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REGULARITY AT THE BOUNDARY FOR SOLUTIONS OF HYPO-ELLIPTIC EQUATIONS

TADATO MATSUZAWA

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0. Introduction

Peetre [7] considered the Dirichlet problem

\[(0.1) \quad P(x, D)u = f \quad \text{in} \quad x_n > 0 \]
\[(0.2) \quad \frac{\partial^j u}{\partial x_n^j} = 0 \quad \text{on} \quad x_n = 0, \quad 0 \leq j < r. \]

where \( P(x, D) \) is formally hypo-elliptic and \( f \) is infinitely differentiable in \( x_n \geq 0 \). He obtained a sufficient condition in order that every solution \( u \) of the problem (0.1), (0.2) should be infinitely differentiable in \( x_n \geq 0 \), that is, a sufficient condition that the Dirichlet problem (0.1), (0.2) should be hypo-elliptic at the boundary \( x_n = 0 \).

In this paper we shall prove the hypo-analyticity at the boundary \( x_n = 0 \) for the above problem under the same condition on \( P(x, D) \). The proof relies upon mainly the results of Friberg [2] and Schechter [8].

In §1 we give some definitions and state our results. In §2 the proof of Theorem 1.1 is given. §3 is devoted to the proof of Theorem 1.2.

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1. Definitions and Results

1.1. Let \( E^n \) be the \( n \)-dimentional Euclidian space; for convenience set \( x=(x_1, \ldots, x_{n-1}) \), \( y=x_n \) and denote by \( (x, y) \) a point of \( E^n \). The half spaces \( y > 0 \) and \( y \geq 0 \) are denoted by \( E^*_n \) and \( E^*_n \), respectively.

Let \( \alpha=(\alpha_1, \cdots, \alpha_n) \) be a multi-index of non-negative integers with length \(|\alpha|=\alpha_1+\cdots+\alpha_n\). Let \( D_j=\frac{1}{i} \frac{\partial}{\partial x_j}, 1 \leq j \leq n \), and set
We consider a hypo-elliptic differential operator of the form

\[(1.1) \quad P(D) = P(D_x, D_y) = D^m + \sum_{0 \leq j \leq m-1} a_{\beta,j} D^\beta D_x^j, \quad m \geq 1,\]

where the coefficients \(a_{\beta,j}\) are complex numbers and \(p=\text{order of } P(D)\). The polynomial corresponding to \(P(D_x, D_y)\) is

\[(1.2) \quad P(\xi, \eta) = \eta^m + \sum_{0 \leq j \leq m-1} a_{\beta,j}^{\xi,\eta} \eta^j,\]

where \(\xi= (\xi_1, \ldots, \xi_n)^T\). We shall also employ the usual notation

\[P^\alpha(\xi, \eta) = \frac{\partial^{|\alpha|} P(\xi, \eta)}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}},\]

for a multi-index \(\alpha\).

Let the linear differential operator \(P(D)\) with constant coefficients be a hypo-elliptic operator. It is known that there exists a constant \(d \geq 1\) such that

\[(1.3) \quad \sum_{\alpha} |P^\alpha(\xi, \eta)| (1+|\xi| + |\eta|)^{|\alpha|/d} \leq K_1 |P(\xi, \eta)|, \quad |\xi| + |\eta| \geq K_2 \]

for some positive constants \(K_1, K_2\), where \(\xi\) and \(\eta\) are real and \(|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2\).

**Definition 1.1.** If (1.3) holds for a hypo-elliptic operator \(P(D)\), then \(P(D)\) is called a hypo-elliptic operator of type \(d\).

For a hypo-elliptic operator \(P(D)\) the followings are known:

(i) An operator \(P(D)\) is elliptic if and only if it is of type \(d\) for any \(d \geq 1\).

(ii) If a hypo-elliptic operator is of type \(d'\), then for any \(d \geq d'\) it is of type \(d\).

(iii) There are constants \(K_1, K_2\) such that

\[\sum_{\alpha} |P^\alpha(\xi, \eta)| \leq K_1 |P(\xi, \eta)|, \quad |\xi| \geq K_2, \quad \xi \in E^{n-1}.\]

(c.f. Schechter [8], Hypothesis 1.)

(iv) For each real vector \(\xi\) let \(\tau_1(\xi), \ldots, \tau_m(\xi)\) be the roots of \(P(\xi, Z)=0\). The number of \(\tau_m(\xi)\) with positive imaginary parts is constant in the set \(|\xi| \geq K_2\) for \(n>2\). (c.f. [4])

In the case of \(n=2\), we make the following Assumption 1.

1) In a hypo-elliptic operator the coefficients of the highest power of \(\eta\) is independent of \(\xi\). (See Hörmander [3])
Assumption 1. \( P(\xi, \eta) \) is of determined type \( r, 1 \leq r \leq m \). That is, the number \( r \) of roots \( \tau_k(\xi) \) with positive imaginary parts is constant in \( |\xi| \geq K_2 \).

By rearrangement if necessary we assume that

(1.4) \( \text{Im } \tau_k(\xi) > 0, \quad 1 \leq k \leq r \)

(1.5) \( \text{Im } \tau_k(\xi) < 0, \quad r < k \leq m \).

1.2. Set

\[
P_+ = \prod_{k=1}^r (\eta - \tau_k(\xi)), \quad P_- = \frac{P}{P_+}
\]

for a hypo-elliptic operator \( P(D) \) of the form (1.1). We make the following additional assumption.

Assumption 2. Let \( Q(\xi, \eta) \) be any polynomial of degree \( < r \) in \( \eta \). Expand \( Q(\xi, \eta)/P(\xi, \eta) \) in partial fractions:

(1.6) \[
Q(\xi, \eta) = \frac{Q_+(\xi, \eta)}{P_+(\xi, \eta)} + \frac{Q_-(\xi, \eta)}{P_-(\xi, \eta)}.
\]

Then the inequality

(1.7) \[
\int_{-\infty}^{\infty} \left| \frac{Q_-(\xi, \eta)}{P_-(\xi, \eta)} \right|^2 d\eta \leq C \int_{-\infty}^{\infty} \left| \frac{Q_+(\xi, \eta)}{P_+(\xi, \eta)} \right|^2 d\eta
\]

holds in \( |\xi| \geq K_2 \) with some constant \( C \).

This is the condition settled by Peetre [7]. The inequality (1.7) holds whenever \( P(D) \) is an elliptic operator satisfying Assumption 1. (c.f. Peetre [7]). Another example of a hypo-elliptic operator satisfying (1.7) is given by

\[
P(D) = (D_y + iA')(D_y - \Delta'),
\]

where

\[
\Delta' = D_1^2 + \cdots + D_{n-1}^2.
\]

This operator is not quasi-elliptic.

1.3. Let \( C_0^\infty(\mathbb{R}^n) \) be the set of all complex valued functions which are infinitely differentiable in \( \mathbb{R}^n \) and vanish at \( (x, y) \) with \( |x|^2 + y^2 \) sufficiently large. Parseval's formula implies that

(1.8) \[
\int_{-\infty}^{\infty} \int_{|\xi| < \infty} |v(\xi, y)|^2 d\xi dy = \int_{-\infty}^{\infty} \int_{|\xi| < \infty} |v(x, y)|^2 dx dy, \quad v \in C_0^\infty(\mathbb{R}^n),
\]

2) We use the same symbol \( C \) to express different constants.
where \( v(\xi, y) \) is the Fourier transform of \( v(x, y) \) with respect to the variables \( x_1, \ldots, x_{n-1} \):

\[
v(\xi, y) = (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-i\xi \cdot x} v(x, y) \, dx,
\]

where \( \langle \xi, x \rangle = \xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1} \).

A polynomial \( R(\xi, \eta) \) is said to be weaker than \( P(\xi, \eta) \) if there exists a constant \( C(>0) \) such that

\[
|R(\xi, \eta)| \leq C \sum_n |P^n(\xi, \eta)|
\]

for all \( \xi, \eta \). The corresponding operator \( R(D) \) is said to be weaker than \( P(D) \).

By Schechter's result [8] we have easily the following whose proof is omitted here.

