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REGULARITY AT THE BOUNDARY FOR SOLUTIONS OF HYPO-ELLIPTIC EQUATIONS

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0. Introduction

Peetre [7] considered the Dirichlet problem

$$(0.1) \qquad P(x, D)u = f \qquad \text{in} \quad x_n > 0$$

$$(0.2) \qquad \frac{\partial^j u}{\partial x_n{}^j} = 0 . \qquad \text{on} \quad x_n = 0,\ 0 \leqq j < r .$$

where $P(x, D)$ is formally hypo-elliptic and f is infinitely differentiable in $x_n \geqq 0$. He obtained a sufficient condition in order that every solution u of the problem (0.1), (0.2) should be infinitely differentiable in $x_n \geqq 0$, that is, a sufficient condition that the Dirichlet problem (0.1), (0.2) should be hypo-elliptic at the boundary $x_n = 0$.

In this paper we shall prove the hypo-analyticity at the boundary $x_n = 0$ for the above problem under the same condition on $P(x, D)$. The proof relies upon mainly the results of Friberg [2] and Schechter [8].

In §1 we give some difinitions and state our results. In §2 the proof of Theorem 1.1 is given. §3 is devoted to the proof of Theorem 1.2.

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1. Difinitions and Results

1.1. Let E^n be the n-dimentional Euclidian space; for convenience set $x = (x_1, \cdots, x_{n-1})$, $y = x_n$ and denote by (x, y) a point of E^n. The half spaces $y > 0$ and $y \geqq 0$ are denoted by E_+^n and $\bar{E}_+^n$, respectively.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index of non-negative integers with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, $1 \leqq j \leqq n$, and set

$$D_x=(D_1,\cdots,D_{n-1}),\ D_y=D_n,\ D=(D_1\cdots,D_n)\,.$$

We consider a hypo-elliptic differential operator of the form

$$(1.1)\qquad P(D)=P(D_x,D_y)=D_y^m+\sum_{\substack{0\leqq j\leqq m-1\\ |\beta|+j\leqq p}} a_{\beta,j}D_x^\beta D_y^j,\ m\geqq 1\,,$$

where the coefficients $a_{\beta,j}$ are complex numbers and p=order of $P(D)$. The polynomial corresponding to $P(D_x, D_y)$ is

$$(1.2)\qquad P(\xi,\eta)=\eta^m+\sum_{\substack{0\leqq j\leqq m-1\\ |\beta|+j\leqq p}} a_{\beta,j}\xi^\beta\eta^j\,,$$

where $\xi=(\xi_1,\cdots,\xi_{n-1})$[1]. We shall also employ the usual notation

$$P^{\alpha}(\xi,\eta)=\frac{\partial^{|\alpha|}P(\xi,\eta)}{\partial\xi_1^{\alpha_1}\cdots\partial\xi_{n-1}^{\alpha_{n-1}}\partial\eta^{\alpha_n}}$$

for a multi-index α.

Let the linear differential operator $P(D)$ with constant coefficients be a hypo-elliptic operator. It is known that there exists a constant $d\geqq 1$ such that

$$(1.3)\qquad \sum_\alpha |P^\alpha(\xi,\eta)|(1+|\xi|+|\eta|)^{|\alpha|/d}\leqq K_1|P(\xi,\eta)|,\ |\xi|+|\eta|\geqq K_2$$

for some positive constants K_1, K_2, where ξ and η are real and $|\xi|^2=\xi_1^2+\cdots+\xi_{n-1}^2$.

DEFINITION 1.1. If (1.3) holds for a hypo-elliptic operator $P(D)$, then $P(D)$ is called a hypo-elliptic operator of type d.

For a hypo-elliptic operator $P(D)$ the followings are known:

(i) An operator $P(D)$ is elliptic if and only if it is of type d for any $d\geqq 1$.

(ii) If a hypo-elliptic operator is of type d', then for any $d\geqq d'$ it is of type d.

(iii) There are constans K_1, K_2 such that

$$\sum_\alpha P^\alpha(\xi,\eta)|\leqq K_1|P(\xi,\eta)|,\qquad |\xi|\geqq K_2,\ \xi\in E^{n-1}\,.$$

(c.f. Schechter [8], Hypothesis 1.)

(iv) For each real vector ξ let $\tau_1(\xi),\cdots,\tau_m(\xi)$ be the roots of $P(\xi, Z)=0$.

The number of $\tau_k(\xi)$ with positive imaginary parts is constant in the set $|\xi|\geqq K_2$ for $n>2$. (c.f. [4])

In the case of $n=2$, we make the following Assumption 1.

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1) In a hypo-elliptic operator the coefficients of the highest power of η is independent of ξ. (See Hörmander [3])

Assumption 1. $P(\xi, \eta)$ is of determined type r, $1 \leqq r \leqq m$. That is, the number r of roots $\tau_k(\xi)$ with positive imaginary parts is constant in $|\xi| \geqq K_2$. By rearrangement if necessary we assume that

$$(1.4) \qquad Im\ \tau_k(\xi) > 0, \qquad 1 \leqq k \leqq r$$

$$(1.5) \qquad Im\ \tau_k(\xi) < 0, \qquad r < k \leqq m .$$

1.2. Set

$$P_+ = \prod_{k=1}^{r} (\eta - \tau_k(\xi)), \qquad P_- = P/P_+$$

for a hypo-elliptic operator $P(D)$ of the form (1.1). We make the following additional assumption.

Assumption 2. Let $Q(\xi, \eta)$ be any polynomial of degree $< r$ in η. Expand $Q(\xi, \eta)/P(\xi, \eta)$ in partial fractions:

$$(1.6) \qquad \frac{Q(\xi, \eta)}{P(\xi, \eta)} = \frac{Q_+(\xi, \eta)}{P_+(\xi, \eta)} + \frac{Q_-(\xi, \eta)}{P_-(\xi, \eta)} .$$

Then the inequality

$$(1.7) \qquad \int_{-\infty}^{\infty} \left| \frac{Q_-(\xi, \eta)}{P_-(\xi, \eta)} \right|^2 d\eta \leqq C \int_{-\infty}^{\infty} \left| \frac{Q_+(\xi, \eta)}{P_+(\xi, \eta)} \right|^2 d\eta$$

holds in $|\xi| \geqq K_2$ with some constant C.[2)]

This is the condition settled by Peetre [7]. The inequality (1.7) holds whenever $P(D)$ is an elliptic operator satisfying Assumption 1. (c.f. Peetre [7]). Another example of a hypo-elliptic operator satisfying (1.7) is given by

$$P(D) = (D_y + i\Delta'^2)(D_y - \Delta') ,$$

where

$$\Delta' = D_1^2 + \cdots + D_{n-1}^2 .$$

This operator is not quasi-elliptic.

1.3. Let $C_0^\infty(\bar{E}_+^n)$ be the set of all complex valued functions which are infinitely differentiable in $\bar{E}_+^n$ and vanish at (x, y) with $|x|^2 + y^2$ sufficiently large. Parseval's formula implies that

$$(1.8) \qquad \|v, \bar{E}_+^n\| = \left(\int_0^\infty \int_{|x|<\infty} |v(x, y)|^2 dx\, dy \right)^{1/2} = \left(\int_0^\infty \int_{|\xi|<\infty} |v(\xi, y)|^2 d\xi\, dy \right)^{1/2} ,$$
$$v \in C_0^\infty(\bar{E}_+^n) ,$$

2) We use the same symbol C to express different constants.

where $v(\xi, y)$ is the Fourier transform of $v(x, y)$ with respect to the variables $x_1, \cdots, x_{n-1}$:

$$v(\xi, y) = (2\pi)^{-(n-1)/2} \int_{E^{n-1}} e^{-i<\xi,x>} v(x, y)\, dx ,$$

where $\langle \xi, x \rangle = \xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1}$.

A polynomial $R(\xi, \eta)$ is said to be weaker than $P(\xi, \eta)$ if there exists a constant $C(>0)$ such that

$$|R(\xi, \eta)| \leqq C \sum_{\alpha} |P^{\alpha}(\xi, \eta)|$$

for all real ξ, η. The corresponding operator $R(D)$ is said to be weaker than $P(D)$.

By Schechter's result [8] we have easily the following whose proof is omitted here.

