclude with an example taken from Fuchsian groups. Let  $G = PSL(2, \mathbf{R})$  considered as the group of Möbius transformations

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{R}$$

of the complex plane. Let  $H$  be the discrete subgroup of translations

$$h(z) = z + n, \quad n \in \mathbf{Z}.$$

Then it is easy to check that the normalizer  $NH$  of  $H$  in  $G$  consists of translations

$$n(z) = z + b, \quad b \in \mathbf{R},$$

whereas  $SH$  equals the affine transformations

$$s(z) = mz + b, \quad m = 1, 2, \dots, b \in \mathbf{R}.$$

In particular  $NH/H$  is a circle  $S^1$  and  $SH/H = \mathbf{N} \times S^1$ .

Taking  $X = G/H$  gives an example of a finite  $G$ -CW-complex (a zero-cell) with  $X^H/NH$  a countable discrete set and therefore of infinite Euler characteristic.

---

### References

- [tD] T. tom Dieck: Transformation Groups, de Gruyter Studies in Mathematics 8, Walter de Gruyter, Berlin-New York, 1987.
- [H] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, New York-San Francisco-London, 1962.
- [K] K. Kawakubo: The Theory of Transformation Groups, Oxford University Press, Oxford-New York-Tokyo, 1991.
- [LL] E. Laitinen and W. Lück: *Equivariant Lefschetz classes*, Osaka J. Math. 26 (1989), 491–525.

Department of Mathematics,  
P.O. Box 4 (Hallituskatu 15)  
00014 University of Helsinki, Finland