**Proposition 1.1** Let \( R(D) \) be any operator weaker than \( P(D) \). Under our assumption on \( P(D) \), there exists a constant \( C \) such that

\[
\|R(D)v, E^n\| \leq C(\|P(D)v, E^n\| + \|v, E^n\|)
\]

for all \( v \in C_\infty(E^n) \) satisfying the Dirichlet condition

\[
D_j^2 v(x, 0) = 0, \quad 0 \leq j \leq r - 1.
\]

**Definition 1.2.** Let \( \Omega \) be a domain in \( \mathbb{E}^n \). We call \( u(x) \) a function of the class \( G(d, d'; \Omega) \) if \( u \) is a \( C^\infty \)-function on \( \Omega \) and if for each compact set \( K \) in \( \Omega \) there exists two constants \( C_0, C_1 \) such that

\[
\|D_\xi^\sigma D_\eta^\tau u(x, y), K\|_\infty \leq C_0 C_1^{\sigma(d+1)} |\sigma|^{d(d+1)} k^d k^\tau
\]

or

\[
\|D_\xi^\sigma D_\eta^\tau u(x, y), K\|_\infty \leq C_0 C_1^{\sigma(d+1)} \Pi_{i=1}^{\sigma-1} (\sigma_i + 1) d^{\sigma_i(k+1)} k^{d'k}
\]

for any \( \sigma (\sigma_n = 0) \) and for any integer \( k (\geq 0) \), where \( \|w, K\|_\infty \) means the essential maximum of \( |w| \) in \( K \). We set \( G(d; \Omega) = G(d, d; \Omega) \).

Let \( \Omega \) be an open set in \( E_n \). It is supposed that the boundary of \( \Omega \) contains an open set \( \omega \) in the plane \( y = 0 \).
Now we can state our results.

**Theorem 1.1.** Let $P(D)$ be a hypo-elliptic operator of the form (1.1) and of type $d \geq p$, satisfying Assumptions 1 and 2. Consider the Dirichlet problem

\begin{align*}
(1.11) & \quad P(D)u(x, y) = f(x, y) \quad \text{in } \Omega \\
(1.12) & \quad \frac{\partial^j u(x, 0)}{\partial y^j} = 0, \quad j = 0, \ldots, r-1 \quad \text{on } \omega
\end{align*}

with $f \in G(d, (p-m+1)d; \Omega \cup \omega)$. Then any function $u \in C^\infty(\Omega \cup \omega)$ satisfying (1.11), (1.12) is a function in $G(d, (p-m+1)d; \Omega \cup \omega)$.

The conclusion of Theorem 1.1 can be extended to operators with variable coefficients. For convenience, assume the origin $(0, 0)$ is contained in the (interior of) plane boundary $\omega$. We now deal with an operator of the form

\begin{equation}
(1.13) \quad P(x, y, D_x, D_y) = D_v^m + \sum_{0 \leq j \leq m-1} \sum_{|\beta| + j \leq p} a_{\beta,j}(x, y) D_x^\beta D_y^j,
\end{equation}

where $a_{\beta,j}(x, y)$ are complex valued functions defined on $\Omega \cup \omega$ and infinitely differentiable. We add following two assumptions on $P$.

**Assumption 3.** $P(x, y, D_x, D_y)$ has constant strength in $\Omega \cup \omega$, that is,

\[
\sum_\alpha |P^\alpha(x, y, \xi, \eta)| \leq C(x, y, x', y')
\]

for $(x, y), (x', y') \in \Omega \cup \omega, (\xi, \eta) \in \mathbb{E}^n$.

**Assumption 4.** Set $P_0(D) = P(0, 0, D_x, D_y)$. Then $P_0(D)$ is a hypo-elliptic operator of type $d \geq p$ of the form

\[
D_v^m + \sum_{0 \leq j \leq m-1} \sum_{|\beta| + j \leq p} a_{\beta,j}(0, 0) D_x^\beta D_y^j
\]

and satisfies Assumptions 1 and 2.

Then we can prove the following

**Theorem 1.2.** Consider the Dirichlet problem

\begin{align*}
(1.14) & \quad P(x, y, D_x, D_y)u(x, y) = f(x, y) \quad \text{in } \Omega \\
(1.15) & \quad D_y^j u(x, 0) = 0, \quad 0 \leq j \leq r-1 \quad \text{on } \omega
\end{align*}

with $f \in G(d, (p-m+1)d; \Omega \cup \omega), a_{\beta,j} \in G(d, (p-m+1)d; \Omega \cup \omega)$, where $d \geq p$. Then any function $u \in H^p(\Omega \cup \omega)$ satisfying (1.14),

3) For the notation $H^p(\Omega \cup \omega)$, see [5].
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(1.15) is a function in $G(d, (p-m+1)d; \Omega \cup \omega)$ for some sufficiently small hemisphere $\Omega \cup \omega = \{(x, y) | x^2 + y^2 \leq r_o, y \geq 0\}$.

In the elliptic case, that is, in the case of type 1 a slight modification of the proof of Morrey-Nirenberg [6] together with the use of the coerciveness estimate obtained in [1] gives the following more detailed and complete theorem.

**Theorem 1.3.** Let $P(x, y, D_x, D_y)$ be a properly elliptic operator defined in $\Omega \cup \omega$ with order $2m$. Consider the Dirichlet problem (1.14), (1.15) with $f \in G(d; \Omega \cup \omega)$ and with all the coefficients in $G(d; \Omega \cup \omega)$ for $d \geq 1$. Then all the solutions $u$ of the problem (1.14), (1.15) are in $G(d; \Omega \cup \omega)$.

2. Proof of Theorem 1.1.

2.1. As a special case of Hörmander’s results [4] we see that any solution $u \in C^m(\Omega \cup \omega)$ of the problem (1.11), (1.12) is infinitely differentiable up to the boundary $\omega$. We shall only estimate the derivatives of the solutions $u$ up to the boundary.

Now take $v \in C^\infty_0(\Omega \cup \omega)$ satisfying the Dirichlet condition (1.12) and regard it as a function in $C^\infty_0(\mathbb{R}^2)$. We consider $v(\xi, y)$ (See (1.8)) as a function of $y \geq 0$ with a vector parameter $\xi$. Following Schechter [8], we let $H^m(E')$ denote the completion of $C^m_0(E')$ with respect to the norm

$$||u||_m = \left( \sum_{\alpha=0}^{m} \int_{-\infty}^{\infty} |D^\alpha_y u(\xi, y)|^2 dy \right)^{1/2}.$$

The first step is to extend $v(\xi, y)$ to the function in $H^m(E')$ by a method due to Morrey-Nirenberg [6], Peetre [7] and Schechter [8].

For $|\xi| \leq K_x$, set

$$v_i(\xi, y) = \begin{cases} v(\xi, y), & y \geq 0 \\ \sum_{k=1}^{m} \lambda_k v(\xi, -ky), & y < 0 \end{cases},$$

where the $\lambda_k$ are constants chosen so that all the derivatives $D^j v$ for $0 \leq j \leq m-1$ are continuous at $y=0$. Here $\lambda_k$ depends only on $m$. It holds that

$$(\xi^\alpha v(\xi, y))_i = \xi^\alpha v_i(\xi, y)$$

for any multi-index $\alpha$ satisfying $\alpha_m = 0$.

Next, for $|\xi| > K_x$, we extend $v(\xi, y)$ by the method due to Schechter [8] and denote the resulting function by $v_i(\xi, y)$. Thus $v_i(\xi, y)$ is defined in $|\xi| < \infty$ and $|y| < \infty$. We also note that it is easily verified that

$$(\xi^\alpha v(\xi, y))_i = \xi^\alpha v_i(\xi, y)$$

for any $\alpha$, $\alpha_m = 0$.