Proposition 1.1 *Let $R(D)$ be any operator weaker than $P(D)$. Under our assumption on $P(D)$, there exists a constant C such that*

$$(1.9) \qquad \|R(D)v, \bar{E}^n_+\| \leqq C(\|P(D)v, \bar{E}^n_+\| + \|v, \bar{E}^n_+\|)$$

for all $v \in C^{\infty}_0(\bar{E}^n_+)$ satisfying the Dirichlet condition

$$D^j_y v(x, 0) = 0, \qquad 0 \leqq j \leqq r-1 .$$

DEFINITION 1. 2. Let Ω be a domain in E^n. We call $u(x)$ a function of the class $G(d, d'; \Omega)$ if u is a C^∞-function on Ω and if for each compact set K in Ω there exists two constants C_0, C_1 such that

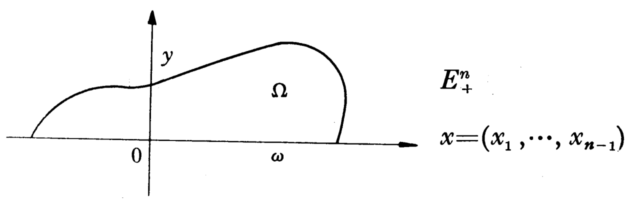
$$(1.10) \qquad \|D^{\sigma}_x D^k_y u(x, y), K\|_\infty \leqq C_0 C_1^{|\sigma|+k} |\sigma|^{d|\sigma|} k^{d'k}$$

or

$$(1.10') \qquad \|D^{\sigma}_x D^k_y u(x, y), K\|_\infty \leqq C_0 C_1^{|\sigma|+k} \prod_{i=1}^{n-1} (\sigma_i + 1)^{d\sigma_i} (k+1)^{d'k}$$

for any σ ($\sigma_n = 0$) and for any integer k ($\geqslant 0$), where $\|w, K\|_\infty$ means the essential maximum of $|w|$ in K. We set $G(d; \Omega) = G(d, d; \Omega)$.

Let Ω be an open set in E^n_+. It is supposed that the boundary of Ω contains an open set ω ($\neq \phi$) in the plane $y=0$.



Now we can state our results.

Theorem 1. 1. *Let $P(D)$ be a hypo-elliptic operator of the form* (1. 1) *and of type $d \geqq p$, satisfying Assumptions* 1 *and* 2. *Consider the Dirichlet problem*

$$P(D)u(x, y) = f(x, y) \quad \text{in } \Omega \tag{1. 11}$$

$$\frac{\partial^j u(x, 0)}{\partial y^j} = 0\,, \quad j = 0\,,\cdots, r-1 \text{ on } \omega \tag{1. 12}$$

with $f \in G(d, (p-m+1)d; \Omega \cup \omega)$. Then any function $u \in C^p(\Omega \cup \omega)$ satisfying (1. 11), (1. 12) *is a function in $G(d, (p-m+1)d; \Omega \cup \omega)$.*

The conclusion of Theorem 1. 1 can be extended to operators with variable coefficients. For convenience, assume the origin $(0, 0)$ is contained in the (interior of) plane boundary ω. We now deal with an operator of the form

$$P(x, y, D_x, D_y) = D_y^m + \sum_{\substack{0 \leqq j \leqq m-1 \\ |\beta| + j \leqq p}} a_{\beta, j}(x, y) D_x^\beta D_y^j\,, \tag{1. 13}$$

where $a_{\beta, j}(x, y)$ are complex valued functions defined on $\Omega \cup \omega$ and infinitely differentiable. We add following two assumptions on P.

Assumption 3. $P(x, y, D_x, D_y)$ has constant strength in $\Omega \cup \omega$, that is,

$$\frac{\sum_\alpha |P^\alpha(x, y, \xi, \eta)|}{\sum_\alpha |P^\alpha(x', y', \xi, \eta)|} \leqq C(x, y, x', y')$$

for (x, y), $(x', y') \in \Omega \cup \omega$, $(\xi, \eta) \in E^n$.

Assumption 4. Set $P_0(D) = P(0, 0, D_x, D_y)$. Then $P_0(D)$ is a hypo-elliptic operator of type $d \geqq p$ of the form

$$D_y^m + \sum_{\substack{0 \leqq j \leqq m-1 \\ |\beta| + j \leqq p}} a_{\beta, j}(0, 0) D_x^\beta D_y^j$$

and satisfies Assumptions 1 and 2.

Then we can prove the following

Theorem 1. 2. *Consider the Dirichlet problem*

$$P(x, y, D_x, D_y) u(x, y) = f(x, y) \quad \text{in } \Omega\,. \tag{1. 14}$$

$$D_y^j u(x, 0) = 0,\ 0 \leqq j \leqq r-1 \quad \text{on } \omega \tag{1. 15}$$

with $f \in G(d, (p-m+1)d; \Omega \cup \omega)$, $a_{\beta, j} \in G(d, (p-m+1)d; \Omega \cup \omega)$, where $d \geqq p$. Then any function $u \in H^p(\Omega \cup \omega)$[3] *satisfying* (1. 14),

3) For the notation $H^p(\Omega \cup \omega)$, see [5].

(1. 15) *is a function in* $G(d, (p-m+1)d; \Omega_0 \cup \omega_0)$ *for some sufficiently small hemisphere* $\Omega_0 \cup \omega_0 = \{(x, y) \mid |x|^2 + y^2 \leqq r_0, y \geqq 0\}$.

In the elliptic case, that is, in the case of type 1 a slight modification of the proof of Morrey-Nirenberg [6] together with the use of the coerciveness estimate obtained in [1] gives the following more detailed and complete theorem.

Theorem 1. 3. *Let* $P(x, y, D_x, D_y)$ *be a properly elliptic operator defined in* $\Omega \cup \omega$ *with order* $2m$. *Consider the Dirichlet problem* (1. 14), (1. 15) *with* $f \in G(d; \Omega \cup \omega)$ *and with all the coefficients in* $G(d; \Omega \cup \omega)$ *for* $d \geqq 1$. *Then all the solutions* u *of the problem* (1. 14), (1. 15) *are in* $G(d; \Omega \cup \omega)$.

2. Proof of Theorem 1. 1.

2. 1. As a special case of Hörmander's results [4] we see that any solution $u \in C^p(\Omega \cup \omega)$ of the problem (1. 11), (1. 12) is infinitely differentiable up to the boundary ω. We shall only estimate the derivatives of the solutions u up to the boundary.

Now take $v \in C_0^\infty(\Omega \cup \omega)$ satisfying the Dirichlet condition (1. 12) and regard it as a function in $C_0^\infty(\bar{E}_+^n)$. We consider $v(\xi, y)$ (See (1. 8)) as a function of $y \geq 0$ with a vector parameter ξ. Following Schechter [8], we let $H^m(E^1)$ denote the completion of $C_0^\infty(E^1)$ with respect to the norm

$$||u||_m = \Big(\sum_{k=0}^{m} \int_{-\infty}^{\infty} |D_y^k u(\xi, y)|^2 dy\Big)^{1/2}.$$

The first step is to extend $v(\xi, y)$ to the function in $H^m(E^1)$ by a method due to Morrey-Nirenberg [6], Peetre [7] and Schechter [8].

For $|\xi| \leqq K_2$, set

$$v_1(\xi, y) = \begin{cases} v(\xi, y), & y \geqq 0 \\ \sum_{k=1}^{m} \lambda_k v(\xi, -ky), & y < 0, \end{cases}$$

where the λ_k are constants chosen so that all the dreivatives $D_y^j v$ for $0 \leqq j \leqq m-1$ are continuous at $y=0$. Here λ_k depends only on m. It holds that

$$(\xi^\alpha v(\xi, y))_1 = \xi^\alpha v_1(\xi, y)$$

for any multi-index α satisfying $\alpha_n = 0$.

Next, for $|\xi| > K_2$, we extend $v(\xi, y)$ by the method due to Schechter [8] and denote the resulting function by $v_1(\xi, y)$. Thus $v_1(\xi, y)$ is defined in $|\xi| < \infty$ and $|y| < \infty$. We also note that it is easily verfied that

$$(\xi^\alpha v(\xi, y))_1 = \xi^\alpha v_1(\xi, y) \qquad \text{for any } \alpha, \ \alpha_n = 0.$$

According to the result of Schechter [8] there exists a constant C independent of v so that the following inequality holds:

$$\text{(2. 1)} \qquad \int_{-\infty}^{\infty} |P(\xi, D_y)v_1(\xi, y)|^2 dy \leqq C \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy\,, \quad |\xi| > K_2\,.$$

Furthermore, for any $R(D)$ weaker than $P(D)$, we can obtain the following inequality

$$\text{(2. 2)} \qquad \int_{-\infty}^{\infty} |R(\xi, D_y)v_1(\xi, y)|^2 dy < C \Big\{ \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy + \int_0^{\infty} |v(\xi, y)|^2 dy \Big\}\,, \quad |\xi| \leqq K_2,\ v(\xi, y) \in C_0^{\infty}(\bar{E}_+^1)\,.$$

Proof of (2. 2). For $|\xi| \leqq K_2$ we have

$$\int_0^{\infty} |D_y^m v(\xi, y)|^2 dy \leqq \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy + C_1 \sum_{k=0}^{m-1} \int_0^{\infty} |D_y^k v(\xi, y)|^2 dy$$

where C_1 is an upperbound for the coefficients of $P(\xi, D_y)$ on the set $|\xi| \leqq K_2$. Thus

$$\sum_{k=0}^{m} \int_0^{\infty} |D_y^k v(\xi, y)|^2 dy \leqq C_2 \Big\{ \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy + \sum_{k=0}^{m-1} \int_0^{\infty} |D_y^k v(\xi, y)|^2 dy \Big\}$$

