Study on commutative ring-theoretic and algebro-geometric properties of toric rings via combinatorics

Submitted to
Graduate School of Information Science and Technology
Osaka University

April 2024

Koji MATSUSHITA

Papers related to the thesis

1. T. Hall, M. Kölbl, K. Matsushita and S. Miyashita, Nearly Gorenstein Polytopes, *Electron. J. Combin.*, **30** (2023), no. 4, Paper No. 4.42.
2. A. Higashitani and K. Matsushita, Conic divisorial ideals and non-commutative crepant resolutions of edge rings of complete multipartite graphs, *J. Algebra* **594** (2022), 685–711.
3. A. Higashitani and K. Matsushita, Three families of toric rings arising from posets or graphs with small class groups, *J. Pure Appl. Algebra*, **226** no. 10, (2022), 107079.
4. A. Higashitani and K. Matsushita, Levelness versus almost Gorensteinness of edge rings of complete multipartite graphs, *Communications in Algebra*, **50** (6):2637–2652, 2022.
5. K. Matsushita, Torsionfreeness for divisor class groups of toric rings of integral polytopes, *J. Algebra*, **644** (2024), 749–760.
6. K. Matsushita, Toric rings of $(0,1)$-polytopes with small rank, *Illinois J. Math.*, to appear.
7. K. Matsushita, Conic divisorial ideals of toric rings and applications to Hibi rings and stable set rings, arXiv:2210.02031.
8. K. Matsushita and S. Miyashita, Conditions of multiplicity and applications for almost Gorenstein graded rings, arXiv:2311.17387.

Contents

Chapter 1

Introduction

Toric rings are affine semigroup rings arising from integral polytopes or rational polyhedral cones. They are an object of interest in various fields such as commutative algebra, combinatorics, algebraic geometry and representation theory, and have been studied in various ways from different perspectives (see e.g., the books [9, 13, 29]). Toric rings satisfy nice commutative ring-theoretical properties and their invariants may be easily computed. For example, it is known that normality provides various important properties to toric rings and some invariants of normal toric rings can be described in the terms associated with underlying polytopes or cones. In fact, it is known that if a toric ring R is normal, then R is Cohen–Macaulay ([44]), and that R is normal if and only if R is a Krull ring (cf. [84, Theorem 9.8.13]). Moreover, the canonical modules and the divisor class groups of normal toric rings can be easily described by using the terms of affine semigroups associated with underlying polytopes or cones.

In this thesis, we study commutative ring-theoretic and algebro-geometric properties of toric rings by using the term of their underlying combinatorial objects. In particular, we focus on the following three topics:

- Divisor class groups;
- Generalizations of Gorenstein graded rings;
- Conic divisorial ideals and non-commutative crepant resolutions.

Divisor class groups

The divisor class group $\mathrm{Cl}(R)$ of a toric ring R, which is a finitely generated abelian group, is one of the most interesting invariants. Divisor class groups of Krull semigroup rings (including normal toric rings) were studied by Chouinard in [11]. Moreover, divisor class groups of Ehrhart rings of rational polytopes, which are normal toric rings, were also investigated by Hashimoto-Hibi-Noma [28]. In the paper [28], a sufficient condition for divisor class groups of Ehrhart rings to be torsionfree is given. As will be discussed later, for the description of conic divisorial ideals of toric rings and the construction of non-commutative crepant resolutions, it is important to determine whether the divisor class group is torsionfree or not. It is known that the rank of $\mathrm{Cl}(R)$ of a toric ring R coincides with $F - \dim R$, where F denotes the number of facets of the underlying polytope or cone of R.

One of the goals of this thesis is to investigate the torsionfreeness of the divisor class groups of the toric rings of integral polytopes. In addition, we would like to analyze the normality and the isomorphism classes of toric rings focusing on the ranks of their divisor class groups. Since it is difficult to do that for general toric rings, we simplify it by restricting them to the toric rings of $(0,1)$-polytopes. Here, $(0,1)$-polytope is the convex hull of a finite set of $(0,1)$-vectors. It arises from various combinatorial objects such as partially ordered sets (posets, for short), graphs and matroids. In particular, we mainly deal with the following three families of $(0,1)$-polytopes and their toric rings:

- Hibi rings, which are the toric rings of order polytopes ([32, 78]);
- Stable set rings, which are the toric rings of stable set polytopes ([12]);
- Edge rings, which are the toric rings of edge polytopes ([64, 69]).

We also study the divisor class groups and the relationships of these toric rings.

Generalizations of Gorenstein graded rings

Cohen–Macaulay (local or graded) rings and Gorenstein (local or graded) rings are the most important properties and play crucial roles in the theory of commutative algebras. Similar to what we have said above, we see an important question as when toric rings have these properties. One important result is the characterization of Gorenstein Ehrhart rings ([16]).

On the other hand, there are quite many examples which are Cohen–Macaulay but not Gorenstein. Recently, many researchers have introduced good "intermediate" classes of those two properties, that is, many classes of Cohen–Macaulay graded rings which are not Gorenstein have been defined and those theories have been developed. Especially, the following classes of generalizations of Gorenstein graded rings have been well studied:

- Level graded rings ([74]);
- Almost Gorenstein graded rings ([22]);
- Nearly Gorenstein graded rings ([30]).

For example, in [56, 57, 30], it is characterized when Hibi rings are level, almost Gorenstein or nearly Gorenstein, respectively. In addition, almost Gorensteinness of a special family of edge rings has been investigated in [39, 1]. In [54, 55], these properties are compared through some semi-standard graded rings. Moreover, it turns out that h-vectors of standard graded rings help us to examine these properties (see e.g., [86, 38]).

In this thesis, we will characterize when certain classes of toric rings satisfy those properties in terms of underlying combinatorial objects, and measure how different those properties are from each other.

Conic divisorial ideals and non-commutative crepant resolutions

Let R be a normal toric ring, which is Cohen–Macaulay as described in Section 2.1. The elements of $\mathrm{Cl}(R)$ are identified with the isomorphism classes of the divisorial ideals of R. Recently, conic divisorial ideals, which are a certain class of divisorial ideals and a special kind of maximal Cohen–Macaulay (MCM, for short) modules of rank one, are well studied (see, e.g., [7, 8]). Indeed, they play beneficial roles in the theory of non-commutative algebraic geometry as well as commutative rings with positive characteristic. For example, conic divisorial ideals are used to analyze the structure of Frobenius push-forward of R. In particular, some invariants such as (generalized) F-signatures and Hilbert–Kunz multiplicities can be computed by using information on the conic divisorial ideals. Moreover, the endomorphism of the direct sum of all conic divisorial ideals of R is a non-commutative resolution (NCR, for short) ([18, 70]). This means that any normal toric ring has an NCR. Furthermore, we can construct non-commutative crepant resolutions (NCCRs, for short) for some toric rings, which were introduced by Van den Bergh ([82]), by considering the endomorphism ring of the direct sum of some conic divisorial ideals. While toric rings always have NCRs as mentioned above, the existence of an NCCR does not hold in general. It was shown in [15] that if a Cohen-Macaulay normal domain R has an NCCR, then R is $\mathbb{Q}$-Gorenstein. This implies that if the divisor class group of R is torsionfree and R is not Gorenstein, then R does not have an NCCR (this is why it is important to decide whether divisor class groups are torsionfree).

The existence of an NCCR for certain classes is one of the most well-studied problems in this area. For example, concerning the case of toric rings, the following results are known:

- an NCCR of each quotient singularity by a finite abelian group (which is a toric ring associated with a simplicial cone) is given (see, e.g., [46, 82]);
- Gorenstein toric rings whose class groups are $\mathbb{Z}$ have an NCCR ([82]);
- Gorenstein Hibi rings whose class groups are $\mathbb{Z}^2$ have an NCCR ([61]);
- NCCRs of 3-dimensional Gorenstein toric rings can be obtained via the theory of dimer models (see [6, 45, 72]);
- an NCCR of Segre products of polynomial rings which have the same variables is constructed ([43]);
- quasi-symmetric toric rings have an NCCR ([70]);
- there are other results on the existence of an NCCR for toric rings (see [70, 71]).

In the construction of NCCRs, it is natural and important to classify MCM divisorial ideals (including conic ones) of certain classes of toric rings. Indeed, it has been investigated in some classes of toric rings. For example, a classification of MCM divisorial ideals is given in the case of toric rings whose divisor class group are $\mathbb{Z}$ or $\mathbb{Z}^2$ ([77, 81]). Moreover, a description of the MCM divisorial ideals of weakly-symmetric toric rings is given ([73]).

In this thesis, we provide an idea to determine conic divisorial ideals and we give a description of the conic divisorial ideals of certain classes of toric rings by using this idea. Moreover, by using the description of the conic divisorial ideals, we construct an NCCR of some toric rings.

Structure of this thesis

The organization of this thesis is as follows. We divide this thesis into four parts. Each part, except for Part I, includes the author's results on each topic.

- Part I is devoted to the introduction of toric rings and the preparation of notions and notation related to them that will be used in the following parts. There are two chapters in Part I. In Chapter 2, we will recall the definitions of toric rings and related combinatorial objects. In Chapter 3, we will introduce certain classes of toric rings that we will focus on in this thesis.

- Part II is devoted to the studies on divisor class groups of toric rings. There are four chapters in Part II. In Chapter 4, we will recall a computation of the divisor class groups of toric rings and Gale-diagrams of polytopes. In Chapter 5, we will discuss what kind of toric rings have torsionfree divisor class groups. In Chapter 6, we will characterize when each of the three families of toric rings (Hibi rings, stable set rings and edge rings) has a small divisor class group and examine their relationships. In Chapter 7, we will study the toric rings of $(0, 1)$-polytopes with small rank. This part contains the results of [41, 50, 52].

- Part III is devoted to the studies on generalizations of Gorenstein graded rings and there are four chapters. In Chapter 8, we will recall the definitions of level rings, almost Gorenstein graded rings and nearly Gorenstein graded rings, and some known results on them. In Chapter 9, we will characterize when the edge ring of a complete multipartite graph is level or almost Gorenstein. In Chapter 10, we will discuss the almost Gorensteinness of graded rings derived from conditions of their multiplicities and provide an application to toric rings. In Chapter 11, we will give necessary conditions and sufficient conditions on integral polytopes for their Ehrhart rings to be nearly Gorenstein. This part contains the results of [42, 53, 27].

- Part IV is devoted to the studies on conic divisorial ideals and NCCRs of toric rings. There are two parts in Part IV. In Chapter 12, we will recall the notions of conic divisorial ideals, discuss how to determine the conic divisorial ideals of toric rings and give a description of the conic divisorial ideals of several toric rings. In Chapter 13, we will introduce a method for constructing NCCRs of toric rings and present an NCCR for certain toric rings. This part contains the results of [40, 51].

Acknowledgments

First of all, I am deeply grateful to my supervisor, Prof. Akihiro Higashitani, for introducing me to this area of research, giving me helpful suggestions and guiding me in writing this thesis.

I would like to thank to the members of my dissertation committee Prof. Yusuke Nakajima, Prof. Daisuke Furihata and Prof. Yoshie Sugiyama for their insightful feedback and constructive criticism.

I am indebted to my colleagues especially Sora Miyashita, Nobukazu Kowaki, Max Kölbl and Nayana Shibu Deepthi for having a lot of discussions in mathematics with me and encouraging each other.

I appreciate all of my acquaintances which I cannot mention here for their important influence on my growth.

Since April 2022, I have been supported by JSPS Research Fellowship for Young Scientists.

Finally, I am thankful to my family for their support.

Part I

Introduction to toric rings

Chapter 2

Preliminaries

The goal of this chapter is to define toric rings and review their fundamental properties. Moreover, we prepare notions and notation of combinatorial objects such as polytopes and graphs. The concepts introduced here will be used in our study in later parts.

2.1 Toric rings

In this section, we recall the toric rings of polytopes or cones. We refer the readers to e.g., [9, 29, 84], for the introduction.

Throughout this thesis, let $\Bbbk$ be an algebraically closed field $\Bbbk$ with characteristic 0. First, we introduce toric rings of integral polytopes. An *integral polytope* (or *lattice polytope*) $P \subset \mathbb{R}^d$ is a polytope whose vertices sit in $\mathbb{Z}^d$. For an integral polytope $P \subset \mathbb{R}^d$, we define ϕ_P as the morphism of $\Bbbk$-algebras:

$$\phi_P : \Bbbk[x_{\mathbf{v}} : \mathbf{v} \in P \cap \mathbb{Z}^d] \to \Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}, t_{d+1}], \text{ induced by } \phi_P(x_{\mathbf{v}}) = \mathbf{t}^{\mathbf{v}} t_{d+1},$$

where $\mathbf{t}^{\mathbf{v}} = t_1^{v_1} \cdots t_d^{v_d}$ for $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$. Then, the kernel of ϕ_P, denoted by I_P, is called the *toric ideal* of P. Moreover, the image of ϕ_P, denoted by $\Bbbk[P]$, is called the *toric ring* of P. Note that $\Bbbk[P] \cong \Bbbk[x_{\mathbf{v}} : \mathbf{v} \in P \cap \mathbb{Z}^d]/I_P$.

It is well known that the toric ideal I_P is generated by homogeneous binomials. The toric ring $\Bbbk[P]$ is a standard graded $\Bbbk$-subalgebra of $\Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}, t_{d+1}]$ by setting $\deg(\mathbf{t}^{\mathbf{v}} t_{d+1}) = 1$ for each $\mathbf{v} \in P \cap \mathbb{Z}^d$. The Krull dimension of $\Bbbk[P]$ is equal to the dimension of P plus 1.

For an integral polytope $P \subset \mathbb{R}^d$, let $\mathcal{A}_P = \{(\mathbf{v}, 1) \in \mathbb{Z}^{d+1} : \mathbf{v} \in P \cap \mathbb{Z}^d\}$ and let $\mathbb{Z}_{\geq 0}\mathcal{A}_P = \{a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n : \mathbf{x}_1, \dots, \mathbf{x}_n \in \mathcal{A}_P, a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}\}$. We define $\mathbb{Z}\mathcal{A}_P$ and $\mathbb{R}_{\geq 0}\mathcal{A}_P$ analogously. In particular, we let $C_P = \mathbb{R}_{\geq 0}\mathcal{A}_P$ and call it the *cone over* P. We say that P is normal if $\mathbb{Z}_{\geq 0}\mathcal{A}_P = \mathbb{Z}\mathcal{A}_P \cap C_P$.

The toric rings of normal polytopes have many good properties as follows:

Theorem 2.1.1 (cf. [84, Theorem 9.8.13])**.** *Let P be an integral polytope. Then the following are equivalent:*

(i) P is normal;

(ii) $\Bbbk[P]$ is normal;

(iii) $\Bbbk[P]$ is a Krull ring.

Therefore, $\Bbbk[P]$ is Cohen–Macaulay if P is normal (see [44]).

Theorem 2.1.2 ([75])**.** *Let P be a normal integral polytope. Then the ideal*

$$(t_1^{v_1}\cdots t_d^{v_d}t_{d+1}^{v_{d+1}} : (v_1,\dots,v_d,v_{d+1}) \in \mathrm{int}(C_P))$$

is isomorphic to the canonical module of $\Bbbk[P]$, where $\mathrm{int}(-)$ denotes the relative interior of a polytope or cone.

Normal toric rings can be regarded as toric rings arising from cones. To explain it, we introduce the definition of the toric ring of a cone.

Let $\mathsf{M} \cong \mathbb{Z}^d$ be a lattice of rank d and let $\mathsf{N} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{M},\mathbb{Z})$ be the dual lattice of M. We set $\mathsf{M}_{\mathbb{R}} = \mathsf{M}\otimes_{\mathbb{Z}}\mathbb{R}$ and $\mathsf{N}_{\mathbb{R}} = \mathsf{N}\otimes_{\mathbb{Z}}\mathbb{R}$ and denote the natural pairing by $\langle -,-\rangle : \mathsf{M}_{\mathbb{R}}\times\mathsf{N}_{\mathbb{R}} \to \mathbb{R}$ (when we consider the case $\mathsf{M} = \mathsf{N} = \mathbb{Z}^d$, we identify it with the usual inner product). We consider a strongly convex rational polyhedral cone

$$\tau = \mathrm{cone}(v_1,\cdots,v_n) = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n \subset \mathsf{N}_{\mathbb{R}}$$

of dimension d generated by $v_1,\cdots,v_n \in \mathsf{N}$ where $d \leq n$. We assume this system of generators is minimal and the generators are primitive, i.e., $\epsilon v_i \notin \mathsf{N}$ for any $0 < \epsilon < 1$. For each generator, we define a linear form $\sigma_i(-) := \langle -, v_i\rangle$ and denote $\sigma(-) = (\sigma_1(-),\cdots,\sigma_n(-))$. We consider the dual cone $\tau^\vee$:

$$\tau^\vee = \{\mathbf{x} \in \mathsf{M}_{\mathbb{R}} \mid \sigma_i(\mathbf{x}) \geq 0 \text{ for all } i \in [n]\},$$

where we let $[n] := \{1,\dots,n\}$. We now define the toric ring of $\tau^\vee$ with respect to M

$$R = \Bbbk[\tau^\vee \cap \mathsf{M}] = \Bbbk[t_1^{\alpha_1}\cdots t_d^{\alpha_d} : (\alpha_1,\cdots,\alpha_d) \in \tau^\vee \cap \mathsf{M}]. \tag{2.1.1}$$

The toric rings of cones are always normal and Cohen-Macaulay while that of polytopes are not necessarily so. If an integral polytope P is normal, then we can regard $\Bbbk[P]$ as the toric ring of the cone C_P with respect to the lattice $\mathbb{Z}\mathcal{A}_P$.

For each $\mathbf{a} = (a_1,\cdots,a_n) \in \mathbb{R}^n$, we set

$$\mathbb{T}(\mathbf{a}) = \{\mathbf{x} \in \mathsf{M} : \sigma_i(\mathbf{x}) \geq a_i \text{ for all } i \in [n]\}.$$

Then, we define the module $T(\mathbf{a})$ generated by all monomials whose exponent vector is in $\mathbb{T}(\mathbf{a})$. By the definition, we have $\mathbb{T}(0) = \tau^\vee \cap \mathsf{M}$ and $T(0) = R$. Moreover, we note some facts associated with the module $T(\mathbf{a})$ (see e.g., [9, Section 4.F]):

- Since $\sigma_i(\mathbf{x}) \in \mathbb{Z}$ for any $i \in [n]$ and any $\mathbf{x} \in \mathsf{M}$, we can see that $T(\mathbf{a}) = T(\ulcorner\mathbf{a}\urcorner)$, where $\ulcorner\ \urcorner$ means the round up and $\ulcorner\mathbf{a}\urcorner = (\ulcorner a_1\urcorner,\dots,\ulcorner a_n\urcorner)$.
- The module $T(\mathbf{a})$ is a divisorial ideal and any divisorial ideal of R takes this form. Therefore, we can identify each $\mathbf{a} \in \mathbb{Z}^n$ with the divisorial ideal $T(\mathbf{a})$.
- It is known that the isomorphic classes of divisorial ideals of R one-to-one correspond to the elements of the divisor class group $\mathrm{Cl}(R)$ of R. We see that for $\mathbf{a},\mathbf{a}' \in \mathbb{Z}^n$, $T(\mathbf{a}) \cong T(\mathbf{a}')$ if and only if there exists $\mathbf{y} \in \mathsf{M}$ such that $a_i = a_i' + \sigma_i(\mathbf{y})$ for all $i \in [n]$. Thus, we have $\mathrm{Cl}(R) \cong \mathbb{Z}^n/\sigma(\mathsf{M})$.

2.2 Polytopes

In this section, we recall some basic definitions and properties of polytopes. We refer the readers to e.g., [25, 87], for the introduction.

First, we introduce some classes of polytopes:

- A *pyramid* $P \subset \mathbb{R}^d$ is the convex hull of the union of a polytope $Q \subset \mathbb{R}^d$ (*basis* of P) and a point $v_0 \in \mathbb{R}^d$ (*apex* of P), where v_0 does not belong to the affine hull of Q. Note that the basis of a pyramid P is a facet of P.
- A polytope $P \subset \mathbb{R}^d$ is *simple* if each vertex of P is contained in precisely $\dim P$ facets.
- A polytopes P is called a $(0,1)$-*polytope* if its all vertices are $(0,1)$-vectors.

We can get the following proposition by observing the toric ideal:

Proposition 2.2.1. *Let $P \subset \mathbb{R}^d$ be a $(0,1)$-pyramid with basis Q. Then, $\Bbbk[P]$ is the polynomial extension of $\Bbbk[Q]$. In particular, we have*

$$\mathrm{Cl}(\Bbbk[P]) \cong \mathrm{Cl}(\Bbbk[Q]).$$

For a polytope P, let $\Psi(P)$ denote the set of facets of P. The *rank* of P is defined by

$$\operatorname{rank} P = F - (\dim P + 1).$$

Note that $\operatorname{rank} P$ is a nonnegative integer.

We define the *product* of two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ as

$$P \times Q = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in P, \mathbf{y} \in Q\} \subset \mathbb{R}^{d+e}.$$

We can see that $P \times Q$ is a polytope of dimension $\dim(P) + \dim(Q)$, whose nonempty faces are the products of nonempty faces (including itself) of P and Q. In particular, the number of facets of $P \times Q$ is equal to $|\Psi(P)| + |\Psi(Q)|$, and hence we have $\operatorname{rank} P \times Q = \operatorname{rank} P + \operatorname{rank} Q + 1$.

The toric ring of the product of two integral polytopes corresponds to the "Segre product" of these toric rings; let P_1 and P_2 be two integral polytopes, then $\Bbbk[P_1 \times P_2]$ is isomorphic to the Segre product of $\Bbbk[P_1]$ and $\Bbbk[P_2]$. Here, for two standard $\Bbbk$-algebras $R = \bigoplus_{n \geq 0} R_n$ and $S = \bigoplus_{n \geq 0} S_n$, we define their *Segre product* $R \# S$ as the graded $\Bbbk$-algebra:

$$R \# S = (R_0 \otimes_{\Bbbk} S_0) \oplus (R_1 \otimes_{\Bbbk} S_1) \oplus \cdots \subset R \otimes_{\Bbbk} S.$$

We denote a homogeneous element $x \otimes_{\Bbbk} y \in R_i \otimes_{\Bbbk} S_i$ by $x \# y$.

Simple $(0,1)$-polytopes have the following trivial structure:

Lemma 2.2.2 ([47, Theorem 1])**.** *A $(0,1)$-polytope $P \subset \mathbb{R}^d$ is simple if and only if it is equal to a product of $(0,1)$-simplices.*

Let A and B be subsets of $\mathbb{R}^d$. Their *Minkowski sum* is defined as

$$A + B := \{x + y \ : \ x \in A, y \in B\}.$$

Note that we can regard $P \times Q$ as the Minkowski sum of polytopes, as follows. Let

$$P' = \left\{ (p, \underbrace{0, \dots, 0}_{e}) \in \mathbb{R}^{d+e} \ : \ p \in P \right\} \text{ and } Q' = \left\{ (\underbrace{0, \dots, 0}_{d}, q) \in \mathbb{R}^{d+e} \ : \ q \in Q \right\}.$$

Then, we can see that $P \times Q = P' + Q'$. Conversely, suppose two polytopes $P', Q' \subset \mathbb{R}^d$ satisfy the following condition: for all $i \in [d]$, we have that $\pi_i(P') = \{0\}$ or $\pi_i(Q') = \{0\}$, where $\pi_i : \mathbb{R}^d \to \mathbb{R}$ is the projection onto the i-th coordinate. Then we can regard $P' + Q'$ as the product of two polytopes.

Next, we introduce a special class of polytopes, which are called *compressed polytopes*. To define it, we recall associated terms.

Let $P \subset \mathbb{R}^d$ be an integral polytope. A *simplex belonging to* P is a subset F of $P \cap \mathbb{Z}^d$ for which $Q = \text{conv}(F)$ is a simplex of $\mathbb{R}^d$, i.e., $\dim Q = |F| - 1$. A *maximal* simplex belonging to P is a simplex belonging to P whose dimension is equal to $\dim P$. Every simplex belonging to P is a subset of a maximal simplex belonging to P. A maximal simplex F belonging to P is called *fundamental* if $\mathbb{Z}F = \mathbb{Z}(P \cap \mathbb{Z}^d)$. In other words, F attains the minimal volume among all simplices formed by taking convex hulls of points in the lattice by $P \cap \mathbb{Z}^d$.

A collection Δ of simplices belonging to P is called a *triangulation* of P if the following conditions are satisfied:

- $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$;
- F and G belong to Δ, then $\text{conv}(F) \cap \text{conv}(G) = \text{conv}(F \cap G)$;
- $P = \bigcup_{F \in \Delta} \text{conv}(F)$.

A triangulation Δ of an integral polytope $P \subset \mathbb{R}^d$ is called *unimodular* if every maximal simplex in the triangulation is fundamental.

Let $P \subset \mathbb{R}^d$ be an integral polytope and $p_1, \dots, p_k$ an ordered list of the integral points in P. The *pulling triangulation* $\Delta_{\text{pull}}(P)$ induced by this ordering is constructed recursively as follows: If $p_1, \dots, p_k$ are affinely independent, then $\Delta_{\text{pull}}(P) = \{\{p_1, \dots, p_k\}\}$. Otherwise, we set

$$\Delta_{\text{pull}}(P) = \bigcup_{F} \{\{p_1\} \cup \sigma : \sigma \in \Delta_{\text{pull}}(F)\}$$

where the union runs over all facets F of P not containing p_1, and the ordering of the integral points in F is the ordering induced by the ordering of the integral vectors in P.

We are now ready to define a compressed polytope.

Definition 2.2.3 ([76])**.** An integral polytope P is *compressed* if every pulling triangulation of P using the integral points in P is unimodular.

Note that if P possesses a unimodular triangulation, then P is normal (cf. [29, Corollary 4.12]). In particular, P is normal if P is compressed.

A characterization of the compressed integral polytopes is known in terms of their facet defining inequalities. For $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, we denote by $H^+(\mathbf{a}; b)$ (resp. $H(\mathbf{a}; b)$) a closed half-space $\{\mathbf{u} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{a} \rangle + b \geq 0\}$ (resp. an affine hyperplane $\{\mathbf{u} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{a} \rangle + b = 0\}$). In particular, we denote the linear hyperplane $H(\mathbf{a}; 0)$ by $H_{\mathbf{a}}$.

For each $F \in \Psi(P)$, there exist a vector $\mathbf{a}_F \in \mathbb{Q}^d$ and a rational number b_F with the following conditions:

(d1) $H(\mathbf{a}_F; b_F)$ is a support hyperplane associated with F and $P \subset H^+(\mathbf{a}_F; b_F)$;

(d2) $d_F(\mathbf{v}) \in \mathbb{Z}$ for any $\mathbf{v} \in P \cap \mathbb{Z}^d$;

(d3) $\sum_{\mathbf{v} \in P \cap \mathbb{Z}^d} d_F(\mathbf{v})\mathbb{Z} = \mathbb{Z}$,

where $d_F(\mathbf{v}) = \langle \mathbf{v}, \mathbf{a}_F \rangle + b_F$ for $\mathbf{v} \in \mathbb{Z}^d$. We can see that $d_F(\mathbf{v})$ for $\mathbf{v} \in P \cap \mathbb{Z}^d$ is independent of the choice of $\mathbf{a}_F$ and b_F.

Lemma 2.2.4 ([79, Theorem 2.4]). *Let P be an integral polytope. Then the following conditions are equivalent:*

(i) P is compressed.

(ii) For any $F \in \Psi(P)$, $|\{d_F(v) : v \in P \cap \mathbb{Z}^d, d_F(v) \neq 0\}| = 1$.

Finally, we give a theorem which enables us to determine when two toric rings are isomorphic.

We say that P has the *integer decomposition property* (*IDP*, for short) if for any $n \in \mathbb{Z}_{>0}$ and any $\mathbf{a} \in nP \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_n \in P \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_n$. It is known that P is normal if P has IDP. Moreover, if $\mathbb{Z}\mathcal{A}_P = \mathbb{Z}^{d+1}$, then P is normal if and only if P has IDP (cf.[29, Theorem 4.7]).

Theorem 2.2.5 ([9, Theorem 5.22]). *For two IDP polytopes P and P', the toric rings $\Bbbk[P]$ and $\Bbbk[P']$ are isomorphic as graded algebras if and only if P and P' are unimodularly equivalent.*

Here, we say that two integral polytopes $P, P' \subset \mathbb{R}^d$ are *unimodularly equivalent* if there are an integral vector $\mathbf{v} \in \mathbb{Z}^d$ and a unimodular transformation $f \in \mathrm{GL}_d(\mathbb{Z})$ such that $P' = f(P) + \mathbf{v}$.

2.3 Graph theory

At the end of this chapter, we prepare some notions and notation on (directed) graphs. We refer the reader to e.g., [17, 21] for the introduction to graph theory.

Throughout this thesis, we assume that graphs are finite and have no loops and no multiple edges. Let G be a graph on the vertex set $V(G)$ with the edge set $E(G)$. For a subset $W \subset V(G)$, let G_W denote the induced subgraph with respect to W. For a vertex v, we denote by $G \setminus v$ instead of $G_{V(G)\setminus\{v\}}$. Similarly, for $S \subset V(G)$, we denote by $G \setminus S$ instead of $G_{V(G)\setminus S}$.

For a subgraph G' of G and $S \subset V(G)$, we define $G' + S$ to be the subgraph of G on the vertex set $V(G') \cup S$ with the edge set $E(G') \cup \{\{v, w\} : v \in S, w \in V(G'), \{v, w\} \in E(G)\}$. Similarly, for $v \in V(G)$, we denote by $G' + v$ instead of $G' + \{v\}$. Given $v \in V(G)$, let $N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$. For $S \subset V(G)$, let $N_G(S) = \bigcup_{v \in S} N_G(v)$.

We recall what bipartite graphs and complete multipartite graphs are. A graph G is called *bipartite* if $V(G)$ can be decomposed into two sets V_1, V_2, called the partition, such that $E(G) \subset V_1 \times V_2$. Let $K_{r_1,\ldots,r_n}$ be the graph on the vertex set $\bigsqcup_{k=1}^n V_k$, $|V_k| = r_k$ for $k = 1, \ldots, n$ and $1 \leq r_1 \leq \cdots \leq r_n$, with the edge set $\{\{u, v\} : u \in V_i, v \in V_j, 1 \leq i < j \leq n\}$. This graph $K_{r_1,\ldots,r_n}$ is called the *complete multipartite graph* with type $(r_1, \ldots, r_n)$.

We always denote the number of vertices of $K_{r_1,\dots,r_n}$ by d, i.e., $d = \sum_{i=1}^{n} r_i$. In the case $n = 2$, we call K_{r_1,r_2} a *complete bipartite graph.* In the case $r_1 = \cdots = r_n = 1$, we call $K_{\underbrace{1,\dots,1}_{n}}$, denoted by K_n, a *complete graph.*

For a graph G, a *path* is a non-empty subgraph $P = p_0p_1 \cdots p_k$ of G on the vertex set $V(P) = \{p_0, p_1, \dots, p_k\}$ with the edge set $E(P) = \{\{p_0, p_1\}, \{p_1, p_2\}, \dots, \{p_{k-1}, p_k\}\}$, where p_i's are all distinct. Then we say that the vertices p_0 and p_k are *connected by* P and p_0 and p_k are called its *end vertices* or *ends.* The *interior* of P, denoted by P°, is the vertices except for p_0, p_k. A *cycle* is a non-empty subgraph $C = p_0p_1 \cdots p_kp_0$ on the vertex set $V(C) = \{p_0, p_1, \dots, p_k\}$ with the edge set $E(C) = \{\{p_0, p_1\}, \{p_1, p_2\}, \dots, \{p_{k-1}, p_k\}, \{p_k, p_0\}\}$, where p_i's are all distinct.

For an edge e which is not an edge of a path P (resp. a cycle C), e is called a *chord* of P (resp. C) if e joins two vertices of P which are not end vertices (resp. two vertices of C). A path (resp. cycle) which has no chord is called *primitive* (resp. a *circuit*). We also say that a graph is *chordal* if each of its cycles of length at least 4 has a chord. If G is a graph with induced subgraphs G_1, G_2 and S, such that $V(G) = V(G_1) \cup V(G_2)$ and $V(S) = V(G_1) \cap V(G_2)$, we say that G arises from G_1 and G_2 by *pasting* these graphs together along S. A characterization of chordal graphs is known as follows:

Proposition 2.3.1 (cf. [17, Proposition 5.5.1])**.** *A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from a complete graph.*

We say that $T \subset V(G)$ is an *independent set* or a *stable set* (resp. a *clique*) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in T$. Note that the empty set and each singleton are regarded as independent sets, and we call such independent sets *trivial.*

We introduce two ways to construct a new graph from two given graphs G_1 and G_2:

- Suppose that $V(G_1) \cap V(G_2)$ is a clique of both G_1 and G_2. Then the *clique sum* $G_1 \sharp G_2$ of G_1 and G_2 along $V(G_1) \cap V(G_2)$ is the graph on the vertex set $V(G_1) \cup V(G_2)$ with the edge set $E(G_1) \cup E(G_2)$.
- Suppose that $V(G_1) \cap V(G_2) = \emptyset$. Then the *join* $G_1 + G_2$ of G_1 and G_2 is the graph on the vertex set $V(G_1) \cup V(G_2)$ with the edge set $E(G_1) \cup E(G_2) \cup \{\{i, j\} : i \in V(G_1), j \in V(G_2)\}$.

For a graph G, the *chromatic number* $\chi(G)$ of G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The *clique number* $\omega(G)$ of G is the maximum cardinality of a clique of G. Note that $\chi(G) \geq \omega(G)$ holds for any graph G. We say that a graph G is *perfect* if for any induced subgraph H of G, the equality $\chi(H) = \omega(H)$ holds. It is known that every chordal graph is perfect (cf. [17, Proposition 5.5.2]).

For a connected graph G, a subgraph T of G is called a *spanning tree* if T is a connected graph with $V(T) = V(G)$ and contains no cycles. For each $e \in E(G) \setminus E(T)$, there is a unique cycle C_e in $T + e$, where $T + e$ is the subgraph of G on the vertex set $V(T)$ with the edge set $E(T) \cup \{e\}$. We call C_e the *fundamental cycle* of e with respect to T.