According to the result of Schechter [8] there exists a constant $C$ independent
of \( v \) so that the following inequality holds:

\[(2.1) \quad \int_{-\infty}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy \leq C \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy, \quad |\xi| > K_2.\]

Furthermore, for any \( R(D) \) weaker than \( P(D) \), we can obtain the following inequality

\[(2.2) \quad \int_{-\infty}^{\infty} |R(\xi, D_y) v(\xi, y)|^2 \, dy < C \left\{ \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy + \int_{0}^{\infty} |v(\xi, y)|^2 \, dy \right\}, \quad |\xi| \leq K_2, \quad v(\xi, y) \in C_0^\infty(\mathbb{E}_1^0).\]

Proof of (2.2). For \( |\xi| \leq K_2 \) we have

\[
\int_{0}^{\infty} |D_\xi v(\xi, y)|^2 \, dy \leq \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy + C \sum_{k=0}^{n-1} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy
\]

where \( C_1 \) is an upper bound for the coefficients of \( P(\xi, D_y) \) on the set \( |\xi| \leq K_2 \). Thus

\[
\sum_{k=0}^{n} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy \leq C_1 \left\{ \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy + \sum_{k=0}^{n-1} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy \right\}
\]

On the other hand

\[
\int_{-\infty}^{\infty} |R(\xi, D_y) v_\xi(\xi, y)|^2 \, dy \leq C_2 \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} |D_\xi^k v_\xi(\xi, y)|^2 \, dy, \quad |\xi| \leq K_2,
\]

where \( C_2 \) is an upper bound for the coefficients of \( R(\xi, D_y) \) on the set \( |\xi| \leq K_2 \). Thus, from the construction of \( v_\xi(\xi, y) \) on the set \( |\xi| \leq K_2 \), we have

\[
\int_{-\infty}^{\infty} |R(\xi, D_y) v_\xi(\xi, y)|^2 \, dy \leq C_2 \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} |D_\xi^k v_\xi(\xi, y)|^2 \, dy \leq C_2 \sum_{k=0}^{n} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy
\]

\[
\leq C_6 \left\{ \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy + \sum_{k=0}^{n-1} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy \right\}
\]

Employing the well known inequality

\[
\sum_{k=0}^{n-1} \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy \leq \varepsilon \int_{0}^{\infty} |D_\xi^k v(\xi, y)|^2 \, dy + C(\varepsilon) \int_{0}^{\infty} |v(\xi, y)|^2 \, dy,
\]

and taking \( \varepsilon \) so small that \( \varepsilon C_6 \leq \frac{1}{2} C_4 \), we have

\[
\int_{-\infty}^{\infty} |R(\xi, D_y) v_\xi(\xi, y)|^2 \, dy \leq C_6 \left\{ \int_{0}^{\infty} |P(\xi, D_y) v(\xi, y)|^2 \, dy + \int_{0}^{\infty} |v(\xi, y)|^2 \, dy \right\}
\]

for all \( v(\xi, y) \in C_0^\infty(\mathbb{E}_1^0) \) and \( |\xi| \leq K_2 \), where \( C_6 \) depends only on the coefficients of \( R(\xi, D_y) \) and \( P(\xi, D_y) \) on \( |\xi| \leq K_2 \).
2.2. Now we prove some lemmas for later use.

**Lemma 2.1** *(c.f. Friberg [2])*.

Let \( P(\xi, \eta) \) be hypo-elliptic of type \( d \geq p \). Then, for any \( \varepsilon > 0 \), there exists a constant \( C = C(\varepsilon) \) such that

\[
(2.3) \quad h^{p-|\alpha|+d} | P^a(\xi, \eta) | | \xi^i | \leq \varepsilon h^{p+d} | P(\xi, \eta) | | \xi^i |
+ C(\varepsilon) h^p (| P(\xi, \eta) | + \| P^a(\xi, \eta) \|)
\]

where \( \alpha \neq 0, 0 < h \leq 1, 1 \leq i \leq n-1 \).

The proof is easily obtained by a simplification of that in [2].

**Lemma 2.2** Let \( P(\xi) \) be that in Theorem 1.1 and let \( d \geq p \). Then

\[
(2.4) \quad h^{p-|\alpha|+d} | P^a(\xi, \eta) | | \xi^i | \leq \varepsilon h^{p+d} (| P(\xi) D_{\xi^i} \eta | + | P(\xi) D_{\xi^i} \eta | + | P(\xi) D_{\xi^i} \eta | + | P(\xi) D_{\xi^i} \eta |)
+ C(\varepsilon) h^p (| P(\xi) | + \| P^a(\xi, \eta) \|)
\]

for any \( v \in C^\infty_0(\mathbb{R}^n) \) satisfying the Dirichlet condition (1.12) and \( 0 < h \leq 1 \).

*Proof.* Using the Parseval's formula and the inequalities (2.1), (2.2) with \( R=P \) or \( P^a \) we have

\[
\begin{align*}
& h^{p-|\alpha|+d} | P^a D_{\xi^i} \eta |^2 = h^{p-|\alpha|+d} \int_{|\xi^i| < \varepsilon} \int_0^\infty | P^a(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, d\eta \, d\xi \\
& \leq h^{p-|\alpha|+d} \int_{|\xi^i| < \varepsilon} \int_0^\infty | P^a(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, d\eta \, d\xi = \\
& h^{p-|\alpha|+d} \int_{\mathbb{R}^n} | P^a(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, de \, d\xi \leq \varepsilon h^{p+d} \int_{\mathbb{R}^n} | P(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, de \, d\xi
+ C(\varepsilon) h^p \int_{\mathbb{R}^n} | P^a(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, de \, d\xi
\end{align*}
\]

\[
\begin{align*}
& = \varepsilon h^{p+d} \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | P(\xi, \eta) D_{\xi^i} \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi \\
& + C(\varepsilon) h^p \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | P(\xi, \eta) D_{\xi^i} \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi
\end{align*}
\]

\[
\begin{align*}
& \leq \varepsilon \cdot C h^{p+d} \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | P(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi \\
& + \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | \xi^i \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi \\
& + C(\varepsilon) h^p \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | P(\xi, \eta) \xi^i \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi \\
& + \int_{|\xi^i| < \varepsilon} \left[ \int_0^\infty | \eta(\xi, \eta) |^2 \, d\eta \right] \, d\xi
\end{align*}
\]
which proves Lemma 2.2.

**Lemma 2.3** (c.f. Friberg [2]). For every compact set $K \subset \bar{E}_n^+$ and for every $h>0$, there are a function $\psi=\psi_{K,h}$ and constants $C_a$ independent of $h$ such that $\psi \in C^\alpha(K_h)$, $\psi \equiv 1$ on $K$ and

\[
||D^a\psi||_\infty \leq C_a h^{-|\alpha|}
\]

for every $\alpha$,

where $K_h = \{x \in \bar{E}_n^+ \mid \text{dis.} \ (x, K) \leq h\}$.

This can be shown by Friberg's argument and the proof is omitted here.

From now on, we employ the method developed by Friberg [2] to estimate tangential derivatives. So we introduce some notations used by Friberg in a slightly different way: $V$ will represent the hemisphere \{(x, y)|x^2 + \cdots + x_{n-1}^2 + y^2 < R^2, y > 0\} contained in $\Omega$, and $V_- = \{(x, y)|x^2 + \cdots + x_{n-1}^2 + y^2 < (R-r)^2, y > 0\}$, $0 < r < R$. Let $t$ be a given positive number, and let

\[
(D^a P^a u)_t = t^{\sigma_1 + p - |\alpha|} D^a P^a u, \quad u \in C^\infty(V).
\]

We set for arbitrary $l \geq 0$,

\[
||(D^a P^a u)_t||_{l+d+|\alpha|}, \quad V
\]

\[
= \sup_{0 < r \leq R} r^{l+d+\sigma_1 + p - |\alpha|} ||(D^a P^a u)_t||_{l+d+\sigma_1 + p - |\alpha|}, \quad V_v
\]

\[
2.3. \quad \text{The following lemma is essential in our proof of Theorem 1.1.}
\]

**Lemma 2.4** There exists a constant $C$ such that

\[
\sum_{|\alpha|=0} ||(D^a P^a u)_t||_{l+d+|\alpha|}, \quad V \leq C \{||D_\sigma Pu||_{l+d+|\alpha|}, \quad V \}
\]

\[
+ \sum_{|\alpha|=0} ||(P^a u)_t||_{l+d+|\alpha|}, \quad V \}, \quad 1 \leq i \leq n-1,
\]

for all $u \in C^\infty(\Omega \cup \omega)$ satisfying the Dirichlet condition (1.12), provided that $0 < t \leq \frac{t_0}{l+d}$.