On the other hand

$$\int_{-\infty}^{\infty} |R(\xi, D_y)v_1(\xi, y)|^2 dy \leqq C_3 \sum_{k=0}^{m} \int_{-\infty}^{\infty} |D_y^k v_1(\xi, y)|^2 dy\,, \quad |\xi| \leqq K_2\,,$$

where C_3 is an upper bound for the coefficients of $R(\xi, D_y)$ on the set $|\xi| \leqq K_2$. Thus, from the construction of $v_1(\xi, y)$ on the set $|\xi| \leqq K_2$, we have

$$\int_{-\infty}^{\infty} |R(\xi, D_y)v_1(\xi, y)|^2 dy \leqq C_3 \sum_{k=0}^{m} \int_{-\infty}^{\infty} |D_y^k v_1(\xi, y)|^2 dy \leqq C_4 \sum_{k=0}^{m} \int_0^{\infty} |D_y^k v(\xi, y)|^2 dy$$
$$\leqq C_5 \Big\{ \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy + \sum_{k=0}^{m-1} \int_0^{\infty} |D_y^k v(\xi, y)|^2 dy \Big\}$$

Employing the well known inequality

$$\sum_{k=0}^{m-1} \int_0^{\infty} |D_y^k v(\varepsilon, y)|^2 dy \leqq \varepsilon \int_0^{\infty} |D_y^m v(\xi, y)|^2 dy + C(\varepsilon) \int_0^{\infty} |v(\xi, y)|^2 dy\,,$$

and taking ε so small that $\varepsilon C_5 \leqq \frac{1}{2} C_4$, we have

$$\int_{-\infty}^{\infty} |R(\xi, D_y)v_1(\xi, y)|^2 dy \leqq C_6 \Big\{ \int_0^{\infty} |P(\xi, D_y)v(\xi, y)|^2 dy + \int_0^{\infty} |v(\xi, y)|^2 dy \Big\}$$

for all $v(\xi, y) \in C_0^{\infty}(\bar{E}_+^1)$ and $|\xi| \leqq K_2$, where C_6 depends only on the coefficients of $R(\xi, D_y)$ and $P(\xi, D_y)$ on $|\xi| \leqq K_2$.

2.2. Now we prove some lemmas for later use.

Lemma 2.1 (*c.f. Friberg* [2]). *Let $P(\xi, \eta)$ be hypo-elliptic of type $d \geqq p$. Then, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ usch that*

$$h^{p-|\alpha|+d}|P^{\alpha}(\xi, \eta)|\,|\xi_i| \leq \varepsilon h^{p+d}|P(\xi, \eta)|\,|\xi_i| + C(\varepsilon) h^{p}(|P(\xi, \eta)| + |P^{\alpha}(\xi, \eta)|) \tag{2.3}$$

where $\alpha \neq 0$, $0 < h \leqq 1$, $1 \leqq i \leqq n-1$.

The proof is easily obtained by a simplification of that in [2].

Lemma 2.2 *Let $P(D)$ be that in Theorem* 1.1 *and let $d \geqq p$. Then*

$$\begin{aligned} h^{p-|\alpha|+d}\|P^{\alpha}D_i v, E^n_+\| &\leqq \varepsilon h^{p+d}(\|P(D)D_i v, E^n_+\| + \|D_i v, E^n_+\|) \\ &+ C(\varepsilon) h^{p}(\|P(D)v, E^n_+\| + \|v, E^n_+\|) \quad \alpha \neq 0,\ 1 \leqq i \leqq n-1 \end{aligned} \tag{2.4}$$

for any $v \in C_0^{\infty}(\bar{E}^n_+)$ satisfying the Dirichlet condition (1.12) *and* $0 < h \leqq 1$.

Proof. Using the Parseval's formula and the inequalities (2.1), (2.2) with $R = P$ or P^{α} we have

$$\begin{aligned}
&h^{2(p-|\alpha|+d)}\|P^{\alpha}D_i v, E^n_+\|^2 = h^{2(p-|\alpha|+d)}\int_{|\xi|<\infty}\Big[\int_0^{\infty}|P^{\alpha}(\xi, D_y)\xi_i v(\xi, y)|^2 dy\Big] d\xi \\
&\leqq h^{2(p-|\alpha|+d)}\int_{|\xi|<\infty}\Big[\int_{-\infty}^{\infty}|P^{\alpha}(\xi, D_y)\xi_i v_1(\xi, y)|^2 dy\Big] d\xi = \\
&h^{2(p-|\alpha|+d)}\int_{E^n}|P^{\alpha}(\xi, \eta)\xi_i v_1(\xi, \eta)|^2 d\eta\, d\xi \leqq \varepsilon h^{2(p+d)}\int_{E^n}|P(\xi, \eta)\xi_i v_1(\xi, \eta)|^2 d\eta\, d\xi \\
&\quad + C(\varepsilon) h^{2p}\int_{E^n}|P(\xi, \eta) v_1(\xi, \eta)|^2 d\eta\, d\xi + \\
&\quad + C(\varepsilon) h^{2p}\int_{E^n}|P^{\alpha}(\xi, \eta) v_1(\xi, \eta)|^2 d\eta\, d\xi \\
&= \varepsilon h^{2(p+d)}\int_{|\xi|<\infty}\Big[\int_{-\infty}^{\infty}|P(\xi, D_y)\xi_i v_1(\xi, y)|^2 dy\Big] d\xi + \\
&\quad + C(\varepsilon) h^{2p}\int_{|\xi|<\infty}\Big[\int_{-\infty}^{\infty}|P(\xi, D_y) v_1(\xi, y)|^2 dy\Big] d\xi + \\
&\quad + \int_{|\xi|<\infty}\Big[\int_{-\infty}^{\infty}|P^{\alpha}(\xi, D_y) v_1(\xi, y)|^2 dy\Big] d\xi\} \\
&\leqq \varepsilon \cdot C h^{2(p+d)}\Big\{\int_{|\xi|<\infty}\Big[\int_0^{\infty}|P(\xi, D_y)\xi_1 v(\xi, y)|^2 dy\Big] d\xi + \\
&\quad + \int_{|\xi|<\infty}\Big[\int_0^{\infty}|\xi_i v(\xi, y)|^2 dy\Big] d\xi + \\
&\quad + C\cdot C(\varepsilon) h^{2p}\Big\{\int_{|\xi|<\infty}\Big[\int_0^{\infty}|P(\xi, D_y) v(\xi, y)|^2 dy\Big] d\xi + \\
&\quad + \int_{|\xi|<\infty}\Big[\int_0^{\infty}|v(\xi, y)|^2 dy\Big] d\xi\,,
\end{aligned}$$

which proves Lemma 2. 2.

Lemma 2. 3 (*c.f. Friberg* [2]). *For every compact set* $K\subset\bar{E}_+^n$ *and for every* $h>0$, *there are a function* $\psi=\psi_{K,h}$ *and constants* C_α *independent of* h *such that* $\psi\in C_0^\infty(K_h)$, $\psi\equiv 1$ *on* K *and*

$$(2.5)\qquad \|D^\alpha\psi\|_\infty\leqq C_\alpha h^{-|\alpha|} \text{ for every } \alpha,$$

where $K_h=\{x\in\bar{E}_+^n \mid dis.(x, K)\leqq h\}$.

This can be shown by Friberg's argument and the proof is omitted here.

From now on, we employ the method developed by Friberg [2] to estimate tangential derivatives. So we introduce some notations used by Friberg in a slightly different way: V will represent the hemisphere $\{(x, y)\mid x_1^2+\cdots+x_{n-1}^2+y^2<R^2, y>0\}$ contained in Ω, and $V_{-r}=\{(x, y)\mid x_1^2+\cdots x_{n-1}^2+y^2<(R-r)^2, y>0\}$, $0<r<R$. Let t be a given positive number, and let

$$(2.6)\qquad (D_x^\sigma P^\alpha u)_t = t^{d|\sigma|+p-|\alpha|}D_x^\sigma P^\alpha u,\ u\in C^\infty(V).$$

We set for arbitrary $l\geqq 0$,

$$(2.7)\qquad \begin{aligned}&\|(D_x^\sigma P^\alpha u)_t;\, l+d|\sigma|+p-|\alpha|,\, V\|\\ &=\sup_{0<r<R} r^{l+d|\sigma|+p-|\alpha|}\|(D_x^\sigma P^\alpha u)_t,\, V_{-r}\|\end{aligned}$$

2. 3. The following lemma is essential in our proof of Theorem 1. 1.

Lemma 2. 4 *There exists a constant C such that*

$$(2.8)\qquad \begin{aligned}&\sum_{|\alpha|\neq 0}\|(D_iP^\alpha u)_t;\, l+d+p-|\alpha|,\, V\|\leqq C\{\|(D_iPu)_t;\\ &l+d+p,\, V\|+\|(Pu)_t;\, l+p,\, V\|\\ &+\sum_{\alpha\neq 0}\|(P^\alpha u)_t;\, l+p-|\alpha|,\, V\|\},\qquad 1\leqq i\leqq n-1,\end{aligned}$$

for all $u\in C^\infty(\Omega\cup\omega)$ *satisfying the Dirichlet condition* (1. 12), *provided that* $0<t\leqq\dfrac{t_0}{l+d}$.