The *flow space* of a directed graph A is the subspace of $\mathbb{R}^{E(A)}$ generated by the vectors $x \in \mathbb{R}^{E(A)}$ such that $D_A x = 0$, where D_A is the *incidence matrix* of A, which is the $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of A, respectively, such that the ve-entry of D_A is equal to 1 if the vertex v is the head of the edge e, -1 if v is the tail of e, and 0 otherwise. Let $C = v_0 v_1 \cdots v_m v_0$ be a cycle in A. Using the orientation of A, the cycle C determines an element $\mathbf{v}(C) \in \mathbb{R}^{E(A)}$ as follows:

$$\mathbf{v}(C)^{(e)} = \begin{cases} 0 & \text{if } e \notin E(C), \\ 1 & \text{if } e = \{v_i, v_{i+1}\} \text{ and } v_{i+1} \text{ is the head of } e, \\ -1 & \text{if } e = \{v_i, v_{i+1}\} \text{ and } v_{i+1} \text{ is the tail of } e. \end{cases}$$

We refer to $\mathbf{v}(C)$ as the *signed characteristic vector* of C. It is known that the signed characteristic vectors of the fundamental cycles with respect to a spanning tree of A form bases of the flow space of A. For a cycle C in A, we set

$$\operatorname{supp}^+(C) = \{e \in E(C) : \mathbf{v}(C)^{(e)} > 0\} \quad \text{and} \quad \operatorname{supp}^-(C) = \{e \in E(C) : \mathbf{v}(C)^{(e)} < 0\}.$$

Chapter 3

Some classes of toric rings

In this chapter, we introduce certain classes of toric rings; Ehrhart rings, Hibi rings, stable set rings and edge rings, which will be studied in later parts.

3.1 Ehrhart rings

Here, we recall the Ehrhart rings of integral polytopes. Let $P \subset \mathbb{R}^d$ be an integral polytope. The *Ehrhart ring* $A(P)$ of P is the toric ring of the cone C_P with respect to $\mathbb{Z}^{d+1}$, i.e.,

$$A(P) = \Bbbk[C_P \cap \mathbb{Z}^{d+1}] = \Bbbk[\mathbf{t}^x s^k : k \in \mathbb{Z}_{>0} \text{ and } x \in kP \cap \mathbb{Z}^d],$$

where $\mathbf{t}^x = t_1^{x_1} \cdots t_d^{x_d}$ and $x = (x_1, \ldots, x_d) \in kP \cap \mathbb{Z}^d$. Note that the Ehrhart ring of P is a normal affine semigroup ring, and hence it is Cohen-Macaulay. Moreover, we can regard $A(P)$ as an graded $\Bbbk$-algebra by setting $\deg(\mathbf{t}^x s^k) = k$ for each $x \in kP \cap \mathbb{Z}^d$.

We can see that $\Bbbk[P] = A(P)$ if and only if P has IDP

In addition, we recall the definitions of (polar) duality and reflexivity of polytopes and the characterization when Ehrhart rings are Gorenstein. Let $P \subset \mathbb{R}^d$ be a polytope. Its *(polar) dual* is

$$P^* = \left\{ n \in \mathbb{R}^d \ : \ \langle n, x \rangle \geq -1 \text{ for all } x \in P \right\}.$$

We call P *reflexive* if both P and P^* are integral polytopes.

Theorem 3.1.1 ([16, Theorem 1.1])**.** *The following are equivalent:*

(i) The Ehrhart ring $A(P)$ is Gorenstein.

(ii) There exist $\delta \in \mathbb{Z}_{>0}$ and $\mathbf{v} \in \mathbb{Z}^d$ such that $\delta P - \mathbf{v}$ is reflexive.

3.2 Three families of toric rings

In this thesis, "three families" of toric rings means Hibi rings, stable set rings and edge rings, which are toric rings arising from posets or graphs. We will mention these definitions and their properties.

3.2.1 Hibi rings

In this subsection, we recall Hibi rings and order polytopes of posets, which are introduced in [32] and [78], respectively.

Let Π be a finite poset equipped with a partial order $\prec$. For a subset $I \subset \Pi$, we say that I is a *poset ideal* of Π if $p \in I$ and $q \prec p$ imply $q \in I$. For a subset $A \subset \Pi$, we call A an *antichain* of Π if $p \not\prec q$ and $q \not\prec p$ for any $p, q \in A$ with $p \neq q$. Note that $\emptyset$ is regarded as a poset ideal and an antichain.

For a poset $\Pi = \{p_1, \dots, p_d\}$, let

$$\mathcal{O}_\Pi = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \ x_i \geq x_j \text{ if } p_i \prec p_j \text{ in } \Pi, \ \ 0 \leq x_i \leq 1 \text{ for } i = 1, \dots, d\}.$$

A convex polytope $\mathcal{O}_\Pi$ is called the *order polytope* of Π. It is known that $\mathcal{O}_\Pi$ is a $(0,1)$-polytope and the vertices of $\mathcal{O}_\Pi$ one-to-one correspond to the poset ideals of Π ([78]). In fact, a $(0,1)$-vector $(a_1, \dots, a_d)$ is a vertex of $\mathcal{O}_\Pi$ if and only if $\{p_i \in \Pi : a_i = 1\}$ is a poset ideal. The toric ring $\Bbbk[\mathcal{O}_\Pi]$ is called the *Hibi ring* of Π. We denote the Hibi ring of Π by $\Bbbk[\Pi]$ instead of $\Bbbk[\mathcal{O}_\Pi]$ for short.

It is known that

- order polytopes are compressed ([66]). In particular, order polytopes have IDP;
- $\Bbbk[\Pi]$ is Gorenstein if and only if Π is *pure*, i.e., all of the maximal chains in Π have the same length ([32]).

The Hibi ring of a poset can be described as the toric ring arising from a rational polyhedral cone as follows. Let $\Pi = \{p_1, \cdots, p_{d-1}\}$. For $p_i, p_j \in \Pi$ with $p_j \prec p_i$, we say that p_i *covers* p_j if there is $p \in \Pi$ with $p_j \preceq p \preceq p_i$ then $p = p_j$ or $p = p_i$. Thus, the edge $\{p_i, p_j\}$ of the Hasse diagram $\mathcal{H}(\Pi)$ of Π if and only if p_i covers p_j or p_j covers p_i. Set $\widehat{\Pi} = \Pi \cup \{\hat{0}, \hat{1}\}$, where $\hat{0}$ (resp. $\hat{1}$) is the unique minimal (resp. maximal) element not belonging to Π. Let us denote $p_0 = \hat{0}$ and $p_d = \hat{1}$. For each edge $e = \{p_i, p_j\}$ of $\mathcal{H}(\widehat{\Pi})$ with $p_i \prec p_j$, let σ_e be a linear form in $\mathbb{R}^d$ defined by

$$\sigma_e(\mathbf{x}) = \begin{cases} x_i - x_j & \text{if } j \neq d, \\ x_i & \text{if } j = d \end{cases}$$

for $\mathbf{x} = (x_0, x_1, \cdots, x_{d-1})$. Let $\tau_\Pi = \text{cone}(\sigma_e : e \text{ is an edge of } \mathcal{H}(\widehat{\Pi})) \subset \mathbb{R}^d$. Then, we can see that $\Bbbk[\Pi] = \Bbbk[\tau_\Pi^\vee \cap \mathbb{Z}^d]$. Let $e_1, \cdots, e_n$ be all the edges of $\mathcal{H}(\widehat{\Pi})$. We set a linear form $\sigma_\Pi : \mathbb{R}^d \to \mathbb{R}^n$ by

$$\sigma_\Pi(\mathbf{x}) = (\sigma_{e_1}(\mathbf{x}), \cdots, \sigma_{e_n}(\mathbf{x})) \tag{3.2.1}$$

for $\mathbf{x} \in \mathbb{R}^d$.

Let Π and Π' be two posets with $\Pi \cap \Pi' = \emptyset$. The *disjoint union* of Π and Π' is the poset $\Pi + \Pi'$ on $\Pi \cup \Pi'$ such that $x \preceq y$ in $\Pi + \Pi'$ if (a) $x, y \in \Pi$ and $x \preceq y$ in Π, or (b) $x, y \in \Pi'$ and $x \preceq y$ in Π'. The *ordinal sum* of Π and Π' is the poset $\Pi \oplus \Pi'$ on $\Pi \cup \Pi' \cup \{z\}$ such that $x \preceq y$ in $\Pi \oplus \Pi'$ if (a) $x, y \in \Pi$ and $x \preceq y$ in Π, (b) $x, y \in \Pi'$ and $x \preceq y$ in Π', (c) $x \in \Pi$ and $y = z$, or (d) $x = z$ and $y \in \Pi'$, where z is a new element which is not contained in $\Pi \cup \Pi'$. By observing poset ideals of $\Pi + \Pi'$ and $\Pi \oplus \Pi'$, the following proposition holds:

Proposition 3.2.1. *Let Π and Π' be two posets with $\Pi \cap \Pi' = \emptyset$.*

(i) One has $\Bbbk[\Pi + \Pi'] \cong \Bbbk[\Pi] \# \Bbbk[\Pi']$.

(ii) One has $\Bbbk[\Pi \oplus \Pi'] \cong \Bbbk[\Pi' \oplus \Pi] \cong \Bbbk[\Pi] \otimes_{\Bbbk} \Bbbk[\Pi']$.

We also recall another polytope arising from Π, which is defined as follows:

$$\mathcal{C}_\Pi = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for } i = 1, \dots, d, \\ x_{i_1} + \cdots + x_{i_k} \leq 1 \text{ for } p_{i_1} \prec \cdots \prec p_{i_k} \text{ in } \Pi\}.$$

A convex polytope $\mathcal{C}_\Pi$ is called the *chain polytope* of Π, which is introduced in [78]. Similarly to order polytopes, it is known that $\mathcal{C}_\Pi$ is a $(0,1)$-polytope and the vertices of $\mathcal{C}_\Pi$ one-to-one correspond to the antichains of Π ([78]).

In general, the order polytope and the chain polytope of Π are not unimodularly equivalent, but the following is known:

Theorem 3.2.2 ([35, Theorem 2.1])**.** *Let Π be a poset. Then $\mathcal{O}_\Pi$ and $\mathcal{C}_\Pi$ are unimodularly equivalent if and only if Π does not contain the "X-shape" subposet.*

Here, the "X-shape" poset is the poset $\{z_1, z_2, z_3, z_4, z_5\}$ equipped with the partial orders $z_1 \prec z_3 \prec z_4$ and $z_2 \prec z_3 \prec z_5$.

3.2.2 Stable set rings

In this subsection, we recall stable set rings, which are the toric rings of stable set polytopes of graphs introduced in [12].

Let G be a graph on the vertex set $V(G) = [d]$ with the edge set $E(G)$. Given a subset $W \subset V(G)$, let $\rho(W) = \sum_{i \in W} \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith unit vector of $\mathbb{R}^d$ for $i \in [d]$ and we let $\rho(\emptyset)$ be the origin of $\mathbb{R}^d$. We define an integral polytope associated with a graph G as follows:

$$\mathrm{Stab}_G = \mathrm{conv}(\{\rho(S) : S \text{ is a stable set}\}).$$

We call Stab_G the *stable set polytope* of G.

In what follows, we treat the stable set rings of *perfect graphs*. The reason why we focus on perfect graphs is derived from the following:

- A graph G is perfect if and only if Stab_G is compressed ([23, 66]). In particular, Stab_G has IDP if G is perfect.
- Suppose that G is perfect. Then $\Bbbk[\mathrm{Stab}_G]$ is Gorenstein if and only if G all maximal cliques of G have the same cardinality ([67, Theorem 2.1]).
- The facets of Stab_G are completely characterized when G is perfect ([12, Theorem 3.1]). More concretely, the facets of Stab_G are exactly defined by the following hyperplanes:

$$\begin{aligned} &H(\mathbf{e}_i; 0) \quad \text{for each } i \in [d]; \\ &H\left(-\sum_{j \in Q} \mathbf{e}_j; 1\right) \quad \text{for each maximal clique } Q. \end{aligned} \tag{3.2.2}$$

From those properties, the stable set ring of a perfect graph can be described as the toric ring arising from a rational polyhedral cone as well as Hibi rings. For a perfect graph G with maximal cliques $Q_0, Q_1, \dots, Q_n$ and $i \in \{0, 1, \dots, n+d\}$, let σ_i be a linear form in $\mathbb{R}^{d+1}$ defined by

$$\sigma_i(\mathbf{x}) = \begin{cases} x_0 - \sum_{j \in Q_i} x_j & \text{if } i \in \{0, 1, \dots, n\}, \\ x_{i-n} & \text{if } i \in \{n+1, \dots, n+d\} \end{cases}$$

for $\mathbf{x} = (x_0, x_1, \cdots, x_d)$. Let $\tau_G = \text{cone}(\sigma_i : i \in \{0, 1, \dots, n+d\}) \subset \mathbb{R}^{d+1}$. Then, we can see that $\Bbbk[\text{Stab}_G] = \Bbbk[\tau_G^\vee \cap \mathbb{Z}^d]$. We set a linear form $\sigma_G : \mathbb{R}^{d+1} \to \mathbb{R}^{n+d+1}$ by

$$\sigma_G(\mathbf{x}) = (\sigma_0(\mathbf{x}), \cdots, \sigma_{n+d}(\mathbf{x})) \in \mathbb{R}^{n+d+1} \tag{3.2.3}$$

for $\mathbf{x} \in \mathbb{R}^{d+1}$.

Given a poset Π, we define the *comparability graph* of P, denoted by $G(\Pi)$, as a graph on the vertex set $V(G(\Pi)) = [d]$ with the edge set

$$E(G(\Pi)) = \{\{i, j\} : p_i \text{ and } p_j \text{ are comparable in } \Pi\}.$$

It is known that $G(\Pi)$ is perfect for any Π (see e.g. [17, Section 5.5]) and the cliques of $G(\Pi)$ one-to-one correspond to the chains of Π. Moreover, we see that $\mathcal{C}_\Pi = \text{Stab}_{G(\Pi)}$ and if Π does not contain the X-shape, then the stable set ring $\Bbbk[\mathcal{C}_\Pi] = \Bbbk[\text{Stab}_{G(\Pi)}]$ is isomorphic to $\Bbbk[\Pi]$ by Theorem 3.2.2.

3.2.3 Edge rings

In this subsection, we recall edge rings and edge polytopes of graphs, which began to be studied by Ohsugi–Hibi ([64]) and Simis–Vasconcelos–Villarreal ([69]).

For a positive integer d, consider a graph G on the vertex set $V(G) = [d]$ with the edge set $E(G)$. We define an integral polytope associated to G as follows:

$$P_G = \text{conv}(\{\rho(\{v, w\}) : \{v, w\} \in E(G)\}).$$

We call P_G the *edge polytope* of G.

Moreover, we also define the edge ring of G, denoted by $\Bbbk[G]$, as a subalgebra of the polynomial ring $\Bbbk[t_1, \dots, t_d]$ in d variables over a field $\Bbbk$ as follows:

$$\Bbbk[G] = \Bbbk[t_i t_j : \{i, j\} \in E(G)].$$

Note that the edge ring of G is nothing but the toric ring of P_G. We have that $\dim P_G = d - b(G) - 1$, where $b(G)$ is the number of bipartite connected components of G (see [84, Proposition 10.4.1]). Thus, $\dim \Bbbk[G] = d - b(G)$.

It is known that $\Bbbk[G]$ is normal if and only if G satisfies the *odd cycle condition*, i.e., for each pair of odd cycles C and C' with no common vertex, there is an edge $\{v, v'\}$ with $v \in V(C)$ and $v' \in V(C')$ (see [84, Corollary 10.3.11]).

The following terminologies are used in [64]:

- We call a vertex v of G *regular* (resp. *ordinary*) if each connected component of $G \setminus v$ contains an odd cycle (resp. if $G \setminus v$ is connected). Note that a non-ordinary vertex is usually called a *cut vertex* of G.
- Given an independent set $T \subset V(G)$, let $B(T)$ denote the bipartite graph on $T \cup N_G(T)$ with the edge set $\{\{v, w\} : v \in T, w \in N_G(T)\} \cap E(G)$.
- When G has at least one odd cycle, a non-empty set $T \subset V(G)$ is said to be a *fundamental set* if the following conditions are satisfied:
  - $B(T)$ is connected;
  - $V(B(T)) = V(G)$, or each connected component of $G \setminus V(B(T))$ contains an odd cycle.
- When G is a bipartite graph, a non-empty set $T \subset V(G)$ is said to be an *acceptable set* if the following conditions are satisfied:
  - $B(T)$ is connected;
  - $G \setminus V(B(T))$ is a connected graph with at least one edge.

Given $i \in [d]$, we denote by H_i (resp. H_i^+) instead of $H_{\mathbf{e}_i}$ (resp. $H_{\mathbf{e}_i}^+$). In addition, given an independent set $T \subset [d]$, let

$$H_T = H\left(\sum_{j \in N_G(T)} \mathbf{e}_j - \sum_{i \in T} \mathbf{e}_i; 0\right).$$

We define H_T^+ analogously. It is proved in [64, Theorem 1.7] that for any connected non-bipartite (resp. bipartite) graph G, each facet of P_G is defined by a supporting hyperplane H_i for some regular (resp. ordinary) vertex i or H_T for some fundamental (resp. acceptable) set. We denote these facets by F_i and F_T, respectively.

Let $\widetilde{\Psi} = \widetilde{\Psi}_r \cup \widetilde{\Psi}_f$ (resp. $\widetilde{\Psi} = \widetilde{\Psi}_o \cup \widetilde{\Psi}_a$) if G is non-bipartite (resp. bipartite), where

$$\begin{aligned}
&\widetilde{\Psi}_r = \{\mathbf{e}_i : i \text{ is a regular vertex}\}, \ \widetilde{\Psi}_o = \{\mathbf{e}_i : i \text{ is an ordinary vertex}\},\\
&\widetilde{\Psi}_f = \left\{\ell_T = \sum_{j \in N_G(T)} \mathbf{e}_j - \sum_{i \in T} \mathbf{e}_i : T \text{ is a fundamental set with } V(B(T)) \neq V(G)\right\} \cup \\
&\left\{\ell_T = \frac{1}{2}\left(\sum_{j \in N_G(T)} \mathbf{e}_j - \sum_{i \in T} \mathbf{e}_i\right) : T \text{ is a fundamental set with } V(B(T)) = V(G)\right\},\\
&\widetilde{\Psi}_a = \left\{\ell_T = \sum_{j \in N_G(T)} \mathbf{e}_j - \sum_{i \in T} \mathbf{e}_i : T \subset V_1 \text{ is an acceptable set}\right\}.
\end{aligned} \tag{3.2.4}$$

Then we have $\langle -, \mathbf{e}_i \rangle = d_{F_i}(-)$ (resp. $\langle -, \ell_T \rangle = d_{F_T}(-)$) for an appropriate vertex i (resp. independent set T). Note that $\frac{1}{2}$ appears in the case of $V(B(T)) = V(G)$ since $\langle \ell_T, \rho(e) \rangle = 0$ or 2 in this case, while $\langle \ell_T, \rho(e) \rangle = 1$ happens otherwise.

Although $\widetilde{\Psi}$ describes all supporting hyperplanes of the facets of P_G, it might happen that H_i and H_T define the same facet for some i and T if G is bipartite. The following proposition tells us an irreducible facet representation of P_G of a bipartite graph G:

Proposition 3.2.3. *Let G be a connected bipartite graph that has the partition $V(G) = V_1 \sqcup V_2$ and let $\widetilde{\Psi}' = \{\mathbf{e}_i : i \text{ is an ordinary vertex}\} \cup \{\ell_T : T \subset V_1 \text{ is an acceptable set}\}$. Then $\left\{H_{\mathbf{n}} : \mathbf{n} \in \widetilde{\Psi}'\right\}$ is the irredundant set of supporting hyperplanes of the facets of P_G.*

Proof. We show that we can choose the set of acceptable sets T as a subset of V_1 and it is irredundant. It easily follows that either $T \subset V_1$ or $T \subset V_2$ holds if T is acceptable. If $T \subset V_1$ is acceptable, then $B(T)$ and $G \setminus V(B(T))$ are connected with at least one edge. Therefore, set $T' = V_2 \setminus N_G(T)$ and we can see that $B(T') = G \setminus V(B(T))$ and $G \setminus V(B(T')) = B(T)$, so T' is an acceptable set contained in V_2. Conversely, if $S \subset V_2$ is acceptable, then there exists an acceptable set $S' \subset V_1$ with $S = V_2 \setminus N_G(S')$. Thus, acceptable sets contained in V_1 one-to-one correspond to ones contained in V_2. Moreover, for an acceptable set $T \subset V_1$, H_T and $H_{T'}$ define the same facet since P_G is contained in the hyperplane

$$\left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i \in V_1} x_i = \sum_{j \in V_2} x_j = 1\right\}.$$

This implies that $\displaystyle\sum_{j \in N_G(T)} x_j - \sum_{i \in T} x_i = \sum_{i \in N_G(T')} x_i - \sum_{j \in T'} x_j.$ □

Part II

Divisor class groups

Chapter 4

Preliminaries

In this part, we will study the divisor class groups of toric rings. More precisely, we will discuss the torsionfreeness of divisor class groups and the toric rings which have small divisor class groups.

4.1 Computation of divisor class groups of toric rings

In this section, we give an algorithm to compute divisor class groups of toric rings of normal integral polytopes. We use theories in [84, Section 9.8].

Let P be a normal integral polytope and assume that C_P has the irreducible representation:

$$C_P = \mathrm{aff}(\mathcal{A}_P) \cap \left(\bigcap_{F \in \Psi(P)} H^+_{\mathbf{c}_F} \right), \tag{4.1.1}$$

where $\mathrm{aff}(\mathcal{A}_P)$ denotes the affine hull of $\mathcal{A}_P$ and $\mathbf{c}_F := (\mathbf{a}_F, b_F) \in \mathbb{Q}^{d+1}$. Given $\mathbf{v} \in P \cap \mathbb{Z}^d$, we define $\mathbf{w_v}$ belonging to a free abelian group $\mathcal{F} = \bigoplus_{F \in \Psi(P)} \mathbb{Z}\epsilon_F$ with its basis $\{\epsilon_F\}_{F \in \Psi(P)}$ as follows:

$$\mathbf{w_v} = \sum_{F \in \Psi(P)} \langle (\mathbf{v}, 1), \mathbf{c}_F \rangle \epsilon_F = \sum_{F \in \Psi(P)} d_F(\mathbf{v}) \epsilon_F. \tag{4.1.2}$$

Let

$$\mathcal{S} = \sum_{\mathbf{v} \in P \cap \mathbb{Z}^d} \mathbb{Z}\mathbf{w_v} = \left\{ \sum_{F \in \Psi(P)} \langle \mathbf{v}', \mathbf{c}_F \rangle \epsilon_F : \mathbf{v}' \in \mathbb{Z}\mathcal{A}_P \right\} \tag{4.1.3}$$

and let $\mathcal{M}_P$ be the matrix whose column vectors consist of $\mathbf{w_v}$ for $\mathbf{v} \in P \cap \mathbb{Z}^d$, that is, $\mathcal{M}_P = (d_F(\mathbf{v}))_{F \in \Psi(P), \mathbf{v} \in P \cap \mathbb{Z}^d}$. Then we can compute the divisor class group of $\Bbbk[P]$ as follows:

Theorem 4.1.1 (cf. [84, Theorem 9.8.19])**.** *Work with the same notation as above and suppose that P is normal. Then, we have*

$$\mathrm{Cl}(\Bbbk[P]) \cong \mathcal{F}/\mathcal{S}.$$

In particular, we have

$$\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}^r \oplus \mathbb{Z}/s_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/s_m\mathbb{Z},$$

where $m = \dim P + 1$*,* $r = |\Psi(P)| - m$ *and* $s_1, \ldots, s_m$ *are positive integers appearing in the diagonal of the Smith normal form of* $\mathcal{M}_P$*. Moreover, the rank of* $\mathrm{Cl}(\Bbbk[P])$ *coincides with that of* P*.*

The integers $s_1, \ldots, s_m$ are called the *invariant factors* of $\mathcal{M}_P$. It is known that $s_i = g_i(\mathcal{M}_P)/g_{i-1}(\mathcal{M}_P)$ where $g_i(\mathcal{M}_P)$ denotes the greatest common divisor of all $i \times i$ minors of $\mathcal{M}_P$ and $g_0(\mathcal{M}_P) = 1$ (see, e.g., [62]).

Suppose that $P \subset \mathbb{R}^d$ is a d-dimensional normal integral polytope and satisfies $\mathbb{Z}\mathcal{A}(P) = \mathbb{Z}^{d+1}$. Moreover, we assume that $\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}^r$. We fix an isomorphism $\iota : \mathcal{F}/\mathcal{S} \to \mathbb{Z}^r$ and let $\beta_F := \iota(\epsilon_F)$ for each $F \in \Psi(P)$. We call β_F's the *weights* of $\Bbbk[P]$. As we will see in the following section, weights of $\Bbbk[P]$ tell us the combinatorial type of P.

4.2 Gale-diagrams

In this section, we recall the notation of Gale-diagrams of a polytope, which helps us to classify the isomorphic classes of toric rings. We refer the reader to e.g., [25, Sections 5.4 and 6.3] and [87, Section 6] for the introduction to Gale-transforms and Gale-diagrams.

Throughout this section, let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with the vertex set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and suppose that P has the irreducible representation (4.1.1).

We consider

$$D(P) = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \sum_{i \in [n]} \alpha_i \mathbf{v}_i = \mathbf{0} \text{ and } \sum_{i \in [n]} \alpha_i = 0 \right\}.$$

We can see that $D(P)$ is an $(n-d-1)$-dimensional subspace of $\mathbb{R}^d$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_{n-d-1}$ be a basis of $D(P)$. Moreover, for $i \in [n]$, let $\overline{\mathbf{b}_i}$ denote the i-th column vector of the $(n-d-1) \times n$ matrix $\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n-d-1} \end{pmatrix}$, that is, $\overline{\mathbf{b}_i} = (\mathbf{b}_1^{(i)}, \ldots, \mathbf{b}_{n-d-1}^{(i)})$, where for a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}^{(i)}$ denotes the i-th coordinate of $\mathbf{v}$.

Then, the n-tuple $(\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_n})$ is called the *Gale-transform* of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (or of P). Furthermore, let

$$\widehat{\mathbf{b}_i} = \begin{cases} \overline{\mathbf{b}_i}/||\overline{\mathbf{b}_i}|| & \text{if } \overline{\mathbf{b}_i} \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \overline{\mathbf{b}_i} = \mathbf{0} \end{cases}$$

for each $i \in [n]$, where $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ for a vector $\mathbf{v} \in \mathbb{R}^n$. Then, the n-tuple $(\widehat{\mathbf{b}_1}, \ldots, \widehat{\mathbf{b}_n})$ is called the *Gale-diagram* of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (or of P). Gale-transforms and Gale-diagrams depend on the choice of the basis of $D(P)$.

Especially, we are interested in the case $n-d-1 = 2$, that is, P just has $d+3$ vertices. In this case, we can draw a *standard* (or *reduced*) Gale-diagram from the Gale-diagram $\widehat{\mathbf{b}_1}, \ldots, \widehat{\mathbf{b}_n}$, which consists of the unit circle centered at the origin of $\mathbb{R}^2$ and diameters having at least one endpoint with multiplicity. See [25, Section 6.3] for the precise way to draw it.

We say that two polytopes Q and Q' are *combinatorially equivalent* or of the same *combinatorial type* (resp. *dual* to each other) if there exists a one-to-one mapping Φ between the set of all faces of Q and the set of all faces of Q' such that Φ is inclusion-preserving (resp. inclusion-reversing). Note that the classes of simplicial polytopes, which are polytopes whose facets are simplices, and simple polytopes are dual to each other.

According to [25, Sections 5.4 and 6.3], the following facts are known:

- Two d-dimensional polytopes with $d+3$ vertices are combinatorially equivalent if and only if their standard Gale-diagrams are orthogonally equivalent (i.e., isomorphic under an orthogonal linear transformation of $\mathbb{R}^2$ onto itself).
- P is simplicial if and only if no diameter of the standard Gale-diagram has both endpoints.

Clearly, all polytopes which are dual to P have the same combinatorial type. In what follows, whenever we consider dual polytopes, we focus on their combinatorial type, so we call them *the* dual polytope of P.

For the remainder of this subsection, we will explain how to obtain a Gale-diagram of the dual polytope P (the following results are a part of the author's work [52]). It is known that if the origin is an interior point of P, then P^* is a polytope and is dual to P. Moreover, $\{\mathbf{a}_F/b_F \ : F \in \Psi(P)\}$ is the vertex set of P^*.

The following theorem means that we can get a Gale-diagram of the dual polytope of P from weights of its toric ring.

Theorem 4.2.1. *Suppose that P is normal, $\mathbb{Z}\mathcal{A}_P = \mathbb{Z}^{d+1}$ and $\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}^r$. Let $\{\beta_F\}_{F\in\Psi(P)}$ be weights of $\Bbbk[P]$. Then, $(\beta_F/\|\beta_F\|)_{F\in\Psi(P)}$ is a Gale-diagram of the dual polytope of P.*

Before we give the proof, we prepare the following lemma:

Lemma 4.2.2. *Work with the same assumption and notation as Theorem 4.2.1. Then, we have $\sum_{F\in\Psi(P)} \beta_F^{(j)}\mathbf{a}_F = \mathbf{0}$ and $\sum_{F\in\Psi(P)} \beta_F^{(j)} b_F = 0$ for each $j \in [r]$.*

Proof. Let $\mathbf{e}_i$ be the i-th unit vector of $\mathbb{Z}^{d+1}$. Since $\mathbf{e}_i \in \mathbb{Z}\mathcal{A}_P$ for each $i \in [d+1]$, $\sum_{F\in\Psi(P)}\langle \mathbf{e}_i, \mathbf{c}_F\rangle \epsilon_F$ belongs to $\mathcal{S}$ from (4.1.3). Thus, by considering its image in $\mathbb{Z}^r$, we can obtain that $\sum_{F\in\Psi(P)} \mathbf{c}_F^{(i)}\beta_F = \mathbf{0}$ for each $i \in [d+1]$. This fact is equivalent to $\sum_{F\in\Psi(P)} \beta_F^{(j)}\mathbf{c}_F = \mathbf{0}$ for each $j \in [r]$. Therefore, we obtain the desired equations. □

Proof of Theorem 4.2.1. Let $\mathbf{u} \in \mathbb{R}^d$ be a vector such that the interior of $P+\mathbf{u}$ has $\mathbf{0}$. Then, $Q = (P+\mathbf{u})^*$ is the dual polytope of P. Moreover, $P+\mathbf{u}$ has the irreducible representation

$$P+\mathbf{u} = \bigcap_{F\in\Psi(P)} H^+(\mathbf{a}_F; b'_F),$$

where $b'_F = b_F - \langle \mathbf{a}_F, \mathbf{u} \rangle$. Note that $b'_F \neq 0$ for any $F \in \Psi(P)$ since $\mathbf{0}$ belongs to the interior of $P + \mathbf{u}$.

We show that $(b'_F \beta_F)_{F \in \Psi(P)}$ is a Gale-transform of Q. For each $j \in [r]$, we define the vector $\mathbf{b}_j$ of $\mathbb{R}^{|\Psi(P)|}$ as $\mathbf{b}_j := (b'_F \beta_F^{(j)})_{F \in \Psi(P)}$. Then, we can see that

$$\overline{\mathbf{b}_j} = (b'_F \beta_F^{(1)}, \dots, b'_F \beta_F^{(r)}) = b'_F \beta_F \quad \text{and} \quad \widehat{\mathbf{b}_j} = \beta_F / \|\beta_F\|.$$

Hence, it is enough to show that $\mathbf{b}_j$'s satisfy the following:

(i) for each $j \in [r]$, $\mathbf{b}_j$ is in $D(Q)$ and (ii) $\mathbf{b}_1, \dots, \mathbf{b}_r$ form a basis of $D(Q)$.

(i) Note that the vertex set of Q is $\{\mathbf{a}_F / b'_F \ : F \in \Psi(P)\}$. From Lemma 4.2.2, for each $j \in [r]$,

$$\sum_{F \in \Psi(P)} b'_F \beta_F^{(j)} \cdot \mathbf{a}_F / b'_F = \sum_{F \in \Psi(P)} \beta_F^{(j)} \mathbf{a}_F = \mathbf{0}.$$

Moreover,

$$\begin{aligned} \sum_{F \in \Psi(P)} b'_F \beta_F^{(j)} &= \sum_{F \in \Psi(P)} b_F \beta_F^{(j)} - \sum_{F \in \Psi(P)} \langle \mathbf{a}_F, \mathbf{u} \rangle \beta_F^{(j)} \\ &= 0 - \langle \sum_{F \in \Psi(P)} \beta_F^{(j)} \mathbf{a}_F, \mathbf{u} \rangle = -\langle \mathbf{0}, \mathbf{u} \rangle = 0. \end{aligned}$$

Therefore, we have $\mathbf{b}_j \in D(Q)$ for all $j \in [r]$.

(ii) Note that $D(Q)$ is an r-dimensional subspace of $\mathbb{R}^{|\Psi(P)|}$. Thus, it is enough to show that $\mathbf{b}_1, \dots, \mathbf{b}_r$ are linearly independent. Suppose that there exists $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}^r$ such that

$$a_1 \mathbf{b}_1 + \cdots + a_r \mathbf{b}_r = 0.$$

Then, for each $F \in \Psi(P)$, one has

$$a_1 (b'_F \beta_F^{(1)}) + \cdots + a_r (b'_F \beta_F^{(r)}) = b'_F \langle \mathbf{a}, \beta_F \rangle = 0.$$

Since $b'_F \neq 0$, we have $\langle \mathbf{a}, \beta_F \rangle = 0$ for any $F \in \Psi(P)$, equivalently, β_F's lie on the hyperplane $H_{\mathbf{a}}$. This is a contradiction to the fact that β_F's span $\mathcal{F}/\mathcal{S} \cong \mathbb{Z}^r$ as a semigroup ([11, Theorem 2]). □

Chapter 5

Torsionfreeness

In this chapter, we discuss the torsionfreeness of the divisor class groups of toric rings. In particular, we focus on three families of toric rings; Hibi rings, stable set rings and edge rings, and investigate their torsionfreeness. Moreover, we will give a sufficient condition for the divisor class group of the toric ring of an integral polytope P to be torsionfree in terms of P.

5.1 Divisor class groups of three families of toric rings

In this section, we discuss descriptions of the divisor class groups of Hibi rings, stable set rings and edge rings in terms of the underlying posets or graphs. As their corollary, we see that their divisor class groups are torsionfree. The contents of this section are contained in the author's paper [41] with A. Higashitani.

5.1.1 Divisor class groups of Hibi rings

First, we consider the divisor class groups of Hibi rings. In [28], the description of divisor class groups of Hibi rings is provided, which we describe below:

Theorem 5.1.1 ([28])**.** *Let Π be a poset with $|\Pi| = d$ and let n be the number of the edges of the Hasse diagram of $\widehat{\Pi}$. Then we have*

$$\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^{n-d-1}.$$

In particular, $\mathrm{Cl}(\Bbbk[\Pi])$ is torsionfree.