Proof. Let $K$ be a hemisphere \{(x, y)|x^2 + \cdots + x_{n-1}^2 + y^2 \leq r^2 < R^2, y \geq 0\}, contained in $V^*$ ($V^* \equiv V \cup (\bar{V} \cap \omega^{n-1})$), and let $h$ be so small that $K_h \subset V^*$. Then we see by Lemma 2.3 that there is a function $\psi=\psi_{K,h} \in C^\alpha(K_h)$ such that $\psi \equiv 1$ on $K$ and $||D^a\psi||_\infty \leq C_a h^{-|\alpha|}$ for any $\alpha$. Thus for every $u \in C^\infty(V^*)$ satisfying the Dirichlet condition (1.12) the product $v=\psi \cdot u$ belongs to $C^\alpha(K_h)$ and $v$ also satisfies (1.12). So we can apply Lemma 2.2 to $v$. Since $u \equiv v$ on $K$, it follows that for $i$, $1 \leq i \leq n-1$,
By using the Leibniz' formula, we investigate the terms on the right hand side of (2.9).

On the first term we have

\[
P D_i(\psi u) = P(D)(\psi \cdot D_i u + \nabla_i \psi \cdot u) = (P(D)D_i u) \cdot \psi + \sum_{\beta \neq 0} P^\beta (D)D_i u \cdot \frac{D^\beta \psi}{\beta!} +
\]

\[
+ P(D)u \cdot D_i \psi + \sum_{\beta \neq 0} P^\beta u \cdot \frac{D^\beta D_i \psi}{\beta!}.
\]

Hence it follows that

\[
||PD_i(\psi u), K_h|| \leq C(||PD_i u, K_h|| + \sum_{\beta \neq 0} h^{-\beta} ||P^\beta D_i u, K_h|| +
\]

\[
+h^{-1}||Pu, K_h|| + \sum_{\beta \neq 0} h^{-1+\beta} ||P^\beta u, K_h||).
\]

Since \(0 < h \leq 1\), we have

\[
h^{\beta+d}||PD_i(\psi u), K_h|| \leq C(h^{\beta+d}||PD_i u, K_h|| + \sum_{\beta \neq 0} h^{\beta-\beta+d} ||P^\beta D_i u, K_h|| +
\]

\[
+h^{\beta}||Pu, K_h|| + \sum_{\beta \neq 0} h^{\beta-\beta} ||P^\beta u, K_h||).
\]

Similarly for the second term, we get

\[
h^{\beta+d}||D_i(\psi u), K_h|| \leq h^{\beta+d-1}||u, K_h|| + h^{\beta+d}||D_i u, K_h||.
\]

On the third term it holds that

\[
h^\beta ||P(\psi u), K_h|| \leq C(h^\beta ||Pu, K_h|| + \sum_{\beta \neq 0} h^{\beta-\beta} ||P^\beta u, K_h||).
\]

Finally on the fourth term, we obtain

\[
h^\beta ||\psi u, K_h|| \leq h^\beta ||u, K_h||.
\]

These four estimates imply that

(2.10) \[h^{\alpha+\beta+d}||P^\alpha D_i u, K|| \leq \epsilon(h^{\beta+d}||PD_i u, K_h|| + \sum_{\beta \neq 0} h^{\beta-\beta+d} ||P^\beta D_i u, K_h||) +
\]

\[+ C(\epsilon)(h^{\beta}||Pu, K_h|| + \sum_{\beta \neq 0} h^{\beta-\beta} ||P^\beta u, K_h||), \alpha \neq 0.
\]

Now the summation of (2.10) for all \(\alpha \neq 0\) yields

(2.11) \[\sum_{\alpha \neq 0} h^{\alpha+\beta+d}||P^\alpha D_i u, K|| \leq \epsilon \sum_{\alpha \neq 0} h^{\beta-\beta+d} ||P^\alpha D_i u, K_h|| +
\]

\[+ C(\epsilon)(h^{\beta+d}||PD_i u, K_h|| + h^\beta ||Pu, K_h|| + \sum_{\alpha \neq 0} h^{\beta-\beta} ||P^\alpha u, K_h||).
\]
Suppose that \( t_0 \) is so small that \( t_0 \cdot R \leq d \). Let \( h = t_0 r \), where \( 0 < r \leq R \) and \( 0 < t \leq \frac{t_0}{1+d} \). If \( l = 0 \) and if \( r \leq R \), then \( h \leq \frac{t_0 \cdot R}{l+d} \leq 1 \). If, in addition, \( t_0 < 1 \), then \( 0 < r(1-t) \leq R \). Let \( K = V_{-r} \). Then \( K_\delta = V_{-r(1-\delta)} \). We rewrite (2.11) in these notations and get

\[
(2.12) \quad \sum_{\alpha \leq 0} |(rt)^{p-|\alpha|+d}| |P^\alpha D\mu, V_{-r}| \leq \varepsilon \sum_{\alpha \leq 0} |(rt)^{p-|\alpha|+d}| |P^\alpha D\mu, V_{-r(1-\delta)}| + \\
\quad + C(\varepsilon) \left\{ |(rt)^{p+|\alpha|+d}| |PD\mu, V_{-r(1-\delta)}| + \right. \\
\quad \left. + (rt)^{p} |P\mu, V_{-r(1-\delta)}| + \\
\quad \left. + \sum_{\alpha \leq 0} |(rt)^{p-|\alpha|}| |P^\alpha \mu, V_{-r(1-\delta)}| \right\}.
\]

Multiply the above inequality by \( t^n r^n \) (\( n \geq 0 \)). We have

\[
\sum_{\alpha \leq 0} |P^\alpha D\mu, V_{-r}| |(rt)^{l+p-|\alpha|+d}| \leq \varepsilon \sum_{\alpha \leq 0} |P^\alpha D\mu, V_{-r(1-\delta)}| |(r(1-t))^{l+p-|\alpha|+d} \left( \frac{t}{1-t} \right)^{l+p-|\alpha|+d} \\
\quad + C(\varepsilon) \left\{ |PD\mu, V_{-r(1-\delta)}| |(r(1-t))^{l+p+d} \left( \frac{t}{1-t} \right)^{l+p+d} \\
\quad + |P\mu, V_{-r(1-\delta)}| |(r(1-t))^{l+p+d} \left( \frac{r}{1-t} \right)^{l+p-|\alpha|} \right\} \leq \\
\quad \leq \varepsilon \sum_{\alpha \leq 0} |(P^\alpha D\mu)_i; l+p-|\alpha|+d, V| \frac{1}{(1-t)^{l+p-|\alpha|+d}} \\
\quad + C(\varepsilon) \left\{ |(PD\mu)_i; l+p+d, V| \frac{1}{(1-t)^{l+p+d}} + |(P\mu)_i; l+p, V| \frac{1}{(1-t)^{l+p}} \\
\quad + \sum_{\alpha \leq 0} |(P^\alpha \mu)_i; l+p-|\alpha|, V| \frac{1}{(1-t)^{l+p-|\alpha|}} \right\}.
\]

Hence

\[
\sum_{\alpha \leq 0} |(P^\alpha D\mu)_i; l+p-|\alpha|+d, V| \leq \varepsilon \sum_{\alpha \leq 0} |(P^\alpha D\mu)_i; l+p-|\alpha|+d, V| \frac{1}{(1-t)^{l+p-|\alpha|}} \\
\quad + C(\varepsilon) \left\{ |(PD\mu)_i; l+p+d, V| \frac{1}{(1-t)^{l+p+d}} + |(P\mu)_i; l+p, V| \frac{1}{(1-t)^{l+p}} \\
\quad + \sum_{\alpha \leq 0} |(P^\alpha \mu)_i; l+p-|\alpha|, V| \frac{1}{(1-t)^{l+p-|\alpha|}} \right\}.
\]

On the other hand there is a constant \( c > 0 \) such that

\[
\frac{1}{1-t} < e^{ct} \quad \text{for any positive} \quad t \leq t_0 \quad (t_0 < 1),
\]

from which
\[
\left(\frac{1}{1-t}\right)^{l+p-|\alpha|+d} \leq e^{ct_{0}(l+p-|\alpha|+d)} \leq e^{ct_{0}(l+p-|\alpha|+d)} \leq e^{ct_{0}}
\]
and
\[
\left(\frac{1}{1-t}\right)^{l+p-|\alpha|} \leq e^{ct_{0}}.
\]

Hence it follows that
\[
(1-\varepsilon e^{t_{0}}) \sum_{\sigma_{n}=0} \|P^*D_{\alpha}u\|_{l+p-|\alpha|+d, V} \| \leq C(\varepsilon) e^{t_{0}} \{\|P_{D_{\alpha}u}\|_{l+p-d, V} + \|P_{u}\|_{l+p, V} + \sum_{\sigma_{n}=0} \|P^*u\|_{l+p-|\alpha|, V}\}.
\]

By taking \(\varepsilon\) small enough here, we get (2.8).