Proof. Let K be a hemisphere $\{(x, y)\mid x_1^2+\cdots+x_{n-1}^2+y^2\leqq r^2<R^2, y\geqq 0\}$, contained in V^* ($V^*\equiv V\cup(\bar{V}\cap\omega^{n-1})$), and let h be so small that $K_h\subset V^*$. Then we see by Lemma 2. 3 that there is a function $\psi=\psi_{k,h}\in C_0^\infty(K_h)$ such that $\psi\equiv 1$ on K and $\|D^\alpha\psi\|_\infty\leqq C_\alpha h^{-|\alpha|}$ for any α. Thus for every $u\in C^\infty(V^*)$ satisfying the Dirichlet condition (1. 12) the product $v=\psi\cdot u$ belongs to $C_0^\infty(K_h)$ and v also satisfies (1. 12). So we can apply Lemma 2. 2 to v. Since $u\equiv v$ on K, it follows that for i, $1\leqq i\leqq n-1$,

(2. 9) $$h^{p-|\alpha|+d}\|P^{\alpha}D_i u, K\| \leqq h^{p-|\alpha|+d}\|PD_i(\psi u), K_h\| \leqq$$
$$\leqq \varepsilon h^{p+d}(\|PD_i(\psi u), K_h\| + \|D_i v, K_h\|) +$$
$$+ C(\varepsilon) h^{p}(\|P(\psi u), K_h\| + \|\psi u, K_h\|),\ \alpha \neq 0,\ 0 < h \leqq 1 .$$

By using the Leibniz' formula, we investigate the terms on the right hand side of (2. 9).

On the first term we have

$$PD_i(\psi u) = P(D)(\psi \cdot D_i u + D_i \psi \cdot u) = (P(D) D_i u) \cdot \psi + \sum_{\beta \neq 0} P^{\beta}(D) D_i u \cdot \frac{D^{\alpha}\psi}{\beta !} +$$
$$+ P(D)u \cdot D_i \psi + \sum_{\beta \neq 0} P^{\beta} u \cdot \frac{D^{\beta} D_i \psi}{\beta !} .$$

Hence it follows that

$$\|PD_i(\psi u), K_h\| \leqq C(\|PD_i u, K_h\| + \sum_{\beta \neq 0} h^{-|\beta|}\|P^{\beta}D_i u, K_h\| +$$
$$+ h^{-1}\|Pu, K_h\| + \sum_{\beta \neq 0} h^{-(1+|\beta|)}\|P^{\beta}u, K_h\|) .$$

Since $0 < h \leqq 1$, we have

$$h^{p+d}\|PD_i(\psi u), K_h\| \leqq C(h^{p+d}\|PD_i u, K_h\| + \sum_{\beta \neq 0} h^{p-|\beta|+d}\|P^{\beta}D_i u, K_h\| +$$
$$+ h^{p}\|Pu, K_h\| + \sum_{\beta \neq 0} h^{p-|\beta|}\|P^{\beta}u, K_h\|) .$$

Similarly for the second term, we get

$$h^{p+d}\|D_i(\psi u), K_h\| \leqq h^{p+d-1}\|u, K_h\| + h^{p+d}\|D_i u, K_h\| .$$

On the third term it holds that

$$h^{p}\|P(\psi u), K_h\| \leqq C(h^{p}\|Pu, K_h\| + \sum_{\beta \neq 0} h^{p-|\beta|}\|P^{\beta}u, K_h\|) .$$

Finally on the fourth term, we obtain

$$h^{p}\|\psi u, K_h\| \leqq h^{p}\|u, K_h\| .$$

These four estimates imply that

(2. 10) $$h^{p-|\alpha|+d}\|P^{\alpha}D_i u, K\| \leqq \varepsilon(h^{p+d}\|PD_i u, K_h\| + \sum_{\beta \neq 0} h^{p-|\beta|+d}\|P^{\beta}D_i u, K_h\|) +$$
$$+ C(\varepsilon)(h^{p}\|Pu, K_h\| + \sum_{\beta \neq 0} h^{p-|\beta|}\|P^{\beta}u, K_h\|),\ \alpha \neq 0 .$$

Now the summation of (2. 10) for all $\alpha \neq 0$ yields

(2. 11) $$\sum_{\alpha \neq 0} h^{p-|\alpha|+d}\|P^{\alpha}D_i u, K\| \leqq \varepsilon \sum_{\alpha \neq 0} h^{p-|\alpha|+d}\|P^{\alpha}D_i u, K_h\| +$$
$$+ C(\varepsilon)(h^{p+d}\|PD_i u, K_h\| + h^{p}\|Pu, K_h\| + \sum_{\alpha \neq 0} h^{p-|\alpha|}\|P^{\alpha}u, K_h\|) .$$

Suppose that t_0 is so small that $t_0 \cdot R \leqq d$. Let $h = tr$, where $0 < r \leqq R$ and $0 < t \leqq \frac{t_0}{l+d}$. If $l \geqq 0$ and if $r \leqq R$, then $h \leqq \frac{t_0 \cdot R}{l+d} \leqq 1$. If, in addition, $t_0 < 1$, then $0 < r(1-t) \leqq R$. Let $K = V_{-r}$. Then $K_h = V_{-r(1-t)}$. We rewrite (2. 11) in these notations and get

$$\begin{aligned}(2.\,12)\quad \sum_{\alpha \neq 0} (rt)^{p-|\alpha|+d} \|P^{\alpha} D_i u,\ V_{-r}\| &\leqq \varepsilon \sum_{\alpha \neq 0} (rt)^{p-|\alpha|+d} \|P^{\alpha} D_i u,\ V_{-r(1-t)}\| + \\ &+ C(\varepsilon)\{(rt)^{p+d} \|PD_i u,\ V_{-r(1-t)}\| + (rt)^{p} \|Pu,\ V_{-r(1-t)}\| + \\ &+ \sum_{\alpha \neq 0} (rt)^{p-|\alpha|} \|P^{\alpha} u,\ V_{-r(1-t)}\|\}\,.\end{aligned}$$

Multiply the above inequality by $t^l r^l$ ($l \geqq 0$). We have

$$\begin{aligned}\sum_{\alpha \neq 0} \|P^{\alpha} D_i u,\ V_{-r}\| (rt)^{l+p-|\alpha|+d} &\leqq \varepsilon \sum_{\alpha \neq 0} \|P^{\alpha} D_i u,\ V_{-r(1-t)}\| (r(1-t))^{l+p-|\alpha|+d} \Big(\frac{t}{1-t}\Big)^{l+p-|\alpha|+d} \\ &+ C(\varepsilon)\Big\{\|PD_i u,\ V_{-r(1-t)}\| (r(1-t))^{l+p+d} \Big(\frac{t}{1-t}\Big)^{l+p+d} \\ &+ \|Pu,\ V_{-r(1-t)}\| (r(1-t))^{l+p} \Big(\frac{t}{1-t}\Big)^{l+p} \\ &+ \sum_{\alpha \neq 0} \|P^{\alpha} u,\ V_{-r(1-t)}\| (r(1-t))^{l+p-|\alpha|} \Big(\frac{r}{1-t}\Big)^{l+p-|\alpha|}\Big\} \leqq \\ &\leqq \varepsilon \sum_{\alpha \neq 0} \|(P^{\alpha} D_i u)_t;\ l+p-|\alpha|+d,\ V\| \frac{1}{(1-t)^{l+p-|\alpha|+d}} \\ & C(\varepsilon)\Big\{\|(PD_i u)_t;\ l+p+d,\ V\| \frac{1}{(1-t)^{l+p+d}} + \|(Pu)_t;\ l+p,\ V\| \frac{1}{(1-t)^{l+p}} \\ &+ \sum_{\alpha \neq 0} \|(P^{\alpha} u)_t;\ l+p-|\alpha|,\ V\| \frac{1}{(1-t)^{l+p-|\alpha|}}\Big\}\,.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\alpha \neq 0} \|(P^{\alpha} D_t u)_t,\ l+p-|\alpha|+d,\ V\| &\leqq \sum_{\alpha \neq 0} \|(P^{\alpha} D_t u)_t;\ l+p-|\alpha|+d,\ V\| \frac{1}{(1-t)^{l+p-|\alpha|}} \\ &+ C(\varepsilon)\Big\{\|(PD_i u)_t;\ l+p+d,\ V\| \frac{1}{(1-t)^{l+p+d}} + \|(Pu)_t;\ l+p,\ V\| \frac{1}{(1-t)^{l+p}} \\ &+ \sum_{\alpha \neq 0} (P^{\alpha} u)_t;\ l+p-|\alpha|,\ V\| \frac{1}{(1-t)^{l+p-|\alpha|}}\Big\}\,.\end{aligned}$$

On the other hand there is a constant $c > 0$ such that

$$\frac{1}{1-t} < e^{ct} \quad \text{for any positive } t \leqq t_0 \ (t_0 < 1)\,,$$

from which

$$\left(\frac{1}{1-t}\right)^{l+p-|\alpha|+d} \leq e^{ct(l+p-|\alpha|+d)} < e^{ct_0\frac{(l+p-|\alpha|+d)}{l+d}} \leqq e^{ct_0}$$

and

$$\left(\frac{1}{1-t}\right)^{l+p-|\alpha|} \leqq e^{ct_0}.$$

Hence it follows that

$$(1-\varepsilon e^{ct_0}) \sum_{\alpha \neq 0} \|(P^\alpha D_i u)_t, l+p-|\alpha|+d, V\| \leqq C(\varepsilon) e^{ct_0} \{\|(PD_i u)_t; l+p+d, V\| + \\ +\|(Pu)_t; l+p, V\| + \sum \|(P^\alpha u)_t; l+p-|\alpha|, V\|\}.$$

By taking ε small enough here, we get (2. 8).