5.1.2 Divisor class groups of stable set rings

Next, we discuss the divisor class groups of stable set rings of perfect graphs.

Proposition 5.1.2. *Let G be a perfect graph with maximal cliques $Q_0, Q_1, \dots, Q_n$. Then $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^n$. In particular, $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G])$ is torsionfree.*

Proof. From Theorem 4.1.1 and the facet description (3.2.2), the rank of $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G])$ is equal to $|\Phi(\mathrm{Stab}_G)| - (d+1) = n$. Moreover, Stab_G contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d$ and has the facets defined by the hyperplanes $H_{\mathbf{e}_i}$ for each $i \in [d]$ and $H\left(-\sum_{j \in Q_0} \mathbf{e}_j; 1\right)$. Therefore,

$\mathcal{M}_{\mathrm{Stab}_G}$ has a $(d+1)\times(d+1)$ submatrix $\mathcal{N}$ which is a triangular matrix whose diagonal entries are equal to 1. Since $\det(\mathcal{N})=1$, we have $g_{d+1}(\mathcal{M}_P)=1$, implying that $\mathrm{Cl}(\mathrm{Stab}_G)$ is torsionfree. □

5.1.3 Divisor class groups of edge rings

Finally, we discuss the divisor class groups of edge rings of connected graphs satisfying the odd cycle condition.

The following lemma will be used for the proofs of our results in many times.

Lemma 5.1.3. *Let G be a non-bipartite connected graph.*

(1) Suppose that S is an independent set of G such that $B(S)$ is connected. Then there exists a fundamental set T such that $S\subset T$ and $V(B(T))=V(G)$.

(2) Let $C=p_0p_1\cdots p_{2k}p_0$ be a primitive odd cycle of length $2k+1$ in G. Then, for each $i=0,\dots,2k$, there exists a fundamental set T_i such that $E(C)\setminus\{p_i,p_{i+1}\}\subset E(B(T_i))$ and $\{p_i,p_{i+1}\}\notin E(B(T_i))$, where $p_{2k+1}=p_0$. In particular, G has at least $2k+1$ fundamental sets.

Proof. (1) If $V(G)=V(B(S))$, then S itself satisfies the required property. Suppose that $V(B(S))\subsetneq V(G)$. Then there exists $v\in V(G)\setminus V(B(S))$ such that v and w are adjacent for some $w\in N_G(S)$ since G is connected. Thus, $S'=S\cup\{v\}$ is an independent set and $B(S')$ is connected. We repeat this application and we eventually obtain S' which satisfies that $B(S')$ is connected and $V(B(S'))=V(G)$, as required.
(2) Fix $i=0$. By setting $S=\{p_2,p_4,\dots,p_{2k}\}$, we can see that S is an independent set since C is primitive and $B(S)$ is a connected graph with $E(C)\setminus\{p_0,p_1\}\subset E(B(S))$ and $\{p_i,p_{i+1}\}\notin E(B(S))$. A proof directly follows from (1). □

Remark 5.1.4. Let G be a non-bipartite connected graph with a cut vertex v and let $C_1,\dots,C_n$ be connected components of $G\setminus v$. For $i=1,\dots,n$, let $G_i=C_i+v$. Suppose that G_1 contains an odd cycle and let T be a fundamental set in G_1.

If $v\in V(B(T))$, then there exists a fundamental set T' in G with $V(B(T'))=V(B(T))\cup\bigcup_{i=2}^n V(G_i)$. We can construct it similarly to Lemma 5.1.3 (1). We call this fundamental set T' an *induced fundamental set* of T. Note that an induced fundamental set is not unique but for distinct fundamental sets T and S in G_1 with $v\in V(B(T))\cap V(B(S))$, their induced fundamental sets are distinct. Moreover, if v is a regular vertex in G, then there exists a fundamental set T'' in G with $V(B(T''))=\bigcup_{i=2}^n V(G_i)$ in the same way. We regard T'' as an induced fundamental set of the empty set although the empty set is not fundamental.

If $v\notin V(B(T))$, then T is also a fundamental set in G. Therefore, we can observe that $|\widetilde{\Psi}_f(G)|\geq|\widetilde{\Psi}_f(G_1)|$ and $|\widetilde{\Psi}_f(G)|\geq|\widetilde{\Psi}_f(G_1)|+1$ if v is regular in G.

Lemma 5.1.5. *Let G be a graph.*

(1) Let $e_1,\dots,e_{2k}$ be the edges of an even cycle in G. Then

$$\mathbf{w}_{\rho(e_1)},\dots,\mathbf{w}_{\rho(e_{2k})}$$

are linearly dependent, where $\mathbf{w}_{\mathbf{v}}$ is the same as defined in (4.1.2).

(2) Let C and C' be two odd cycles and let $e_1, \ldots, e_{2k+1}$ (resp. $e'_1, \ldots, e'_{2k'+1}$) be the edges of C (resp. C').

(2-1) Assume that C and C' have a unique common vertex. Then

$$\mathbf{w}_{\rho(e_1)}, \ldots, \mathbf{w}_{\rho(e_{2k+1})}, \mathbf{w}_{\rho(e'_1)}, \ldots, \mathbf{w}_{\rho(e'_{2k'+1})}$$

are linearly dependent.

(2-2) Assume that C and C' have no common vertex but there is a path whose edges are $f_1, \ldots, f_m$ between C and C' connecting them. Then

$$\mathbf{w}_{\rho(e_1)}, \ldots, \mathbf{w}_{\rho(e_{2k+1})}, \mathbf{w}_{\rho(e'_1)}, \ldots, \mathbf{w}_{\rho(e'_{2k'+1})}, \mathbf{w}_{\rho(f_1)}, \ldots, \mathbf{w}_{\rho(f_m)}$$

are linearly dependent.

Proof. (1) We see that

$$\sum_{i=1}^{2k}(-1)^i \mathbf{w}_{\rho(e_i)} = \sum_{i=1}^{2k}(-1)^i \sum_{\ell \in \widetilde{\Psi}} \langle \ell, \rho(e_i) \rangle \epsilon_\ell = \sum_{\ell \in \widetilde{\Psi}} \langle \ell, \sum_{i=1}^{2k}(-1)^i \rho(e_i) \rangle \epsilon_\ell = \sum_{\ell \in \widetilde{\Psi}} \langle \ell, \mathbf{0} \rangle \epsilon_\ell = \mathbf{0}.$$

(2) In the case (2-1), let $e_1 \cap e_{2k+1} \cap e'_1 \cap e'_{2k'+1}$ be the unique common vertex of C and C'. In the case (2-2), let P be the path consisting of $f_1, \ldots, f_m$ which connects the vertex $e_1 \cap e_{2k+1}$ of C and $e'_1 \cap e'_{2k'+1}$ of C'. Then we see the following:

$$\sum_{i=1}^{2k+1} (-1)^i \mathbf{w}_{\rho(e_i)} - \sum_{i=1}^{2k'+1} (-1)^i \mathbf{w}_{\rho(e'_i)} = \mathbf{0};$$

$$\sum_{i=1}^{2k+1} (-1)^i \mathbf{w}_{\rho(e_i)} - \sum_{i=1}^{2k'+1} (-1)^i \mathbf{w}_{\rho(e'_i)} - 2\sum_{j=1}^{m}(-1)^j \mathbf{w}_{\rho(f_j)} = \mathbf{0} \text{ if } m \text{ is even;}$$

$$\sum_{i=1}^{2k+1} (-1)^i \mathbf{w}_{\rho(e_i)} + \sum_{i=1}^{2k'+1} (-1)^i \mathbf{w}_{\rho(e'_i)} - 2\sum_{j=1}^{m}(-1)^j \mathbf{w}_{\rho(f_j)} = \mathbf{0} \text{ if } m \text{ is odd.}$$

□

A *block* of a graph G means a maximal 2-connected component of G. Thus, a block contains no cut vertex. Let A denote the set of cut vertices of G, and $\mathcal{B}$ the set of its blocks. We then have a natural bipartite graph on the vertex set $A \sqcup \mathcal{B}$ with the edge set $\{\{a, B\} : a \in B \text{ for } a \in A \text{ and } B \in \mathcal{B}\}$. We call this bipartite graph the *block graph* of G, denoted by $\mathrm{Block}(G)$. Note that $\mathrm{Block}(G)$ is a tree if G is connected.

Proposition 5.1.6 (cf. [84, Proposition 10.1.48])**.** *Let G be a graph.*

(1) Let $G_1, \ldots, G_n$ be the connected components of G. Then we have $\Bbbk[G] \cong \Bbbk[G_1] \otimes \cdots \otimes \Bbbk[G_n]$. Therefore, $\mathrm{Cl}(\Bbbk[G]) \cong \mathrm{Cl}(\Bbbk[G_1]) \oplus \cdots \oplus \mathrm{Cl}(\Bbbk[G_n])$.

(2) Suppose that G is connected and let $B_1, \ldots, B_n$ be the blocks of G. If there is at most one non-bipartite block among them, then we have $\Bbbk[G] \cong \Bbbk[B_1] \otimes \cdots \otimes \Bbbk[B_n]$. Therefore, $\mathrm{Cl}(\Bbbk[G]) \cong \mathrm{Cl}(\Bbbk[B_1]) \oplus \cdots \oplus \mathrm{Cl}(\Bbbk[B_n])$.

Now, we are ready to discuss the description of $\mathrm{Cl}(\Bbbk[G])$ and show its torsionfreeness for G satisfying the odd cycle condition.

Theorem 5.1.7. *Let G be a connected graph satisfying the odd cycle condition. Then $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^{|\widetilde{\Psi}|-\dim \Bbbk[G]}$. In particular, $\mathrm{Cl}(\Bbbk[G])$ is torsionfree.*

Proof. By Theorem 4.1.1, it is enough to show that the invariant factors $s_1, \dots, s_m$ of $\mathcal{M}_{P_G}$ are 1.

The case where G is bipartite:

We may assume that G is 2-connected by Proposition 5.1.6. Take a spanning tree T of G. For any $e' \in E(G) \setminus E(T)$, the subgraph T' obtained by adding e' to T has an even cycle containing e'. We see that $\mathbf{w}_{\rho(e)}$'s for $e \in E(T')$ are linearly dependent by Lemma 5.1.5, so we can erase the columns corresponding to the edges $e' \in E(G) \setminus E(T)$ in $\mathcal{M}_{P_G}$ by using $e \in T$. Moreover, we consider the row corresponding to (the supporting hyperplane of) the ordinary vertex v whose degree is 1 in T. Since G is 2-connected, i.e., every vertex in G is ordinary, the entry corresponding to the edge e_0 which joins v is 1 and the other entries are all 0 in the row. Therefore, $\mathbf{w}_{\rho(e_0)}$ can be transformed into a unit vector. We repeat this transformation for $T \setminus v$. Then we can see that $\mathbf{w}_{\rho(e)}$'s for $e \in E(T)$ are linearly independent, that is, $\operatorname{rank} \mathcal{M}_{P_G} = |E(T)| = d - 1 = \dim \Bbbk[G]$ and $d_1 = \cdots = d_s = 1$.

The case where G is non-bipartite:

Let $B_1, \dots, B_n$ be the blocks of G. We prove the assertion by induction on n.

Let G' be a connected subgraph G' of G satisfying the following properties:

- G' is a spanning subgraph of G;
- G' has d edges;
- G' has exactly one primitive odd cycle $C = p_0 \cdots p_{2k} p_0$.

In the case $n = 1$, for any $e' \in E(G) \setminus E(G')$, consider the subgraph G'' obtained by adding e' to G'. Then G'' satisfies one of the following conditions:

(i) G'' contains an even cycle;

(ii) G'' contains two odd cycles and they have a unique common vertex;

(iii) G'' contains two odd cycles C' and C'' with no common vertex but there is a path between C' and C'' connecting them.

We can see that $\mathbf{w}_{\rho(e)}$'s for $e \in E(G'')$ are linearly dependent by Lemma 5.1.5. Moreover, since G is 2-connected, i.e., every vertex in G except for $V(C)$ is regular, $\mathbf{w}_{\rho(e)}$'s for $e \in E(G') \setminus E(C)$ can be transformed into a unit vector by the same discussions above. For $\{p_i, p_{i+1}\}$ $(i = 0, \dots, 2k)$, take a fundamental set T satisfying Lemma 5.1.3 (2). Then the entry corresponding to the edge $\{p_i, p_{i+1}\}$ is 1 and the other entries are all 0 in the row corresponding to (the supporting hyperplane of) the fundamental set T. Thus, $\mathbf{w}_{\rho(\{p_i, p_{i+1}\})}$ can be transformed into a unit vector. Hence, we conclude that $\operatorname{rank} \mathcal{M}_{P_G} = |G'| = d = \dim \Bbbk[G]$ and $s_1 = \cdots = s_m = 1$.

Let $n \geq 2$. Then there exists B_i containing a unique primitive odd cycle C such that $G'_{V(B_j)}$ is a tree for $j \neq i$. We may assume that $i = 1$. Note that all vertices in G are regular on G except for cut vertices of G and $p_0, \dots, p_{2k}$. Then we can find a cut vertex v of G such that the subgraph $\mathrm{Block}(G) \setminus v$ of $\mathrm{Block}(G)$ has a unique connected component containing B_1 and the other components are isolated vertices; these are blocks $B_{i_1}, \dots, B_{i_l}$ such that $B'_{i_j} = G'_{V(B_{i_j})}$ are trees. Since every vertex in $\bigcup_{j \in [l]} V(B_{i_j})$ is regular except for v, $\mathbf{w}_{\rho(e)}$'s for $e \in \bigcup_{j \in [l]} E(B'_{i_j})$ can be transformed into a unit vector. Let $H = G \setminus \left(\bigcup_{j \in [l]} V(B_{i_j}) \setminus \{v\} \right)$. As mentioned in Remark 5.1.4, if a vertex $u \neq v$ on H is regular, then u is also regular on G, and if S is a fundamental set on H, then S or an induced fundamental S' is fundamental on G. Although v is not regular on G, it might happen that v is regular on H. If v is regular on H, we can take an induced fundamental set U of the empty set on G. In the row corresponding to (the supporting hyperplane of) the fundamental set U, the entries corresponding to the edges joining v on H is 1 and the other entries are all 0. Thus, we can regard a fundamental set U on G as a regular vertex on H. Therefore, we can see that $\mathbf{w}_{\rho(e)}$'s for $e \in E(H) \cap E(G')$ can be transformed into unit vectors by induction. □

5.2 A sufficient condition for divisor class groups to be torsionfree

In this section, we give a sufficient condition for $\mathrm{Cl}(\Bbbk[P])$ to be torsionfree in terms of P, where P is a normal integral polytope. The contents of this section are contained in the author's paper [50].

Let $P \subset \mathbb{R}^d$ be an integral polytope and let k_P be a maximal nonnegative integer satisfying the following statement:

($\diamond$) There exist distinct integral points $\mathbf{v}_1, \dots, \mathbf{v}_{k_P} \in P \cap \mathbb{Z}^d$ and distinct facets $F_1, \dots, F_{k_P}$ of P such that $\mathbf{v}_i \in \bigcap_{l=1}^{i-1} F_l$ for each $1 < i \leq k_P$ and $d_{F_i}(\mathbf{v}_i) = 1$ for each $1 \leq i \leq k_P$.

Example 5.2.1. (a) Let $P_1 = \mathrm{conv}(\{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\})$. See Figure 5.1. We can see that $k_{P_1} = 3$. Indeed, let $\mathbf{v}_1 = (1,1)$, $\mathbf{v}_2 = (1,0)$, $\mathbf{v}_3 = (0,0)$, $F_1 = \mathrm{conv}(\{(0,0),(1,0)\})$, $F_2 = \mathrm{conv}(\{(0,0),(0,1)\})$ and $F_3 = \mathrm{conv}(\{(0,1),(1,2)\})$. Then we can check easily that integral points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in P_1$ and facets $F_1, F_2, F_3 \in \Psi(P_1)$ satisfy the statement ($\diamond$).

(b) Let $P_2 = \mathrm{conv}(\{(1,0),(0,1),(2,1),(1,2)\})$. See Figure 5.2. Then we have $d_F(\mathbf{v}_1) = 1$ for every facet F of P_2 and the integral point $\mathbf{v}_1 = (1,1) \in P_2$. However, for any integral points $\mathbf{v}$ in P_2 except $\mathbf{v}_1$, the equation $d_F(\mathbf{v}) = 1$ does not hold. Thus we can obtain that $k_{P_2} = 1$.

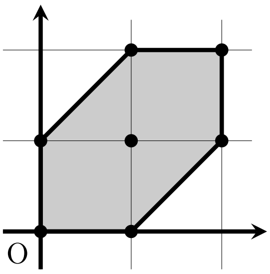

Figure 5.1: The polytope P_1

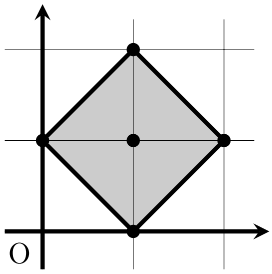

Figure 5.2: The polytope P_2

Theorem 5.2.2. *Let $P \subset \mathbb{R}^d$ be a normal integral polytope and let $s_1, \dots, s_r$ be the invariant factors of $\mathcal{M}_P$. Then $s_1 = \cdots = s_{k_P} = 1$. In particular,* $\mathrm{Cl}(\Bbbk[P])$ *is torsionfree if $k_P = \dim P + 1$.*

Proof. Assume that $\mathbf{v}_1, \dots, \mathbf{v}_{k_P} \in P \cap \mathbb{Z}^d$ and $F_1, \dots, F_{k_P} \in \Psi(P)$ satisfy the statement ($\diamond$). Then $k_P \times k_P$ submatrix $(d_{F_i}(\mathbf{v}_j))$ of $\mathcal{M}_P$ is a triangular matrix whose diagonal entries are equal to 1 since $d_{F_1}(\mathbf{v}_i) = \cdots = d_{F_{i-1}}(\mathbf{v}_i) = 0$ and $d_{F_i}(\mathbf{v}_i) = 1$ for each $1 \leq i \leq k_P$. Thus we have $\det\big((d_{F_i}(\mathbf{v}_j))\big) = 1$, and so $g_{k_P}(\mathcal{M}_P) = 1$. This implies $s_1 = \cdots = s_{k_P} = 1$. Moreover, it follows directly from Theorem 4.1.1 that $\mathrm{Cl}(\Bbbk[P])$ is torsionfree if $k_P = \dim P + 1$. $\square$

This theorem gives a sufficient condition for the divisor class group of toric rings to be torsionfree. However, it is not necessary, namely, there exists a normal integral polytope P such that $\mathrm{Cl}(\Bbbk[P])$ is torsionfree, but $k_P < \dim P + 1$.

Example 5.2.3. Let $P_3 = \mathrm{conv}(\{(0,0),(1,4),(2,5),(3,1)\})$. See Figure 5.3. This integral polytope is normal, and we can see that $k_{P_3} = 1$. However, we can compute $s_1 = s_2 = s_3 = 1$.

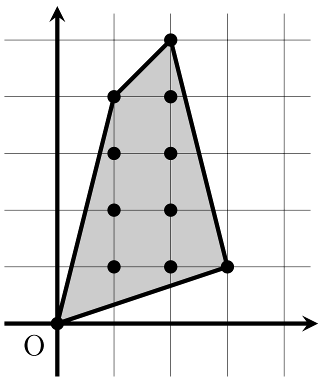

Figure 5.3: The polytope P_3

Theorem 5.2.2 can be applied to compressed polytopes as follows:

Corollary 5.2.4. *If an integral polytope $P \subset \mathbb{R}^d$ is compressed, then* $\mathrm{Cl}(\Bbbk[P])$ *is torsionfree.*

Proof. Let $n = \dim P$. We can choose $F_1, \ldots, F_{n+1} \in \Psi(P)$ with $\dim \bigcap_{l=1}^{i} F_l = n - i$ for each $1 \leq i \leq n$ and $\bigcap_{l=1}^{n} F_l \nsubseteq F_{n+1}$, and choose $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in P \cap \mathbb{Z}^d$ with $\mathbf{v}_i \notin F_i$ for each $1 \leq i \leq n+1$ and $\mathbf{v}_{i+1} \in \bigcap_{l=1}^{i} F_l$ for each $1 \leq i \leq n$. By Lemma 2.2.4 and $\sum_{\mathbf{v} \in P \cap \mathbb{Z}^d} d_F(\mathbf{v})\mathbb{Z} = \mathbb{Z}$ for any $F \in \Psi(P)$, one has $d_{F_i}(\mathbf{v}_i) = 1$ for each $1 \leq i \leq n$. Therefore the $\mathbf{v}_i$'s and F_i's satisfy the statement ($\diamond$). $\square$

Compressed polytopes appear in several places. For example, as mentioned in Section 3.2, order polytopes and stable set polytopes of perfect graphs are compressed. Moreover, we can see that edge polytopes of bipartite graphs and complete multipartite graphs are compressed from their facet defining inequalities. Therefore, Corollary 5.2.4 enable us to recover Theorem 5.1.1, Proposition 5.1.2 and certain parts of Theorem 5.1.7.

Chapter 6

Three families of toric rings of polytopes with small rank

In this chapter, we will characterize each of the three families of toric rings whose divisor class groups are rank 1 or 2, and examine their relationships. The contents of this chapter are contained in the author's paper [41] with A. Higashitani.

6.1 Hibi rings with small divisor class groups

We define four posets as follows.

(i) For $s_1, s_2 \in \mathbb{Z}_{>0}$, let $\Pi_1(s_1, s_2) = \{p_1, \dots, p_{s_1}, p_{s_1+1}, \dots, p_{s_1+s_2}\}$ be the poset equipped with the partial orders $p_1 \prec \cdots \prec p_{s_1}$ and $p_{s_1+1} \prec \cdots \prec p_{s_1+s_2}$. Figure 6.1 shows the Hasse diagram of $\Pi_1(s_1, s_2)$.

(ii) For $s_1, s_2, s_3 \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}_{\geq 0}$, let $\Pi_2(s_1, s_2, s_3, t) = \{p_1, \dots, p_d\}$ ($d = s_1+s_2+s_3+t$) be the poset equipped with the partial orders

- $p_1 \prec \cdots \prec p_t$,
- $p_t \prec p_{t+1} \prec \cdots \prec p_{t+s_1}$ and $p_t \prec p_{t+s_1+1} \prec \cdots \prec p_{t+s_1+s_2}$ ($p_1 \prec \cdots \prec p_{s_1}$ and $p_{s_1+1} \prec \cdots \prec p_{s_1+s_2}$ if $t = 0$) and
- $p_{t+s_1+s_2+1} \prec \cdots \prec p_d$.

Figure 6.2 shows the Hasse diagram of $\Pi_2(s_1, s_2, s_3, t)$ and Figure 6.3 is the case $t = 0$.

(iii) Moreover, for $s_1, s_2, t_1, t_2 \in \mathbb{Z}_{>0}$ and $t_3 \in \mathbb{Z}_{\geq 0}$, let $\Pi_3(s_1, s_2, t_1, t_2, t_3) = \{p_1, \dots, p_d\}$ ($d = s_1 + s_2 + t_1 + t_2 + t_3$) be the poset equipped with the partial orders

- $p_1 \prec \cdots \prec p_{t_1} \prec p_{t_1+1} \cdots \prec p_{t_1+s_1}$,
- $p_{t_1+s_1+1} \prec \cdots \prec p_{t_1+s_1+s_2} \prec p_{s_1+t_1+s_2+1} \cdots \prec p_{t_1+s_1+s_2+t_2}$ and
- $p_{t_1} \prec p_{t_1+s_1+s_2+t_2+1} \cdots \prec p_d \prec p_{t_1+s_1+s_2+1}$.

Figure 6.4 shows the Hasse diagram of $\Pi_3(s_1, s_2, t_1, t_2, t_3)$.

(iv) Furthermore, for $s_1, s_2, t_1, t_2 \in \mathbb{Z}_{>0}$, let $\Pi_4(s_1, s_2, t_1, t_2) = \{p_1, \dots, p_{d+1}\}$ $(d = s_1 + s_2 + t_1 + t_2)$ be the poset equipped with the partial orders

- $p_1 \prec \cdots \prec p_{t_1} \prec p_{d+1}$, $p_{t_1+1} \prec \cdots \prec p_{t_1+t_2} \prec p_{d+1}$ and
- $p_{d+1} \prec p_{t_1+t_2+1} \prec \cdots \prec p_{t_1+t_2+s_1}$, $p_{d+1} \prec p_{t_1+t_2+s_1+1} \prec \cdots \prec p_d$.

Figure 6.5 shows the Hasse diagram of $\Pi_4(s_1, s_2, t_1, t_2)$.

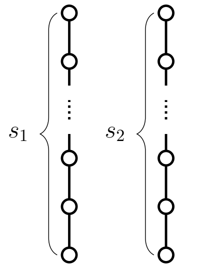

Figure 6.1: The poset Π_1

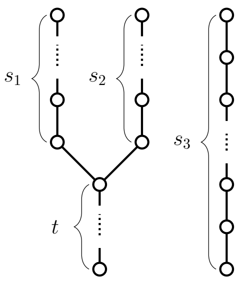

Figure 6.2: The poset Π_2

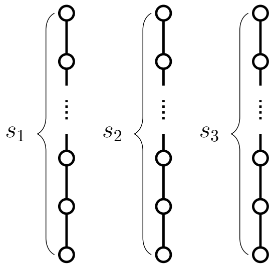

Figure 6.3:
The poset Π_2 with $t = 0$

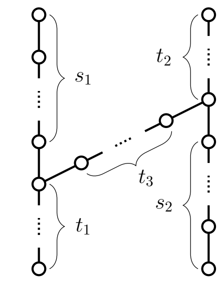

Figure 6.4: The poset Π_3

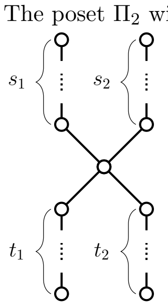

Figure 6.5: The poset Π_4

In [61], Gorenstein Hibi rings $\Bbbk[\Pi]$ with $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}$ or $\mathbb{Z}^2$ are discussed and the characterization of the associated posets is given. We can see that [61, Example 3.1] and the proof of [61, Lemma 3.2] works even for non-pure posets. Thus, we can characterize the Hibi rings $\Bbbk[\Pi]$ with $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}$ or $\mathbb{Z}^2$ as follows:

Proposition 6.1.1 (cf. [61, Example 3.1 and Lemma 3.2])**.** *Let Π be a poset. Assume that $\Bbbk[\Pi]$ is not a polynomial extension of a Hibi ring.*

(1) If $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}$, then $\mathcal{O}_\Pi$ is unimodularly equivalent to $\mathcal{O}_{\Pi_1(s_1,s_2)}$ for some s_1, s_2 with $d = s_1 + s_2$.

(2) *If* $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^2$, *then* $\mathcal{O}_\Pi$ *is unimodularly equivalent to* $\mathcal{O}_{\Pi_2(s_1,s_2,s_3,t)}$ *for some* s_1, s_2, s_3, t *with* $d = s_1 + s_2 + s_3 + t$, $\mathcal{O}_{\Pi_3(s_1,s_2,t_1,t_2,t_3)}$ *for some* s_1, s_2, t_1, t_2, t_3 *with* $d = s_1+s_2+t_1+t_2+t_3$ *or* $\mathcal{O}_{\Pi_4(s_1,s_2,t_1,t_2)}$ *for some* s_1, s_2, t_1, t_2 *with* $d = s_1+s_2+t_1+t_2$.

Given a poset Π, we define the *comparability graph* of Π, denoted by $G(\Pi)$, as a graph on the vertex set $V(G(\Pi)) = [d]$ with the edge set

$$E(G(\Pi)) = \{\{i, j\} : p_i \text{ and } p_j \text{ are comparable in } \Pi\}.$$

It is known that $G(\Pi)$ is perfect for any Π (see e.g. [17, Section 5.5]). Moreover, we see that $\mathcal{C}_\Pi = \mathrm{Stab}_{G(\Pi)}$.

Proposition 6.1.2. *Let* Π *be* $\Pi_1(s_1, s_2)$ *or* $\Pi_2(s_1, s_2, s_3, t)$ *or* $\Pi_3(s_1, s_2, t_1, t_2, t_3)$. *Then* $\mathcal{O}_\Pi$ *is unimodularly equivalent to* $\mathcal{C}_\Pi = \mathrm{Stab}_{G(\Pi)}$.

Proof. This directly follows from Theorem 3.2.2. □

6.2 Stable set rings with small divisor class groups

For stable set rings, if their class groups are isomorphic $\mathbb{Z}$ or $\mathbb{Z}^2$, then we see that we can associate Hibi rings as follows:

Theorem 6.2.1. *Let* G *be a perfect graph. Assume that* $\Bbbk[\mathrm{Stab}_G]$ *is not a polynomial extension of a stable set ring.*

(1) *Assume that* $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}$. *Then* Stab_G *is unimodularly equivalent to* $\mathcal{O}_{\Pi_1(s_1,s_2)}$ *for some* $s_1, s_2 \in \mathbb{Z}_{>0}$.

(2) *Assume that* $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^2$. *Then* Stab_G *is unimodularly equivalent to* $\mathcal{O}_{\Pi_2(s_1,s_2,s_3,t)}$ *for some* $s_1, s_2, s_3 \in \mathbb{Z}_{>0}$ *and* $t \in \mathbb{Z}_{\geq 0}$, *or* $\mathcal{O}_{\Pi_3(s_1,s_2,t_1,t_2,t_3)}$ *for some* $s_1, s_2 \in \mathbb{Z}_{>0}$ *and* $t_1, t_2, t_3 \in \mathbb{Z}_{\geq 0}$,

Proof. Let $Q_0, Q_1, \dots, Q_n$ be the maximal cliques of G. Then $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^n$ by Proposition 5.1.2. If $v \in \bigcap_{i=0}^n Q_i \neq \emptyset$, then v is adjacent to any vertex in G, so we see that Stab_G is a $(0, 1)$-pyramid with basis $\mathrm{Stab}_{G \setminus v}$. In particular, $\Bbbk[\mathrm{Stab}_G] \cong \Bbbk[\mathrm{Stab}_{G\setminus v}][x_v]$ from Proposition 2.2.1. Thus, we may assume that $\bigcap_{i=0}^n Q_i = \emptyset$.

Let $n = 1$. We can see that $G = G(\Pi_1(s_1, s_2))$, where $s_1 = |Q_0|$ and $s_2 = |Q_1|$, by observing (3.2.2) for $G(\Pi_1(s_1, s_2))$ and the definition of $\mathcal{C}_{\Pi_1(s_1,s_2)}$. It follows from Theorem 3.2.2 that $\Bbbk[\mathcal{O}(\Pi_1(s_1, s_2))] \cong \Bbbk[\mathcal{C}(\Pi_1(s_1, s_2))] = \Bbbk[\mathrm{Stab}(G(\Pi_1(s_1, s_2)))]$.

Let $n = 2$.

(i) If $Q_0 \cap Q_1 = Q_0 \cap Q_2 = Q_1 \cap Q_2 = \emptyset$, then we can see that $G = G(\Pi_2(s_1, s_2, s_3, 0))$, where $s_1 = |Q_0|$, $s_2 = |Q_1|$ and $s_3 = |Q_2|$.

(ii) If $Q_0 \cap Q_1 = Q_0 \cap Q_2 = \emptyset$ and $Q_1 \cap Q_2 \neq \emptyset$, then we can see that $G = G(\Pi_2(s_1, s_2, s_3, t))$, where $s_1 = |Q_1 \setminus Q_2|$, $s_2 = |Q_2 \setminus Q_1|$, $s_3 = |Q_0|$ and $t = |Q_1 \cap Q_2|$.

(iii) If $Q_0 \cap Q_1, Q_0 \cap Q_2 \neq \emptyset$ and $Q_1 \cap Q_2 = \emptyset$, then we can see that $G = G(\Pi_3(s_1, s_2, t_1, t_2, t_3))$, where $s_1 = |Q_1 \setminus Q_0|$, $s_2 = |Q_2 \setminus Q_0|$, $t_1 = |Q_0 \cap Q_1|$, $t_2 = |Q_0 \cap Q_2|$ and $t_3 = |Q_0 \setminus (Q_1 \cup Q_2)|$.

(iv) If $Q_0 \cap Q_1, Q_0 \cap Q_2, Q_1 \cap Q_2 \neq \emptyset$, then we see that $Q = (Q_0 \cap Q_1) \cup (Q_0 \cap Q_2) \cup (Q_1 \cap Q_2)$ is also a maximal clique which is different from Q_0, Q_1, Q_2. This contradicts to $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^2$ by Proposition 5.1.2.

□

6.3 Edge rings with small divisor class groups

The goal of this section is to give a complete description of G satisfying the odd cycle condition with $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}$ or $\mathbb{Z}^2$. Throughout this section, we let G be a connected graph satisfying the odd cycle condition. We discuss G by dividing it into whether G is bipartite or not.

Proposition 6.3.1. *Let* $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^t$. *If G contains at least two non-bipartite blocks, then* $t \geq 4$.

Proof. Let $B_1, \ldots, B_m$ be the blocks of G, where $m \geq 2$, and assume that at least two of them are non-bipartite. We prove the assertion by induction on m.

Let $m = 2$. Then B_1 and B_2 are non-bipartite. Thus, B_1 and B_2 have primitive odd cycle $C_1 = p_0 \cdots p_{2k_1} p_0$ and $C_2 = q_0 \cdots q_{2k_2} q_0$ $(1 \leq k_1 \leq k_2)$, respectively. Let $v \in V(B_1) \cap V(B_2)$ be a unique cut vertex. Then we see that every vertex in $V(G) \setminus \{v\}$ is regular, implying that $|\Psi_r| \geq |V(G)| - 1 = d - 1$ and G has $|\Psi_f| \geq \min\{|V(C_1), V(C_2)|\} = 2k_1 + 1$ by Lemma 5.1.3 (2).

- Suppose that $v \notin V(C_1) \cup V(C_2)$. Then there is a path containing v which connects $V(C_1)$ and $V(C_2)$. This is a contradiction to what G satisfies the odd cycle condition.