2.4. Now we need the following notation similar to Friberg [2]:

\[(2.13) \quad A_{i}(P^*D_{\alpha}^*u) = ||(P^*D_{\alpha}^*u)\|_{l+d-|\sigma|+p-|\alpha|, V}, \quad \sigma \leq 1, \quad \sigma_{n}=0,
\]

\[A_{i+1}(P^*u) = \max_{i \geq 0} A_{i}(P^*D_{\alpha}^*u), \quad i \geq 0,
\]

and

\[(2.14) \quad B_{i}(u) = \max_{i \geq 0} A_{i}(P^*u), \quad i \geq 0,
\]

\[(2.15) \quad ||u; d, \lambda; l, V|| \sup_{\sigma_{n}=0} \sum_{i=0}^{\sigma-1} \left(\frac{\lambda}{\sigma_{i}+1}\right)^{d_{i}} \cdot ||D_{\alpha}^*u; l+d-|\sigma|, V||, \quad u \in C^{\infty}(V), \quad \lambda > 0.
\]

We can prove the following

**Theorem 2.1** Let \(P(D)\) and \(d(\geq p)\) be those in Theorem 1.1. Let \(V\) be the same as above and \(l \geq 0\) a given number. Then there are positive constants \(c\) and \(C\) such that

\[(2.16) \quad \sum_{\alpha \geq 0} ||P^*u; d, c; l+p-|\alpha|, V|| \leq C \{||Pu; d, \lambda, l+p, V|| + \sum_{\alpha \geq 0} ||P^*u; l+p-|\alpha|, V||\}
\]

for all \(u \in C^{\infty}(V^*)\) satisfying the Dirichlet condition (1.12).

To prove the theorem we need several lemmas as in Friberg [2].

**Lemma 2.5** Let \(P(D)\) and \(d\) be those in Theorem 1.1. Then there is a constant \(C > 1\) such that

\[(2.17) \quad B_{i}(u) \leq \max_{i \geq 0} \{\max_{i \geq 0} C^{i+1} A_{i}(Pu), C^{j} B_{i}(u)\},
\]
for \( j=1, 2, \cdots \) and for all \( u \in C^\infty(V^*) \) satisfying (1.12), provided that \( 0<t \leq \frac{t_0}{l+d \cdot j} \).

Proof. We note that (2.8) is equivalent to

\begin{equation}
B_0(u) \leq \max \{CA_\delta(Pu), CB_\delta(u)\}
\end{equation}

for some positive constant \( C \). The inequality (2.18) shows that (2.17) is true when \( j=1 \) and \( 0<t \leq \frac{t_0}{l+d} \). Since we can replace \( u \) and the parameter \( l \) in (2.17) by \( \sigma^{|\sigma|} D_\sigma^u \) and by \( l+d |\sigma| \) respectively \((|\sigma| \leq 1, \sigma_n = 0)\), we get

\begin{equation}
B_\delta(u) \leq \max \{CA_\delta(Pu), CB_\delta(u)\}.
\end{equation}

Again by (2.18) we obtain

\begin{equation}
B_\delta(u) \leq \max \{C^2 A_\delta(Pu^2), C^2 B_\delta(u)\}.
\end{equation}

The inequalities (2.19) and (2.20) prove that (2.17) is valid for \( j=2 \), provided that \( 0<t \leq \frac{t}{l+2d} \). Proceeding in this way, we can prove (2.17) for all \( j \).

\textbf{Lemma 2.6} Let \( A_0 \) be defined by (2.13) with \( t = \frac{1}{l+d} \), for \( l \) fixed, and \( t_i \leq t_0 \). Then there are constants \( c<1 \) and \( C_1 \) such that

\begin{equation}
C_1^{-1} ||P^\alpha u; d, c \cdot \lambda; l+p-|\alpha|, V|| \leq \sup C^{-j} A_j(P^\alpha u) \leq C_1 ||P^\alpha u; d, \lambda; l+p-|\alpha|, V||
\end{equation}

for any \( \alpha \), if \( C>1 \) and if

\begin{equation}
\lambda = \frac{t_1}{d \cdot C^{l/d}}.
\end{equation}

Proof. Put \( N=||P^\alpha u; d, \lambda; l+p-|\alpha|, V|| \), where \( \lambda = \frac{t_1}{d \cdot C^{l/d}} \), and suppose that \( t = \frac{t_1}{l+d \cdot j} \). Then

\begin{equation}
\max C^{-j} A_j(P^\alpha u) \leq \max C^{-|\sigma|} \left( \frac{t_1}{l+d \cdot j} \right)^{d(|\sigma|+p-|\alpha|)} ||P^\alpha D_\sigma^u; l+d \cdot j \cdot \sigma|+p-|\alpha|, V|| \leq \max C^{-|\sigma|} \left( \frac{t_1}{l+d \cdot j} \right)^{d(|\sigma|+p-|\alpha|) \cdot \frac{\sum_{i=1}^{\sigma} (\sigma_i+1)}{\lambda} \cdot N} \leq \max \left( \frac{d \cdot j \cdot (j+1)}{l+d \cdot i} \right)^{d \cdot j} \cdot N \leq \left( 1+ \frac{j}{j+1} \right)^{d \cdot j} \cdot N \ll \epsilon^d \cdot N.
\end{equation}
This proves the one half of Lemma 2.6.

Next, by the definition of $A_j(P^su)$ in (2.13), it holds

\[(2.23) \quad ||(P^sD_x)_{l+d\cdot j} - |\alpha|, V|| \leq A_j(P^su) \quad |\sigma| = j, \sigma_n = 0\]

for any $j \geq 0$. Let us put $t = \frac{t_1}{l+d|\sigma|} \frac{t_1}{l+d\cdot i}$. Then (2.23) yields

\[
\left( \frac{t_1}{l+d\cdot j} \right)^{d_s+j+p-|\alpha|} ||P^sD^\sigma_x u; l+d\cdot j+p-|\alpha|, V|| \leq A_j(P^su); |\sigma| = j.
\]

Hence, for $c(>0)$ determined later;

\[
C^{-i} \left( \frac{t_1}{l+d\cdot j} \right)^{d_s+j+p-|\alpha|} \prod_{i=1}^{s-1} \left( \frac{\sigma_i+1}{c\cdot \lambda} \right)^{d_s} \prod_{i=1}^{s-1} \left( \frac{c\lambda}{\sigma_i+1} \right)^{d_s} ||P^sD^\sigma_x u; l+d\cdot j+p-|\alpha|, V|| \leq C^{-i}A_j(P^su).
\]

Substituting $\lambda = \frac{t_1}{l+d\cdot j}$ and noting $C^{-i} = \frac{1}{C^{1/d}|\sigma|}$, we have

\[
\left( \frac{t_1}{l+d\cdot j} \right)^{d_s+j+p-|\alpha|} \prod_{i=1}^{s-1} \left( \frac{d(\sigma_i+1)}{c\cdot \lambda} \right)^{d_s} ||P^sD^\sigma_x u; l+d\cdot j+p-|\alpha|, V|| \leq C^{-i}A_j(P^su).
\]

Put $K = \left( \frac{t_1}{l+d\cdot j} \right)^{d_s+j+p-|\alpha|} \prod_{i=1}^{s-1} \left( \frac{d(\sigma_i+1)}{c\cdot \lambda} \right)^{d_s}$. Then

\[
K^{-1} = \left( \frac{l+d\cdot j}{t_1} \right)^{d_s+j+p-|\alpha|} \prod_{i=1}^{s-1} \left( \frac{c\lambda}{d(\sigma_i+1)} \right)^{d_s} \left( \frac{l+d\cdot j}{t_1} \right)^{d_s} \prod_{i=1}^{s-1} \left\{ \frac{(l+d\cdot i)}{d(\sigma_i+1)} \right\}^{d_s}
\]

is finite if $c$ is sufficiently small. This completes the proof of the lemma.