2.4. Now we need the following notation similar to Friberg [2]:

$$\text{(2. 13)} \quad A_0(P^\alpha D_x^\sigma u) = \|(P^\alpha D_x^\sigma u)_t; l+d|\sigma|+p-|\alpha|, V\|, \ |\sigma| \leqq 1, \ \sigma_n = 0, \\ A_{i+1}(P^\alpha u) = \max_{\substack{|\sigma| \leqq 1 \\ \sigma_n = 0}} A_i(P^\alpha D_x^\sigma u), \qquad i \geqq 0,$$

$$\text{(2. 14)} \quad B_i(u) = \max_{\beta \neq 0} A_i(P^\beta u), \qquad i \geqq 0,$$

and

$$\text{(2. 15)} \quad \|u; d, \lambda; l, V\| \sup_{\substack{\sigma \geq 0 \\ \sigma_n = 0}} \prod_{i=1}^{n-1} \left(\frac{\lambda}{\sigma_i + 1}\right)^{d\sigma_i} \cdot \|D_x^\sigma u; l+d|\sigma|, V\|, \ u \in C^\infty(V), \\ \lambda > 0.$$

We can prove the following

Theorem 2. 1 *Let $P(D)$ and $d(\geqq p)$ be those in Theorem* 1. 1. *Let V be the same as above and $l \geqq 0$ a given number. Then there are positive constants c and C such that*

$$\text{(2. 16)} \quad \sum_{\alpha \neq 0} \|P^\alpha u; d, c\lambda; l+p-|\alpha|, V\| \leqq C\{\|Pu; d, \lambda, l+p, V\| + \\ + \sum_{\alpha \neq 0} \|P^\alpha u; l+p-|\alpha|, V\|\}$$

for all $u \in C^\infty(V^*)$ satisfying the Dirichlet condition (1. 12).

To prove the theorem we need several lemmas as in Friberg [2].

Lemma 2.5 *Let $P(D)$ and d be those in Theorem* 1. 1. *Then there is a constant $C > 1$ such that*

$$\text{(2. 17)} \quad B_j(u) \leqq \max_{s+k=j} \{\max C^{s+1} A_k(Pu), \ C^j B_0(u)\},$$

for $j=1, 2, \cdots$ *and for all* $u\in C^{\infty}(V^*)$ *satisfying* (1. 12), *provided that* $0<t\leqq\dfrac{t_0}{l+d\cdot j}$.

Proof. We note that (2. 8) is eqiuvalent to

$$B_0(u)\leqq \max\{CA_1(Pu),\ CB_0(u)\} \tag{2. 18}$$

for some positive constant C. The inequality (2. 18) shows that (2. 17) is true when $j=1$ and $0<t\leqq\dfrac{t_0}{l+d}$. Since we can replace u and the parameter l in (2. 17) by $t^{d|\sigma|}D_x^{\sigma}u$ and by $l+d|\sigma|$ respectively ($|\sigma|\leq 1$, $\sigma_n=0$), we get

$$B_2(u)\leqq \max\{CA_2(Pu),\ CB_1(u)\}\ . \tag{2. 19}$$

Again by (2. 18) we obtain

$$CB_1(u)\leqq \max\{C^2A_1(Pu^2)_0\ C^2B_0(u)\}\ . \tag{2. 20}$$

The inequalities (2. 19) and (2. 20) prove that (2. 17) is valid for $j=2$, provided that $0<t\leqq\dfrac{t}{l+2d}$. Proceeding in this way, we can prove (2. 17) for all j.

Lemma 2. 6 *Let* A_0 *be defined by* (2. 13) *with* $t=\dfrac{1}{l+dj}$, *for* l *fixed, and* $t_i\leqq t_0$. *Then there are constants* $c<1$ *and* C_1 *such that*

$$\begin{aligned} C_1^{-1}||P^{\alpha}u;\, d,\, c\cdot\lambda;\, l+p|\alpha|,\, V||&\leqq \sup C^{-j}A_j(P^{\alpha}u)\leqq \\ &\leqq C_1||P^{\alpha}u;\, d,\, \lambda;\, l+p-|\alpha|,\, V|| \end{aligned} \tag{2. 21}$$

for any α, *if* $C>1$ *and if*

$$\lambda=\frac{t_1}{d\cdot C^{1/d}}\,. \tag{2. 22}$$

Proof. Put $N=||P^{\alpha}u;\, d,\, \lambda;\, l+p-|\alpha|,\, V||$, where $\lambda=\dfrac{t_1}{dC^{1/d}}$, and suppose that $t=\dfrac{t_1}{l+d\cdot j}$. Then

$$\begin{aligned} \max C^{-j}A_j(P^{\alpha}u)&\leqq \max_{\substack{(\sigma_n=0)\\ |\sigma|\leqq j}} C^{-|\sigma|}\Big(\frac{t_1}{l+d\cdot j}\Big)^{d|\sigma|+p-|\alpha|}||P^{\alpha}D_x^{\sigma}u;\\ &\quad l+d|\sigma|+p-|\alpha|,\, V||\leqq \max_{|\sigma|\leqq j} C^{-|\sigma|}\Big(\frac{t_1}{l+d\cdot j}\Big)^{d|\sigma|+p-|\alpha|}\prod_{i=1}^{n-1}\Big(\frac{\sigma_i+1}{\lambda}\Big)^{d\sigma_i}\cdot N\leqq \\ &\leqq \max_{|\sigma|\leqq j}\Big(\frac{t_1}{C^{1/d}}\Big)^{d|\sigma|}\cdot\Big(\frac{t_1}{l+j\cdot d}\Big)^{d|\sigma|+p-|\alpha|}\prod_{i=1}^{n-1}\Big(\frac{\sigma_i+\lambda}{\lambda}\Big)^{d\sigma_i}\cdot N \\ &\leqq \max_{|\sigma|\leqq j}\Big\{\frac{d(\sigma_i+1)}{l+d\cdot i}\Big\}^{d\sigma_i}\cdot N \\ &\leqq\Big\{\frac{d(j+1)}{l+d\cdot i}\Big\}^{d\cdot j}\cdot N\leqq\Big\{\frac{d(j+1)}{d\cdot j}\Big\}^{d\cdot j}\cdot N=\Big(1+\frac{1}{j}\Big)^{d\cdot j}\cdot N<e^d\cdot N\,. \end{aligned}$$

This proves the one half of Lemma 2. 6.

Next, by the definition of $A_i(P^\alpha u)$ in (2. 13), it holds

$$(2.23)\qquad \|(P^\alpha D_x)_t;\, l+d\cdot j-|\alpha|,\, V\|\leqq A_j(P^\alpha u) \qquad |\sigma|=j,\ \sigma_n=0$$

for any $j\geqq 0$. Let us put $t=\dfrac{t_1}{l+d|\sigma|}=\dfrac{t_1}{l+d\cdot i}$. Then (2. 23) yields

$$\left(\frac{t_1}{l+d\cdot j}\right)^{d\cdot j+p-|\alpha|}\|P^\alpha D_x^\sigma u;\, l+d\cdot j+p-|\alpha|,\, V\|\leqq A_j(P^\alpha u);\ |\sigma|=j\,.$$

Hence, for $c(>0)$ determined later;

$$C^{-j}\left(\frac{t_1}{l+d\cdot j}\right)^{d\cdot j+p-|\alpha|}\prod_{i=1}^{n-1}\left(\frac{\sigma_i+1}{c\cdot\lambda}\right)^{d\sigma_i}\prod_{i=1}^{n-1}\left(\frac{c\lambda}{\sigma_i+1}\right)^{d\sigma_i}\|P^\alpha D_x^\sigma u;\, l+d\cdot j+p-|\alpha|,\, V\|\leqq$$
$$\leqq C^{-j}A_j(P^\alpha u)\,.$$

Substituting $\lambda=\dfrac{t_1}{dC^{1/d}}$ and noting $C^{-j}=\dfrac{1}{C^{d/d|\sigma|}}$, we have

$$\left(\frac{t_1}{l+d\cdot j}\right)^{d\cdot l+p-|\alpha|}\prod_{i=1}^{n-1}\left(\frac{d(\sigma_i+1)}{ct_1}\right)^{d\sigma i}\|P^\alpha D_x^\sigma;\, l+d\cdot j+p-|\alpha|,\, V\|\leqq C^{-l}A_j(P^\alpha u)\,.$$

Put $K=\left(\dfrac{t_1}{l+d\cdot j}\right)^{d\cdot j+p-|\alpha|}\displaystyle\prod_{i=1}^{n-1}\left(\dfrac{d(\sigma_i+1)}{ct_1}\right)^{d\sigma i}$. Then

$$K^{-1}=\left(\frac{l+d\cdot j}{t_1}\right)^{d\cdot j+p+|\alpha|}\prod_{i=1}^{n-1}\left(\frac{ct_1}{d(\sigma_i+1)}\right)^{d\sigma i}=\left(\frac{l+d\cdot j}{t_1}\right)^{p-|\alpha|}\prod_{i=1}^{n-1}\left\{\frac{c(l+d\cdot i)}{d(\sigma_i+1)}\right\}^{d\sigma i}$$

is finite if c is sufficiently small. This completes the proof of the lemma.