- Suppose that $v \in V(C_1) \setminus V(C_2)$. Let, say, $v = p_0$. Then we can take two fundamental sets on G as follows. Let $S_1 = \{p_1, p_3, \ldots, p_{2k_1-1}\}$ and $S_2 = \{p_2, p_4, \ldots, p_{2k_1}\}$. Then there exist fundamental sets T_1 and T_2 such that $S_i \subset T_i$ and $V(B(T_i)) = V(B_1)$ for $i = 1, 2$ by Lemma 5.1.3 (1). Namely, we can get two (or more) fundamental sets. Hence,

$$t = |\Psi| - \dim \Bbbk[G] = |\Psi_f| + |\Psi_r| - d \geq (2k_1 + 1) + 2 + (d - 1) - d \geq 4.$$

- Suppose that $v \in V(C_1) \cap V(C_2)$. Let, say, $v = p_0 = q_0$. Then we can also take two (or more) fundamental sets on G as follows. Let $U_1 = \{q_1, q_3, \ldots, q_{2k_2-1}\}$ and $U_2 = \{q_2, q_4, \ldots, q_{2k_2}\}$ and take S_1 and S_2 above. Then there exist fundamental sets $T'_{i,j}$ for $i = 1, 2$ and $j = 1, 2$ such that $S_i \cup U_j \subset T'_{i,j}$ and $V(B(T'_{i,j})) = V(G)$ by Lemma 5.1.3 (1). Hence, as above, we obtain that $t \geq 4$.

Suppose that $m \geq 3$. Take a block B_i whose degree is 1 on Block(G). Then B_i has a unique cut vertex u on G. Let $H = G \setminus (V(B_i) \setminus \{u\})$ and $b = |V(B_i)|$. Note that H has an odd cycle by assumption and every vertex in $B_i \setminus u$ is regular on G. Thus, we have

$$|\Psi_r(G)| = \begin{cases} |\Psi_r(H)| + (b-1), & \text{if (i) } u \text{ is non-regular in } H \text{ and in } G, \\ |\Psi_r(H)| + (b-1) - 1, & \text{if (ii) } u \text{ is regular in } H \text{ and non-regular in } G. \end{cases}$$

Notice that if u is regular in H and G, then $B_i \setminus u$ and all connected components of $H \setminus u$ contain an odd cycle, a contradiction by the same reason as discussed above. Moreover, it never happens that u is non-regular on H and regular on G.

In the case of (ii), we have $|\Psi_f(G)| \geq |\Psi_f(H)| + 1$ by Remark 5.1.4. Therefore, in the case of (i), we obtain by inductive hypothesis the following:

$$\begin{aligned} t = |\Psi_r(G)| + |\Psi_f(G)| - d &\geq (|\Psi_r(H)| + (b-1) - 1) + (|\Psi_f(H)| + 1) - d \\ &= |\Psi_r(H)| + |\Psi_f(H)| - (d - (b-1)) = |\Psi(H)| - \dim \Bbbk[H] \\ &\geq 4. \end{aligned}$$

□

Lemma 6.3.2. *Let G be a bipartite graph with the partition $V(G) = V_1 \sqcup V_2$. If G is not a complete bipartite graph, then there exists an acceptable set contained in V_1.*

Proof. Let $n_1 = |V_1|$ and $n_2 = |V_2|$. Note that $n_1, n_2 \geq 2$ since G is connected and non-complete bipartite. Take a vertex $v_0 \in V_1$ such that $\deg(v_0) = \min\{\deg(v) : v \in V_1\}$. Then $\deg(v_0) < n_2$. Moreover, $G \setminus V(B(\{v_0\}))$ contains connected components $C_1, \ldots, C_n$ which have at least one edge, and it might have some isolated vertices $v_1, \ldots, v_m$ in V_1. For $i \in [n]$, let $A_i = \{v_0, v_1, \ldots, v_m\} \cup \left(\bigcup_{j \in [n], j \neq i} V(C_j) \cap V_1\right)$. Then each A_i is acceptable. In fact, $B(A_i)$ is connected since G is connected, and $G \setminus V(B(A_i)) = C_i$ is a connected graph with at least one edge. □

We define two graphs $K_{s_1,s_2}^{t_1,t_2}$ and $K_{1,s_1,s_2}^{t_1,t_2}$ as follows:

Definition 6.3.3. Let s_1, s_2, t_1, t_2 be integers with $0 \leq t_1 < s_1$ and $0 \leq t_2 < s_2$.

- Let $K_{s_1,s_2}^{t_1,t_2}$ denote the bipartite graph on the vertex set $V(K_{s_1,s_2}^{t_1,t_2}) = [d]$ ($d = s_1 + s_2 + t_1 + t_2$) with the edge set

$$\begin{aligned} E(K_{s_1,s_2}^{t_1,t_2}) = &\{\{i,j\} : 1 \leq i \leq s_1 + t_1, s_1 + t_1 + t_2 + 1 \leq j \leq d\} \\ &\cup \{\{i,j\} : 1 \leq i \leq s_1, s_1 + t_1 + 1 \leq j \leq d\}. \end{aligned}$$

  See Figure 6.6.

- Let $K_{1,s_1,s_2}^{t_1,t_2}$ denote the graph on the vertex set $V(K_{1,s_1,s_2}^{t_1,t_2}) = [d+1]$ ($d = s_1 + s_2 + t_1 + t_2$) with the edge set

$$E(K_{1,s_1,s_2}^{t_1,t_2}) = E(K_{s_1,s_2}^{t_1,t_2}) \cup \{\{i, d+1\} : 1 \leq i \leq s_1 \text{ or } s_1 + t_1 + t_2 + 1 \leq i \leq d\}.$$

  See Figure 6.7.

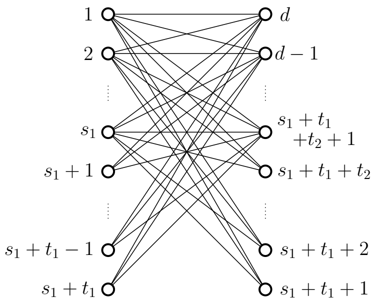

Figure 6.6:
The graph $K^{t_1,t_2}_{s_1,s_2}$

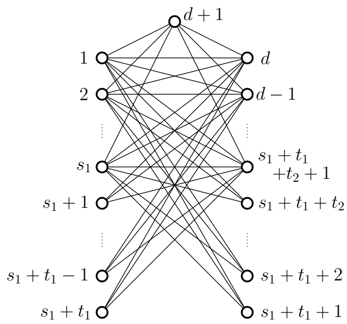

Figure 6.7:
The graph K^{t_1,t_2}_{1,s_1,s_2}

Note that $K^{t_1,t_2}_{s_1,s_2}$ (resp. K^{t_1,t_2}_{1,s_1,s_2}) is a complete bipartite graph K_{s_1,s_2} (resp. a complete 3-partite graph K_{1,s_1,s_2}) minus the edges of K_{t_1,t_2}. Thus, $K^{t_1,t_2}_{s_1,s_2}$ is bipartite, but K^{t_1,t_2}_{1,s_1,s_2} is not. When $t_1 = t_2 = 0$, we regard $K^{t_1,t_2}_{s_1,s_2}$ (resp. K^{t_1,t_2}_{1,s_1,s_2}) as K_{s_1,s_2} (resp. K_{1,s_1,s_2}) itself.

First, we discuss the case of bipartite graphs. We give the characterization of which $\mathrm{Cl}(\Bbbk[G])$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$ in terms of G for bipartite graphs. By Proposition 5.1.6, we may assume that G is 2-connected.

Theorem 6.3.4. *Let G be a 2-connected bipartite graph with its partition $V(G) = V_1 \sqcup V_2$.*

(1) $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}$ *if and only if G is a complete bipartite graph K_{s_1,s_2} with $s_1, s_2 \geq 2$.*

(2) $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^2$ *if and only if G is a bipartite graph $K^{t_1,t_2}_{s_1,s_2}$ for some $t_1, t_2 \geq 1$ and $s_1, s_2 \geq 2$.*

Proof. (1) Since every vertex in G is ordinary, we see that $\mathrm{rank}(\mathrm{Cl}(\Bbbk[G])) = |\Psi| - \dim \Bbbk[G] = |\Psi_o| + |\Psi_a| - (d-1) = |\Psi_a| + 1$ (see Theorem 5.1.7). If G is not a complete bipartite, then G contains an acceptable set by Lemma 6.3.2 and we have $t \geq 2$. Therefore, we can see that G is a complete bipartite and $s_1, s_2 \geq 2$ since G is 2-connected. Conversely, if G is a complete bipartite graph K_{s_1,s_2} with $s_1, s_2 \geq 2$, then it is easy to check that $\mathrm{Cl}(K_{s_1,s_2}) \cong \mathbb{Z}$.
(2) Assume that $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^2$. By (1), G cannot be a complete bipartite graph. Thus, we can take $v_0, v_1, \ldots, v_m$, $C_1, \ldots, C_n$ and $A_1, \ldots, A_n$ mentioned in Lemma 6.3.2. We can see that $n = 1$ since $t = |\Psi_a| + 1 = 2$. Moreover, we see that $B(\{v_0, v_1 \ldots, v_m\})$ is a complete bipartite by definition of $v_0, v_1, \ldots, v_m$. Note that $A_1 = \{v_0, v_1, \ldots, v_m\}$. Thus, it is enough to show that C_1 and G_W are complete bipartite graphs, where $W = (V(C_1) \cap V_1) \cup N_G(v_0)$.

If C_1 is not a complete bipartite graph, then we can take an acceptable set $A \subset V_1$ of C_1 by Lemma 6.3.2 and A' is an acceptable set of G, where

$$A' = \begin{cases} A & \text{if } N_G(A) \cap N_G(v_0) = \emptyset, \\ A \cup A_1 & \text{if } N_G(A) \cap N_G(v_0) \neq \emptyset, \end{cases}$$

a contradiction. Similarly, if G_W is not a complete bipartite graph, then we can take an acceptable set of G by the same way in Lemma 6.3.2. Let $s_1 = |V(C_1) \cap V_1|$, $s_2 = |N_G(A_1)|$, $t_1 = |A_1|$ and $t_2 = |V(C_1) \cap V_2|$. Then G coincide with $K^{t_1,t_2}_{s_1,s_2}$ and we see that $s_1, s_2 \geq 2$ since G is 2-connected. Conversely, the subset $\{s_1+1, \ldots, s_1+t_1\}$ of $V(K^{t_1,t_2}_{s_1,s_2})$ is a unique acceptable set of $K^{t_1,t_2}_{s_1,s_2}$ and we have $\mathrm{Cl}(\Bbbk[K^{t_1,t_2}_{s_1,s_2}]) \cong \mathbb{Z}^2$. □

Next, we discuss non-bipartite graphs.

Lemma 6.3.5. *Let G be a 2-connected graph with primitive odd cycles $C_i = p_{i,0} \cdots p_{i,2k_i} p_{i,0}$ for $i \in [m]$, where $1 \leq k_1 \leq \cdots \leq k_m$, and let $P = x_0 x_1 \cdots x_l$ with $l \geq 2$ be a primitive path whose end vertices x_0, x_l are in $V(C_m)$ and $x_k \notin V(C_m)$ for all $k \in [l-1]$.*

(1) For $j \in \{0, 1, \dots, 2k_m\}$, $p_{m,j}$ is non-regular in G if and only if $p_{m,j} \in V(C_i)$ for all $i \in [m]$.

(2) Suppose that $x_0 = p_{m,0}$ and $x_l = p_{m,j}$ ($j \neq 1, 2k_m$). Then C_m has a regular vertex in G.

(3) Suppose that $\{x_0, x_l\} = \{p_{m,j}, p_{m,j+1}\}$ for $j \in \{0, 1, \dots, 2k_m\}$, where $p_{2k_m+1} = p_0$ and $l = 2l' + 1$. Then there are two different fundamental sets T_1, T_2 such that $E(C_m) \setminus \{p_{m,j}, p_{m,j+1}\} \subset E(B(T_i))$ and $\{p_{m,j}, p_{m,j+1}\} \notin E(B(T_i))$ for $i = 1, 2$.

Proof. (1) If there exists $i \in [m]$ such that $p_{m,j} \notin V(C_i)$, then the connected graph $G \setminus p_{m,j}$ contains C_i as a subgraph. Hence, $p_{m,j}$ is regular in G. Conversely, if $p_{m,j} \in V(C_i)$ for all $i \in [m]$, then the connected graph $G \setminus p_{m,j}$ has no odd cycles. Thus, $p_{m,j}$ is non-regular.
(2) Let $C = x_0 x_1 \cdots x_l p_{m,j-1} p_{m,j-2} \cdots p_{m,0}$ and $C' = x_0 \cdots x_l p_{m,j+1} p_{m,j+2} \cdots p_{m,2k_m} p_{m,0}$. Then C or C' is a primitive odd cycle because C_m is a primitive odd cycle. Therefore, $p_{m,1}, \dots, p_{m,j-1}$ or $p_{m,j+1}, \dots, p_{m,2k_m}$ are regular vertices in $V(C_m)$.
(3) We may assume that $j = 0$. Let $S_1 = \{p_{m,2}, p_{m,4}, \dots, p_{m,2k_m}, x_1, x_3, \cdots, x_{2l'-1}\}$ and $S_2 = \{p_{m,2}, p_{m,4}, \dots, p_{m,2k_m}, x_2, x_4, \cdots, x_{2l'}\}$ are independent sets and $N_G(S_i)$ is connected for $i = 1, 2$. Therefore, the statement immediately follows from Lemma 5.1.3 (1). □

Theorem 6.3.6. *Let G be a 2-connected non-bipartite graph.*

(1) $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}$ *if and only if G is obtained by one of the following two ways.*

For the complete bipartite graph K_{s_1,s_2} with $s_1, s_2 \geq 2$,

- *(1-1) choose i and j from the different partition, respectively, and connect them by a path of even length at least 2 (see* Figure 6.8*); or*
- *(1-2) choose i and j from the same partition and connect them by a path of odd length (see* Figure 6.9*).*

(2) $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^2$ *if and only if G is obtained by one of the following six ways.*

For the complete bipartite graph K_{s_1,s_2} and K_{t_1,t_2} with $s_1, s_2, t_1, t_2 \geq 2$;

- *(2-1) choose i and j (resp. k and l) from the different partition of K_{s_1,s_2} (resp. K_{t_1,t_2}), respectively, and connect i and k by a path $P_{i,k}$, j and l by a path $P_{j,l}$ such that the sum of the lengths of $P_{i,k}$ and $P_{j,l}$ is odd (see* Figure 6.10*); or*
- *(2-2) choose i and j from the same partition of K_{s_1,s_2} and choose k and l from the different partition of K_{t_1,t_2}, respectively, and connect i and k by a path $P_{i,k}$, j and l by a path $P_{j,l}$ such that the sum of the lengths of $P_{i,k}$ and $P_{j,l}$ is even (see* Figure 6.11*); or*

(2-3) *choose i and j (resp. k and l) from the same partition of K_{s_1,s_2} (resp. K_{t_1,t_2}), respectively, and connect i and k by a path $P_{i,k}$, j and l by a path $P_{j,l}$ such that the sum of the lengths of $P_{i,k}$ and $P_{j,l}$ is odd (see* Figure 6.12*);*

where if the length of the path is allowed to be 0*, then identify i and k (or j and l).*

For the bipartite graph $K_{s_1,s_2}^{t_1,t_2}$ with $s_1, s_2 \geq 2$;

(2-4) *choose i and j from the different partition, respectively, and connect them by a path of even length at least* 2 *(see* Figure 6.13*); or*

(2-5) *choose i and j from the same partition and connect them by a path of odd length (see* Figure 6.14*); or*

(2-6) *G coincides with $K_{1,s_1,s_2}^{t_1,t_2}$ with $s_1, s_2 \geq 2$ (see* Figure 6.7*).*

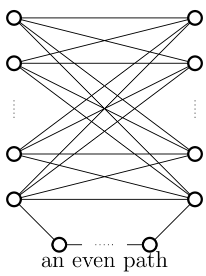

Figure 6.8:
The graph given by (1-1)

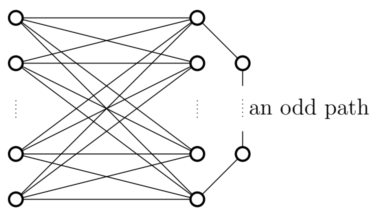

Figure 6.9:
The graph given by (1-2)

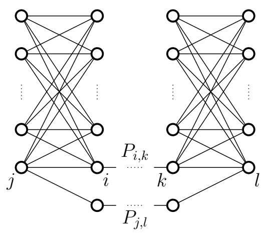

Figure 6.10:
The graph given by (2-1)

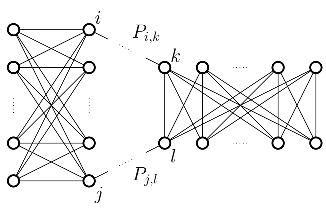

Figure 6.11:
The graph given by (2-2)

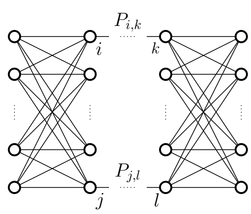

Figure 6.12:
The graph given by (2-3)

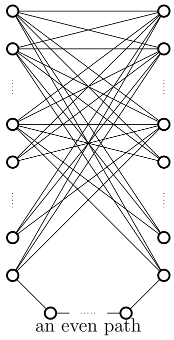

Figure 6.13:
The graph given by (2-4)

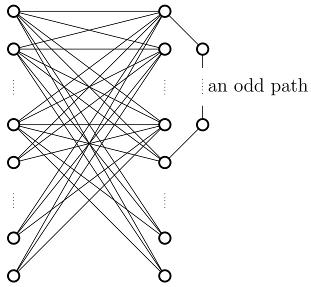

Figure 6.14:
The graph given by (2-5)

Remark 6.3.7. Regarding the above constructions, although those graphs are not bipartite due to the additional paths appearing in each case of (1-1),(1-2) and (2-1)—(2-5), we observe that every odd cycle in each graph passes through those additional paths. Namely, if C and C' are odd cycles in a given graph as above, then C and C' always share the additional paths.

On the other hand, it is well-known that the toric ideal of $\Bbbk[G]$ is generated by the binomials corresponding to primitive even closed walks appearing in G. See, e.g. [29, Section 5.3], for the details.

Hence, for the graphs G constructed like Theorem 6.3.6, we see that the variables corresponding to the edges of the additional paths never appear in generators of the toric ideal of G. This means that $\Bbbk[G]$ is isomorphic to the polynomial extension of $\Bbbk[G']$, where G' is the graph obtained by removing all the edges in the additional paths, i.e., G' is K_{s_1,s_2} or two copies of K_{s_1,s_2} or $K_{s_1,s_2}^{t_1,t_2}$ by construction.

Proof of Theorem 6.3.6. First, suppose that G satisfies one of (1-1),(1-2),(2-1)–(2-6). Then we can see that $\mathrm{Cl}(\Bbbk[G])$ is isomorphic to $\mathrm{Cl}(\Bbbk[K_{s_1,s_2}])$, $\mathrm{Cl}(\Bbbk[K_{s_1,s_2}]) \oplus \mathrm{Cl}(\Bbbk[K_{t_1,t_2}])$, $\mathrm{Cl}(\Bbbk[K_{s_1,s_2}^{t_1,t_2}])$ or $\mathrm{Cl}(\Bbbk[K_{1,s_1,s_2}^{t_1,t_2}])$, and those are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$ by Theorem 6.3.4.

(1) Since $v \in V(G) \setminus V(C_m)$ is regular, that is, $|\Psi_r| \geq d - (2k_m + 1)$ and $|\Psi_f| \geq 2k_m + 1$ by Lemma 5.1.3, we see that G should contain one extra fundamental set or one extra regular vertex.

Suppose that G contains one extra fundamental. Then $p_{m,0}, \dots, p_{m,2k_m}$ are non-regular and we have $C_1 = \cdots = C_m$ by Lemma 6.3.5 (1). By $G \neq C_m$, there exists a primitive odd path $P = x_0x_1 \cdots x_l$ whose end vertices x_0, x_l are in $V(C_m)$ and $x_k \notin V(C_m)$ for all $k \in [l-1]$. Furthermore, from Lemma 6.3.5 (2) and (3), we can see that vertices on C_m whose degree are at least 3 are just only x_0 and x_l. We may assume that $\{x_0, x_l\} = \{p_0, p_{2k_m}\}$. Consider the path $Q = p_{m,0}p_{m,1} \cdots p_{m,2k_m}$ and the graph G' given by removing Q° from G. We can see that G' contains no odd cycles, that is, G' is bipartite and the edges on Q does not appear as generators of toric ideal of $\Bbbk[G]$. Since $\mathrm{Cl}(\Bbbk[G]) \cong \mathrm{Cl}(\Bbbk[G']) \cong \mathbb{Z}$, G' is a complete bipartite graph K_{s_1,s_2} with $s_1, s_2 \geq 2$ by Theorem 6.3.4 and we see that G is obtained by (1-1).

Suppose that G has one extra regular vertex. We may assume that it is $p_{m,0}$. As above, by Lemma 6.3.5, we can observe that $\{p_{m,1}, p_{m,2}, \dots, p_{m,2k_m}\} \subset V(C_i)$ for all $i \in [m]$ and so vertices on C_m whose degree are at least 3 are just only $p_{m,2k_m}$, $p_{m,0}$ and $p_{m,1}$. Consider the path $Q = p_{m,1}p_{m,2}\cdots p_{m,2k_m}$ and the graph G' given by removing Q° from G. We can see that G' has no odd cycles, that is, G' is bipartite and the edges on Q does not appear as generators of toric ideal of $\Bbbk[G]$. Since $\mathrm{Cl}(\Bbbk[G]) \cong \mathrm{Cl}(\Bbbk[G'])\mathbb{Z}$, G' is a complete bipartite graph K_{s_1,s_2} with $s_1, s_2 \geq 2$ by Theorem 6.3.4 and we see that G is obtained by (1-2).

(2) Similarly to (1), G has

(i) two extra fundamental sets,

(ii) one extra vertex and one extra fundamental set, or

(iii) two extra regular vertices.

Suppose that (i). Then $p_{m,0}, \ldots, p_{m,2k_m}$ are non-regular and we have $C_1 = \cdots = C_m$ by Lemma 6.3.5 (1). If there exists just one type of paths $P_i = x_{i,0}\cdots x_{i,l_i}$ whose end vertices $x_{i,0}, x_{l_i}$ are in $V(C_m)$ and $x_{i,k} \notin V(C_m)$ for all $k \in [l_i - 1]$, G is obtained by (2-4). Suppose that there exist two types of paths P_1, P_2. We may assume that $\{x_{1,0}, x_{1,l_1}\} = \{p_{m,0}, p_{m,1}\}$ and $\{x_{2,0}, x_{2,l_2}\} = \{p_{m,j}, p_{m,j+1}\}$. Consider two paths $Q_1 = p_{m,0}\cdots p_{m,j}$ and $Q_2 = p_{m,j+1}\cdots p_{m,2k_m}p_{m,0}$ and the graph G' given by removing Q_1° and Q_2° from G. We can observe that G' has two connected components G_1, G_2 and they have no odd cycles, that is, they are bipartite. Therefore, we have $\mathrm{Cl}(\Bbbk[G]) \cong \mathrm{Cl}(\Bbbk[G_1]) \oplus \mathrm{Cl}(\Bbbk[G_2]) \cong \mathbb{Z}^2$ and so G_1, G_2 are complete bipartite graphs K_{s_1,s_2}, K_{t_1,t_2} with $s_1, s_2, t_1, t_2 \geq 2$. This G is obtained by (2-1).

Suppose that (ii). We may assume that it is $p_{m,0}$. We observe that $\{p_{m,1}, \ldots, p_{m,2k_m}\} \subset V(C_i)$ for all $i \in [m]$, and $p_{m,2k_m}$, $p_{m,0}$ and $p_{m,1}$ have degree 3 or more. If the other vertices have degree 2, then G is obtained by (2-5). If there exist the other vertices whose degree is at least 3, then there exists a primitive odd path $P = x_0 \cdots x_l$ with end vertices $\{x_0, x_l\} = \{p_{m,j}, p_{m,j+1}\}$ for $j \in [2k_m - 1]$. Then this G is obtained by (2-2).

Suppose that (iii). We may assume that $p_{m,0}$ and $p_{m,j}$ are regular. If $k_1 < k_m$, $k_1 = k_m - 1$ because $\{p_{m,1}, \ldots, \hat{p}_{m,j}, \ldots, p_{m,2k_m}\} \subset C_i$ for all $i \in [m]$. However, then C_m has a chord, a contradiction. Thus, $k_1 = k_m$. If $j \neq 1, 2k_m$, the vertices on C_m whose degree are at least 3 are $p_{m,2k_m}, p_{m,0}, p_{m,1}, p_{m,j-1}, p_{m,j}$ and $p_{m,j+1}$. This G is obtained by (2-3).

Suppose that $j = 1$ or $2k_m$. We may assume that $j = 1$. If $k_m \geq 2$, the vertices on C_m whose degree are at least 3 are $p_{m,2k_m}, p_{m,0}, p_{m,1}, p_{m,2}$. Hence, This G is obtained by (2-4).

Suppose that $j = 1$ and $k_m = 1$. Note that $G \setminus p_{m,2}$ is bipartite. Let V_1 and V_2 be the partition of the bipartite graph $G \setminus p_{m,2}$, let $S_i = N_G(p_{m,2}) \cap V_i$ for $i = 1, 2$ and let $T_i = V_i \setminus U_i$. We show that $G \setminus p_{m,2}$ coincides with $K_{s_1,s_2}^{t_1,t_2}$, where $s_i = |S_i| \geq 2$ and $t_i = |T_i|$ for $i = 1, 2$.

Note that all vertices except for $p_{m,2}$ are regular, V_1 and V_2 are fundamental sets since $G \setminus p_{m,2}$ is connected, and there exists a fundamental set T containing $p_{m,2}$. If $\{v_1, v_2\} \notin E(G)$ for some $v_1 \in S_1$, $v_2 \in S_2$, then $\{v_1 v_2\}$ is an independent set and $B(\{v_1, v_2\})$ is connected. Thus, we can obtain a fundamental set containing $\{v_1 v_2\}$ and it is different from V_1, V_2, T. It is a contradiction to $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^2$. If $\{u_1, u_2\} \in E(G)$ for some $u_1 \in T_1$, $u_2 \in T_2$, $\{p_{m,2}, u_i\}$ is an independent set and we can obtain an independent set I_i by adding $\{p_{m,2}, u_i\}$ to some vertices in T_i such that $B(I_i)$ is connected for $i = 1, 2$, a contradiction by the same reason. Then we have $T = \{p_{m,2}\} \cup T_1 \cup T_2$. Finally, if $\{w_1, w_2\} \notin E(G)$ for some $w_1 \in T_1$ and $w_2 \in S_2$, then $\{w_1, w_2\}$ is an independent set and we can obtain an independent set I by adding $\{w_1, w_2\}$ to some vertices in S_2 such that $B(I)$ is connected, a contradiction by the same reason. Therefore, G satisfies (2-6). □

6.4 The relationships among $\mathbf{Order}_n$, $\mathbf{Stab}_n$ and $\mathbf{Edge}_n$

Let $\mathbf{Order}_n$, $\mathbf{Stab}_n$ and $\mathbf{Edge}_n$ be the sets of unimodular equivalence classes of order polytopes, stable set polytopes of perfect graphs and normal edge polytopes such that the associated toric rings have the class groups of rank n, respectively. This section is devoted to the discussions on the relationships among $\mathbf{Order}_n$, $\mathbf{Stab}_n$ and $\mathbf{Edge}_n$ in the cases $n = 1, 2, 3$ by using the results in the previous sections.

6.4.1 The case $n = 1$

Proposition 6.4.1. *Let R be the Segre product of the polynomial rings $\Bbbk[x_1, \dots, x_s]$ and $\Bbbk[y_1, \dots, y_t]$ for some $s, t \in \mathbb{Z}_{>0}$. Note that $\mathrm{Cl}(R) \cong \mathbb{Z}$. Then R is isomorphic to $\Bbbk[\Pi]$, $\Bbbk[\mathrm{Stab}_G]$ and $\Bbbk[H]$ for some poset Π and some graphs G, H.*

Conversely, for $S = \Bbbk[\Pi]$ or $\Bbbk[\mathrm{Stab}_G]$ or $\Bbbk[H]$ for some poset Π or some graphs G, H with $\mathrm{Cl}(S) \cong \mathbb{Z}$ such that S is not a polynomial extension, S is isomorphic to the Segre product of the polynomial rings $\Bbbk[x_1, \dots, x_s]$ and $\Bbbk[y_1, \dots, y_t]$ for some $s, t \in \mathbb{Z}_{>0}$.

In particular, we have $\mathbf{Order}_1 = \mathbf{Stab}_1 = \mathbf{Edge}_1$.

Proof. These statements follow from Proposition 6.1.1 (1), Theorems 6.2.1 (1), 6.3.4 (1) and 6.3.6 (1), and we can see that $\Bbbk[P_{K_{s_1+1,s_2+1}}]$ are isomorphic to the Segre product of the polynomial rings $\Bbbk[x_1, \dots, x_m]$ and $\Bbbk[y_1, \dots, y_n]$. Moreover, the procedures (1-1) and (1-2) in Theorem 6.3.6 (1) correspond to the polynomial extension. □

6.4.2 The case $n = 2$

Lemma 6.4.2. *Let s_1, s_2, t_1, t_2 be positive integers and let $d = s_1 + s_2 + t_1 + t_2$.*

(1) The edge polytope $P_{K^{t_1,t_2}_{s_1+1,s_2+1}}$ is unimodularly equivalent to the order polytope $\mathcal{O}_{\Pi_3(s_1,s_2,t_1,t_2,0)}$.

(2) The edge polytope $P_{K^{t_1-1,t_2-1}_{1,s_1+1,s_2+1}}$ is unimodularly equivalent to the order polytope $\mathcal{O}_{\Pi_3(s_1,s_2,t_1,t_2,0)}$.

In particular, $P_{K^{t_1,t_2}_{s_1+1,s_2+1}}$ and $P_{K^{t_1-1,t_2-1}_{1,s_1+1,s_2+1}}$ are unimodularly equivalent.

Proof. It is enough to show that $P_{K^{t_1,t_2}_{s_1+1,s_2+1}}$ (resp. $P_{K^{t_1-1,t_2-1}_{1,s_1+1,s_2+1}}$) is unimodularly equivalent to $\mathcal{C}(\Pi_3(s_1, s_2, t_1, t_2, 0))$ (resp. $\mathcal{C}(\Pi_3(s_1, s_2, t_1, t_2, 0))$).

(1) By Definition 6.3.3, it is straightforward to see that the vertices of $P_{K^{t_1,t_2}_{s_1+1,s_2+1}}$ one-to-one correspond to the antichains of $\Pi_3(s_1, s_2, t_1, t_2, 0)$ by considering the projection

$\mathbb{R}^{d+2} \to \mathbb{R}^d$ which ignores the 1-th and d-th coordinates and this projection gives a unimodular transformation between $P_{K^{t_1,t_2}_{s_1+1,s_2+1}}$ and $\mathcal{C}(\Pi_3(s_1,s_2,t_1,t_2,0))$.

(2) Consider the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ by ignoring the $(d+1)$-th coordinate. Then the set of vertices of $P_{K^{t_1-1,t_2-1}_{1,s_1+1,s_2+1}}$ becomes $\{\mathbf{e}_i+\mathbf{e}_j : 1 \le i \le s_1+t_1, s_1+t_1+t_2 \le j \le d\} \cup \{\mathbf{e}_i+\mathbf{e}_j : 1 \le i \le s_1+1, s_1+t_1+1 \le j \le d\} \cup \{\mathbf{e}_k : 1 \le k \le s_1+1 \text{ or } s_1+t_1+t_2 \le k \le d\}$. By applying a unimodular transformation $\begin{pmatrix} 1 & 1 & \cdots & 1 & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & 1 & 1 & \cdots & 1 \end{pmatrix}$ to those vertices (from the left-hand side) and translating them by $-\mathbf{e}_1 - \mathbf{e}_d$ and applying a unimodular transformation $\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$, the vertices become as follows:

$\mathbf{e}_i+\mathbf{e}_j \mapsto \mathbf{e}_1+\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_d \mapsto \mathbf{e}_i+\mathbf{e}_j \mapsto \mathbf{e}_i+\mathbf{e}_j$
$(1 < i \le s_1+t_1,\ s_1+t_1+t_2 \le j < d \text{ or } 1 < i \le s_1+1,\ s_1+t_1+1 \le j < d)$
$\mathbf{e}_i+\mathbf{e}_d \mapsto \mathbf{e}_1+\mathbf{e}_i+\mathbf{e}_d \mapsto \mathbf{e}_i \mapsto \mathbf{e}_i\ (1 < i \le s_1+t_1)$
$\mathbf{e}_1+\mathbf{e}_j \mapsto \mathbf{e}_1+\mathbf{e}_j+\mathbf{e}_d \mapsto \mathbf{e}_j \mapsto \mathbf{e}_j\ (s_1+t_1+1 \le j < d), \quad \mathbf{e}_1+\mathbf{e}_d \mapsto \mathbf{0}$
$\mathbf{e}_k \mapsto \mathbf{e}_1+\mathbf{e}_k \mapsto \mathbf{e}_k-\mathbf{e}_d \mapsto \mathbf{e}_k+\mathbf{e}_d\ (1 < k \le s_1+1), \ \ \mathbf{e}_k \mapsto \mathbf{e}_1+\mathbf{e}_k\ (s_1+t_1+t_2 \le k < d)$
$\mathbf{e}_1 \mapsto \mathbf{e}_d, \quad \mathbf{e}_d \mapsto \mathbf{e}_1.$

We can directly see that these lattice points one-to-one correspond to the antichains of $\Pi_3(s_1,s_2,t_1,t_2,0)$. □

Proposition 6.4.3. (1) *Let G be a perfect graph with $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^2$. Then Stab_G is unimodularly equivalent to $\mathcal{O}_\Pi$ for some poset Π. In particular, we have* $\mathbf{Stab}_2 \subset \mathbf{Order}_2$.
(2) *Let G be a 2-connected graph with $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^2$. Then P_G is unimodularly equivalent to $\mathcal{O}_\Pi$ for some poset Π. In particular, we have* $\mathbf{Edge}_2 \subset \mathbf{Order}_2$.
(3) *Let Π be a poset with $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^2$. Then $\mathcal{O}_\Pi$ is unimodularly equivalent to $\mathcal{C}_{G(\Pi)}$ or P_G for some G. In particular,* $\mathbf{Order}_2 \subset \mathbf{Stab}_2 \cup \mathbf{Edge}_2$.
(4) *There exist a graph G and a graph H with $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathrm{Cl}(\Bbbk[H]) \cong \mathbb{Z}^2$ such that* $\mathrm{Stab}_G \notin \mathbf{Edge}_2$ *and* $P_H \notin \mathbf{Stab}_2$*, respectively.*

Proof. The statement (1) directly follows from Theorem 6.2.1 (2). The statement (2) follows from Theorems 6.3.4 (2), 6.3.6 (2) and Lemma 6.4.2.