2.4. Proof of Theorem 2.1. We can now complete the proof of Theorem 2.1. Take the inequality (2.21), divide both side by $C^j$ and put $t = \frac{t_1}{l+d\cdot j}$, $t_1 \leq t_0$. Then we have

\[
C^{-i}B_j(u) \leq \max \{ \max_{0 \leq k \leq j} \{ C^{-k}A_k(u), B_k(u) \} \}.
\]

Therefore, it follows from Lemma 2.6 that

\[(2.24) \quad \max_{\sigma \neq 0} \{ ||P^su; d, \lambda; l+p-|\alpha|, V|| \} \leq C \max \{ ||Pu; d, \lambda; l+p, V||, \max_{\sigma \neq 0} ||P^su; l+p-|\alpha|, V|| \}\]

for all $u \in C^\infty(V^*)$ satisfying (1.12). The inequality (2.24) is equivalent to (2.16). Thus we have Theorem 2.1.

2.5. Now we can prove Theorem 1.1. Let $f(x, y)$ be in $G(d, (p-m+1)$
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Then for any hemisphere \( K = \{(x, y) | x^2 + y^2 \leq r, y \geq 0\} \subset V^* \), there are constants \( C_0, C_1 \) such that \( \|D^2_z f, K\| \leq C_0C_1^{\sigma |d|} |\sigma| |d|^{|d|} \) for any \( \sigma(\sigma_n = 0) \).

If the inequality (2.16) is established once, then it turns out by (2.15) that for the above \( K \) and for a solution \( u \in C^\infty(V^*) \) of the problem (1.11), (1.12), there are new constants \( C_0, C_1 \) such that

\[
(2.25) \quad \|D^\beta_z P^\beta u, K\| \leq C_0C_1^{\sigma |d^\beta|} |\sigma| |d^\beta|^{|d^\beta|}
\]

for any \( \beta \neq 0 \), and for any \( \sigma(\sigma_n = 0) \).

We note \( \frac{\partial^m P(\xi, \eta)}{\partial \eta^m} = m! \). Therefore, (2.25) implies

\[
(2.26) \quad \|D^\beta_z u, K\| \leq C_0C_1^{\sigma |d^\beta|} |\sigma| |d^\beta|^{|d^\beta|}
\]

for new constants \( C_0 \) and \( C_1 \).

Next we note \( \frac{m!P(\xi, \eta)}{\partial \eta^{m-1}} = m! + P_1(\xi) \), where \( P_1(\xi) \) is a polynomial in \( \xi \) only. From (2.25) it follows

\[
(2.27) \quad \|D^\beta_z D^\beta_{\bar{z}} u + D^\beta_z P(D_z)u, K\| \leq C_0C_1^{\sigma |d^\beta|} |\sigma| |d^\beta|^{|d^\beta|}.
\]

On the other hand, again by the inequality (2.16) and Fribreg's results (Ch. 2 in [2]) for new constants \( C_0, C_1 \), we obtain

\[
\|D^\beta_z P(D_z)u, K\| \leq C_0C_1^{\sigma |d^\beta|} |\sigma| |d^\beta|^{|d^\beta|}.
\]

Hence we have for new constants \( C_0, C_1 \),

\[
(2.28) \quad \|D^\beta_z D^\beta_{\bar{z}} u, K\| \leq C_0C_1^{\sigma |d^\beta|} |\sigma| |d^\beta|^{|d^\beta|}
\]

for any \( \sigma(\sigma_n = 0) \).

Repeating the process \( m \) times we can obtain for new constants \( C_0, C_1 \)

\[
(2.29) \quad \|D^\beta_z D^\beta_{\bar{z}} u, K\| \leq C_0C_1^{\sigma |d^\beta|}, 0 \leq j \leq m - 1,
\]

for any \( \sigma(\sigma_n = 0) \).

Thus we may assume that for some constants \( C_0, C_1 \), \( \geq 1 \)

\[
(2.30) \quad \|D^\beta_z D^\beta_{\bar{z}} u, K\| \leq C_0C_1^{\sigma |d^\beta|}, 0 \leq j \leq m - 1,
\]

for any \( \sigma(\sigma_n = 0) \) and for any \( \beta, |\beta| \leq \rho \).

\[
\|D^\beta_z D^\beta_{\bar{z}} u, K\| \leq C_0C_1^{\sigma |d^\beta| + k} |\sigma| |d^\beta| |k|^{\rho - m + 1} |d|^{|d|} \]

for any \( \sigma(\sigma_n = 0) \) and for any \( k \).

Now the equation \( P(D)u = f \) can be written in the form

\[
(2.31) \quad D^\beta_z u = \sum_{0 \leq j < m - 1} a_{\beta, j} D^\beta_{\bar{z}} D^\beta_z u + f.
\]

Put \( 1 + \sum |a_{\beta, j}| = B \geq \rho \). Differentiating (2.30) with respect to \( x \)-variables and applying (2.30), we have
Again differentiating (2. 31) we have

\[ D_x^p D_y q u = - \sum_{j=m-1}^{j-1} a_{\beta_j} D_x^p D_y q u + \sum_{j=m-2}^{j-2} a_{\beta_j} D_x^p D_y q u + D_x^p D_y q f, \]

where we consider \( a_{\beta_j} = 0 \) when \( j < 0 \). Applying (2. 32) we have

\[ (2. 33) \quad ||D_x^p D_y q u, K|| \leq B \cdot C_q C_1^{|\sigma| + p - m + 1} \cdot \sigma \cdot d^{\sigma} + B \cdot C_q C_1^{|\sigma| + p - m + 1} \cdot \sigma \cdot d^{\sigma}. \]

Repeating the procedure we can obtain by a simple induction argument on \( k \)

\[ (2. 34) \quad ||D_x^p D_y q u, K|| \leq (B + 1)^{k+1} C_q C_1^{|\sigma| + k + p - m + 1} \cdot \sigma \cdot d^{\sigma} + (B + 1)^{k+1} C_q C_1^{|\sigma| + k} \cdot \sigma \cdot d^{\sigma} + \ldots \]

for all \( k, 0 \leq k \leq m \).

Suppose now that (2. 34) holds for any \( k \leq k_0 \leq m \). Since

\[ (2. 35) \quad D_x^p D_y q u = - \sum_{j=m-1}^{j-1} a_{\beta_j} D_x^p D_y q u + \ldots \]

we have by (2. 34)

\[ (2. 36) \quad ||D_x^p D_y q u, K|| \leq B \cdot (B + 1)^{k_0+1} C_q C_1^{|\sigma| + k_0 + 1} \cdot (p - m + 1) + \ldots \]

Since

\[ \sum_{i=1}^{k_0+1} (B + 1)^i \cdot C_q C_1^{|\sigma| + k_0 + 1} \cdot (p - m + 1) + \ldots \]

we have by (2. 34)
Hence we arrive at the conclusion that there are two constants $C_0$, $C_1$ such that

$$
(I \sigma + (k_0 + 1)(p-m+1)+m)
$$

$$
+ (B+1)^{k_0+2} C_1 |\sigma|^{d|\sigma|+(k_0+1)(p-m+1)+m}
$$

for any $\sigma(\sigma_n=0)$ and for any $k$.

We apply the Sobolev's Lemma to the inequality (2.37) and obtain Theorem 1.1. We omit the details here. (c.f. Friberg [2], Lemma 2.2.2.)

3. Proof of Theorem 1.2

3.1 The proof can be obtained in a quite similar manner to the proof of Theorem 1.1 by applying the method developed by Friberg for the formally hypo-elliptic equations (Ch. 4 in [2]).

Lemma 3.1 Let $Q(D)$ be a linear differential operator with constant coefficients weaker than $P_0(D)=P(0, 0, D_x, D_y)$. Let $p$ be the order of $P_0(D)$. Then it holds

$$
(t^{d|\sigma|+p}) |D_x^\sigma P_0 u| l + d |\sigma| + p, V |
$$

$$
\leq C \sum_{s} t^{d|\sigma|+p-|\sigma|} |D_x^\sigma P_0 u| l + d + p - |\alpha|, V |
$$

for all $u \in C^\omega(V^*)$ satisfying (1.12), all $d \geq 1$, all $\sigma \geq 0$ ($\sigma_n=0$), all $l>0$, and for all $t$ with $0 \leq t \leq \frac{t_0}{l + d |\sigma| + p}$.