2. 4. Proof of Theorem 2. 1. We can now complete the proof of Theorem 2. 1. Take the inequality (2. 21), devide both side by C^j and put $t=\dfrac{t_1}{l+d\cdot j}$, $t_1\leqq t_0$. Then we have

$$C^{-j}B_j(u)\leqq \max\{\max_{0\leqq k\leqq j} C^{-k}A_k(Pu),\, B_0(u)\}\,.$$

Therefore, it follows from Lemma 2.6 that

$$(2.24)\qquad \max_{\alpha\neq 0}\{\|P^\alpha u;\, d,\, c\lambda;\, l+p-|\alpha|,\, V\|\}\leqq$$
$$\leqq C\max\{\|Pu;\, d,\, \lambda;\, l+p,\, V\|,\, \max_{\alpha\neq 0}\|P^\alpha u;\, l+p-|\alpha|,\, V\|\}$$

for all $u\in C^\infty(V^*)$ satisfying (1. 12). The inequality (2. 24) is equivalent to (2. 16). Thus we have Theorem 2. 1.

2. 5. Now we can prove Theorem 1. 1. Let $f(x, y)$ be in $G(d, (p-m+1)$

d; V^*) $(d \geqq p)$. Then for any hemisphere $K=\{(x, y) \mid |x|^2+y^2 \leqq r,\ y \geqq 0\} \subset V^*$, there are constants C_0, C_1 such that $\|D_x^\sigma f, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|}$ for any $\sigma(\sigma_n=0)$. If the inequality (2. 16) is established once, then it turns out by (2. 15) that for the above K and for a solution $u \in C^\infty(V^*)$ of the problem (1. 11), (1. 12), there are new constants C_0, C_1 such that

$$(2.25) \quad \|D_x^\sigma P^\beta u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|}$$

for any $\beta \neq 0$, and for any $\sigma(\sigma_n=0)$.

We note $\dfrac{\partial^m P(\xi, \eta)}{\partial \eta^m} = m!$. Therefore, (2. 25) implies

$$(2.26) \quad \|D_x^\sigma u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|} \qquad \text{for any } \sigma \geqq 0\ (\sigma_n=0),$$

for new constants C_0 and C_1.

Next we note $\dfrac{\partial^{m-1} P(\xi, \eta)}{\partial \eta^{m-1}} = m!\,\eta + P_1(\xi)$, where $P_1(\xi)$ is a polynomial in ξ only. From (2. 25) it follows

$$(2.27) \quad \|D_x^\sigma D_y u + D_x^\sigma P_1(D_x) u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|}.$$

On the other hand, again by the inequality (2. 16) and Fribreg's results (Ch. 2 in [2]) for new constants C_0, C_1, we obtain

$$\|D_x^\sigma P_1(D_x) u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|}.$$

Hence we have for new constants C_0, C_1,

$$(2.28) \quad \|D_x^\sigma D_y u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|} \text{ for any } \sigma(\sigma_n=0).$$

Repeating the process m times we can obtain for new constants C_0, C_1

$$(2.29) \quad \|D_x^\sigma D_y^j u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|},\ 0 \leqq j \leqq m-1, \text{ for any } \sigma(\sigma_n=0).$$

Thus we may assume that for some constants C_0, C_1 $(\geqq 1)$

$$(2.30) \quad \|D_x^\sigma D_x^\beta D_y^j u, K\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|},\ 0 \leqq j \leqq m-1, \qquad \text{for any } \sigma(\sigma_n=0) \text{ and for any } \beta,\ |\beta| \leqq p.$$

$$\|D_x^\sigma D_y^k f, K\| \leqq C_0 C_1^{|\sigma|+k} |\sigma|^{d|\sigma|} k^{(p-m+1)d} \quad \text{for any } \sigma(\sigma_n=0) \text{ and for any } k.$$

Now the equation $P(D)u=f$ can be written in the form

$$(2.31) \quad D_y^m u = -\sum_{\substack{0<j<m-1 \\ |\beta|+j<p}} a_{\beta,j} D_x^\beta D_y^j u + f.$$

Put $1+\sum |a_{\beta,j}| = B$ (>1). Differentiating (2. 30) with respect to x-variables and applying (2. 30), we have

$$(2.32)\qquad \|D_x^\sigma D_y^m u, K\| \leqq \sum_{\substack{0\leq j\leq m-1\\ |\beta|+j\leq p}} |a_{\beta,j}|\,\|D_x^\sigma D_x^\beta D_y^j u, K\| + C_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|}$$
$$\leqq B\cdot C_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|} + C_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|}.$$

Again differentiating (2. 31) we have

$$D_x^\sigma D_y^{m+1}u = -\sum_{\substack{j=m-1\\ |\beta|\leq p-m+1}} a_{\beta,j} D_x^\sigma D_x^\beta D_y^m u + \sum_{\substack{j\leq m-2\\ |\beta|\leq p-m+2}} a_{\beta,j} D_x^\sigma D_x^\beta D_y^{j+1} u$$
$$+ D_x^\sigma D_j f,$$

where we consider $a_{\beta,j}=0$ when $j<0$. Applying (2. 32) we have

$$(2.33)\qquad \|D_x^\sigma D_y^{m+1}u, K\| \leqq B^2 C_0 C_1^{|\sigma|+p-m+1}(|\sigma|+p-m+1)^{d(|\sigma|+p-m+1)}$$
$$+ B C_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|} + C_0 C_1^{|\sigma|+1}|\sigma|^{d|\sigma|}$$

Repeating the procedure we can obtain by a simple induction argument on k

$$(2.34)\qquad \|D_x^\sigma D_y^{m+k}u, K\| \leqq (B+1)^{k+1} C_0 C_1^{|\sigma|+k(p-m+1)+m}(|\sigma|+k(p-m+1)$$
$$+m)^{d(|\sigma|+k(p-m+1)+m)}(B+1)^{k+1} C_0 C_1^{|\sigma|+k}|\sigma|^{d|\sigma|}\cdot k^{(p-m+1)k}$$
for all k, $0\leqq k\leqq m$.

Suppose now that (2. 34) holds for any $k\leqq k_0\leqq m$. Since

$$(2.35)\qquad D_x^\sigma D_y^{m+k_0+1}u = -\sum_{\substack{j=m-1\\ |\beta|<p-m+1}} a_{\beta,j} D_x^\sigma D_x^\beta D_y^{m+k_0} u + \cdots +$$
$$-\sum_{\substack{j=0\\ |\beta|\leq p}} a_{\beta,j} D_x^\sigma D_x^\beta D_y^{k_0+1} u + D_x^\sigma D_y^{k_0+1} f,$$

we have by (2. 34)

$$(2.36)\qquad \|D_x^\sigma D_y^{m+k_0+1}u, K\| \leqq B(B+1)^{k_0+1} C_0 C_1^{|\sigma|+(k_0+1)(p-m+1)+m}$$
$$(|\sigma|+(k_0+1)(p-m+1)+m)^{d(|\sigma|+(k_0+1)(p-m+1)+m)}$$
$$+B(B+1)^{k_0+1} C_0 C_1^{|k|+k_0}|\sigma|^{d|\sigma|} k_0^{(p-m+1)k_0} +$$
$$\cdots$$
$$+B(B+1)^{k_0-m+1} C_0 C_1^{|\sigma|+(k_0+1)(p-m+1)+m}$$
$$(|\sigma|+(k_0+1)(p-m+1)+m)^{d(|\sigma|+(k_0+1)(p-m+)+m)}$$
$$+C_0 C_1^{|\sigma|+k_0+1}|\sigma|^{d|\sigma|}(k_0+1)^{(p-m+1)(k_0+1)}$$
$$\leqq B\cdot \sum_{i=1}^{k_0+1}(B+1)^i\cdot C_0 C_1^{|\sigma|+(k_0+1)(p-m+1)+m}$$
$$(|\sigma|+(k_0+1)(p-m+1)+m)^{d(|\sigma|+(k_0+1)(p-m+1)+m)}$$
$$+B\cdot \sum_{i=1}^{k_0+1}(B+1)^i C_0 C_1^{|\sigma|+k_0+1}|\sigma|^{d|\sigma|}(k_0+1)^{(p-m+1)(k_0+1)}$$

$$\leqq (B+1)^{k_0+2} C_0 C_1^{|\sigma|+(k_0+1)(p-m+1)+m}$$
$$(|\sigma|+(k_0+1)(p-m+1)+m)^{d(|\sigma|+(k_0+1)(p-m+1)+m)}$$
$$+(B+1)^{k_0+2} C_0 C_1^{|\sigma|+k_0+1} |\sigma|^{d|\sigma|} (k_0+1)^{(p-m+1)(k_0+1)}$$

Hence we arrive at the conclusion that there are two constants C_0, C_1 such that

$$\text{(2. 37)} \qquad \|D_x^\sigma D_y^{m+k} u, K\| \leqq C_0 C_1^{|\sigma|+m+k} |\sigma| (m+k)^{d|\sigma|(p-m+1)d(m+k)}$$

for any $\sigma(\sigma_n=0)$ and for any k.