(3) By Propositions 6.1.1 and 6.1.2, it is enough to consider the case $\Pi = \Pi_4(s_1,s_2,t_1,t_2)$ for some $s_1,s_2,t_1,t_2 \in \mathbb{Z}_{>0}$. Let K be the bipartite graph on the vertex set $[d+3]$ with

the edge set

$$\begin{aligned} E(K) = \{\{i, j\} : \ & i \in \{1, \dots, t_1, d+2\}, j \in \{t_1+1, \dots, t_1+t_2, d+3\} \text{ or} \\ & i \in \{t_1+t_2+1, \dots, t_1+t_2+s_1, d+3\}, j \in \{t_1+t_2+s_1+1, \dots, d, d+1\}\}. \end{aligned}$$

Note that K is obtained by identifying some vertex of K_{s_1+1,s_2+1} and some vertex of K_{t_1+1,t_2+1} (see Figure 6.15).

Moreover, let $I_p = \{q \in \Pi_4 : q \prec p\}$ for $p \in \Pi_4$. Note that for any poset ideal I of Π_4, I coincides with the empty set, I_p or $I_p \cup I_q$ for some $p, q \in \Pi_4$. We can see that by consider the projection $\mathbb{R}^{d+3} \to \mathbb{R}^{d+1}$ ignoring the $(d+2)$-th and $(d+3)$-th coordinates and by applying a unimodular transformation

$$\begin{pmatrix} 1 & \cdots & 1 & & & & & & & 1 & \cdots & 1 & 1 \\ & \ddots & \vdots & & & & & & & 1 & \cdots & 1 & 1 \\ & & 1 & & & & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & \ddots & \vdots & & & & & & & \\ & & & & & 1 & & & & 1 & \cdots & 1 & 1 \\ & & & & & & 1 & \cdots & 1 & & & & \\ & & & & & & & \ddots & \vdots & & & & \\ & & & & & & & & 1 & & & & \\ & & & & & & & & & 1 & \cdots & 1 & \\ & & & & & & & & & & \ddots & \vdots & \\ & & & & & & & & & & & 1 & \\ & & & & & & & & & 1 & \cdots & 1 & 1 \end{pmatrix}$$

to vertices of P_K, the vertices become as follows:

$$\begin{aligned} & \mathbf{e}_i + \mathbf{e}_{d+3} \mapsto \mathbf{e}_i \mapsto \sum_{p_k \in I_{p_i}} \mathbf{e}_k \ (1 \le i \le t_1 \text{ or } t_1+t_2+s_1+1 \le i \le d+1), \\ & \mathbf{e}_i + \mathbf{e}_{d+2} \mapsto \mathbf{e}_i \mapsto \sum_{p_k \in I_{p_i}} \mathbf{e}_k \ (t_1+1 \le i \le t_1+t_2), \quad \mathbf{e}_{d+2} + \mathbf{e}_{d+3} \mapsto 0, \\ & \mathbf{e}_i + \mathbf{e}_{d+1} \mapsto \sum_{p_k \in I_{p_i}} \mathbf{e}_k \ (t_1+t_2+1 \le i \le t_1+t_2+s_1), \\ & \mathbf{e}_i + \mathbf{e}_j \mapsto \sum_{p_k \in I_{p_i} \cup I_{p_j}} \mathbf{e}_k, \end{aligned}$$

$(1 \le i \le t_1, \ t_1+1 \le j \le t_1+t_2$ or $t_1+t_2+1 \le i \le t_1+t_2+s_1, \ t_1+t_2+s_1+1 \le j \le d)$.

We can directly see that these lattice points one-to-one correspond to the poset ideals of $\Pi_4(s_1, s_2, t_1, t_2)$.

(4) Let $G = G(\Pi_2(1,1,1,2))$ (see Figure 6.16) and let H be the graph on the vertex set $\{1, \dots, 7\}$ with the edge set $E(G) = \{12, 17, 26, 34, 47, 56, 57, 67\}$ (see Figure 6.17). Then we have $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathrm{Cl}(\Bbbk[P_G]) \cong \mathbb{Z}^2$ by construction.

If $\mathrm{Stab}_G \in \mathbf{Edge}_2$, that is, there exists a graph G' such that $P_{G'}$ is unimodularly equivalent to Stab_G, then G' satisfies that G' is bipartite and has 7 vertices and 12 edges or G' is non-bipartite and has 6 vertices and 12 edges. We can check by `MAGMA` ([5]) that for any such graphs G', $P_{G'}$ is not unimodularly equivalent to Stab_G.

Similarly, if $P_H \in \mathbf{Stab}_2$, that is, there exists a graph H' such that $\mathrm{Stab}_{H'}$ is unimodularly equivalent to P_G, then H' has 5 vertices and 8 independent sets. Similarly, we can check by `MAGMA` that for any such graphs H', $\mathrm{Stab}_{H'}$ is not unimodularly equivalent to P_H. □

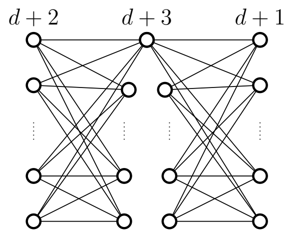

Figure 6.15: The graph K

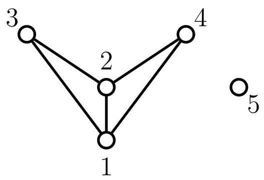

Figure 6.16:
The graph $G(\Pi_2(1,1,1,2))$

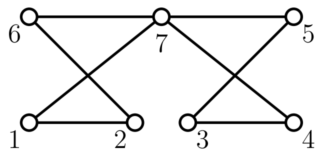

Figure 6.17: The graph H

6.4.3 The case $n = 3$

We conclude this chapter by providing examples showing that there is no inclusion among $\mathbf{Order}_3$, $\mathbf{Stab}_3$ and $\mathbf{Edge}_3$.

We define the following three objects: a poset Π, a perfect graph Γ and a connected graph G.

- Let $\Pi = \{z_1, \ldots, z_6\}$ equipped with the partial orders $z_1 \prec z_3 \prec z_4$ and $z_2 \prec z_3 \prec z_5$. Namely, Π is the disjoint union of the "X-shape" poset and one point. See Figure 6.18. Then we see from (5.1.1) that $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^3$.

- Let Γ be the graph on the vertex set $\{1, \ldots, 6\}$ with the edge set

$$E(\Gamma) = \{15, 16, 24, 26, 34, 35, 45, 46, 56\},$$

  See Figure 6.19. Then Γ is perfect since Γ is chordal. Moreover, Γ contains four maximal cliques: $\{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}$ and $\{4, 5, 6\}$. Thus, we see that $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_\Gamma]) \cong \mathbb{Z}^3$.

- Let $G = K_{2,2,2}$ be the complete tripartite graph. Namely, $V(G) = \{1, \ldots, 6\}$ with

$$E(G) = \{13, 14, 15, 16, 23, 24, 25, 26, 35, 36, 45, 46\}.$$

  See Figure 6.20. The class groups of the edge rings of complete multipartite graphs are investigated in [40]. By [40, Theorem 1.3], we see that $\mathrm{Cl}(\Bbbk[G]) \cong \mathbb{Z}^3$.

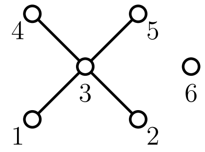

Figure 6.18: The poset Π

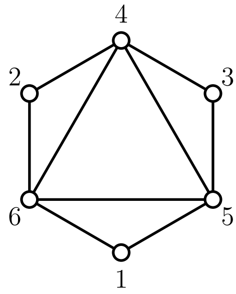

Figure 6.19: The graph Γ

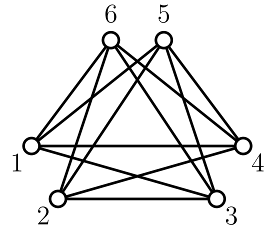

Figure 6.20: The graph $K_{2,2,2}$

We can see that $\mathcal{O}_\Pi \notin \mathbf{Stab}_3 \cup \mathbf{Edge}_3$, $\mathrm{Stab}_\Gamma \notin \mathbf{Order}_3 \cup \mathbf{Edge}_3$ and $P_G \notin \mathbf{Order}_3 \cup \mathbf{Stab}_3$ as follows.

$\mathcal{O}_\Pi \notin \mathbf{Stab}_3 \cup \mathbf{Edge}_3$: Consider $\mathcal{O}_\Pi$.

Suppose that there exists a perfect graph Γ' such that $\mathrm{Stab}_{\Gamma'}$ is unimodularly equivalent to $\mathcal{O}_\Pi$. Then Γ' has 6 vertices and non-trivial 4 independent sets. Since such graphs are finitely many, we can check by `MAGMA` that their stable set polytopes are not unimodularly equivalent to $\mathcal{O}_\Pi$.

Similarly, suppose that there exists a graph G' such that $P_{G'}$ is unimodularly equivalent to $\mathcal{O}_\Pi$. Then G' is a bipartite graph on 8 vertices or a non-bipartite graph on 7 vertices. Since $\mathrm{Cl}(\Bbbk[G']) \cong \mathbb{Z}^3$, G' contains at most one non-bipartite block by Proposition 6.3.1. We can also check that edge polytopes of such graphs are not unimodularly equivalent to $\mathcal{O}_\Pi$.

Proofs of $\mathrm{Stab}_\Gamma \notin \mathbf{Order}_3 \cup \mathbf{Edge}_3$ and $P_G \notin \mathbf{Order}_3 \cup \mathbf{Stab}_3$ can be performed in the similar way to the above discussions.

Chapter 7

Toric rings of $(0,1)$-polytopes with small rank

In this chapter, we focus on the toric rings of $(0,1)$-polytopes with small rank (in other words, we deal with toric rings whose divisor class groups have small rank). We study their normality, the torsionfreeness of their divisor class groups and the classification of their isomorphism classes. The contents of this chapter are contained in the author's paper [52].

We say that a normal integral polytope P is *torsionfree* if so is $\mathrm{Cl}(\Bbbk[P])$. Let $(\mathbf{0},\mathbf{1})_n$ be the set of the isomorphism classes of the toric rings of $(0,1)$-polytopes with rank n.

We consider the following problems:

- Examine the normality of $(0,1)$-polytopes with small rank.
- Are normal $(0,1)$-polytopes always torsionfree?
- Determine the set $(\mathbf{0},\mathbf{1})_n$. Also, does the relationship $(\mathbf{0},\mathbf{1})_n = \mathbf{Order}_n$ hold if $n \leq 2$?
- Does the combinatorial equivalence of two $(0,1)$-polytopes imply the isomorphism of their toric rings?

Let P be a $(0,1)$-polytope. We give complete or partial answers to these problems in each case;

$$\text{(r1)} \quad \operatorname{rank} P = 0 \text{ or } 1, \qquad \text{(r2)} \quad \operatorname{rank} P = 2, \qquad \text{(r3)} \quad \operatorname{rank} P \geq 3.$$

7.1 Case (r1)

First, we discuss the case $\operatorname{rank} P = 0$, this case is trivial. Notice that P has rank 0 if and only if P is a simplex. Moreover, it is known that the toric ring of d-dimensional $(0,1)$-simplex is isomorphic to the polynomial ring with $d+1$ variables over $\Bbbk$ (cf. [84, Lemma 3.1.5]).

Clearly, polynomial rings are normal and their divisor class groups are torsionfree. Moreover, for two $(0,1)$-simplices P and P', one has $\Bbbk[P] \cong \Bbbk[P']$ if and only if P and

P' are combinatorially equivalent (equivalently, they have the same number of vertices). Therefore, we get the following proposition:

Proposition 7.1.1. *All* $(0,1)$*-polytopes with rank* 0 *are normal and torsionfree. Moreover, the following relationship holds:*

$$(\mathbf{0},\mathbf{1})_0 = \mathbf{Order}_0 = \{\Bbbk[x_1,\ldots,x_k] : k \in \mathbb{Z}_{>0}\}.$$

Furthermore, for two $(0,1)$*-polytopes* P_1 *and* P_2 *with* $\operatorname{rank} P_i = 0$*, they have the same combinatorial type if and only if* $\Bbbk[P_1] \cong \Bbbk[P_2]$*.*

Next, we discuss the case $\operatorname{rank} P = 1$.

Theorem 7.1.2. *Let* $P \subset \mathbb{R}^d$ *be a* $(0,1)$*-polytope. Then the following conditions are equivalent:*

- *(i)* $\operatorname{rank} P = 1$*, that is,* P *has just* $\dim P + 2$ *facets;*
- *(ii)* $\Bbbk[P]$ *is isomorphic to the Segre product of two polynomial rings over* $\Bbbk$ *or its polynomial extension;*
- *(iii)* P *is normal and* $\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}$*.*

In particular, P *is normal and torsionfree if* P *has at most* $\dim P + 2$ *facets.*

Proof. (i)⇒(ii): A polytope P which has $\dim P + 2$ facets is a simple polytope or a pyramid. If P is an integral pyramid with basis Q, then Q has $\dim Q + 2$ facets. Thus Q is a simple polytope or a pyramid again. Therefore, from Proposition 2.2.1, it is enough to consider P in the case where P is simple. From Lemma 2.2.2, P is equal to a product of $(0,1)$-simplices $\Delta_1, \ldots, \Delta_m$. Since Δ_i has $\dim \Delta_i + 1$ facets for each i, we see that P has $\sum_{i=1}^{m}(\dim \Delta_i + 1) = \dim P + m$ facets. Thus, we have $m = 2$, and hence $\Bbbk[P]$ is isomorphic to the Segre product of two polynomial rings or its polynomial extension.

(ii)⇒(iii): It is known that the Segre product of two polynomial rings over $\Bbbk$ is normal and its divisor class group is isomorphic to $\mathbb{Z}$. In fact, the Segre product of some polynomial rings is realized as a Hibi ring (see, e.g., [43, Example 2.6]). Hibi rings are normal ([32]) and the description of their divisor class groups is provided in [5]. Thus, P is normal and $\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}$.

(iii)⇒(i): From Theorem 4.1.1, the rank of $\mathrm{Cl}(\Bbbk[P])$ is equal to $|\Psi(P)| - (\dim P + 1)$. Therefore $|\Psi(P)| = \dim P + 2$. □

Therefore, $(0,1)$-polytopes which have rank 1 are normal and torsionfree. Moreover, it follows from Proposition 6.4.1 that $(\mathbf{0},\mathbf{1})_1$ coincides with $\mathbf{Order}_1$. Furthermore, the equivalence (i) and (ii) imply that for two $(0,1)$-polytopes P and P' with $\operatorname{rank} P = \operatorname{rank} P' = 1$, one has $\Bbbk[P] \cong \Bbbk[P']$ if and only if P and P' are combinatorially equivalent. Hence, the following theorem holds:

Theorem 7.1.3. *All* $(0,1)$*-polytopes with rank* 1 *are normal and torsionfree. Moreover, the following relationship holds:*

$$(\mathbf{0},\mathbf{1})_1 = \mathbf{Order}_1 = \{(\Bbbk[x_1,\ldots,x_{n+1}]\#\Bbbk[y_1,\ldots,y_{m+1}]) \otimes_{\Bbbk} \Bbbk[z_1,\ldots,z_{l-1}] : n,m,l \in \mathbb{Z}_{>0}\}.$$

Furthermore, for two $(0,1)$*-polytopes* P_1 *and* P_2 *with* $\operatorname{rank} P_i = 1$*, they have the same combinatorial type if and only if* $\Bbbk[P_1] \cong \Bbbk[P_2]$*.*

In summary, in Case (r1), our problems have positive answers.

7.2 Case (r2)

7.2.1 A new family of $(0,1)$-polytopes

In this section, we construct a new family of $(0,1)$-polytopes $P_{n_1,\ldots,n_k}$ and study its properties, which is needed to consider our problems. We show that $P_{n_1,\ldots,n_k}$ is normal and torsionfree. Moreover, we see that if $k \geq 3$, then $\Bbbk[P_{n_1,\ldots,n_k}]$ is not isomorphic to any Hibi ring.

Let $n_1, \ldots, n_k$ be positive integers and let $d = n_1 + \cdots + n_k$. Moreover, let $\mathcal{B}_n$ be the standard basis of $\mathbb{R}^n$ and let $\mathcal{B}_{n_1,\ldots,n_k} = \mathcal{B}_{n_1} \times \cdots \times \mathcal{B}_{n_k}$. Then, we define the subset $V_{n_1,\ldots,n_k}$ of $\mathbb{Z}^d$ as

$$V_{n_1,\ldots,n_k} = \{\mathbf{0}\} \cup \mathcal{B}_d \cup \mathcal{B}_{n_1,\ldots,n_k}$$

and define the $(0,1)$-polytope $P_{n_1,\ldots,n_k} = \mathrm{conv}(V_{n_1,\ldots,n_k})$.

Example 7.2.1. We can see that

$$P_{1,1,1} = \mathrm{conv}(\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\})$$

and

$$\Bbbk[P_{1,1,1}] \cong \Bbbk[x_1,x_2,x_3,x_4,x_5]/(x_2x_3x_4 - x_1^2x_5).$$

In addition, we can see that

$$\begin{aligned} P_{1,1,1,2} = \mathrm{conv}(\{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0)\\ &(0,0,0,1,0),(0,0,0,0,1),(1,1,1,1,0),(1,1,1,0,1)\}) \end{aligned}$$

and

$$\Bbbk[P_{1,1,1,2}] \cong \Bbbk[x_1,\ldots,x_8]/(x_6x_7 - x_5x_8, x_2x_3x_4x_6 - x_1^3x_8, x_2x_3x_4x_5 - x_1^3x_7).$$

First, we describe the facet defining inequalities of $P_{n_1,\ldots,n_k}$. By considering the embedding

$$\mathcal{B}_{n_p} \hookrightarrow \mathbb{R}^d \qquad \mathbf{i}_p \mapsto (\underbrace{0,\ldots\ldots,0}_{n_1+\cdots+n_{p-1}},\ \mathbf{i}_p\ ,\underbrace{0,\ldots\ldots,0}_{n_{p+1}+\cdots+n_k}),$$

we regard $\mathbf{i}_p \in \mathcal{B}_{n_p}$ as the element of $\mathbb{R}^d$.

For $p \in [k]$, let $\mathbf{f}_p = (k-2)\mathbf{1}_p - \sum_{q \in [k]\setminus\{p\}} \mathbf{1}_q$, where $\mathbf{1}_p = \sum_{\mathbf{i}_p \in \mathcal{B}_{n_p}} \mathbf{i}_p$.

Proposition 7.2.2. *The $(0,1)$-polytope $P_{n_1,\ldots,n_k}$ has the irreducible representation:*

$$P_{n_1,\ldots,n_k} = \Big(\bigcap_{\mathbf{e} \in \mathcal{B}_d} H^+_{\mathbf{e}}\Big) \cap \Big(\bigcap_{p \in [k]} H^+(\mathbf{f}_p; 1)\Big).$$

Proof. If $k=1$, then $P_{n_1} = \mathrm{conv}(\{\mathbf{0}\} \cup \mathcal{B}_{n_1})$ and we can see easily that P_{n_1} has the above irreducible representation. In what follows, suppose that $k \geq 2$.

We can find d affinely independent vectors in $V_{n_1,\ldots,n_k}$ on each hyperplane. Thus, these define facets of $P_{n_1,\ldots,n_k}$.

Let $P' = \left(\bigcap_{\mathbf{e}\in\mathcal{B}_d} H^+_{\mathbf{e}}\right) \cap \left(\bigcap_{p\in[k]} H^+(\mathbf{f}_p;1)\right)$ and we show $P_{n_1,\ldots,n_k} = P'$. It is clear that $P_{n_1,\ldots,n_k} \subset P'$ since $V_{n_1,\ldots,n_k} \subset P'$. Also, each vertex of $P_{n_1,\ldots,n_k}$ is on d affinely independent hyperplanes $H_{\mathbf{e}}$ or $H(\mathbf{f}_p;1)$, that is, the set of vertices of P', denoted by V, contains $V_{n_1,\ldots,n_k}$. Therefore, it is enough to show that $V \subset V_{n_1,\ldots,n_k}$.

Note that if $\mathbf{x} \in P'$, then $\mathbf{x} \in H^+(\mathbf{f}_p;1)$, i.e., $\langle \mathbf{x}, \mathbf{f}_p\rangle + 1 \geq 0$ for all $p \in [k]$. Thus, for any $p \in [k]$, we have

$$\sum_{q\in[k]\setminus\{p\}} (\langle \mathbf{x}, \mathbf{f}_q\rangle + 1) = -(k-1)\langle \mathbf{x}, \mathbf{1}_p\rangle + (k-1) \geq 0 \ \Rightarrow\ \langle \mathbf{x}, \mathbf{1}_p\rangle \leq 1. \tag{7.2.1}$$

Moreover, since $\mathbf{x} \in H^+_{\mathbf{e}}$ for all $\mathbf{e} \in \mathcal{B}_d$, we see that $V \subset P' \subset [0,1]^d$.

Assume that there exists an element $\mathbf{v}$ in $V \setminus V_{n_1,\ldots,n_k}$. First, suppose that $\mathbf{v} \in \{0,1\}^d$. In this case, we see that $\langle \mathbf{v}, \mathbf{1}_p\rangle = 0$ or 1 holds for all $p \in [k]$ from (7.2.1). One of the following three cases happens;

(i) $\langle \mathbf{v}, \mathbf{1}_p\rangle = 0$ for all $p \in [k]$ or $\langle \mathbf{v}, \mathbf{1}_p\rangle = 1$ for all $p \in [k]$.

(ii) $\langle \mathbf{v}, \mathbf{1}_p\rangle = 1$ for some $p \in [k]$ and $\langle \mathbf{v}, \mathbf{1}_q\rangle = 0$ for all $q \in [k] \setminus \{p\}$.

(iii) There exist $p_1, p_2, p_3 \in [k]$ such that $\langle \mathbf{v}, \mathbf{1}_{p_1}\rangle = \langle \mathbf{v}, \mathbf{1}_{p_2}\rangle = 1$ and $\langle \mathbf{v}, \mathbf{1}_{p_3}\rangle = 0$.

In the cases (i) and (ii), $\mathbf{v}$ must lie in $V_{n_1,\ldots,n_k}$. In the case (iii), we have $\langle \mathbf{v}, \mathbf{f}_{p_3}\rangle + 1 < 0$, that is, $\mathbf{v} \notin H^+(\mathbf{f}_{p_3};1)$, a contradiction.

Now, suppose that $\mathbf{v} \notin \{0,1\}^d$. If there are two elements $\mathbf{i}_p, \mathbf{j}_p \in \mathcal{B}_{n_p}$ with $\langle \mathbf{v}, \mathbf{i}_p\rangle, \langle \mathbf{v}, \mathbf{j}_p\rangle > 0$ for some $p \in [k]$, then $\mathbf{v}$ is not a vertex of P'. Indeed, let

$$\mathbf{v}' = \mathbf{v} + \epsilon \mathbf{i}_p - \epsilon \mathbf{j}_p, \quad \mathbf{v}'' = \mathbf{v} - \epsilon \mathbf{i}_p + \epsilon \mathbf{j}_p,$$

where $\epsilon > 0$ is sufficiently small. Then, we can check that $\mathbf{v}', \mathbf{v}'' \in P'$ and $\mathbf{v} = \frac{1}{2}(\mathbf{v}' + \mathbf{v}'')$. Therefore, we assume that for each $p \in [k]$, $\langle \mathbf{v}, \mathbf{i}_p\rangle \geq 0$ for some $\mathbf{i}_p \in \mathcal{B}_{n_p}$ and $\langle \mathbf{v}, \mathbf{j}_p\rangle = 0$ for all $\mathbf{j}_p \in \mathcal{B}_{n_p} \setminus \{\mathbf{i}_p\}$. Let $t = \sum_{p\in[k]}\langle \mathbf{v}, \mathbf{i}_p\rangle$. In the case $t \leq 1$, let $t_p = \langle \mathbf{v}, \mathbf{i}_p\rangle$ for $p \in [k]$ and $t_{k+1} = 1 - t$. Then, we have $\mathbf{v} = \sum_{p\in[k]} t_p \mathbf{i}_p + t_{k+1}\mathbf{0}$ and $\sum_{p\in[k+1]} t_p = 1$, and hence $\mathbf{v} \notin V$. Finally, we consider the case of $t > 1$. If there exists $p \in [k]$ such that $\langle \mathbf{v}, \mathbf{i}_p\rangle = 0$ then we have $\langle \mathbf{v}, \mathbf{f}_p\rangle + 1 = -\sum_{q\in[k]\setminus\{p\}}\langle \mathbf{v}, \mathbf{i}_q\rangle + 1 = -t + 1 < 0$, a contradiction. Thus, we can see that $\langle \mathbf{v}, \mathbf{i}_p\rangle > 0$ for all $p \in [k]$. Moreover, since $\mathbf{v}$ must lie on d affinely independent hyperplanes $H_{\mathbf{e}}$ or $H(\mathbf{f}_p;1)$, we have $\mathbf{v} \in H(\mathbf{f}_p;1)$ for all $p \in [k]$, that is, $\langle \mathbf{v}, \mathbf{f}_p\rangle + 1 = 0$ holds for any $p \in [k]$. We may assume that $\langle \mathbf{v}, \mathbf{i}_1\rangle = \min\{\langle \mathbf{v}, \mathbf{i}_p\rangle : p \in [k]\}$, and let $t_p = \begin{cases} \langle \mathbf{v}, \mathbf{i}_p\rangle - \langle \mathbf{v}, \mathbf{i}_1\rangle & \text{if } p \neq 1, \\ \langle \mathbf{v}, \mathbf{i}_1\rangle & \text{if } p = 1 \end{cases}$ for $p \in [k]$. Then, we obtain

$$\mathbf{v} = \sum_{p\in[k]\setminus\{1\}} t_p \mathbf{i}_p + t_1 \sum_{p\in[k]} \mathbf{i}_p$$

and

$$\sum_{p\in[k]} t_p = -(k-2)\langle \mathbf{v}, \mathbf{i}_1\rangle + \sum_{p\in[k]\setminus\{1\}} \langle \mathbf{v}, \mathbf{i}_p\rangle = -(k-2)\langle \mathbf{v}, \mathbf{1}_1\rangle + \sum_{p\in[k]\setminus\{1\}} \langle \mathbf{v}, \mathbf{1}_p\rangle = -\langle \mathbf{v}, \mathbf{f}_1\rangle = 1.$$

Therefore, v is not a vertex of P', and hence V coincides with $V_{n_1,\ldots,n_k}$.

□

Example 7.2.3. We consider $P_{1,1,1}$. Then,

$$\begin{aligned} x_i &\geq 0 \quad (i = 1, 2, 3), \\ \langle \mathbf{f}_p, \mathbf{x} \rangle + 1 = x_p - (x_{q_1} + x_{q_2}) + 1 &\geq 0 \quad (p = 1, 2, 3 \text{ and } \{p, q_1, q_2\} = \{1, 2, 3\}) \end{aligned}$$

are the facet defining inequalities of $P_{1,1,1}$.

Moreover, we consider $P_{1,1,1,2}$. Then, the following are the facet defining inequalities of $P_{1,1,1,2}$:

$$\begin{aligned} x_i &\geq 0 \quad (i = 1, \ldots, 5), \\ 2x_p - (x_{q_1} + \cdots + x_{q_4}) + 1 &\geq 0 \quad (p = 1, 2, 3 \text{ and } \{p, q_1, \ldots, q_4\} = \{1, \ldots, 5\}) \text{ and} \\ 2(x_4 + x_5) - (x_1 + x_2 + x_3) + 1 &\geq 0. \end{aligned}$$

Next, we investigate the initial ideal of the toric ideal $I_{P_{n_1,\ldots,n_k}}$ with respect to a monomial order and provide its Gröbner basis, which allows us to study the normality. For the fundamental materials on initial ideals and Gröbner bases, consult, e.g., [29].

For $x_{\mathbf{v}} \in T = \Bbbk[x_{\mathbf{v}} : \mathbf{v} \in P_{n_1,\ldots,n_k} \cap \mathbb{Z}^d]$, we denote by $x_{\mathbf{i}_1,\ldots,\mathbf{i}_k}$ instead of $x_{(\mathbf{i}_1,\ldots,\mathbf{i}_k)}$ for $(\mathbf{i}_1, \ldots, \mathbf{i}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$. Let $<$ denote the reverse lexicographic order on T induced by the ordering of the variables as follows:

- $x_{\mathbf{e}} < x_{\mathbf{i}_1,\ldots,\mathbf{i}_k}$ for any $\mathbf{e} \in \{\mathbf{0}\} \cup \mathcal{B}_d$ and any $(\mathbf{i}_1, \ldots, \mathbf{i}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$;
- For $\mathbf{e}, \mathbf{e}' \in \{\mathbf{0}\} \cup \mathcal{B}_d$ (resp. for $(\mathbf{i}_1, \ldots, \mathbf{i}_k), (\mathbf{j}_1, \ldots, \mathbf{j}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$), $x_{\mathbf{e}} < x_{\mathbf{e}'}$ (resp. $x_{\mathbf{i}_1,\ldots,\mathbf{i}_k} < x_{\mathbf{j}_1,\ldots,\mathbf{j}_k}$) if and only if $\mathbf{e} < \mathbf{e}'$ (resp. $(\mathbf{i}_1, \ldots, \mathbf{i}_k) < (\mathbf{j}_1, \ldots, \mathbf{j}_k)$), which means that the leftmost nonzero component of the vector $\mathbf{e}' - \mathbf{e}$ (resp. $(\mathbf{j}_1 - \mathbf{i}_1, \ldots, \mathbf{j}_k - \mathbf{i}_k)$) is positive.

Moreover, let $\mathcal{G}_{n_1,\ldots,n_k}$ be the sets of the following binomials in T:

(b1) $x_{\mathbf{i}_1} x_{\mathbf{i}_2} \cdots x_{\mathbf{i}_k} - x_{\mathbf{0}}^{k-1} x_{\mathbf{i}_1,\ldots,\mathbf{i}_k}$ for $(\mathbf{i}_1, \ldots, \mathbf{i}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$;

(b2) $x_{\mathbf{j}_p} x_{\mathbf{i}_1,\ldots,\mathbf{i}_p,\ldots,\mathbf{i}_k} - x_{\mathbf{i}_p} x_{\mathbf{i}_1,\ldots,\mathbf{j}_p,\ldots,\mathbf{i}_k}$ for $(\mathbf{i}_1, \ldots, \mathbf{i}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$, $\mathbf{j}_p \in \mathcal{B}_{n_p}$ with $\mathbf{i}_p < \mathbf{j}_p$;

(b3) $x_{\mathbf{i}_1,\ldots,\mathbf{i}_k} x_{\mathbf{j}_1,\ldots,\mathbf{j}_k} - x_{\mathbf{i}'_1,\ldots,\mathbf{i}'_k} x_{\mathbf{j}'_1,\ldots,\mathbf{j}'_k}$ for $(\mathbf{i}_1, \ldots, \mathbf{i}_k), (\mathbf{j}_1, \ldots, \mathbf{j}_k) \in \mathcal{B}_{n_1,\ldots,n_k}$,

where we define $\mathbf{i}'_p = \begin{cases} \mathbf{i}_p & \text{if } x_{\mathbf{i}_p} < x_{\mathbf{j}_p}, \\ \mathbf{j}_p & \text{else} \end{cases}$ and define $\mathbf{j}'_p$ as satisfying $\{\mathbf{i}'_p, \mathbf{j}'_p\} = \{\mathbf{i}_p, \mathbf{j}_p\}$. Note that each leading term of these binomials is the initial monomial with respect to $<$. Furthermore, these binomials belong to the toric ideal of $P_{n_1,\ldots,n_k}$, that is, $\mathcal{G}_{n_1,\ldots,n_k} \subset I_{P_{n_1,\ldots,n_k}}$.

Proposition 7.2.4. *Let the notation be the same as above. Then, $\mathcal{G}_{n_1,\ldots,n_k}$ is a Gröbner basis of $I_{\mathcal{P}_{n_1,\ldots,n_k}}$ with respect to $<$.*

To show this proposition, we use the following lemma:

Lemma 7.2.5 ([29, Theorem 3.11]). *Let $I \subset S = \Bbbk[x_1, \dots, x_n]$ be the toric ideal of an integral polytope and $\mathcal{G} = \{g_1, \dots, g_s\}$ the set of binomials in I. Fix a monomial order $<$ on S and let $\mathrm{in}_<(\mathcal{G})$ be the ideal of S generated by the initial monomials $\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s)$, that is, $\mathrm{in}_<(\mathcal{G}) = (\{\mathrm{in}_<(g) : g \in \mathcal{G}\})$. Then, the following conditions are equivalent:*

(i) $\mathcal{G}$ is a Gröbner basis of I with respect to $<$.

(ii) For monomials $u, v \in S$, if $u \notin \mathrm{in}_<(\mathcal{G})$, $v \notin \mathrm{in}_<(\mathcal{G})$ and $u \neq v$ then $u - v \notin I$.

Proof of Proposition 7.2.4. We show that $\mathcal{G}_{n_1,\dots,n_k}$ satisfies the condition (ii) in Lemma 7.2.5. Let u be a monomial in T. Then, u can be written by

$$x_{\mathbf{0}}^a x_{\mathbf{e}_1} x_{\mathbf{e}_2} \cdots x_{\mathbf{e}_s} x_{\mathbf{i}_{11},\dots,\mathbf{i}_{1k}} \cdots x_{\mathbf{i}_{t1},\dots,\mathbf{i}_{tk}},$$

where $a \in \mathbb{Z}_{\geq 0}$, $\mathbf{e}_1, \dots, \mathbf{e}_s \in \mathcal{B}_d$ and $\mathbf{i}_{1p}, \dots, \mathbf{i}_{tp} \in \mathcal{B}_{n_p}$ for each $p \in [k]$. Let $M_p = \{\mathbf{e}_1, \dots, \mathbf{e}_s\} \cap \mathcal{B}_{n_p}$ and $N_p = |M_p|$ for $p \in [k]$. Then, if $u \notin \mathrm{in}_<(\mathcal{G}_{n_1,\dots,n_k})$, we can see that:

(a) there exists $p \in [k]$ such that $M_p = \emptyset$ from (b1);

(b) for each $p \in [k]$, we have $\mathbf{e}_i \leq \mathbf{i}_{lp}$ for any $\mathbf{e} \in M_p$ and for any $l \in [t]$ from (b2);

(c) $\mathbf{i}_{1p} \leq \mathbf{i}_{2p} \leq \cdots \leq \mathbf{i}_{tp}$ for all $p \in [k]$ by permuting $x_{\mathbf{i}_{11},\dots,\mathbf{i}_{1k}}, \dots, x_{\mathbf{i}_{t1},\dots,\mathbf{i}_{tk}}$ from (b3).