The proof is omitted as it is simpler than that of Lemma 2.4.

Now by the assumption on $P(x, y, D_x, D_y)$ in Theorem 1.2, $P(x, y, D_x, D_y)$ can be written as

$$
P(x, y, D_x, D_y) = P_0(D_x, D_y) + \sum_{i} C_i(x, y) P_0(D_x, D_y),
$$

where $P_0(D)$ is of type $d(\geq p=\text{order of } P_0)$ of the form (1.1) and satisfies assumptions of Theorem 1.1 and further all the $P_0$, are weaker than $P_0$. The coefficients $C_i$ belong to $G(d, (p-m+1)d; \Omega \cup \omega)$, and

$$
|C_i(x, y)| = 0(|x| + y), \quad \text{when } |x| + y \to 0.
$$

Lemma 3.2 (c.f. Lemma 2.4) Let $P(x, y, D_x, D_y)$ be that given in Theorem 1.2, and $\varepsilon>0$ a given number. Set $p=\text{order of } P_0$. Then there exist a hemisphere
$V_0 = \{(x, y) | |x|^2 + y^2 \leq r, y > 0\} \subset V$ and constants $t_0, C$ such that

(3.4) \[
\max_{|\sigma| \leq 1} \sum_{\sigma_n=0} \mathcal{E}^{|\sigma|+|\rho|}||D_{x}^2 P_{n} u; l+d |\sigma| + p, V_0|| \leq \]
\[
C \max_{|\sigma| \leq 1} \sum_{\sigma_n=0} \mathcal{E}^{|\sigma|+|\rho|}||D_{x}^2 P_{n} u; l+d |\sigma| + p, V_0|| +
\]
(3.5) \[
\mathcal{E}^{\sum_{\sigma_n=0} \mathcal{E}^{|\sigma|+|\rho|}||D_{x}^2 P_{n} u; l+d |\sigma| + p, V_0||}
\]
for all $u \in C^\infty(V^\ast)$ satisfying the Dirichlet condition $1.12$ and for all $l \geq 0$ and $0 < t \leq \frac{t_0}{l+d+p}$.

Proof. Set

$A(D_{x}^2 P_{n} u) = \mathcal{E}^{|\rho|}||D_{x}^2 P_{n} u; l+d |\sigma| + p, V_0||$.

Then it follows from (3.2) that

(3.5) \[
A(D_{x}^2 P_{n} u) \leq A(D_{x}^2 P_{n} u) + \sum_{\sigma_n=0} A(D_{x}^2 (C_{n} P_{n} u)) .
\]

For $\sigma, |\sigma| = 1$ ($\sigma_n = 0$)

$A(D_{x}^2 C_{n} P_{n} u) \leq \mathcal{E}^{||D_{x}^2 C_{n} ; d |\sigma| , V_0||} \cdot A(P_{n} u)$
\[
+ ||C_{n} ; 0, V_0|| \cdot A(D_{x}^2 P_{n} u) .
\]

Now let $t = \frac{t_0}{l+d+p}$, with $0 < t \leq t_0$, and take $\mu$ so small that $C_{n} \subset C^\infty(d, \mu ; 0, V_0)$

(For notation $G_{\infty}(d, \mu ; 0, V_0)$, see Ch. 2, in [2]). Then

$\mathcal{E}^{||D_{x}^2 C_{n} ; d |\sigma| , V_0||}$
\[
\leq \Pi \left\{ \frac{t \sigma_{i} + 1}{\mu(l+d+p)} \right\}^{d_{i}} ||C_{i} ; d, \mu ; 0, V_0||_{\infty}
\]
so that

$\mathcal{E}^{||D_{x}^2 C_{n} ; d |\sigma| , V_0||}$
\[
\leq C \Pi \left\{ \frac{t \sigma_{i} + 1}{d} \right\}^{d_{i}},
\]
if $t_0 \leq \varepsilon \mu$. Since $d$ is always $\geq 1$, this shows that

$\sum_{|\sigma| = 0} \mathcal{E}^{||D_{x}^2 C_{n} ; d |\sigma| , V_0||} = 0(\varepsilon)$,

as $\varepsilon$ tends to zero. But $||C_{n} ; 0, V_0||$ can be made as small as we want by taking $V_0$ sufficiently small. (See (3.2)). We have

(3.6) \[
A(D_{x}^2 C_{n} P_{n} u) \leq C_{n} \mathcal{E} \max_{|\sigma| \leq 1} A(D_{x}^2 P_{n} u),
\]
provided that \( C \in G_\infty(d, \mu; 0, V_0) \), \( t \leq \frac{t_0}{l+d+p} \) \((t_0 \leq \varepsilon \mu) \) and \( V_0 \) is sufficiently small. Now let us use Lemma 3.1, with \( Q = P_\sigma \) and with \( V_0 \) instead of \( V \). Then we get

\[
(3.7) \quad A(D^\sigma_\sigma P_\mu u) \leq C \sum_\sigma A(D^\sigma_\sigma P_\sigma u), \quad \text{for any } \sigma(\sigma = 0).
\]

Thus, in view of (3.5), (3.6) and (3.7),

\[
A(D^\sigma_\sigma P_\sigma u) \leq A(D^\sigma_\sigma P_\sigma u) + C_\varepsilon \max \sum_\sigma A(D^\sigma_\sigma P_\sigma u),
\]

for any \( \sigma(|\sigma| \leq 1, \sigma = 0) \), if \( t = \frac{t_1}{l+d+p} \), \( t_1 < t_0 \), and if \( t_0 \) and \( V_0 \) are sufficiently small. This means also that

\[
\max_{|\sigma| \leq 1, (\sigma = 0)} A(D^\sigma_\sigma P_\sigma u) \leq \max_{|\sigma| \leq 1, (\sigma = 0)} A(D^\sigma_\sigma P_\sigma u) + C_\varepsilon \max \sum_\sigma A(D^\sigma_\sigma P_\sigma u).
\]

Suppose now that \( C_\varepsilon \leq \frac{1}{2} \). Then \( 0 < \varepsilon = \frac{C_\varepsilon \varepsilon}{1 - C_\varepsilon} \leq 1 \) and we get

\[
(3.8) \quad \max_{|\sigma| \leq 1, (\sigma = 0)} A(D^\sigma_\sigma P_\sigma u) \leq 2 \max_{|\sigma| \leq 1, (\sigma = 0)} A(D^\sigma_\sigma P_\sigma u) + \varepsilon \max \sum_\sigma A(D^\sigma_\sigma P_\sigma u).
\]

Obviously, (3.8) and (3.4) are equivalent.

Let us define \( A_i(P_\sigma u) \) in terms of \( A_\sigma(D^\sigma_\sigma P_\sigma u) \) as in (2.13). Then it follows from (3.8) (or (3.3)) that for an arbitrary \( \varepsilon > 0 \)

\[
(3.9) \quad A_i(P_\sigma u) \leq C_i A_i(P_\sigma u) + \varepsilon \sum_\sigma A_i(P_\sigma u), \quad \text{for any } i \geq 0,
\]

under the usual conditions on \( u, l, t \) and \( V_0 \). We can also apply Lemma 2.5 to \( P_\sigma \) and obtain the estimate

\[
(3.10) \quad \max_{\sigma \neq 0} A_j(P_\sigma u) \leq \max \left\{ \max_{t+1=0} C^{t+1} A_k(P_\sigma u), C^t \sum_\sigma A_i(P_\sigma u) \right\}
\]

for \( j = 1, 2, \cdots \), and for all \( t \) with \( 0 < t \leq \frac{t_0}{l+d+j} \). From (3.9), we see that (3.10) can be replaced by

\[
(3.11) \quad \max_{\sigma \neq 0} A_j(P_\sigma u) \leq C \max \left\{ \max_{t+1=0} C^{t+1} A_k(P_\sigma u) C^t \sum_\sigma A_i(P_\sigma u) \right\}
\]

for \( j = 1, 2, \cdots \), and for \( 0 < t \leq \frac{t_0}{l+d+j} \), if \( t_0 \) and \( V_0 \) are sufficiently small.
3.2. As a simple application of Lemma 2.6, we can prove the following.