We apply the Sobolev's Lemma to the inequality (2. 37) and obtain Theorem 1. 1. We omit the details here. (c.f. Friberg [2], Lemma 2. 2. 2.)

3. Proof of Theorem 1. 2

3. 1 The proof can be obtained in a quite similar manner to the proof of Theorem 1. 1 by applying the method devoloped by Friberg for the formally partially hypo-elliptic equations (Ch. 4 in [2]).

Lemma 3. 1 *Let $Q(D)$ be a linear differential operator with constant coefficients weaker than $P_0(D)=P(0, 0, D_x, D_y)$. Let p be the order of $P_0(D)$. Then it holds*

$$\text{(3. 1)} \qquad t^{d|\sigma|+p} \|D_x^\sigma Q; l+d|\sigma|+p, V\|$$
$$\leqq C \sum_\alpha t^{d|\sigma|+p-|\alpha|} \|D_x^\sigma P^\alpha u; l+d+p-|\alpha|, V\|$$

for all $u \in C^\infty(V^*)$ satisfying (1. 12), *all $d \geqq 1$, all $\sigma \geqq 0$ $(\sigma_n=0)$, all $l>0$, and for all t with $0<t\leqq \dfrac{t_0}{l+d|\sigma|+p}$.*

The proof is omitted as it is simpler than that of Lemma 2. 4.

Now by the assumption on $P(x, y, D_x, D_y)$ in Theorem 1. 2, $P(x, y, D_x, D_y)$ can be written as

$$\text{(3. 2)} \qquad P(x, y, D_x, D_y) = P_0(D_x, D_y) + \sum_1^N C_\nu(x, y) P_\nu(D_x, D_y),$$

where $P_0(D)$ is of type $d(\geqq p=\text{order of } P_0)$ of the form (1. 1) and satisfies assumptions of Theorem 1. 1 and further all the P_ν, are weaker than P_0. The coefficients C_ν belong to $G(d, (p-m+1)d; \Omega \cup \omega)$, and

$$\text{(3. 3)} \qquad |C_\nu(x, y)| = 0(|x|+y), \qquad \text{when } |x|+y \to 0 .$$

Lemma 3. 2 (*c.f. Lemma* 2. 4) *Let $P(x, y, D_x, D_y)$ be that given in Theorem 1. 2, and $\varepsilon>0$ a given number. Set p=order of P_0. Then there exist a hemisphere*

$V_0=\{(x, y) \mid |x|^2+y^2 \leqq r_0, y>0\} \subset V$ *and constants* t_0, C *such that*

$$(3.4) \qquad \max_{\substack{|\sigma|\leq 1 \\ \sigma_n=0}} t^{d|\sigma|+p}\|D_x^\sigma P_0 u; l+d|\sigma|+p, V_0\| \leqq$$
$$\leqq C \max_{\substack{|\sigma|\leq 1 \\ \sigma_n=0}} t^{d|\sigma|+p}\|D_x^\sigma Pu; l+d|\sigma|+p, V_1\|+$$
$$+\varepsilon \max_{\substack{|\sigma|\leq 1 \\ \sigma_n=0}} \sum_{\beta\neq 0} t^{d|\sigma|+p-|\beta|}\|D_x^\sigma P_0^\beta u; l+d|\sigma|+p-|\beta|, V_1\|,$$

for all $u \in C^\infty(V^*)$ *satisfying the Dirichlet condition* (1.12) *and for all* $l \geqq 0$ *and* $0<t \leqq \dfrac{t_0}{l+d+p}$.

Proof. Set

$$A(D_x^\alpha P_\nu^\alpha u) = t^{d|\sigma|+p-|\alpha|}\|D_x^\alpha P_\nu^\alpha u; l+d|\sigma|+p|\alpha|, V_0\|.$$

Then it follows from (3.2) that

$$(3.5) \qquad A(D_x^\sigma P_0 u) \leqq A(D_x^\sigma Pu)+\sum_\nu A(D_x^\sigma(C_\nu P_\nu u)).$$

For σ, $|\sigma|=1$ $(\sigma_n=0)$

$$A(D_x^\sigma C_\nu P_\nu u) \leqq t^d\|D_x^\sigma C_\nu; d|\sigma|, V_0\|_\infty \cdot A(P_0 u)$$
$$+\|C_\nu; 0, V_0\|_\infty \cdot A(D_x^\sigma P_\nu u).$$

Now let $t=\dfrac{t_0}{l+d+p}$, with $0<t_1 \leqq t_0$, and take μ so small that $C_\nu \in C^\infty(d, \mu; 0, V_0)$ (For notation $G_\infty(d, \mu; 0, V)$, see Ch. 2, in [2]). Then

$$t^{d|\sigma|}\|D_x^\sigma C_\nu; d|\sigma|, V_0\|_\infty \leqq \Pi \left\{\frac{t_1(\sigma_i+1)}{\mu(l+d+p)}\right\}^{d\sigma_i} \|C_\nu; d, \mu; 0, V\|_\infty$$

so that

$$t^{d|\sigma|}\|D_x^\sigma C_\nu; d|\sigma|, V_0\|_\infty \leqq C \prod_{i=1}^{n-1} \left\{\frac{\varepsilon(\sigma_i+1)}{d}\right\}^{d\sigma_i},$$

if $t_0 \leqq \varepsilon\mu$. Since d is always $\geqq 1$, this shows that

$$\sum_{\substack{|\sigma|=1 \\ \sigma_n=0}} t^{d|\sigma|}\|D_x^\sigma C_\nu; d|\sigma|, V_0\|_\infty = 0(\varepsilon),$$

as ε tends to zero. But $\|C_\nu; 0, V_0\|_\infty$ can be made as small as we want by taking V_0 sufficiently small. (See (3.2)). We have

$$(3.6) \qquad A(D_x^\sigma C_\nu P_\nu u) \leqq C_1 \varepsilon \max_{\substack{|\sigma|\leq 1 \\ \sigma_n=0}} A(D_x^\sigma P_\nu u),$$

provided that $C_\nu \in G_\infty(d, \mu; 0, V_0)$, $t \leqq \frac{t_0}{l+d+p}$ $(t_0 \leq \varepsilon\mu)$ and V_0 is sufficiently small. Now let us use Lemma 3. 1, with $Q=P_\nu$ and with V_0 instead of V. Then we get

$$(3.7) \qquad A(D_x^\sigma Pu) \leqq C \sum_\alpha A(D_x^\sigma P_0^\alpha u), \qquad \text{for any } \sigma(\sigma_n=0).$$

Thus, in view of (3. 5), (3. 6) and (3. 7),

$$A(D_x^\sigma P_0 u) \leqq A(D_x^\sigma Pu) + C_2 \varepsilon \max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} \sum_\alpha A(D_x^\sigma P_0^\alpha u),$$

for any $\sigma(|\sigma| \leq 1, \sigma_n=0)$, if $t=\frac{t_1}{l+d+p}$, $t_1<t_0$, and if t_0 and V_0 are sufficiently small. This means also that

$$\max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} A(D_x^\sigma P_0 u) \leqq \max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} A(D_x^\sigma Pu) + C_2 \max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} \sum_\alpha A(D_x^\sigma P_0^\alpha u).$$

Suppose now that $C_2 \cdot \varepsilon \leqq \frac{1}{2}$. Then $0<\varepsilon_1=\frac{C_1\varepsilon}{1-C_2\varepsilon} \leqq 1$ and we get

$$(3.8) \qquad \max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} A(D_x^\sigma P_0 u) \leqq 2 \max_{\substack{|\sigma| \leq 1 \\ (\sigma_n=0)}} A(D_x^\sigma Pu) + \varepsilon \max_{\substack{|\sigma| < 1 \\ (\sigma_n=0)}} \sum_{\beta \neq 0} A(D_x^\sigma P_0^\beta u).$$

Obivously, (3. 8) and (3. 4) are equivalent.