Let $\phi_{P_{n_1\dots,n_k}}(u) = t_1^{r_1} t_2^{r_2} \cdots t_d^{r_d} t_{d+1}^{r_{d+1}}$. Now, we may assume that $M_1 = \emptyset$. Then, we can see that $t = r_1 + r_2 + \cdots + r_{n_1}$. Similarly, N_p and a can also be represented by r_i's. Moreover, it follows from (b), (c) that $\mathbf{e}_i$ and $\mathbf{i}_{1p}, \dots, \mathbf{i}_{tp}$ can be determined uniquely from $r_1, \dots, r_d, r_{d+1}$. Therefore, we can recover u from $t_1^{r_1} t_2^{r_2} \cdots t_d^{r_d} t_{d+1}^{r_{d+1}}$. This is equivalent to (ii) in Lemma 7.2.5. □

Now, we give the main theorem in this section.

Theorem 7.2.6. *The $(0,1)$-polytope $P_{n_1\dots,n_k}$ has the following properties:*

(i) $P_{n_1,\dots,n_k}$ has IDP.

(ii) $\Bbbk[P_{n_1,\dots,n_k}]$ is Gorenstein if and only if $n_1 = n_2 = \cdots = n_k$.

(iii) $\mathrm{Cl}(\Bbbk[P_{n_1,\dots,n_k}]) \cong \mathbb{Z}^{k-1}$.

(iv) If $k \geq 3$, then $\Bbbk[P_{n_1,\dots,n_k}] \notin \mathbf{Order}_{k-1}$ for any $n_1, \dots, n_k \in \mathbb{Z}_{>0}$.

Proof. (i) It follows from Proposition 7.2.4 that the initial ideal of $I_{n_1,\dots,n_k}$ is squarefree, and hence $P_{n_1,\dots,n_k}$ possesses a regular unimodular triangulation (cf. [29, Theorem 4.17]). This implies that $P_{n_1,\dots,n_k}$ is normal. Moreover, $\mathbb{Z}\mathcal{A}(P_{n_1,\dots,n_k})$ coincides with $\mathbb{Z}^{d+1}$ since $\{\mathbf{0}\} \cup \mathcal{B}_d \subset P_{n_1,\dots,n_k}$, and hence $P_{n_1,\dots,n_k}$ has IDP.

(ii) Since $\Bbbk[P_{n_1,\dots,n_k}]$ is a normal affine semigroup ring, the canonical module $\omega_{\Bbbk[P_{n_1,\dots,n_k}]}$ is isomorphic to the module generated by all monomials whose exponent vector is a lattice point in $\left(\bigcap_{\mathbf{e} \in \mathcal{B}_d} H^+((\mathbf{e}, 0); 1)\right) \cap \left(\bigcap_{p \in [k]} H^+((\mathbf{f}_p, 1); 1)\right)$. By the parallel translation by

$(1,\dots,1,\alpha)$ for some integer α, we can see that it is also isomorphic to the module generated by all monomials whose exponent vector is a lattice point in $\left(\bigcap_{\mathbf{e}\in\mathcal{B}_d} H^+((\mathbf{e},0);0)\right)\cap \left(\bigcap_{p\in[k]} H^+((\mathbf{f}_p,1);m_p)\right)$, where $m_p = 1-\alpha-(k-2)n_p+\sum_{q\in[k]\setminus\{p\}} n_q$.

Thus, if $\Bbbk[P_{n_1,\dots,n_k}]$ is Gorenstein, then the equality $m_1=\cdots=m_k=0$ must hold. This implies $n_1=\cdots=n_k$. Conversely, if $n_1,\dots,n_k$ are equal, then we can see that $\omega_{\Bbbk[P_{n_1,\dots,n_k}]}$ is isomorphic to $\Bbbk[P_{n_1,\dots,n_k}]$ by setting $\alpha=n_1+1$, that is, $\Bbbk[P_{n_1,\dots,n_k}]$ is Gorenstein.

(iii) Let $\mathbf{v}_1,\dots,\mathbf{v}_d$ be vectors in $\mathbb{Z}^d$ satisfying $\{\mathbf{v}_1,\dots,\mathbf{v}_d\}=\mathcal{B}_d$ and let $\mathbf{v}_{d+1}=\mathbf{0}$. Moreover, for $i\in[d]$, let F_i be the facet of $P_{n_1,\dots,n_k}$ defined by $H(\mathbf{v}_i;0)$ and let F_{d+1} be the facet of $P_{n_1,\dots,n_k}$ defined by $H(\mathbf{f}_1;1)$. Then, we can see that these sequences satisfy the statement $(*)$ and we have $k_{P_{n_1,\dots,n_k}}=d+1$. Therefore, we have $\mathrm{Cl}(\Bbbk[P_{n_1,\dots,n_k}])\cong\mathbb{Z}^{k-1}$ from Theorem 5.2.2.

(iv) Since $\mathcal{G}_{n_1,\dots,n_k}$ is a Gröbner basis, the binomials (b1), (b2) and (b3) generate $I_{P_{n_1,\dots,n_k}}$. Consider a minimal generators $\mathcal{G}'$ of $I_{P_{n_1,\dots,n_k}}$ contained these binomials. If $\mathcal{G}'$ does not contain any binomial $x_{\mathbf{i}_1}x_{\mathbf{i}_2}\cdots x_{\mathbf{i}_k}-x_{\mathbf{0}}^{k-1}x_{\mathbf{i}_1,\dots,\mathbf{i}_k}$ in (b1), then there exists an expression:

$$x_{\mathbf{i}_1}x_{\mathbf{i}_2}\cdots x_{\mathbf{i}_k}-x_{\mathbf{0}}^{k-1}x_{\mathbf{i}_1,\dots,\mathbf{i}_k}=\sum_{i=1}^{s}\alpha_i\mathbf{x}^{\mathbf{w}_i}f_i,$$

where $\alpha_i\in\mathbb{Z}$, $\mathbf{x}^{\mathbf{w}_i}$ is a monomial of the polynomial ring $\Bbbk[x_{\mathbf{v}}:\mathbf{v}\in P\cap\mathbb{Z}^d]$ and f_i's are binomials in (b2) or (b3) (cf. [29, Lemma 3.7]). However, it is impossible because the variable $x_{\mathbf{i}_1,\dots,\mathbf{i}_k}$ for some $(\mathbf{i}_1,\dots,\mathbf{i}_k)\in V_{n_1,\dots,n_k}$ must appear in the terms of f_i. Therefore, $\mathcal{G}'$ contains $x_{\mathbf{i}_1}x_{\mathbf{i}_2}\cdots x_{\mathbf{i}_k}-x_{\mathbf{0}}^{k-1}x_{\mathbf{i}_1,\dots,\mathbf{i}_k}$ for some $\mathbf{i}_1,\dots,\mathbf{i}_k$. This implies that $I_{P_{n_1,\dots,n_k}}$ cannot be generated by quadratic binomials.

On the other hand, the toric ideals of Hibi rings are generated by quadratic binomials ([32]). Thus, $\Bbbk[P_{n_1,\dots,n_k}]$ is not isomorphic to any Hibi ring. $\square$

Finally, we compute weights of $\Bbbk[P_{n_1,\dots,n_k}]$. Let $F_{\mathbf{e}}$ denote the facet $P_{n_1,\dots,n_k}\cap H(\mathbf{e};0)$ for $\mathbf{e}\in\mathcal{B}_d$ and let F_i denote the facet $P_{n_1,\dots,n_k}\cap H(\mathbf{f}_p;1)$ for $p\in[k]$. From (4.1.3) and Theorem 7.2.6 (i), the following elements in $\mathcal{F}$ belong to $\mathcal{S}$:

$$\sum_{F\in\Psi(P_{n_1,\dots,n_k})}\langle(\mathbf{i}_p,0),\mathbf{c}_F\rangle\epsilon_F=\epsilon_{F_{\mathbf{i}_p}}+(k-2)\epsilon_{F_p}-\sum_{q\in[k]\setminus\{p\}}\epsilon_{F_q} \tag{7.2.2}$$

for each $p\in[k]$ and $\mathbf{i}_p\in\mathcal{B}_{n_p}$ and

$$\sum_{F\in\Psi(P_{n_1,\dots,n_k})}\langle(\mathbf{0},1),\mathbf{c}_F\rangle\epsilon_F=\sum_{p\in[k]}\epsilon_{F_p}. \tag{7.2.3}$$

We consider the map $\iota:\mathcal{F}/\mathcal{S}\to\mathbb{Z}^{k-1}$; let $\iota(\epsilon_{F_i})=\mathbf{e}_i$ for $i\in[k-1]$, where $\mathbf{e}_i$ denotes the i-th unit vector of $\mathbb{Z}^{k-1}$. This induces an isomorphism $\iota:\mathcal{F}/\mathcal{S}\to\mathbb{Z}^{k-1}$ and we can calculate the remaining weight from (7.2.2) and (7.2.3):

$$\begin{aligned}
\iota(\epsilon_{F_k})&=-(\mathbf{e}_1+\cdots+\mathbf{e}_{k-1});\\
\iota(\epsilon_{F_{\mathbf{i}_p}})&=-(k-1)\mathbf{e}_p \quad \text{for each } p\in[k-1] \text{ and any } \mathbf{i}_p\in\mathcal{B}_{n_p};\\
\iota(\epsilon_{F_{\mathbf{i}_k}})&=(k-1)(\mathbf{e}_1+\cdots+\mathbf{e}_{k-1}) \quad \text{for any } \mathbf{i}_k\in\mathcal{B}_{n_k}.
\end{aligned}$$

In particular, we can get the weights of the case $k = 3$ as follows:

$$(1,0), \quad (0,1), \quad (-1,-1), \quad (-2,0)\times n_1, \quad (0,-2)\times n_2, \quad (2,2)\times n_3. \tag{7.2.4}$$

Here $\times$ stands for the multiplicities.

7.2.2 Approaches using Gale-diagrams

By Proposition 2.2.1, in what follows, we may assume that P is not pyramidal.

Weights of the Hibi rings whose divisor class groups have rank 2 are computed in [61, Sections 3.2 and 3.3]. The following table summarizes the weights of the Hibi rings associated with the posets in Figures 6.2, 6.3, 6.4 and 6.5:

Π_2'	Π_2	Π_3	Π_4
$(1,0)\times s_1+1$ $(0,1)\times s_2+1$ $(-1,-1)\times s_3+1$	$(1,0)\times s_1+1$ $(-1,-1)\times s_2+1$ $(0,1)\times s_3+1$ $(0,-1)\times t$	$(1,0)\times s_1+1$ $(0,1)\times s_2+1$ $(0,-1)\times t_1$ $(-1,0)\times t_2$ $(-1,-1)\times t_3+1$	$(1,0)\times s_1+1$ $(-1,0)\times s_2+1$ $(0,1)\times t_1+1$ $(0,-1)\times t_2+1$

Here Π_2' means that Π_2 with $t = 0$.

By Theorem 4.2.1, we can obtain the standard Gale-diagrams of the dual polytopes of the order polytopes which have rank 2 and P_{n_1,n_2,n_3}. We draw the standard Gale-diagrams as follows; the dual polytopes of $\mathcal{O}_{\Pi_2'}, \mathcal{O}_{\Pi_2}, \mathcal{O}_{\Pi_3}, \mathcal{O}_{\Pi_4}$ and P_{n_1,n_2,n_3} correspond to the standard Gale-diagrams $\mathrm{Gale}_1, \mathrm{Gale}_2, \mathrm{Gale}_3, \mathrm{Gale}_4$ and Gale_5, respectively.

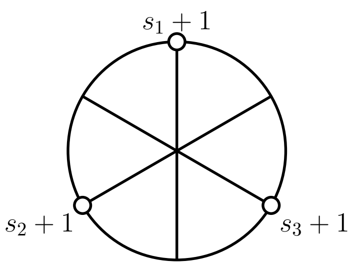

Figure 7.1:
The Gale-diagram Gale_1

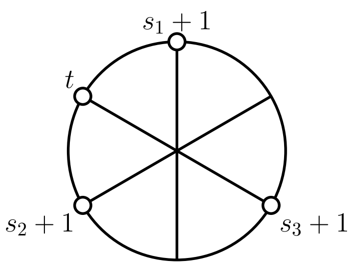

Figure 7.2:
The Gale-diagram Gale_2

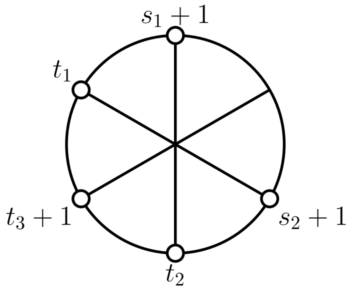

Figure 7.3:
The Gale-diagram Gale_3

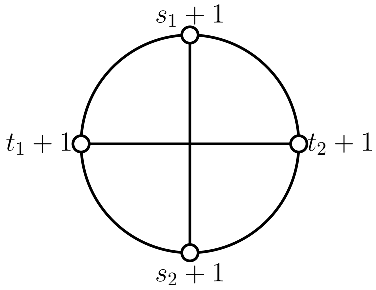

Figure 7.4:
The Gale-diagram Gale_4

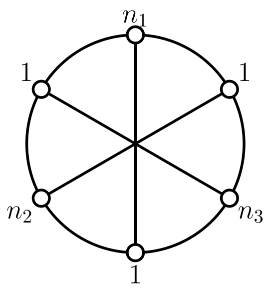

Figure 7.5:
The Gale-diagram Gale_5

Theorem 7.2.7. *Let P be a $(0,1)$-polytope. Then the following are equivalent:*

(i) The standard Gale-diagram of the dual polytope of P is orthogonally equivalent to Gale_1*.*

(ii) The toric ring of P is isomorphic to the Segre product of three polynomial rings over $\Bbbk$*.*

In particular, let P_1 and P_2 be two $(0,1)$-polytopes which have a standard Gale-diagram Gale_1*, then P_1 and P_2 are combinatorially equivalent if and only if* $\Bbbk[P_1] \cong \Bbbk[P_2]$*.*

Proof. (i) $\Rightarrow$ (ii) : Since no diameter of the standard Gale-diagram has both endpoints, the dual polytope of P is simplicial, that is, P is simple. Therefore, P is the product of just three $(0,1)$-simplices from Proposition 2.2.2, and hence $\Bbbk[P]$ is isomorphic to the Segre product of three polynomial rings.

(ii) $\Rightarrow$ (i) : It follows from Proposition 3.2.1 that the Segre product of three polynomial rings can be realized as the Hibi ring of a poset Π_1. Its weights have been already given and we can obtain the standard Gale-diagram of the dual polytope of P which is orthogonally equivalent to Gale_1.

The last statement follows from the equivalence (i) and (ii). □

Theorem 7.2.8. *The following are equivalent:*

(i) The standard Gale-diagram of the dual polytope of P is orthogonally equivalent to Gale_4*.*

(ii) The toric ring of P is isomorphic to $R_1 \otimes_{\Bbbk} R_2$, where R_i is the Segre product of two polynomial rings over $\Bbbk$.

In particular, let P_1 and P_2 be two $(0,1)$-polytopes which have a standard Gale-diagram Gale_4*, then P_1 and P_2 are combinatorially equivalent if and only if $\Bbbk[P_1] \cong \Bbbk[P_2]$.*

Proof. (i) $\Rightarrow$ (ii) : Since the standard Gale-diagram of the dual polytope of P is orthogonally equivalent to Gale_4, P is combinatorially equivalent to $\mathcal{O}_{\Pi_4}$. Therefore, there exists a one-to-one mapping Φ between the set of all faces of $\mathcal{O}_{\Pi_4}$ and the set of all faces of P such that Φ is inclusion preserving.

Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ denote the vertices of $\mathcal{O}_{\Pi_4}$ corresponding to the poset ideals of Π_4 including the element which is comparable with any other element of Π_4, and let $\mathbf{v}_1, \dots, \mathbf{v}_m$ denote the remaining vertices of $\mathcal{O}_{\Pi_4}$. Notice that any facet F of $\mathcal{O}_{\Pi_4}$ contains $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ or $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, that is, $d_F(\mathbf{u}_i) = 0$ for all $i \in [n]$, or $d_F(\mathbf{v}_j) = 0$ for all $j \in [m]$. Moreover, we can see that $O_1 = \text{conv}(\{\mathbf{u}_1, \dots, \mathbf{u}_n\})$ and $O_2 = \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_m\})$ are $(0,1)$-polytopes with $\text{rank}\, O_1 = \text{rank}\, O_2 = 1$.

Let $\mathbf{w}_i = \Phi(\mathbf{u}_i)$, $\mathbf{z}_j = \Phi(\mathbf{v}_j)$, $Q_1 = \Phi(O_1) = \text{conv}(\{\mathbf{w}_1, \dots, \mathbf{w}_n\})$ and $Q_2 = \Phi(O_2) = \text{conv}(\{\mathbf{z}_1, \dots, \mathbf{z}_m\})$. We show that $\Bbbk[P] \cong \Bbbk[Q_1] \otimes_{\Bbbk} \Bbbk[Q_2]$, equivalently, $I_P = RI_{Q_1} + RI_{Q_2}$, where $R = \Bbbk[x_{\mathbf{v}} : \mathbf{v} \in P \cap \mathbb{Z}^d]$. Since $I_P \supset RI_{Q_1}$ and $I_P \supset RI_{Q_2}$, we have $I_P \supset RI_{Q_1} + RI_{Q_2}$. To prove the reverse inclusion, it is enough to show that a binomial $b = x_{\mathbf{w}_{i_1}} \cdots x_{\mathbf{w}_{i_p}} x_{\mathbf{z}_{j_1}} \cdots x_{\mathbf{z}_{j_q}} - x_{\mathbf{w}_{h_1}} \cdots x_{\mathbf{w}_{h_s}} x_{\mathbf{z}_{g_1}} \cdots x_{\mathbf{z}_{g_t}} \in I_P$ belongs to $RI_{Q_1} + RI_{Q_2}$.

In this situation, we have

$$\mathbf{w}'_{i_1} + \cdots + \mathbf{w}'_{i_p} + \mathbf{z}'_{j_1} + \cdots + \mathbf{z}'_{j_q} = \mathbf{w}'_{h_1} + \cdots + \mathbf{w}'_{h_s} + \mathbf{z}'_{g_1} + \cdots + \mathbf{z}'_{g_t},$$

where for $\mathbf{v} \in \mathbb{Z}^d$, we define $\mathbf{v}' = (\mathbf{v}, 1)$.

Since for any $F' \in \Psi(P)$, $d_{F'}(\mathbf{w}_i) = 0$ for all $i \in [n]$, or $d_{F'}(\mathbf{z}_j) = 0$ for all $j \in [m]$, we can see that $\langle \sum_{k \in [p]} \mathbf{w}'_{i_k} - \sum_{l \in [s]} \mathbf{w}'_{h_l}, \mathbf{c}_{F'} \rangle = 0$ for all $F' \in \Psi(P)$. Indeed, if F' contains $\mathbf{w}_i$ for all $i \in [n]$, then

$$\begin{aligned}\langle \sum_{k \in [p]} \mathbf{w}'_{i_k} - \sum_{l \in [s]} \mathbf{w}'_{h_l}, \mathbf{c}_{F'} \rangle &= \sum_{k \in [p]} \langle \mathbf{w}'_{i_k}, \mathbf{c}_{F'} \rangle - \sum_{l \in [s]} \langle \mathbf{w}'_{h_l}, \mathbf{c}_{F'} \rangle \\ &= \sum_{k \in [p]} d_{F'}(\mathbf{w}_{i_k}) - \sum_{l \in [s]} d_{F'}(\mathbf{w}_{h_l}) = 0.\end{aligned}$$

Moreover, if F' contains $\mathbf{z}_j$ for all $j \in [m]$, then

$$\begin{aligned}\langle \sum_{k \in [p]} \mathbf{w}'_{i_k} - \sum_{l \in [s]} \mathbf{w}'_{h_l}, \mathbf{c}_{F'} \rangle &= \langle \sum_{k \in [p]} \mathbf{w}'_{i_k} + \sum_{k \in [q]} \mathbf{z}'_{j_k} - \sum_{l \in [s]} \mathbf{w}'_{h_l} - \sum_{l \in [t]} \mathbf{z}'_{g_l}, \mathbf{c}_{F'} \rangle \\ &= \langle \mathbf{0}, \mathbf{c}_{F'} \rangle = 0.\end{aligned}$$

The homomorphism from $\mathbb{Z}\mathcal{A}_P$ onto $\mathcal{S}$ given by $\mathbf{v}' \mapsto \sum_{F \in \Psi(P)} \langle \mathbf{v}', \mathbf{c}_F \rangle \epsilon_F$ is an isomorphism (cf. [84, Proposition 9.8.17]). Therefore, we have $\sum_{k \in [p]} \mathbf{w}'_{i_k} - \sum_{l \in [s]} \mathbf{w}'_{h_l} = \mathbf{0}$. Similarly, we have $\sum_{k \in [q]} \mathbf{z}'_{j_k} - \sum_{l \in [t]} \mathbf{z}'_{g_l} = \mathbf{0}$, and hence

$$\begin{aligned}b &= x_{\mathbf{w}_{i_1}} \cdots x_{\mathbf{w}_{i_p}} (x_{\mathbf{z}_{j_1}} \cdots x_{\mathbf{z}_{j_q}} - x_{\mathbf{z}_{g_1}} \cdots x_{\mathbf{z}_{g_t}}) + x_{\mathbf{z}_{g_1}} \cdots x_{\mathbf{z}_{g_t}} (x_{\mathbf{w}_{i_1}} \cdots x_{\mathbf{w}_{i_p}} - x_{\mathbf{w}_{h_1}} \cdots x_{\mathbf{w}_{h_s}}) \\ &\in RI_{Q_1} + RI_{Q_2}.\end{aligned}$$

Since Q_1 and Q_2 have rank 1, it follows from Theorem 7.1.2 that $\Bbbk[Q_1]$ and $\Bbbk[Q_2]$ are the Segre products of two polynomial rings, we get desired.

(ii) $\Rightarrow$ (i) : By Proposition 3.2.1 (i) and (ii), $\Bbbk[P]$ can be realized as the Hibi ring of a poset Π_4. Its weights are given and we can see that the standard Gale-diagram of the dual polytope of P is orthogonally equivalent to Gale_4.

The last statement follows from the equivalence (i) and (ii). $\square$

These theorems do not give a complete answer to our problems. We are left with the following questions:

Question 7.2.9. *For any* $(0,1)$*-polytope* P *with* $\operatorname{rank} P = 2$*, is* $\Bbbk[P]$ *isomorphic to a Hibi ring or* $\Bbbk[P_{n_1,n_2,n_3}]$ *for some* $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$*? In other words, does the relationship* $(\mathbf{0},\mathbf{1})_2 = \mathbf{Order}_2 \sqcup \{\Bbbk[P_{n_1,n_2,n_3}] : n_1, n_2, n_3 \in \mathbb{Z}_{>0}\}$ *hold?*

Question 7.2.10. *For any* $(0,1)$*-polytope* P *with* $\operatorname{rank} P = 2$*, is the standard Gale-diagram of the dual polytope of* P *orthogonally equivalent to one of the* Gale_i*'s? Also, let* P *and* P' *be two* $(0,1)$*-polytopes whose dual polytopes have the standard Gale-diagrams* Gale_2*,* Gale_3 *or* Gale_5*, then does the combinatorial equivalence of* P *and* P' *imply the isomorphism of their toric rings?*

If Question 7.2.9 has a positive answer, then all $(0,1)$-polytopes with rank 2 are normal and torsionfree.

7.3 Case (r3)

Finally, we discuss the normality, torsionfreeness and classification in the case $\operatorname{rank} P \geq 3$. In fact, unlike the previous cases, these properties are not guaranteed.

Proposition 7.3.1. *For any positive integer* $r \geq 3$*, there exists a non-normal* $(0,1)$*-polytope* P *with* $\operatorname{rank} P = r$*.*

Proof. Let

$$Q_1 = \operatorname{conv}(\{(0,0,0,0),(1,1,0,0),(1,0,1,0),(0,1,1,0),(0,0,0,1),(1,1,1,1)\}).$$

Then, we can see that Q_1 has rank 3 and is not normal. Indeed, we can check that $\dim Q_1 = 4$ and Q_1 has 8 facets by using `MAGMA`, thus $\operatorname{rank} Q_1 = 3$. Moreover, one has $(1,1,1,0,2) \in \mathbb{Z}\mathcal{A}(Q_1) \cap C_{Q_1}$ while $(1,1,1,0,2) \notin \mathbb{Z}_{\geq 0}\mathcal{A}(Q_1)$.

Furthermore, $Q_1 \times [0,1]^{r-3}$, where $[0,1]^d$ denotes the d dimensional unit cube, is also a non-normal $(0,1)$-polytope and its rank is equal to r. $\square$

Proposition 7.3.2. *For any positive integer* $r \geq 3$*, there exists a non-torsionfree normal* $(0,1)$*-polytope* P *with* $\operatorname{rank} P = r$*.*

Proof. Let

$$Q_2 = \operatorname{conv}(\{(0,0,0,0),(1,0,0,0),(0,1,0,1),(0,0,1,1),(1,1,1,0),(1,1,1,1)\}).$$

Then, we can see that Q_2 has rank 3 and is normal but not torsionfree. Indeed, we can see that $\mathbb{Z}\mathcal{A}(Q_2) = \mathbb{Z}^5$ and Q_2 has IDP by using `MAGMA`, so Q_2 is normal. Moreover, we can compute $\mathcal{M}_{Q_2}$ and its Smith normal form as follows:

$$\mathcal{M}_{Q_2} = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have $\mathrm{Cl}(\Bbbk[Q_2]) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})^3$ by Theorem 4.1.1.

Moreover, $P := Q_2 \times [0,1]^{r-3}$ is also a normal $(0,1)$-polytope and its rank is equal to r. We can calculate $\mathcal{M}_P$ and its Smith normal form as follows:

$$\mathcal{M}_P = \begin{pmatrix} \mathcal{M}_{Q_2} & \cdots & \mathcal{M}_{Q_2} \\ & A_{r-3} & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 2 \\ & & & & \end{pmatrix},$$

where we define $A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ and

$$A_n = \begin{pmatrix} & A_{n-1} & & & A_{n-1} & \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}$$

for $n \geq 2$. Therefore, we obtain $\mathrm{Cl}(\Bbbk[P]) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^2$ by Theorem 4.1.1. □

It seems so hopeless to classify the isomorphism classes in this case because even in the case of the Hibi ring, it is difficult to give a complete classification. In addition, the method using Gale-diagrams is no longer useful. In fact, there exist two $(0,1)$-polytopes which have the same combinatorial type such that their toric rings are not isomorphic to each other.

Proposition 7.3.3. *For any positive integer $r \geq 3$, there exist two $(0,1)$-polytopes P and P' with the same combinatorial type and $\operatorname{rank} P = \operatorname{rank} P' = r$ such that their toric rings are not isomorphic to each other.*

Proof. Actually, Q_1 and Q_2 appearing in Propositions 7.3.1 and 7.3.2 satisfy those conditions. `MAGMA` confirms that these are combinatorially equivalent. On the other hand, $\Bbbk[Q_2]$ is normal, but $\Bbbk[Q_1]$ is not. Therefore, these are not isomorphic.

The same holds for $Q_1 \times [0,1]^{r-3}$ and $Q_2 \times [0,1]^{r-3}$. □

Part III

Generalizations of Gorenstein graded rings

Chapter 8

Preliminaries on commutative algebra

Throughout this chapter, let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen–Macaulay graded ring of dimension d with $R_0 = \Bbbk$. We recall the definitions of levelness and almost Gorensteinness and some properties of those algebras.

Before defining them, we recall some fundamental materials. (Consult, e.g., [10] for the introduction to Cohen–Macaulay graded rings.)

- Let ω_R denote a canonical module of R, let $a(R)$ denote the a-invariant of R, i.e., $a(R) = -\min\{j : (\omega_R)_j \neq 0\}$, and let $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$.
- For a graded R-module M, we use the following notation:
  - Let $\mu_j(M)$ denote the number of minimal generators of M with degree j as an R-module, and let $\mu(M) = \sum_{j \in \mathbb{Z}} \mu_j(M)$, i.e., the number of minimal generators.
  - Let $e(M)$ denote the multiplicity of M. Then the inequality $\mu(M) \leq e(M)$ always holds.
  - Let $M(-\ell)$ denote the R-module whose grading is given by $M(-\ell)_n = M_{n-\ell}$ for any $n \in \mathbb{Z}$.
- Let $r(R)$ denote the Cohen–Macaulay type of R. Note that $r(R) = \mu(\omega_R)$. We see that R is Gorenstein if and only if $r(R) = 1$.
- Let $P_H(M, n)$ denote the Hilbert function of M, i.e., $P_H(M, n) = \dim_\Bbbk M_n$, where $\dim_\Bbbk$ stands for the dimension as a $\Bbbk$-vector space. Note that $P_H(M, n)$ can be described by a polynomial in n of degree $d - 1$ and its leading coefficient coincides with $e(M)/d!$. (See [10, Section 4].) In addition, we denote $\mathrm{H}(M, t)$ the Hilbert series of M, i.e.,
$$\mathrm{H}(M, t) = \sum_{n \in \mathbb{Z}} \dim_\Bbbk M_n t^n.$$
- We say that R is *semi-standard graded* if R is finitely generated $\Bbbk[R_1]$-module. In particular, R is called *standard graded* (or *homogeneous*) if $R = \Bbbk[R_1]$. If R is a

semi-standard graded ring, its Hilbert series is of the form

$$\mathrm{H}(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

where $h_0 + h_1 t + \cdots + h_s t^s \in \mathbb{Z}[t]$. We call this polynomial the *h-polynomial* of R and denote it as $h_R(t)$. Note that $\sum_{i=0}^{s} h_i \neq 0$ and $h_s \neq 0$. We call the index s the *socle degree* of R and the integer sequence $(h_0, h_1, \ldots, h_s)$ the *h-vector* of R. Note that $h_s \leq r(R)$ holds. Moreover, we see that $d + a(R) = s$ and $e(R) = h_0 + h_1 + \cdots + h_s$ (see [10, Section 4.4]).

- Let $\mathrm{tr}_R(M)$ be the sum of the ideals $\phi(M)$ over all $\phi \in \mathrm{Hom}_R(M,R)$, i.e.

$$\mathrm{tr}_R(M) = \sum_{\phi \in \mathrm{Hom}_R(M,R)} \phi(M).$$

  When there is no risk of confusion about the ring, we simply write $\mathrm{tr}(M)$.

Proposition 8.0.1. *If $e(R) \geq 2$, then we have $r(R) < e(R)$.*

Proof. Since R is Cohen–Macaulay, there exists a regular sequence $\mathbf{x}$ on $\mathfrak{m}$ such that $e(R)$ coincides with the length of $R/\mathbf{x}$. Thus we may assume $d = 0$ since $r(R) = r(R/(\mathbf{x}))$. By the assumption, we have $\mathrm{Soc}(R) \neq R$ so $\mathrm{Soc}(R) \subset \mathfrak{m}$, where $\mathrm{Soc}(R) = \{x \in R : x\mathfrak{m} = 0\}$. Therefore, we have $r(R) = \dim_{\Bbbk} \mathrm{Soc}(R) \leq \dim_{\Bbbk} \mathfrak{m} < \dim_{\Bbbk} R = e(R)$ so $r(R) < e(R)$. □

First, we define level rings:

Definition 8.0.2 (Level, [74])**.** We say that R is *level* if all the degrees of the minimal generators of ω_R are the same.

Remark 8.0.3. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of R. Assume that R is level. In this case, if $h_s = 1$, then R is Gorenstein. In fact, since R being level is equivalent to $h_s = r(R)$ (see, e.g., [10, Section 5]), we obtain that $r(R) = 1$.

Regarding the levelness of homogeneous domains, we know the following:

Theorem 8.0.4 ([86, Corollary 3.11])**.** *Let R be a Cohen-Macaulay homogeneous domain with its socle degree s. If $s = 2$, then R is level.*

Next, we give the definition of almost Gorenstein graded rings:

Definition 8.0.5 (Almost Gorenstein, [22, Definition 1.5])**.** We call R *almost Gorenstein* if there exists an exact sequence of graded R-modules

$$0 \to R \to \omega_R(-a(R)) \to C \to 0 \qquad (8.0.1)$$

with $\mu(C) = e(C)$.

We call R *satisfies* $(*)$ if the following condition holds:

$$\text{there exists an } R\text{-monomorphism } \phi : R \hookrightarrow \omega_R(-a(R)) \text{ of degree } 0. \tag{$*$}$$

If R satisfies $(*)$, take an R-monomorphism $\phi : R \hookrightarrow \omega_R(-a(R))$ of degree 0 and we define $\delta_\phi(R)$ as $e(\operatorname{coker}(\phi))$. When there is no risk of confusion about the monomorphism we simply write $\delta_\phi(R)$ as $\delta(R)$. Moreover, we call R is *almost Gorenstein with respect to* ϕ if $\delta_\phi(R) = \mu(\operatorname{coker}(\phi))$.

Note that there always exists a degree-preserving injection ϕ from R to $\omega_R(-a(R))$ if R is a domain ([38, Proposition 2.2]). Moreover, the following propositions say that $\mu(\operatorname{coker}(\phi))$ and $\delta(R)$ do not depend on the choice of ϕ under some assumptions:

Proposition 8.0.6 ([38, Proposition 2.3])**.** *Suppose that R satisfies $(*)$. Then we have $\mu(\operatorname{coker}(\phi)) = r(R) - 1$.*

Proposition 8.0.7 ([38, Proposition 2.4] and [55, Theorem 4.5])**.** *Let R be a Cohen–Macaulay semi-standard graded ring which satisfies $(*)$ and let $(h_0, \dots, h_s)$ be its h-vector. Take any monomorphism $\phi : R \hookrightarrow \omega_R(-a(R))$ with degree 0. Then the Hilbert series of $C := \operatorname{coker}(\phi)$ is*

$$\mathrm{H}(C,t) = \frac{\sum_{j=0}^{s-1}((h_s + \cdots + h_{s-j}) - (h_0 + \cdots + h_j))t^j}{(1-t)^{d-1}}. \tag{8.0.2}$$

In particular, we have

$$\delta_\phi(R) = \sum_{j=0}^{s-1}((h_s + \cdots + h_{s-j}) - (h_0 + \cdots + h_j)) = \sum_{j=0}^{s}(2j - s)h_j.$$

Regarding the almost Gorensteinness of homogeneous domains, we know the following:

Theorem 8.0.8 ([38, Theorem 4.7])**.** *Let R be an almost Gorenstein homogeneous domain and $(h_0, h_1, \dots, h_s)$ its h-vector with $s \geq 2$. Then $h_s = 1$.*

Finally, we recall the definition of nearly Gorenstein:

Definition 8.0.9 ([30, Definition 2.2])**.** We say that R is *nearly Gorenstein* if $\operatorname{tr}(\omega_R) \supseteq \mathfrak{m}$. In particular, R is Gorenstein if and only if $\operatorname{tr}(\omega_R) = R$.