**Theorem 3.1** Let \( P(x, y, D_x, D_y) \) be given as in Theorem 1.2 which satisfies the prescribed condition. Then there are positive constants \( c<1 \) and \( C \) such that

\[
\sum_{\sigma \neq 0} \| P \sigma u; d, c \mu; l + p - |\alpha|, V_0 \| \leq C \left\{ \| Pu; d, \lambda; l + p, V_0 \| + \right. \\
+ \sum_{\alpha \in \mathbb{Z}} \| P \sigma u; l + p - |\alpha|, V_0 \| \}
\]

for all \( u \in C^\infty(V^*) \) satisfying the Dirichlet condition (1.12) and for all \( \lambda > 0 \), provided that \( V_0 = \{(x, y) | x^2 + y^2 < r_0, y > 0 \} \) is a sufficiently small hemisphere.

Similarly to the proof of Theorem 1.1, if the inequality (3.12) is obtained, then from the assumption \( f \in G(d, (p - m + 1)d, V^*) \) and by (2.15) we may assume that for any solution \( u \) of (1.14), (1.15), there are positive constants \( C_0, C_1(\geq 1) \) such that

\[
\| D_{x}^{\alpha}D_{y}^{\beta}D_{x}^{m}u, V_0 \| \leq C_0 C_1^{\sigma - p} |\sigma|^{d[p]} \sigma \| \leq \rho \right. \\
+ \sum_{\alpha \in \mathbb{Z}} \| D_{x}^{\alpha}D_{y}^{\beta}D_{x}^{m}u, V_0 \| \leq C_0 C_1^{\sigma - p} |\sigma|^{d[p]} \sigma \| \leq \rho \}
\]

and

\[
\| D_{x}^{\alpha}D_{y}^{\beta}D_{x}^{m}D_{y}^{n}u, V_0 \| \leq C_0 C_1^{\sigma - p} |\sigma|^{d[p]} \sigma \| \leq \rho \right. \\
+ \sum_{\alpha \in \mathbb{Z}} \| D_{x}^{\alpha}D_{y}^{\beta}D_{x}^{m}D_{y}^{n}u, V_0 \| \leq C_0 C_1^{\sigma - p} |\sigma|^{d[p]} \sigma \| \leq \rho \}
\]

where we put \( p_0 = p - m + 1 \) (\( \geq 1 \)).

Now we can assume \( d > 1 \).\(^4\) Rewrite the equation \( P(x, y, D_x, D_y)u = f \) in the form

\[
D_{x}^{\alpha}u = - \sum_{0 \leq j \leq m - 1} a_{x, j}(x, y) D_{x}^{\alpha}D_{x}^{m}u + f.
\]

We differentiate (3.14) with respect to \( x \)-variables and get

\[
D_{x}^{\alpha}D_{x}^{\alpha}u = - \sum_{0 \leq j \leq m - 1} D_{x}^{\alpha}(a_{x, j}D_{x}D_{y}^{m}u) + D_{x}^{\alpha}f.
\]

Consider each term

\[
D_{x}^{\alpha}(a_{x, j}D_{x}D_{y}^{m}u) = \sum_{\rho \in \mathbb{Z}} \left( \begin{array}{c} \sigma \\ \rho \end{array} \right) D_{x}^{\alpha - \rho} a_{x, j} \cdot D_{x}^{\alpha}D_{y}^{m}D_{x}^{\alpha}u
\]

in the summation. By (3.13) we see

\[
\| D_{x}^{\alpha}(a_{x, j}D_{x}D_{y}^{m}u), V_0 \| \leq \sum_{\rho \in \mathbb{Z}} \left( \begin{array}{c} \sigma \\ \rho \end{array} \right) C_0 C_1^{\sigma - p} |\sigma - \rho|^{d[p]} C_0 C_1^{\sigma - p} |\rho|^{d[p]}.
\]

Now we use the following simple inequalities

\(^4\) We note that all the hypo-elliptic operators of first order and of type 1 are not of determined type.
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\[(3.16) \quad \binom{k}{j}(k-j)^{k-j}j^j \leq k^k \quad \text{for integers } j, k, 0 \leq j \leq k,\]

\[(3.17) \quad \binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|} \quad \text{for } \beta \leq \alpha.\]

For any \(b > 0\), there is a constant \(C' = C'(b, n)\) independent of \(\alpha\) such that

\[(3.18) \quad \sum_{\beta \leq \alpha} \binom{\alpha}{\beta}^{-b} \leq C'.\]

Thus we have

\[(3.19) \quad \|D_\theta^a(a_\beta, jD_\theta D_\theta u), V_\theta \| \leq C^a C_\theta^2 C_\tau^{|\beta|+|\sigma|} |\sigma| \|D_\theta^a u, 0 \leq j \leq m-1, \]

with \(b = d - 1\) and

\[(3.20) \quad \|D_\theta^a D_\theta^b u, V_\theta \| \leq NC'C_\theta^2 C_\tau^{|\beta|+|\sigma|} + C_\theta C_\tau^{|\beta|+|\sigma|},\]

where \(N\) is the number of terms of \(P(x, y, D_x, D_y)u\).

Again differentiating (3.14) we have

\[D_\theta^a D_\theta^b D_\theta^c D_\theta^d D_\theta^e u = - \sum_{j=-m+1}^{m-1} D_\theta^a D_\theta^b (a_\beta, jD_\theta D_\theta^e D_\theta^f u) + \sum_{j=-m+1}^{m-1} D_\theta^a D_\theta^b (a_\beta, jD_\theta D_\theta^e D_\theta^f u) +
\]

\[+ D_\theta^a D_\theta^b D_\theta^c D_\theta^d D_\theta^e D_\theta^f f,\]

where we put \(a_\beta, j = 0\) for \(j < 0\). Consider again each term of the first summation

\[D_\theta^a D_\theta^b (a_\beta, m-1, D_\theta^m D_\theta^e D_\theta^f u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D_\theta^a D_\theta^b a_\beta, m-1, D_\theta^m D_\theta^e D_\theta^f D_\theta^e D_\theta^f u, \alpha = \sigma + (0', 1).\]

By (3.13) and (3.20) we have

\[\|D_\theta^a D_\theta^b (a_\beta, m-1, D_\theta^m D_\theta^e D_\theta^f u), V_\theta \| \leq BC'C_\theta^2 C_\tau^{|\beta|+|\sigma|+|\rho|} |\sigma| \|D_\theta^a D_\theta^b D_\theta^c D_\theta^d D_\theta^e D_\theta^f u, 0 \leq m \leq m,\]

Hence we obtain

\[(3.21) \quad \|D_\theta^a D_\theta^b D_\theta^c D_\theta^d D_\theta^e D_\theta^f u, V_\theta \| \leq (B+1) C_\theta C_\tau^{|\beta|+|\sigma|+|\rho|} |\sigma| \|D_\theta^a D_\theta^b D_\theta^c D_\theta^d D_\theta^e D_\theta^f u, 0 \leq m \leq m,\]

Thus, using the inequalities (3.16), (3.17), (3.18) and the estimates (3.20) (3.21), we can repeat the procedure similar to that in the proof of Theorem 1.1. So, the proof of Theorem 1.2 is obtained.

We omit the proof of Theorem 1.3.
4. Remark. In the case when $m=1$, we can improve Theorem 1.1 in the following form.

Let $P(D)$ be a hypo-elliptic operator of the form

$$P(D) = D_\gamma + \sum_{|\beta| \leq \gamma} a_\beta D_\beta^\beta,$$

satisfying Assumptions 1 and 2. Furthermore let $P(D)$ be a hypo-elliptic operator of type $d(\geq 1)$ in $x$, that is, there exists a constant $C$ independent of real $\xi$ and $\eta$ such that

$$\sum_a |P^a(\xi, \eta)| \left(1 + |\xi| \right)^{m/d} \leq C(1 + |\xi|)^{m/d}.$$

Then any function $u \in C^d(\Omega \cup \omega)$ satisfying (1.11), (1.12) with $f \in G(d, pd; \Omega \cup \omega)$ is also a function in $G(d, pd; \Omega \cup \omega)$.

In Theorem 1.2, the similar to the above is true.

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References