Let us define $A_i(P_0 u)$ in terms of $A_0(D_x^\sigma P_0^\alpha u)$ as in (2. 13). Then it follows from (3. 8) (or (3. 3)) that for an arbitrary $\varepsilon>0$

$$(3.9) \qquad A_i(P_0 u) \leqq C_1 A_i(Pu) + \varepsilon \sum_{\alpha \neq 0} A_i(P_0^\alpha u), \qquad \text{for any } i \geqq 0,$$

under the usual conditions on u, l, t and V_0. We can also apply Lemma 2. 5 to P_0 and obtain the estimate

$$(3.10) \qquad \max_{\alpha \neq 0} A_j(P_0^\alpha u) \leqq \max\left\{\max_{s+k=j} C^{s+1} A_k(P_0 u),\ C^j \sum_{\alpha \neq 0} A_0(Pu)\right\}$$

for $j=1, 2, \cdots$, and for all t with $0<t \leqq \frac{t_0}{l+d \cdot j}$. From (3. 9), we see that (3. 10) can be replaced by

$$(3.11) \qquad \max_{\alpha \neq 0} A_j(P_0^\alpha u) \leqq C_2 \max\left\{\max_{s+k=j} C^{s+1} A_k(Pu) C^j \sum_{\alpha \neq 0} A_0(P_0^\alpha u)\right\}$$

for $j=1, 2, \cdots$, and for $0<t \leqq \frac{t_0}{l+d \cdot j}$, if t_0 and V_0 are sufficiently small.

3.2. As a simple application of Lemma 2. 6, we can prove the following.

Theorem 3. 1 *Let $P(x, y, D_x, D_y)$ be given as in Theorem* 1. 2 *which satisfies the prescribed condition. Then there are positive constants $c<1$ and C such that*

$$(3.12)\quad \sum_{\alpha\neq 0} \|P_0^\alpha u; d, c\mu; l+p-|\alpha|, V_0\| \leqq C\{\|Pu; d, \lambda; l+p, V_0\| + \\ + \sum_{\alpha\neq 0} \|P_0^\alpha u; l+p-|\alpha|, V_0\|\}$$

for all $u\in C^\infty(V^*)$ satisfying the Dirichlet condition (1. 12) *and for all $\lambda>0$, provided that $V_0=\{(x, y)\,|\,|x|^2+y^2<r_0, y>0\}$ is a sufficiently small hemisphere.*

Similarly to the proof of Theorem 1. 1, if the inequality (3. 12) is obtained, then from the assumption $f\in G(d, (p-m+1)d, V^*)$ and by (2. 15) we may assume that for any solution u of (1. 14), (1. 15), there are positive constants C_0, $C_1(\geqq 1)$ such that

$$(3.13)\quad \begin{aligned} &\|D_x^\sigma D_x^\beta D_y^j u, V_0\| \leqq C_0 C_1^{|\sigma|} |\sigma|^{d|\sigma|}, \ |\beta|\leqq p, \quad \sigma(\sigma_n=0), \\ &\|D_x^\sigma D_y^k f, V_0\| \leqq C_0 C_1^{|\sigma|+k} (|\sigma|+k)^{d(|\sigma|+p_0 k)}, \quad \sigma(\sigma_n=0), \end{aligned}$$

and

$$\|D_x^\sigma D_y^k a_{\beta,j}, V_0\| \leqq C_0 C_1^{|\sigma|+k} (|\sigma|+k)^{d(|\sigma|+p_0 k)}, \quad \sigma(\sigma_n=0),$$

where we put $p_0=p-m+1$ $(\geqq 1)$.

Now we can assume $d>1$.[4] Rewrite the equation $P(x, y, D_x, D_y)u=f$ in the form

$$(3.14)\quad D_y^m u = -\sum_{\substack{0\leqq j\leqq m-1\\ |\beta|+j\leqq p}} a_{\beta,j}(x, y) D_x^\beta D_x^j u + f.$$

We differentiate (3. 14) with respect to x-variables and get

$$(3.15)\quad D_x^\sigma D_y^m u = -\sum_{\substack{0\leqq j\leqq m-1\\ |\beta|+j\leqq p}} D_x^\sigma (a_{\beta,j} D_x D_y^j u) + D_x^\sigma f.$$

Consider each term

$$D_x^\sigma (a_{\beta,j} D_x^\beta D_y^j u) = \sum_{\rho\leqq\sigma} \binom{\sigma}{\rho} D_x^{\sigma-\rho} a_{\beta,j}\cdot D_x^\rho D_x^\sigma D_y^j u$$

in the summation. By (3. 13) we see

$$\|D_x^\sigma (a_{\beta,j} D_x^\beta D_y^j u), V_0\| \leqq \sum_{\rho\leqq\sigma} \binom{\sigma}{\rho} C_0 C_1^{|\sigma-\rho|} |\sigma-\rho|^{d|\sigma-\rho|} C_0 C_1^{|\rho|} |\rho|^{d|\rho|}.$$

Now we use the following simple inequalities

4) We note that all the hypo-elliptic operators of first order and of type 1 are not of determined type.

(3. 16) $\binom{k}{j}(k-j)^{k-j}j^j \leqq k^k$ for integers j, k, $0 \leqq j \leqq k$,

(3. 17) $\binom{\alpha}{\beta} \leqq \binom{|\alpha|}{|\beta|}$ for $\beta \leqq \alpha$.

For any $b>0$, there is a constant $C'=C'(b, n)$ independent of α such that

(3. 18) $\sum_{\beta \leqq \alpha} \binom{\alpha}{\beta}^{-b} \leqq C'$

Thus we have

(3. 19) $\|D_x^\sigma(a_{\beta,j}D_x^\beta D_y^j u), V_0\| \leqq C^\sigma C_0^2 C_1^{|\sigma|}|\sigma|^{d|\sigma|}$, $0 \leqq j \leqq m-1$,

with $b=d-1$ and

(3. 20) $\|D_x^\sigma D_y^m u, V_0\| \leqq NC'C_0^2 C_1^{|\sigma|}|\sigma|^{d|\sigma|} + C_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|}$,

where N is the number of terms of $P(x, y, D_x, D_y)u$.

Again differentiating (3. 14) we have

$$D_x^\sigma D_y^{m+1}u = -\sum_{\substack{j=m-1\\|\beta|\leqq p-m+1}} D_x^\sigma D_y(a_{\beta,j}D_x^\beta D_y^{m-1}u) - \sum_{\substack{j<m-1\\|\beta|+j\leqq p}} D_x D_y(a_{\beta,j}D_x^\beta D_y^k u) + \\ + D_x^\sigma D_y f,$$

where we put $a_{\beta,j}\equiv 0$ for $j<0$. Consider again each term of the first summation

$$D_x^\sigma D_y(a_{\beta,m-1}D_x^\beta D_y^{m-1}u) = \sum_{\rho\leqq\alpha}\binom{\alpha}{\rho}D^{\alpha-\rho}a_{\beta,m-1}D^\rho D_x^\beta D_y^{m-1}u,\ \alpha=\sigma+(0', 1).$$

By (3. 13) and (3. 20) we have

$$\|D_x^\sigma D_y(a_{\beta,m-1}D_x^\beta D_y^{m-1}u), V_0\| \leqq BC'C_0^2 C_1^{|\sigma|+p_0}(|\sigma|+p_0)^{d(|\sigma|+p_0)} \\ + BC_0 C_1^{|\sigma|}|\sigma|^{d|\sigma|}.$$

Hence we obtain

(3. 21)
$$\begin{aligned}\|D_x^\sigma D_y^{m+1}u, V_0\| &\leqq (B^2+B)C_0 C_1^{|\sigma|+p_0}(|\sigma|+p_0)^{d|\sigma|+p_0} \\ &\quad +(B+1)C_0 C_1^{|\sigma|+1}(|\sigma|+1)^{d(|\sigma|+p_0)} \\ &\leqq (B+1)^2 C_0 C_1^{|\sigma|+p_0}(|\sigma|+p_0)^{d(|\sigma|+p_0)} + (B+1)C_0 C_1^{|\sigma|+p_0}(|\sigma|+ \\ &\quad +p_0)^{d(|\sigma|+p_0)}.\end{aligned}$$

Thus, using the inequalities (3. 16), (3. 17), (3. 18) and the estimates (3. 20) (3. 21), we can repeat the procedure similar to that in the proof of Theorem 1. 1. So, the proof of Theorem 1. 2 is obtained.

We omit the proof of Theorem 1. 3.

4. Remark. In the case when $m=1$, we can improve Theorem 1. 1 in the following form.

Let $P(D)$ be a hypo-elliptic operator of the form

$$P(D) = D_y + \sum_{|\beta|\leq p} a_\beta D_x^\beta .$$

satisfying Assumptions 1 and 2. Furthermore let $P(D)$ be a hypo-elliptic operator of type $d(\geqq 1)$ in x, that is, there exists a constant C independent of real ξ and η such that

$$\sum_\alpha |P^{\alpha}(\xi, \eta)|(1+|\xi|)^{|\alpha|/d} \leqq C(|P(\xi, \eta)|+1) .$$

Then any function $u \in C^p(\Omega \cup \omega)$ satisfying (1. 11), (1. 12) with $f \in G(d, pd; \Omega \cup \omega)$ is also a function in $G(d, pd; \Omega \cup \omega)$.

In Theorem 1. 2, the similar to the above is true.

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