The following proposition helps us to compute $\operatorname{tr}(\omega_R)$:

Proposition 8.0.10 ([30, Lemma 1.1])**.** *Let R be a ring and I an ideal of R containing a non-zero divisor of R. Let $Q(R)$ be the total quotient ring of fractions of R and $I^{-1} := \{x \in Q(R) \ : \ xI \subseteq R\}$. Then*

$$\operatorname{tr}(I) = I \cdot I^{-1}.$$

We give the following propositions associated with the levelness and nearly Gorensteinness of the Segre product:

Proposition 8.0.11 ([31, Proposition 2.2 and Theorem 2.4])**.** *Let $R_1, \dots, R_s$ be the toric rings of normal integral polytopes which have Krull dimension at least 2. Let $R = R_1 \# R_2 \# \cdots \# R_s$ be the Segre product. Then the following is true.*

$$\omega_R = \omega_{R_1} \# \omega_{R_2} \# \cdots \# \omega_{R_s} \quad \text{and} \quad \omega_R^{-1} = \omega_{R_1}^{-1} \# \omega_{R_2}^{-1} \# \cdots \# \omega_{R_s}^{-1}.$$

Lemma 8.0.12. *Let $R_1, \dots, R_s$ be the toric rings of normal integral polytopes which have Krull dimension at least 2. Let $R = R_1 \# \cdots \# R_s$ be the Segre products. Then the following are true:*

(1) If R is nearly Gorenstein, then R_i is nearly Gorenstein for all i.

(2) If R_i is level for all i, then R is level.

Proof. It suffices to prove the case $s = 2$. Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be $\Bbbk$-basis of $(R_1)_1$ and $\mathbf{y}_1, \dots, \mathbf{y}_m$ be a $\Bbbk$-basis of$(R_2)_1$.

(1): In this case, by using Proposition 8.0.11, we get $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$ and $\omega_R^{-1} \cong \omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$. Then we may identify ω_R and $\omega_R{}^{-1}$ with $\omega_{R_1} \# \omega_{R_2}$ and $\omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$, respectively. It is enough to show that $\mathbf{x}_i \in \operatorname{tr}(\omega_{R_1})$ for any $1 \leq i \leq n$. Since R is nearly Gorenstein, there exist homogeneous elements $\mathbf{v}_1 \# \mathbf{v}_2 \in \omega_{R_1} \# \omega_{R_2}$ and $\mathbf{u}_1 \# \mathbf{u}_2 \in \omega_{R_1}^{-1} \# \omega_{R_2}^{-1}$ such that $\mathbf{x}_i \# \mathbf{y}_1 = (\mathbf{v}_1 \# \mathbf{v}_2)(\mathbf{u}_1 \# \mathbf{u}_2) = (\mathbf{v}_1 \mathbf{u}_1 \# \mathbf{v}_2 \mathbf{u}_2)$, by [54, Proposition 4.2]. Thus, we get $\mathbf{x}_i = \mathbf{v}_1 \mathbf{u}_1 \in \operatorname{tr}(\omega_{R_1})$, so R_1 is nearly Gorenstein. In the same way as above, we can show that R_2 is also nearly Gorenstein.

(2): First, $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$ by Proposition 8.0.11. Let a_1 and a_2 be the a-invariants of R_1 and R_2, respectively, and assume that $a_1 \leq a_2$. Since R_1 and R_2 are level, $\omega_{R_1} \cong \langle f_1, \cdots, f_r \rangle R_1$ and $\omega_{R_2} \cong \langle g_1, \cdots, g_l \rangle R_2$ where $\deg f_i = -a_1$ and $\deg g_j = -a_2$ for all $1 \leq i \leq r$, $1 \leq j \leq l$. Thus, since $\omega_R \cong \omega_{R_1} \# \omega_{R_2}$, we may identify ω_R with $\langle f_1, \cdots, f_r \rangle R_1 \# \langle g_1, \cdots, g_l \rangle R_2$. We set

$$V := \left\{ \mathbf{y}^{\mathbf{b}} g_j \ : \ 1 \leq j \leq l, \, \mathbf{a} \in \mathbb{N}^m, \ \sum_{i=1}^{m} b_i = a_2 - a_1 \right\},$$

where $\mathbf{y}^{\mathbf{a}} := \mathbf{y}_1^{a_1} \cdots \mathbf{y}_m^{a_m}$. Then $\omega_R = \langle f_i \# v \ : \ 1 \leq i \leq r, v \in V \rangle R$. Therefore, R is level. □

Chapter 9

Levelness versus almost Gorensteinness of edge rings of complete multipartite graphs

In this chapter, we characterize when the edge ring of a complete multipartite graph is level or almost Gorenstein. In addition, we compare these properties by using the characterization. The contents of this chapter are contained in the author's paper [42] with A. Higashitani.

9.1 Gorensteinness, levelness and almost Gorensteinness

In this section, we recall the characterization of Gorensteinness of $\Bbbk[K_{r_1,\dots,r_n}]$ and introduce our results; the characterization of levelness and almost Gorensteinness of $\Bbbk[K_{r_1,\dots,r_n}]$.

Proposition 9.1.1 (Characterization of Gorensteinness, [65, Remark 2.8])**.** *Let $1 \leq r_1 \leq \cdots \leq r_n$ and let $d = \sum_{i=1}^{n} r_i$, where $n \geq 2$. Then the edge ring of the complete multipartite graph $K_{r_1,\dots,r_n}$ is Gorenstein if and only if*

- $n = 2$ *and* $(r_1, r_2) \in \{(1, m), (m, m) : m \geq 1\}$;
- $n = 3$ *and* $1 \leq r_1 \leq r_2 \leq r_3 \leq 2$;
- $n = 4$ *and* $r_1 = \cdots = r_4 = 1$.

This proposition is a direct consequence of [16].

The first result, which is the characterization of the levelness of $\Bbbk[K_{r_1,\dots,r_n}]$, will be shown in Section 9.3.2:

Theorem 9.1.2 (Characterization of levelness)**.** *Let $1 \leq r_1 \leq \cdots \leq r_n$ and let $d = \sum_{i=1}^{n} r_i$, where $n \geq 2$. Then the edge ring of the complete multipartite graph $K_{r_1,\dots,r_n}$ is level if and only if n and $(r_1, \dots, r_n)$ satisfy one of the following:*

(i) $n = 2$;
(ii) $n = 3$ *and* $(r_1, r_2, r_3) \in \{(1, 1, m) : m \geq 1\} \cup \{(1, 2, m) : m \geq 2\}$;
(iii) $n = 3$ *and* $(r_1, r_2, r_3) \in \{(2, 2, m) : m \geq 2\} \cup \{(3, 3, 3)\}$;
(iv) $n = 4$ *and* $(r_1, r_2, r_3, r_4) \in \{(1, 1, 1, m) : m \geq 1\}$;

(v) $n = 5$ *and* $r_1 = \cdots = r_5 = 1$.

Note that the first two cases come from the results on Hibi rings. See Proposition 9.2.4.

The second result, which is the characterization of the almost Gorensteinness of $\Bbbk[K_{r_1,\dots,r_n}]$, will be shown in Section 9.3.3:

Theorem 9.1.3 (Characterization of almost Gorensteinness)**.** *Let* $1 \leq r_1 \leq \cdots \leq r_n$ *and let* $d = \sum_{i=1}^n r_i$*, where* $n \geq 2$*. Then the edge ring of the complete multipartite graph* $K_{r_1,\dots,r_n}$ *is almost Gorenstein if and only if* n *and* $(r_1, \dots, r_n)$ *satisfy one of the following:*

(i) $n = 2$ *and* $(r_1, r_2) \in \{(1, m), (m, m) : m \geq 1\} \cup \{(2, m) : m \geq 2\}$*;*
(ii) $n = 3$ *and* $(r_1, r_2, r_3) \in \{(1, 1, m), (1, m, m) : m \geq 1\}$*;*
(iii) $n = 3$ *and* $(r_1, r_2, r_3) = (2, 2, 2)$*;*
(iv) $n = 4$ *and* $(r_1, r_2, r_3, r_4) \in \{(1, 1, m, m) : m \geq 1\}$*;*
(v) $n \geq 4$ *and* $(r_1, \dots, r_{n-1}, r_n) = (1, \dots, 1, n-3)$.
(vi) n *is even with* $n \geq 6$ *and* $r_1 = \cdots = r_n = 1$*;*

Note that the first two cases come from the result on Hibi rings. See Section 9.2 (below Proposition 9.2.4).

As an immediate corollary of those theorems, we obtain the following:

Corollary 9.1.4. *The edge ring of the complete multipartite graph* $K_{r_1,\dots,r_n}$ *is level and almost Gorenstein but not Gorenstein if and only if one of the following holds:*

- $n = 2$ *and* $(r_1, r_2) \in \{(2, m) : m \geq 3\}$*;*
- $n = 3$ *and* $(r_1, r_2, r_3) \in \{(1, 1, m) : m \geq 3\}$.

In particular, in both cases, the edge rings are isomorphic to certain Hibi rings.

Example 9.1.5 ($n = 2$ or $n = 3$)**.** For K_{r_1,r_2}, we see that $\Bbbk[K_{r_1,r_2}]$ is always level. Moreover, $\Bbbk[K_{1,m}]$ and $\Bbbk[K_{m,m}]$ are Gorenstein, while $\Bbbk[K_{2,m}]$ is not Gorenstein but almost Gorenstein if $m \geq 3$.

For K_{r_1,r_2,r_3}, we see that $\Bbbk[K_{r_1,r_2,r_3}]$ is

- level but not Gorenstein for $K_{1,1,m}, K_{1,2,m}, K_{2,2,m}$ with $m \geq 3$ and $K_{3,3,3}$;
- almost Gorenstein but not Gorenstein if and only if $K_{1,1,m}, K_{1,m,m}$ with $m \geq 3$;
- level and almost Gorenstein but not Gorenstein if and only if $K_{1,1,m}$ with $m \geq 3$.

Example 9.1.6 ($n = 4$)**.** For K_{r_1,r_2,r_3,r_4}, we see that $\Bbbk[K_{r_1,r_2,r_3,r_4}]$ is

- level for $K_{1,1,1,m}$;
- almost Gorenstein for $K_{1,1,m,m}$;
- Gorenstein for $K_{1,1,1,1}$.

Example 9.1.7 ($n \geq 5$)**.** In the case $n \geq 5$, $\Bbbk[K_{r_1,\dots,r_n}]$ is never Gorenstein, and it is level only for K_5. On the other hand, it is almost Gorenstein for K_{2m} with $m \geq 3$ and $K_{\underbrace{1,\dots,1}_{n-1},n-3}$. Recall that K_n denotes the complete graph on n vertices.

9.2 In the case of Hibi rings

In this section, we recall the characterization results on Gorensteinness, levelness and almost Gorensteinness of Hibi rings since some edge ring of $K_{r_1,\dots,r_n}$ is isomorphic to a Hibi ring as follows.

For $m, n \in \mathbb{Z}_{>0}$, let $\Pi_{m,n} = \Pi_1(m,n)$ and $\Pi'_{m,n} = \Pi_3(m-1, n-1, 1, 1, 0)$ (Π_1 and Π_3 are defined in Section 6.1). From Proposition 6.4.1 and Lemma 6.4.2, we have the following proposition:

Proposition 9.2.1. *The edge ring $\Bbbk[K_{m+1,n+1}]$ (resp. $\Bbbk[K_{1,m,n}]$) is isomorphic to the Hibi ring $\Bbbk[\Pi_{m,n}]$ (resp. $\Bbbk[\Pi'_{m,n}]$).*

Therefore, in the case where $n = 2$ or $n = 3$ with $r_1 = 1$, the characterization of levelness and almost Gorensteinness of $\Bbbk[K_{r_1,\dots,r_n}]$ can be deduced into those of the Hibi rings $\Bbbk[\Pi_{m,n}]$ and $\Bbbk[\Pi'_{m,n}]$. Hence, in what follows, we give the characterizations of those properties for $\Bbbk[\Pi_{m,n}]$ and $\Bbbk[\Pi'_{m,n}]$, which prove Theorems 9.1.2 and 9.1.3 in the case where $n = 2$ or $n = 3$ with $r_1 = 1$.

Miyazaki gave the characterizations of levelness [56] and almost Gorensteinness [57] of Hibi rings. For explaining those results, we introduce some notions.

- For $x,\ y \in \Pi$ with $x \preceq y$, we set $[x,y]_\Pi := \{z \in \Pi : x \preceq z \preceq y\}$.
- We define $\operatorname{rank}\Pi$ to be the maximal length of the chains in Π, and define $\operatorname{rank}[x,y]_\Pi$ analogously.
- Let $y_1, x_1, y_2, x_2, \dots, y_t, x_t$ be a (possibly empty) sequence of elements in $\widehat{\Pi}$. We say that the sequence $y_1, x_1, y_2, x_2, \dots, y_t, x_t$ satisfies *condition N* if
  (1) $x_1 \neq \hat{0}$,
  (2) $y_1 \succ x_1 \prec y_2 \succ x_2 \prec \cdots \prec y_t \succ x_t$, and
  (3) $y_i \not\succeq x_j$ for any i, j with $1 \leq i < j \leq t$.
- Let
$$r(y_1, x_1, \dots, y_t, x_t) := \sum_{i \in [t]} (\operatorname{rank}[x_{i-1}, y_i]_{\widehat{\Pi}} - \operatorname{rank}[x_i, y_i]_{\widehat{\Pi}}) + \operatorname{rank}[x_t, \hat{1}]_{\widehat{\Pi}},$$
  where we set an empty sum to be 0 and $x_0 = \hat{0}$.
- Given $x \in \Pi$, let
$$\operatorname{star}_\Pi(x) = \{y \in \Pi : y \preceq x \text{ or } x \preceq y\}.$$

Theorem 9.2.2 ([56, Theorem 3.9])**.** *The Hibi ring of Π is level if and only if*
$$r(y_1, x_1, \dots, y_t, x_t) \leq \operatorname{rank}\widehat{\Pi}$$
holds for any sequence of elements in $\widehat{\Pi}$ with condition N.

Corollary 9.2.3 ([56, Corollary 3.10])**.** *If $[x, \hat{1}]_{\widehat{\Pi}}$ is pure for any $x \in \Pi$, then $\Bbbk[\Pi]$ is level.*

By using those results by Miyazaki, we can prove the following:

Proposition 9.2.4. *Let $m \le n$.*
(i) *The Hibi ring $\Bbbk[\Pi_{m,n}]$ of $\Pi_{m,n}$ is level for any m, n.*
(ii) *The Hibi ring $\Bbbk[\Pi'_{m,n}]$ of $\Pi'_{m,n}$ is level if and only if $m = 1$ or $m = 2$.*

Proof. The assertion (i) directly follows from Corollary 9.2.3.

Similarly, we see from Corollary 9.2.3 that $\Bbbk[\Pi'_{1,n}]$ and $\Bbbk[\Pi'_{2,n}]$ are level.

Let $y_1 = p_{m+n}$ and $x_1 = p_1$ and take the sequence y_1, x_1, which satisfies condition N. Then we see that

$$r(y_1, x_1) = \operatorname{rank}[\hat{0}, y_1]_{\widehat{\Pi}} - \operatorname{rank}[x_1, y_1]_{\widehat{\Pi}} + \operatorname{rank}[x_1, \hat{1}]_{\widehat{\Pi}} = n - 1 + m.$$

On the other hand, we have $\operatorname{rank} \widehat{\Pi} = n + 1$. If $m \ge 3$, then $m - 1 > 1$, so we have

$$n + m - 1 = r(x_1, y_1) > \operatorname{rank} \widehat{\Pi} = n + 1.$$

Hence, $\Bbbk[\Pi'_{m,n}]$ is not level when $m \ge 3$ by Theorem 9.2.2. Therefore, the assertion (ii) follows. □

Hence, in the case where $n = 2$ or $n = 3$ with $r_1 = 1$, $\Bbbk[K_{r_1,\dots,r_n}]$ is level if and only if $K_{r_1,\dots,r_n}$ satisfies (1) or (2) in Theorem 9.1.2.

Note that $\Bbbk[K_{1,n}]$ is isomorphic to a polynomial ring with n variables over $\Bbbk$.

Regarding the characterization of almost Gorenstein Hibi rings, see [57, Introduction]. According to it, we see the following:

- Let $m \le n$. Consider the poset $\Pi_{m,n}$.
  - We see that $\Pi_{1,n}$ fits into the case of (1) of [57, Introduction]. Thus, $\Bbbk[\Pi_{1,n}]$ is almost Gorenstein.
  - Since $\Pi_{m,m}$ is pure, we know that $\Bbbk[\Pi_{m,m}]$ is Gorenstein, in particular, almost Gorenstein.
  - If $1 < m < n$, then we see from the characterization that $\Bbbk[\Pi_{m,n}]$ is never almost Gorenstein.
- Let $m \le n$. Consider the poset $\Pi'_{m,n}$.
  - We have $\operatorname{star}_{\Pi'_{1,n}}(p_{n+1}) = \Pi'_{1,n}$ and $\Pi'_{1,n} \setminus \{p_{n+1}\}$ fits into the case of (1) of [57, Introduction]. Thus, $\Bbbk[\Pi_{1,n}]$ is almost Gorenstein.
  - We see that $\Pi'_{m,m}$ fits into the case of (2) (ii) (with $p = 0$). Hence, $\Bbbk[\Pi'_{m,m}]$ is almost Gorenstein.
  - If $1 < m < n$, then we see from the characterization that $\Bbbk[\Pi'_{m,n}]$ is never almost Gorenstein.

9.3 Characterization of levelness and almost Gorensteinness

The goal of this section is to complete the proofs of Theorems 9.1.2 and 9.1.3. As shown in Section 9.2, the case where $n = 2$ or $n = 3$ with $r_1 = 1$ was already done. Thus, in principle, we discuss the case where $n = 3$ with $r_1 \ge 2$ or $n \ge 4$.

9.3.1 Preliminaries for $P_{K_{r_1,\dots,r_n}}$

Before proving the assertions, we recall some geometric information on edge polytopes of complete multipartite graphs.

If $n \geq 3$ with $r_1 \geq 1$ or $n \geq 4$, then all vertices of $K_{r_1,\dots,r_n}$ are regular and each V_k for $k \in [n]$ is a fundamental set. Thus, we have

$$\widetilde{\Psi}_r = \{\mathbf{e}_i : i \in [d]\} \text{ and } \widetilde{\Psi}_f = \left\{ \ell_{V_k} = \frac{1}{2}\left(\sum_{i \in [d]\setminus V_k} \mathbf{e}_i - \sum_{i \in V_k} \mathbf{e}_i \right) : k \in [n] \right\}. \qquad (9.3.1)$$

For $i \in [d]$, let $p_i : \mathbb{R}^d \to \mathbb{R}$ be the i-th projection, and let $p_{V_k} := \sum_{j \in V_k} p_j$ for $k \in [n]$. In addition, for $k \in [n]$, let $f_k = 2\ell_{V_k} = \displaystyle\sum_{i \in [d]\setminus V_k} \mathbf{e}_i - \sum_{j \in V_k} \mathbf{e}_j$.

Note that $K_{r_1,\dots,r_n}$ satisfies the odd cycle condition. Let $(h_0, h_1, \dots, h_s)$ be the h-vector of the edge ring $\Bbbk[K_{r_1,\dots,r_n}]$ and let $\ell = \min\{m \in \mathbb{Z}_{>0} : \operatorname{int}(mP_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d \neq \emptyset\}$. Since the ideal generated by the monomials contained in the interior of $P_{K_{r_1,\dots,r_n}}$ is the canonical module of $\Bbbk[K_{r_1,\dots,r_n}]$, we see that $\ell = -a(\Bbbk[K_{r_1,\dots,r_n}])$. Thus, $d = \ell + s$ holds. In the case where $n = 3$ with $r_1 \geq 2$ or $n \geq 4$, one has $\iota \in \operatorname{int}(mP_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ if and only if $\langle \iota, l \rangle > 0$ holds for all $l \in \widetilde{\Psi}$.

For $\iota \in \operatorname{int}((\ell + k)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ for $k \in \mathbb{Z}_{>0}$, then we say that ι is a *first appearing interior point* in $\operatorname{int}((\ell + k)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ if it cannot be written as a sum of $\iota' \in \operatorname{int}((\ell + i)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ with $0 \leq i < k$ and $\rho(e)$'s with $e \in E(K_{r_1,\dots,r_n})$, where the elements in $\operatorname{int}(\ell P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ are regarded as first appearing interior points. Let $\mu_k(K_{r_1,\dots,r_n})$ denote the number of first appearing interior points in $\operatorname{int}((\ell + k)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ for $k \in \mathbb{Z}_{\geq 0}$. Note that $\mu_0(K_{r_1,\dots,r_n}) = h_s$ holds.

In what follows, for the study of $\Bbbk[K_{r_1,\dots,r_n}]$ with $1 \leq r_1 \leq \cdots \leq r_n$ and $d = \sum_{i=1}^d r_i$, we divide into the following three cases on $K_{r_1,\dots,r_n}$:

(A) $2r_n < d$ and d is even;
(B) $2r_n < d$ and d is odd;
(C) $2r_n \geq d$.

Given a graph G with the edge set $E(G)$, we say that $\mathcal{M} \subset E(G)$ is a *perfect matching* (a.k.a. 1-*factor*) if every vertex of G is incident to exactly one edge of $\mathcal{M}$.

Lemma 9.3.1. *The complete multipartite graph $K_{r_1,\dots,r_n}$ has a perfect matching if and only if d is even and $2r_n \leq d$.*

Proof. Tutte's theorem (see, e.g., [17, Theorem 2.2.1]) claims that a graph G on the vertex set $V(G)$ has a perfect matching if and only if $q(G - U) \leq |U|$ holds for any $U \subset V(G)$, where $q(\cdot)$ denotes the number of connected components with odd cardinality and $G - U$ denotes the induced subgraph of G by $V(G) \setminus U$.

When $U = V(K_{r_1,\dots,r_n})$, the inequality trivially holds. When $U = \emptyset$, we can see that $q(G) = 0$ holds if and only if d is even.

Consider $\emptyset \neq U \subsetneq V(K_{r_1,\dots,r_n})$. If there are two vertices u, v in $V(K_{r_1,\dots,r_n}) \setminus U$ with $u \in V_i$ and $v \in V_j$ with $i \neq j$, the number of connected components of $K_{r_1,\dots,r_n} - U$ is equal to 1, so $q(K_{r_1,\dots,r_n} - U) \le |U|$ holds. Thus, We may assume that there exists $k \in [n]$ with $V(K_{r_1,\dots,r_n}) \setminus U \subset V_k$. In the case $k \le n-1$, it follows from $1 \le r_1 \le \cdots \le r_n$ and $V_n \subset U$ that $q(K_{r_1,\dots,r_n} - U) \le r_k \le r_n \le |U|$. Therefore, by considering the case $k = n$, we conclude the following:

$$q(K_{r_1,\dots,r_n} - U) \le |U| \text{ for any } U \iff r_n \le \sum_{k\in[n-1]} r_k \text{ and } d \text{ is even}$$
$$\iff 2r_n \le d \text{ and } d \text{ is even.}$$

□

Proposition 9.3.2. *Let $(h_0, h_1, \dots, h_s)$ be the h-vector of $\Bbbk[K_{r_1,\dots,r_n}]$.*

(a) In the case of (A), we have $\ell = d/2$ and $h_s = 1$.
(b) In the case of (B), we have $\ell = (d+1)/2$ and $h_s \ge 2$.
(c) In the case of (C), we have $\ell = r_n + 1$ and $h_s \ge 2$.

Proof. In the case of (A), by Lemma 9.3.1, there exists a perfect matching $\mathcal{M} \subset E(K_{r_1,\dots,r_n})$, and we obtain $\rho(\mathcal{M}) := \sum_{e\in\mathcal{M}} \rho(e) = \sum_{i\in[d]} \mathbf{e}_i$. We can see that $\rho(\mathcal{M})$ is the unique element in $\text{int}(|\mathcal{M}|P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ since $\langle \rho(\mathcal{M}), l\rangle > 0$ hold for all $l \in \widetilde{\Psi}$, and it is clear that $\text{int}(mP_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d = \emptyset$ for $m < |\mathcal{M}|$. Therefore, we have $\ell = |\mathcal{M}| = d/2$ and $h_s = 1$.

Next, assume the case of (B). We consider the induced subgraph $K_{r_1,\dots,r_n} - \{v_n\}$ for $v_n \in V_n$. From the assumption, we observe that $d-1$ is even and $2\min\{r_{n-1}, r_n - 1\} \le 2r_n \le d-1$ holds. Hence, we can take a perfect matching $\mathcal{M}'$ of $K_{r_1,\dots,r_n} - \{v_n\}$ by Lemma 9.3.1. Take $v_1 \in V_1$, add the edge $\{v_1, v_n\}$ to $\mathcal{M}'$ and write $\mathcal{M}''$ for it. Then we have $\rho(\mathcal{M}'') = 2\mathbf{e}_{v_1} + \sum_{i\in[d]\setminus\{v_1\}} \mathbf{e}_i$, and $\langle \rho(\mathcal{M}''), l\rangle > 0$ for all $l \in \widetilde{\Psi}$. In fact, for f_1, we observe that

$$\langle \rho(\mathcal{M}''), f_1\rangle = \left(\sum_{i\in[n]\setminus\{1\}} p_{V_i} - p_{V_1}\right)(\rho(\mathcal{M}'')) = \sum_{k\in[n]\setminus\{1\}} r_k - (r_1+1) \ge \sum_{k\in[n]\setminus\{1,2\}} r_k - 1 > 0.$$

Thus, we have $\rho(\mathcal{M}'') \in \text{int}(|\mathcal{M}''|P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ and $\text{int}(mP_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d = \emptyset$ for $m < |\mathcal{M}''|$. Moreover, by exchanging v_1 with another vertex of V_1 or a vertex of V_2 if $r_1 = 1$, we obtain $\rho(\mathcal{M}'') \in \text{int}(|\mathcal{M}''|P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ in the same way. Therefore, we have $\ell = |\mathcal{M}''| = (d+1)/2$ and $h_s \ge 2$.

Finally, assume the case of (C). Let $r'_n := \sum_{i\in[n-1]} r_i$ and let $r''_n := r_n - r'_n$. We join r'_n vertices of V_n to vertices of $[d] \setminus V_n$ one-by-one, and join the remaining r''_n vertices of V_n to v_1, and join a vertex $v_2 \in V_2$ to v_1. Let $\mathcal{E}$ be the set of those edges. Then we have $\rho(\mathcal{E}) = (r''_n + 2)\mathbf{e}_{v_1} + 2\mathbf{e}_{v_2} + \sum_{i\in[d]\setminus\{v_1,v_2\}} \mathbf{e}_i$, and $\langle \rho(\mathcal{E}), l\rangle > 0$ for all $l \in \widetilde{\Psi}$. In fact, for f_1, we observe that

$$\langle \rho(\mathcal{E}), f_1\rangle = \sum_{k\in[n]\setminus\{1\}} r_k + 1 - (r_1 + r''_n + 1) \ge \sum_{k\in[n]\setminus\{1,2,n\}} r_k + r'_n > 0.$$

Thus, we have $\rho(\mathcal{E}) \in \text{int}(|\mathcal{E}|P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ and $\text{int}(mP_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d = \emptyset$ for $m < |\mathcal{E}|$ since $\langle \iota, r\rangle > 0$ for all $\iota \in \text{int}(\ell P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ and $r \in \widetilde{\Psi}$. Moreover, by exchanging v_2 with another vertex of V_2 or a vertex of V_3 if $r_2 = 1$, we obtain $\rho(\mathcal{E}) \in \text{int}(|\mathcal{E}|P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ in the same way. Therefore, we have $\ell = |\mathcal{E}| = r_n + 1$ and $h_s \ge 2$. □

Corollary 9.3.3. *Assume that $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. If $\Bbbk[K_{r_1,\dots,r_n}]$ is almost Gorenstein, then $K_{r_1,\dots,r_n}$ is in the case of (A).*

Proof. In our assumption, we can check $s \geq 2$ by using $s = d - \ell$. Thus, it follows directly from Proposition 9.3.2 and Theorem 8.0.8 that only case (A) is possible. □

9.3.2 Proof of Theorem 9.1.2

This subsection is devoted to proving Theorem 9.1.2. Since the case where $n = 2$ or $n = 3$ with $r_1 = 1$ has been already done in Section 9.2, we assume that $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. Under this assumption, we prove that $\Bbbk[K_{r_1,\dots,r_n}]$ is level if and only if one of the following holds:

$$\begin{aligned} &n = 3 \text{ with } r_1 = r_2 = 2 \text{ or } r_1 = r_2 = r_3 = 3; \\ &n = 4 \text{ with } r_1 = r_2 = r_3 = 1; \\ &n = 5 \text{ with } r_1 = \cdots = r_5 = 1. \end{aligned} \tag{9.3.2}$$

Assume the case of (A). By Proposition 9.3.2, we have $h_s = 1$. Thus, $\Bbbk[K_{r_1,\dots,r_n}]$ is Gorenstein if it is level (see Remark 8.0.3). Hence, $\Bbbk[K_{r_1,\dots,r_n}]$ is level if and only if $K_{r_1,\dots,r_n} = K_{2,2,2}$ or $K_{1,1,1,1}$ by [65, Remark 2.8].

Therefore, in what follows, we consider the cases (B) and (C).

"Only if" part:

Assume the case of (B). Take $\mathcal{M}''$ and $v_1 \in V_1$ as in the proof of Proposition 9.3.2. Then there exists an edge $\{i, j\} \in \mathcal{M}''$ such that $i \notin V_1$ and $j \notin V_1$. Remove such edge from $\mathcal{M}''$ and add $\{v_1, i\}$ and $\{v_1, j\}$ to $\mathcal{M}''$. Write $\mathcal{N}$ for it. Then we have

$$\rho(\mathcal{N}) = 4\mathbf{e}_{v_1} + \sum_{i \in [d] \setminus \{v_1\}} \mathbf{e}_i \in (\ell + 1)P_{K_{r_1,\dots,r_n}} \cap \mathbb{Z}^d.$$

Since there is only one entry which is more than 1, we see that $\rho(\mathcal{N})$ cannot be written as the sum of an element of $\text{int}(\ell P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ and $\rho(e)$ for some $e \in E(K_{r_1,\dots,r_n})$. Hence, once we have $\rho(\mathcal{N}) \in \text{int}((\ell+1)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$, it is not level. Since we know $\langle \rho(\mathcal{N}), r \rangle > 0$ for all $r \in \widetilde{\Psi}_r$, we may observe those of $\widetilde{\Psi}_f$:

$$\langle f_1, \rho(\mathcal{N}) \rangle = \sum_{i \in [n] \setminus \{1\}} r_i - (r_1 + 3) > 0; \tag{9.3.3}$$

$$\langle f_k, \rho(\mathcal{N}) \rangle = \sum_{i \in [n] \setminus \{k\}} r_i + 3 - r_k > 0 \text{ for } k \in [n] \setminus \{1\}. \tag{9.3.4}$$

The inequality (9.3.4) always holds by the assumption (B). The inequality (9.3.3) holds if

$$\begin{aligned} &n \geq 6, \\ &n = 5 \text{ with } r_5 \geq 2, \\ &n = 4 \text{ with } r_4 \geq 3, \text{ or} \\ &n = 3 \text{ with } r_3 \geq 4. \end{aligned}$$

Therefore, in the case of (B),

$$\Bbbk[K_{r_1,\dots,r_n}] \text{ are not level except for } K_{1,1,1,1,1},\ K_{1,1,1,2},\ K_{2,2,3}, \text{ and } K_{3,3,3}.$$

Note that we can confirm that $\Bbbk[K_{1,2,2,2}]$ is not level by using `Macaulay2` ([24]).

Assume the case of (C). Take $\mathcal{E}$, $v_1 \in V_1$, and $v_2 \in V_2$ as in the proof of Proposition 9.3.2. In $\mathcal{E}$, let $v_n \in V_n$ be the vertex adjacent to v_2, and let v_n' be the vertex adjacent to a vertex $v_2' \neq v_2$ of V_2 or a vertex $v_3 \in V_3$ if $r_2 = 1$. Remove $\{v_2, v_n\}$ and $\{v_2', v_n'\}$ from $\mathcal{E}$ and add $\{v_1, v_n\}$, $\{v_1, v_2'\}$ and $\{v_1, v_n'\}$ to $\mathcal{E}$. Write $\mathcal{N}'$ for it. Then we have

$$\rho(\mathcal{N}') = (r_n'' + 5)\mathbf{e}_{v_1} + \sum_{i \in [d]\setminus\{v_1\}} \mathbf{e}_i \in (\ell+1)P_{K_{r_1,\dots,r_n}} \cap \mathbb{Z}^d.$$

Then we see that $\rho(\mathcal{N}')$ cannot be written as a sum of $\mathrm{int}(\ell P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$ and $\rho(e)$ for $e \in E(K_{r_1,\dots,r_n})$. Hence, once we have $\rho(\mathcal{N}') \in \mathrm{int}((\ell+1)P_{K_{r_1,\dots,r_n}}) \cap \mathbb{Z}^d$, it is not level. Since we know $\langle \rho(\mathcal{N}'), r\rangle > 0$ for all $r \in \widetilde{\Psi}_r$, we may observe those of $\widetilde{\Psi}_f$:

$$\langle f_1, \rho(\mathcal{N}')\rangle = \sum_{i \in [n]\setminus\{1\}} r_i - (r_1 + r_n'' + 4) > 0; \tag{9.3.5}$$

$$\langle f_k, \rho(\mathcal{N}')\rangle = \sum_{i \in [n]\setminus\{k\}} r_i + r_n'' + 4 - r_k > 0 \text{ for } k \in [n]\setminus\{1\}. \tag{9.3.6}$$

The inequality (9.3.6) always holds by (C). The inequality (9.3.5) holds if

$$\begin{aligned} &n \geq 5, \text{ or} \\ &n = 4 \text{ with } r_3 \geq 2, \text{ or} \\ &n = 3 \text{ with } r_2 \geq 3. \end{aligned}$$

Thus, in the case of (C),

$$\Bbbk[K_{r_1,\dots,r_n}] \text{ is not level except for } K_{2,2,r_3} \text{ with } r_3 \geq 4 \text{ and } K_{1,1,1,r_4} \text{ with } r_4 \geq 3.$$

Therefore, we obtain that $\Bbbk[K_{r_1,\dots,r_n}]$ is not level if not in the case (9.3.2).

"If" part:

Our remaining task is to show that the edge rings of (9.3.2) are level.

($K_{2,2,r_3}$ with $r_3 \geq 2$)

If $r_3 = 2$, $\Bbbk[K_{2,2,2}]$ is Gorenstein, and if $r_3 = 3$, $\Bbbk[K_{2,2,3}]$ is level by using `Macaulay2`.

Hence, let us assume that $r_3 \geq 4$. Then $K_{r_1,\dots,r_n}$ satisfies (C). Thus, we have $\ell = r_3 + 1$. It is enough to show that for any $k \geq 0$ and $\iota \in \mathrm{int}((\ell+k)P_{K_{2,2,r_3}}) \cap \mathbb{Z}^d$, ι can be written as a sum of an element of $\mathrm{int}(\ell P_{K_{2,2,r_3}}) \cap \mathbb{Z}^d$ and k elements of $P_{K_{2,2,r_3}} \cap \mathbb{Z}^d$, i.e., $\rho(e_1), \dots, \rho(e_k)$ with $e_1, \dots, e_k \in E(K_{2,2,r_3})$. We show this by induction on k. The case $k = 0$ trivially holds.

We have $\left(\sum_{i \in [d]} p_i\right)(\iota) = 2(\ell + k) = 2r_3 + 2k + 2 \geq 2r_3 + 4$, $p_i(\iota) > 0$ for $i \in [d]$, $p_{V_1}(\iota)$, $p_{V_2}(\iota) \geq 2$, and $p_{V_3}(\iota) \geq r_3$. In the case $p_{V_3}(\iota) = r_3$, we can see that $\langle \iota, f_j\rangle > 0$ holds, that is, $p_{V_j}(\iota) \geq 3$ for $j = 1, 2$, and there exist a $v_1 \in V_1$ and a $v_2 \in V_2$ such that

$p_{v_j}(\iota) \geq 2$ for $j = 1, 2$. Let $\iota' := \iota - \rho(\{v_1, v_2\})$. If $\langle \iota', l \rangle > 0$ holds for $l \in \widetilde{\Psi}$, we have $\iota' \in \text{int}((\ell + k - 1)P_{K_{2,2,r_3}}) \cap \mathbb{Z}^d$. It is enough to discuss that of f_3:

$$\langle \iota', f_3 \rangle = \left(\sum_{k \in \{1,2\}} p_{V_k} \right)(\iota) - 2 - p_{V_3}(\iota) = (r_3 + 2k + 2) - 2 - r_3 > 0.$$

In the case $p_{V_3}(\iota) \geq r_3 + 1$, there exists a $v_3 \in V_3$ such that $p_{v_3}(\iota) = 2$. We may assume that $p_{V_1}(\iota) \leq p_{V_2}(\iota)$. Then there is a $v_2' \in V_2$ such that $p_{v_2'}(\iota) \geq 2$. Let $\iota' := \iota - \rho(\{v_2', v_3\})$. If we have $\langle \iota, l \rangle > 0$ for $l \in \widetilde{\Psi}$, we obtain $\iota' \in \text{int}((\ell + k - 1)P_{K_{2,2,r_3}}) \cap \mathbb{Z}^d$. It is enough to discuss that of f_1:

$$\langle \iota', f_1 \rangle = \left(\sum_{k \in \{2,3\}} p_{V_k} \right)(\iota) - 2 - p_{V_1}(\iota) \geq p_{V_3}(\iota) - 2 > 0.$$

Therefore, we obtain the desired result.

($K_{3,3,3}$)

In the same way as above, it is enough to show that for any $k \geq 0$ and $\iota \in \text{int}((\ell + k)P_{K_{3,3,3}}) \cap \mathbb{Z}^d$, ι can be written as a sum of an element of $\text{int}(\ell P_{K_{3,3,3}}) \cap \mathbb{Z}^d$ and $\rho(e_1), \ldots, \rho(e_k)$ with $e_1, \ldots, e_k \in E(K_{3,3,3})$.

By $\ell = 5$, we have $\left(\sum_{i \in [d]} p_i \right)(\iota) = 2(\ell + k) = 2k + 10 \geq 12$. We may assume that $p_{V_1}(\iota) \leq p_{V_2}(\iota) \leq p_{V_3}(\iota)$. If we have $p_{V_1}(\iota) = p_{V_2}(\iota) = 3$, we obtain $p_{V_3}(\iota) = 2k + 4 \geq 6$ and $\langle \iota', f_3 \rangle \leq 0$. This is a contradiction. Thus, we have $4 \leq p_{V_2}(\iota) \leq p_{V_3}(\iota)$. Hence, there exist a $v_2 \in V_2$ and $v_3 \in V_3$ such that $p_{v_j}(\iota) \geq 2$ for $j = 1, 2$. Let $\iota' := \iota - \rho(\{v_2, v_3\})$. If $\langle \iota', l \rangle > 0$ holds for all $l \in \widetilde{\Psi}$, we obtain $\iota' \in \text{int}((\ell + k - 1)P_{K_{3,3,3}}) \cap \mathbb{Z}^d$. It is enough to discuss that of f_1:

$$\langle \iota', f_1 \rangle = \left(\sum_{k \in \{2,3\}} p_{V_k} \right)(\iota) - 2 - p_{V_1}(\iota) = \left(p_{V_2} - p_{V_1} \right)(\iota) + \left(p_{V_3}(\iota) - 2 \right) > 0.$$

Therefore, we obtain the desired result.

($K_{1,1,1,r_4}$ with $r_4 \geq 1$)

We can see that $\Bbbk[K_{1,1,1,1}]$ is Gorenstein, and we can check by `Macaulay2` that $\Bbbk[K_{1,1,1,2}]$ is level. For $r_4 \geq 3$, by Proposition 9.3.2 (c), $K_{1,1,1,r_4}$ is in the case of (C) and $s = d - r_4 - 1 = 2$. It is always level by Theorem 8.0.4.

($\Bbbk[K_{1,1,1,1,1}]$)

We can check by `Macaulay2` that $\Bbbk[K_{1,1,1,1,1}]$ is level.

9.3.3 Proof of Theorem 9.1.3

We still assume the condition $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. This subsection is devoted to giving a proof of Theorem 9.1.3.

We recall a notion of Ehrhart polynomials. Let $P \subset \mathbb{R}^N$ be an integral polytope. For $m \in \mathbb{Z}_{>0}$, consider the number of integer points contained in $mP \cap \mathbb{Z}^N$. Then it is known that such number $|mP \cap \mathbb{Z}^N|$ can be described by a polynomial in m of degree $\dim P$, denoted by $i(P,m)$. The enumerating polynomial $i(P,m)$ is called the *Ehrhart polynomial* of P. For the introduction to the Ehrhart polynomials, see, e.g., [2].

Throughout the remaining parts of this subsection, let $R = \Bbbk[K_{r_1,\dots,r_n}]$. Note that R is normal since $K_{r_1,\dots,r_n}$ satisfies odd cycle condition. Regarding the definition of almost Gorensteinness, let C be the cokernel of the injection $R \to \omega_R(-a(R))$. Note that C is a Cohen–Macaulay R-module of dimension $d-1$. Our goal is to characterize when $e(C) = \mu(C)$ holds. For this, we prepare the following two lemmas.

Lemma 9.3.4. *Assume the case of (A). Then*

$$e(C) = \sum_{k\in[n]} \left(\frac{d}{2} - r_k - 1\right)\binom{d-2}{r_k-1}.$$

Proof. Let $i(P_{K_{r_1,\dots,r_n}}, m) = c_{d-1}m^{d-1} + c_{d-2}m^{d-2} + \cdots + 1$ be the Ehrhart polynomial of $P_{K_{r_1,\dots,r_n}}$. Since R is normal, we see that $P_H(R,m) = i(P_{K_{r_1,\dots,r_n}}, m)$. Note that $i(\mathrm{int}(P_{K_{r_1,\dots,r_n}}), m) = (-1)^{d-1} i(P_{K_{r_1,\dots,r_n}}, -m)$ holds. (See, e.g., [2, Theorem 4.1].) From an exact sequence (8.0.1), we can see that the Hilbert function $H(C,m)$ of C coincides with

$$i(\mathrm{int}(P_{K_{r_1,\dots,r_n}}), m+\ell) - i(P_{K_{r_1,\dots,r_n}}, m),$$

where $\ell = -a(R)$. This implies that the leading coefficient of $H(C,m)$ coincides with $(d-1)\ell c_{d-1} - 2c_{d-2}$. Note that $\dim C = d-1$. Thus,

$$e(C) = (d-2)!((d-1)\ell c_{d-1} - 2c_{d-2}).$$

Here, [65, Theorem 2.6] claims that

$$i(P_{K_{r_1,\dots,r_n}}, m) = \binom{d+2m-1}{d-1} - \sum_{k\in[n]} \sum_{1\le i\le j\le r_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j}.$$

Hence, a direct computation shows that

$$(d-1)!c_{d-1} = 2^{d-1} - \sum_{k\in[n]}\sum_{j\in[r_k]} \binom{d-1}{j-1} \quad \text{(see [65, Corollary 2.7]), and}$$

$$(d-2)!c_{d-2} = 2^{d-3}d - \sum_{k\in[n]}\sum_{j\in[r_k]} \left(\binom{d-2}{j-2} + \frac{d-2}{2}\left(\binom{d-3}{j-3} + \binom{d-3}{j-1}\right)\right).$$

Remark $\ell = d/2$ by Proposition 9.3.2 (a). Therefore, we conclude that

$$e(C) = \sum_{k\in[n]}\sum_{j\in[r_k]} \left(2\binom{d-2}{j-2} + (d-2)\left(\binom{d-3}{j-3} + \binom{d-3}{j-1}\right) - \frac{d}{2}\binom{d-1}{j-1}\right),$$

where we set $\binom{n}{r}$ for $r \in \mathbb{Z}_{<0}$ to be 0. By using

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} \text{ for } n, r \in \mathbb{Z}, \tag{9.3.7}$$

we obtain that

$$e(C) = \sum_{k\in[n]} \sum_{j\in[r_k]} \left(\frac{d}{2}\binom{d-1}{j-1} - 2(d-1)\binom{d-2}{j-1} + 2(d-2)\binom{d-3}{j-1} \right).$$

It is enough to prove that

$$\sum_{j\in[r_k]} \left(\frac{d}{2}\binom{d-1}{j-1} - 2(d-1)\binom{d-2}{j-1} + 2(d-2)\binom{d-3}{j-1} \right) = \left(\frac{d}{2} - r_k - 1\right)\binom{d-2}{r_k-1} \tag{9.3.8}$$

holds. We prove this by induction on r_k. We can directly see that (9.3.8) holds when $r_k = 1$.

Suppose that $r_k > 1$. By the hypothesis of induction, we have

$$\begin{aligned}
&\sum_{j\in[r_k+1]} \left(\frac{d}{2}\binom{d-1}{j-1} - 2(d-1)\binom{d-2}{j-1} + 2(d-2)\binom{d-3}{j-1} \right) \\
&\quad = \left(\frac{d}{2} - r_k - 1\right)\binom{d-2}{r_k-1} + \left(\frac{d}{2}\binom{d-1}{r_k} - 2(d-1)\binom{d-2}{r_k} + 2(d-2)\binom{d-3}{r_k} \right).
\end{aligned}$$

By using $(d - r_k - p)\binom{d-p}{r_k} = (d-p)\binom{d-p-1}{r_k}$ for $p = 1, 2$ and (9.3.7), we obtain that

$$\begin{aligned}
&\left(\frac{d}{2} - r_k - 1\right)\left(\binom{d-1}{r_k} - \binom{d-2}{r_k}\right) \\
&\quad + \left(\frac{d}{2}\binom{d-1}{r_k} - 2(d-1)\binom{d-2}{r_k} + 2(d-r_k-2)\binom{d-2}{r_k} \right) \\
&= \left(d - r_k - 1\right)\binom{d-1}{r_k} - \left(\frac{d}{2} + r_k + 1\right)\binom{d-2}{r_k} \\
&= (d-1)\binom{d-2}{r_k} - \left(\frac{d}{2} + r_k + 1\right)\binom{d-2}{r_k} = \left(\frac{d}{2} - r_k - 2\right)\binom{d-2}{r_k}.
\end{aligned}$$

This completes the proof. □

Lemma 9.3.5. *Assume the case of (A). Then*

$$\mu(C) = \sum_{k\in[n]} \sum_{j\in[\frac{d}{2}-r_k-1]} \binom{r_k - 1 + 2j}{r_k - 1},$$

where we let $\sum_{j\in[\frac{d}{2}-r_k-1]} \binom{r_k-1+2j}{r_k-1} = 0$ *if* $r_k = \frac{d}{2} - 1$.

Proof. Remark $\ell = d/2$.

Since $\mu(C) = \sum_{j \geq \ell} \mu_j(\omega_{K_{r_1,\ldots,r_n}}) - 1 = \sum_{j \geq \ell+1} \mu_j(\omega_{K_{r_1,\ldots,r_n}}) = \sum_{j \geq 1} \mu_j(K_{r_1,\ldots,r_n})$, where we recall that $\mu_j(K_{r_1,\ldots,r_n})$ is the number of first appearing interior points. We compute $\mu_j(K_{r_1,\ldots,r_n})$ for $j \geq 1$.

From Lemma 9.3.6 below, we see that $\iota \in (\ell + j)P_{K_{r_1,\ldots,r_n}} \cap \mathbb{Z}^d$ $(j \geq 1)$ is a first appearing interior point if and only if there exists $k \in [n]$ such that ι satisfies

$$\begin{cases} p_{V_j}(\iota) = r_k + 2j, \\ p_i(\iota) = 1 \text{ for } i \in [d] \setminus V_k, \text{ and} \\ \langle \iota, f_k \rangle = (d - r_k) - (r_k + 2j) > 0, \text{ that is, } 1 \leq j \leq d/2 - r_k - 1. \end{cases}$$

Hence, for j and k respectively, we observe that the number of first appearing interior points is $\binom{r_k - 1 + 2j}{r_k - 1}$, and so $\mu(C) = \sum_{j \geq 1} \mu_j(K_{r_1,\ldots,r_n}) = \sum_{j \geq 1} \sum_{k \in [n]} \binom{r_k - 1 + 2j}{r_k - 1}$. □

Lemma 9.3.6. *Assume the case of (A). Given* $\iota \in \mathrm{int}((\ell + j)P_{K_{r_1,\ldots,r_n}}) \cap \mathbb{Z}^d$ *for each* $j \geq 0$*,* ι *is a first appearing interior point if and only if there exists* $k \in [n]$ *such that*

$$\begin{cases} p_{V_j}(\iota) = r_k + 2j, \\ p_i(\iota) = 1 \text{ for } i \in [d] \setminus V_k. \end{cases} \tag{9.3.9}$$

Proof. **"If" part**: By the condition on ι, we see that ι cannot be written as a sum of an element of $\mathrm{int}((\ell + j')P_{K_{r_1,\ldots,r_n}}) \cap \mathbb{Z}^d$ with $j' < j$ and $\rho(e_1), \ldots, \rho(e_{j'})$ with $e_1, \ldots, e_{j'} \in E(K_{r_1,\ldots,r_n})$. Thus, we obtain the desired result.

"Only if" part: We prove the assertion by induction on $j \geq 0$. If $j = 0$, then $\sum_{i \in [d]} \mathbf{e}_i$ is the unique first appearing interior point in $\mathrm{int}(\ell P_{K_{r_1,\ldots,r_n}}) \cap \mathbb{Z}^d$.

Let $j \geq 1$. Let $r'_k := p_{V_k}(\iota) - r_k$ for $k \in [n]$. By the hypothesis of induction, there are at least two k's with $r'_k \neq 0$. Take these $r'_{k_1} \geq r'_{k_2} \geq \cdots \geq r'_{k_s} > 0$ and $k_p > k_q$ if $r'_{k_p} = r'_{k_q}$. Remark $s \geq 2$. Then there are $v_{k_1} \in V_{k_1}$ and $v_{k_2} \in V_{k_2}$ such that $p_{v_{k_1}}, p_{v_{k_2}} \geq 2$.

If we can have $\iota' = \iota - \rho(\{v_{k_1}, v_{k_2}\}) \in \mathrm{int}((\ell + j - 1)P_{K_{r_1,\ldots,r_n}}) \cap \mathbb{Z}^d$, then ι is not a first appearing interior point. Since $\langle \iota', r \rangle > 0$ for $r \in \widetilde{\Psi}_r$ holds, we may show that $\langle \iota', f_k \rangle > 0$ with $k = \max\{p_{V_j}(\iota) : j \in [n]\}$. We see that k should be one of k_1, k_3 and n.

$(k = k_1)$ We have $\langle \iota', f_{k_1} \rangle = \langle \iota, f_{k_1} \rangle > 0$.

$(k = k_3)$ We see that $p_{V_{k_1}}(\iota) = p_{V_{k_2}}(\iota) = p_{V_{k_3}}(\iota)$. Remark $p_{V_k}(\iota) = r_k + r'_k$. Then

$$\begin{aligned} \langle \iota', f_{k_3} \rangle &= \Bigg(\sum_{i \in [n] \setminus \{k_3\}} p_{V_i} \Bigg)(\iota) - 2 - p_{V_{k_3}}(\iota) \\ &= \Bigg(\sum_{i \in [n] \setminus \{k_1, k_2, k_3\}} p_{V_i} \Bigg)(\iota) + (r_{k_2} - 1) + (r'_{k_2} - 1) > 0. \end{aligned}$$

$(k = n)$ If $r'_n \geq r'_{k_2}$, then we have $n = k_1$ or k_2, so we can deduce the case $k = k_1$ or k_3. Hence, we may assume that $r'_n < r'_{k_2}$. Then we see that

$$\begin{aligned}\langle \iota', f_n \rangle &= \left(\sum_{i \in [n-1]} p_{V_i} \right)(\iota) - 2 - p_{V_n}(\iota) \\ &\geq \left(\sum_{i \in [n-1]} r_i - r_n \right) + (r'_{k_1} - 1) + (r'_{k_2} - r'_n - 1) > 0.\end{aligned}$$

Therefore, we obtain the desired result. □

Lemma 9.3.7. *Let $r_1, \dots, r_n$ be positive integers with $1 \leq r_1 \leq \dots \leq r_n$, let $d = \sum_{i \in [n]} r_i$ be even. Assume that $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. Then n and $(r_1, \dots, r_n)$ satisfy one of the conditions (iii)—(vi) in Theorem 9.1.3 if and only if $r_i \in \{1, d/2 - 1\}$ holds for any $i \in [n]$.*

Proof. Since "only if" part is easy to see, we prove "if" part.

Assume that $r_i \in \{1, d/2 - 1\}$ holds for any $i \in [n]$. Note that $d \geq n$ by definition. Let α be the number of r_i's with $r_i = d/2 - 1$. Then $r_1 = \dots = r_{n-\alpha} = 1$. Thus, we have $d = (n - \alpha) + (d/2 - 1)\alpha$.

The case $\alpha = 0$ is nothing but the case (vi). Note that n should be even by $d = n$. If $\alpha = 1$, then $d = (n - 1) + (d/2 - 1)$ holds by definition, which implies that $d = 2n - 4$. Thus, this corresponds to the case (v).

Suppose $\alpha \geq 2$. When $n \geq 5$, we see that $d = (d/2 - 2)\alpha + n \geq d - 4 + n > d$, a contradiction. Hence, $n = 3$ or $n = 4$.

- Let $n = 3$. Since we assume $r_1 \geq 2$, we may discuss the case $\alpha = 3$. Then $d = 3(d/2 - 1)$ holds, i.e., $d = 6$. Hence, we obtain that $r_1 = r_2 = r_3 = 2$, which is the case (iii).
- Let $n = 4$. If $\alpha = 2$, then we see that $r_3 (= r_4)$ can be arbitrary, which is the case (iv). If $\alpha = 3$ (resp. $\alpha = 4$), then $d = 1 + 3(d/2 - 1)$ (resp. $d = 4(d/2 - 1)$) holds, i.e., $d = 4$. Hence, we obtain that $r_1 = \dots = r_4 = 1$, which is included in (iv).

□

Now, we are ready to give a proof of Theorem 9.1.3. Since the case where $n = 2$ or $n = 3$ with $r_1 = 1$ has been already done in Section 9.2, we assume that $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. Under this assumption, it suffices to prove that $\Bbbk[K_{r_1,\dots,r_n}]$ is almost Gorenstein if and only if d is even and $r_i \in \{1, d/2 - 1\}$ for any $i \in [n]$ by Lemma 9.3.7. Then we may assume the case (A) by Corollary 9.3.3. Remark that $1 \leq r_k \leq d/2 - 1$ holds. By Lemmas 9.3.4 and 9.3.5, we have $e(C) = \sum_{k \in [n]} e_k(C)$ and $\mu(C) = \sum_{k \in [n]} \mu_k(C)$, where

$$e_k(C) := \left(\frac{d}{2} - r_k - 1 \right) \binom{d-2}{r_k - 1} \text{ and } \mu_k(C) = \sum_{j \in [\frac{d}{2} - r_k - 1]} \binom{r_k - 1 + 2j}{r_k - 1} \text{ for each } k \in [n].$$

- If $r_k = 1$, then we have $e_k(C) = \mu_k(C) = d/2 - 2$.

- If $r_k = d/2 - 1$, we have $e_k(C) = \mu_k(C) = 0$.
- If $1 < r_k < d/2 - 1$, then we have

$$\mu_k(C) \le \sum_{j \in [\frac{d}{2} - r_k - 1]} \binom{d - r_k - 3}{r_k - 1} = \left(\frac{d}{2} - r_k - 1\right) \binom{d - r_k - 3}{r_k - 1} < e_k(C).$$

Hence, if there is $k \in [n]$ with $1 < r_k < d/2 - 1$, then $e(C) > \mu(C)$ holds. Therefore, we conclude that $e(C) = \mu(C)$ holds if and only if $r_k = 1$ or $d/2 - 1$ for any $k \in [n]$.

Chapter 10

Conditions of multiplicity and applications for almost Gorenstein graded rings

In this chapter, we will discuss the almost Gorensteinness of graded rings derived from conditions of their multiplicities and provide an application to edge rings and stable set rings. The contents of this chapter are contained in the author's paper [53] with S. Miyashita.

10.1 Conditions for almost Gorenstein rings

In this section, we show the following theorem and discuss conditions appearing in the theorem. Moreover, we apply our theorem to tensor products of semi-standard graded rings and the quotient rings divided by their regular sequences.

Theorem 10.1.1. *Let A, B and R be Cohen–Macaulay positively graded rings. Assume that A, B and R satisfy $(*)$ (see Chapter 8). Moreover, we assume $e(A) > 1$, $e(B) > 1$ and*

$$\delta_\phi(R) \geq e(B)\delta(A) + e(A)\delta(B). \tag{10.1.1}$$

Then the following conditions are equivalent:

(1) R is almost Gorenstein with respect to ϕ and $r(R) \leq r(A)r(B)$;
(2) R is Gorenstein;
(3) A and B are Gorenstein and the equality of (10.1.1) holds.

10.1.1 Proof of Theorem 10.1.1

The following is a key lemma.

Lemma 10.1.2. *Let A and B be Cohen–Macaulay local (or graded) rings with $e(A) > 1$ and $e(B) > 1$. If there exist $\gamma_1, \gamma_2 \in \mathbb{R}_{\geq 0}$ such that $r(A) \leq \gamma_1 + 1$, $r(B) \leq \gamma_2 + 1$ and*

$$e(B)\gamma_1 + e(A)\gamma_2 \leq r(A)r(B) - 1, \tag{10.1.2}$$

then A and B are Gorenstein and $\gamma_i = 0$ for $i = 1, 2$.

Proof. We get the following inequality by the assumptions:

$$e(B)\gamma_1 + e(A)\gamma_2 \leq r(A)r(B) - 1 \leq \gamma_1\gamma_2 + \gamma_1 + \gamma_2. \tag{10.1.3}$$

Thus we have

$$X := (e(A) - 1 - \gamma_1)(e(B) - 1 - \gamma_2) - (e(A) - 1)(e(B) - 1) \geq 0.$$

It follows from $(e(A)-1)(e(B)-1) > 0$ and $X \geq 0$ that $(e(A)-1-\gamma_1)(e(B)-1-\gamma_2) > 0$, equivalently, "$e(A) - 1 - \gamma_1 > 0$ and $e(B) - 1 - \gamma_2 > 0$" or "$e(A) - 1 - \gamma_1 < 0$ and $e(B)-1-\gamma_2 < 0$". Suppose that $e(A)-1-\gamma_1 < 0$ and $e(B)-1-\gamma_2 < 0$. This hypothesis and Proposition 8.0.1 imply that

$$\begin{aligned} e(B)\gamma_1 + e(A)\gamma_2 &> e(B)(e(A) - 1) + e(A)(e(B) - 1) \\ &\geq r(B)(r(A) - 1) + r(A)(r(B) - 1) \\ &\geq r(A)r(B) - 1, \end{aligned}$$

a contradiction to (10.1.2).

Thus we have $e(A) - 1 - \gamma_1 > 0$ and $e(B) - 1 - \gamma_2 > 0$. On the other hand, in this situation, we can see that $X < 0$ unless $\gamma_1 = \gamma_2 = 0$. Then we obtain $\gamma_1 = \gamma_2 = 0$, and hence $r(A) = r(B) = 1$ by our assumption. Therefore, A and B are Gorenstein, as desired. □

Proof of Theorem 10.1.1. It is clear that (3) ⇒ (2) ⇒ (1) since R is Gorenstein if and only if $\delta_\phi(R) = 0$. We show that (1) implies (3). Since R is almost Gorenstein, we have $\delta_\phi(R) = \mu(\operatorname{coker}(\phi)) = r(R) - 1$.Then we have

$$e(B)\delta(A) + e(A)\delta(B) \leq \delta_\phi(R) = r(R) - 1 \leq r(A)r(B) - 1. \tag{10.1.4}$$

Therefore, since $r(A)-1 \leq \delta(A)$ and $r(B)-1 \leq \delta(B)$ by Proposition 8.0.6 and $e(B)\delta(A)+e(A)\delta(B) \leq r(A)r(B) - 1$, A and B are Gorenstein by Lemma 10.1.2. Moreover, in this case, we have $r(A) = r(B) = 1$ and $\delta(A) = \delta(B) = 0$, so we get $e(B)\delta(A) + e(A)\delta(B) = \delta_\phi(R) = 0$ by (10.1.4), as desired. □

Remark 10.1.3. (a) If we add the assumption that R is semi-standard graded to Theorem 10.1.1, (1) does not depend on how ϕ is chosen, that is, (1) holds if and only if R is almost Gorenstein and $r(R) \leq r(A)r(B)$ because of Propositions 8.0.6 and 8.0.7.

(b) The essential part of the proof of the above theorem is (1) ⇒ (3). In the above theorem, the assertion does not hold in general if we drop the assumption that $e(A) > 1$ and $e(B) > 1$; even if R is almost Gorenstein and $r(R) \leq r(A)r(B)$, B is not necessarily Gorenstein when $e(A) = 1$, that is, in the previous theorem, (1) does not imply (3). However, it can be derived that B is almost Gorenstein as follows:

Suppose that $e(A) = 1$. Then we can rewrite (10.1.1) as $\delta(B) \leq \delta(R)$. Since R is almost Gorenstein and $r(R) \leq r(B)$, we have

$$\delta(B) \leq \delta(R) = r(R) - 1 \leq r(B) - 1 \leq \delta(B). \tag{10.1.5}$$

Therefore, we get $\delta(B) = r(B) - 1$, and hence B is almost Gorenstein.

(c) If we drop the assumption $r(R) \leq r(A)r(B)$ from (1), this does not imply (3) in general. In fact, put $R = \mathbb{Q}[s, st^{18}, st^{21}, st^{23}, st^{26}]$, $A = \mathbb{Q}[s, st, st^2]$ and $B = \mathbb{Q}[s, st^9, st^{10}, st^{13}]$. By using `Macaulay2` ([24]), we can check $h_R(t) = 1+3t+5t^2+7t^3+6t^4+3t^5+t^6$, $h_A(t) = 1+t$ and $h_B(t) = 1+2t+3t^2+4t^3+2t^4+t^5$. Moreover, R is almost Gorenstein and the equality of (10.1.1) holds. However, B is not Gorenstein and $r(R) = 3 > 2 = r(A)r(B)$.

10.1.2 Sufficient conditions to satisfy the multiplicity condition

In this subsection, we provide sufficient conditions to satisfy the equality of (10.1.1) for semi-standard graded rings.

Let $h = h(t) = \sum_{i=0}^{s} h_i t^i \in \mathbb{Z}[t]$ and put $e(h) := \sum_{i=0}^{s} h_i = h(1)$, $\delta(h) := \sum_{i=0}^{s}(2i - s)h_i$.

Remark 10.1.4. Let R be a Cohen–Macaulay semi-standard graded ring satisfies $(*)$ and let $h_R(t)$ be its h-polynomial. Then we have $e(R) = e(h_R)$ and $\delta(R) = \delta(h_R)$.

Proposition 10.1.5. *Let $h, g \in \mathbb{Z}[t]$. Then we have $\delta(hg) = e(h)\delta(g) + e(g)\delta(h)$. In particular, if $g = 1 + t + \cdots + t^{a-1}$ for some integer $a > 0$, we have $\delta(hg) = a\delta(h)$.*

Proof. Put $h(t) = a_0 + a_1 t + \cdots + a_n t^n$, $g(t) = b_0 + b_1 t + \cdots + b_m t^m$. Then we have

$$\delta(hg) = \sum_{i=0}^{m+n}\left((2i - (m+n))\sum_{j=0}^{i} a_{i-j}b_j\right) = \sum_{j=0}^{m}\left(\sum_{i=0}^{m+n}(2i-(m+n))a_{i-j}\right)b_j.$$

Now we calculate the coefficient A_j of b_j in $\delta(hg)$ for each $1 \leq j \leq m$. Note that $a_i = 0$ if $i < 0$ or $n < i$. Then we have

$$\begin{aligned} A_j = \sum_{i=j}^{j+n}(2i - (m+n))a_{i-j} &= \sum_{i=0}^{n}(2(i+j) - (m+n))a_i \\ &= \sum_{i=0}^{n}((2i-n)a_i + (2j-m)a_i) = \delta(h) + (2j-m)e(h). \end{aligned}$$

Therefore,

$$\delta(hg) = \sum_{i=0}^{m} A_i b_i = e(g)\delta(h) + e(h)\sum_{i=0}^{m}(2i-m)b_i = e(g)\delta(h) + e(h)\delta(g).$$

□

Corollary 10.1.6. *Let A, B and R be Cohen–Macaulay semi-standard graded rings satisfying $(*)$. If $h_R(t) = h_A(t)h_B(t)$, we have $\delta(R) = e(B)\delta(A) + e(A)\delta(B)$.*

Example 10.1.7. We see that A, B and R in Remark 10.1.3 (c) satisfy $h_A(t)h_B(t) = h_R(t)$. Even by using Corollary 10.1.6, it can be confirmed that $\delta(R) = e(B)\delta(A) + e(A)\delta(B)$.

Now we apply Theorem 10.1.1 to the tensor product of semi-standard graded rings and its quotient rings divided by a homogeneous regular sequence.

Corollary 10.1.8. *Let A and B be semi-standard graded Cohen–Macaulay $\Bbbk$-algebras over a field $\Bbbk$ such that A and B satisfy $(*)$ and $e(A) > 1$ and $e(B) > 1$. In addition, let $T = A \otimes_{\Bbbk} B$ and let $\mathbf{x} = x_1, \ldots, x_n$ be a homogeneous regular sequence on T with $\deg(x_k) = a_k$ for $1 \leq k \leq n$ such that $T/(\mathbf{x})$ satisfies $(*)$. Set $R = T$ or $R = T/(\mathbf{x})$, then the following conditions are equivalent:*

(1) R is almost Gorenstein;
(2) R is Gorenstein;
(3) A and B are Gorenstein.

Proof. Note that $r(R) = r(T) = r(A)r(B)$ (cf. [10, Theorem 3.3.5 (a)] and [30, Theorem 4.2]) and R satisfies $(*)$ since A and B do. First, we consider the case of $R = T$. Since $h_A(t)h_B(t) = h_T(t)$, the assertion follows from by Theorem 10.1.1 and Corollary 10.1.6. Next, we consider the case of $R = T/(\mathbf{x})$. In this case, we can check $h_R(t) = \left(\prod_{i=1}^{n}(1+t+\cdots+t^{a_i-1})\right)h_T(t)$ so we have $\delta(R) = a_1\cdots a_n\delta(T)$ by Proposition 10.1.5. Then we have

$$\delta(R) \geq \delta(T) = e(B)\delta(A) + e(A)\delta(B)$$

by Corollary 10.1.6 so the assertion follows from Theorem 10.1.1. □

We can study polynomial extensions of almost Gorenstein rings by using our main theorem. The semi-standard graded case can be proven easily.

Corollary 10.1.9. *Let R and $S = R[x]$ be Cohen–Macaulay semi-standard graded rings over a field $\Bbbk$. Then the following conditions are equivalent:*

(1) S is almost Gorenstein;
(2) R is almost Gorenstein.

Proof. Note that $S = R \otimes_\Bbbk \Bbbk[x]$ and that R satisfies $(*)$ if and only if S does. We have $r(R) = r(S)$ and $\delta(R) = \delta(S)$ by Corollary 10.1.6. Thus we can see that all the equalities appearing in (10.1.5) hold, so the assertion follows by the observation of Remark 10.1.3 (b). □

10.2 Applications to toric rings

In this section, we provide some applications of our results to toric rings; edge rings and stable set rings. Before that, we discuss toric splittings of toric ideals, which helps us to show our results.

We say that a toric ideal I of a polynomial ring S over $\Bbbk$ has *toric splitting* (or I is a *splittable toric ideal*) if there exist toric ideals I_1 and I_2 of S such that $I = I_1 + I_2$. See [19] for details on toric splittings.

For a polynomial f of S, we denote the set of variables of S appearing in f by $\mathrm{var}(f)$ and for a subset F of S, we define $\mathrm{var}(F) := \bigcup_{f\in F}\mathrm{var}(f)$.

Proposition 10.2.1. *Let I be a toric ideal of S generated by binomials $f_1, \dots, f_n, g_1, \dots, g_m$ and let $B_1 = \{f_1, \dots, f_n\}$ and $B_2 = \{g_1, \dots, g_m\}$. Suppose that I has the toric splitting $I = (B_1) + (B_2)$, that $\mathrm{var}(B_1 \cup B_2)$ contains all variables of S and that $\mathrm{var}(B_1) \cap \mathrm{var}(B_2)$ consists a single variable z. Then we have*

$$S/I \cong \left(\Bbbk[\mathrm{var}(\widetilde{B}_1)]/(\widetilde{B}_1) \otimes_\Bbbk \Bbbk[\mathrm{var}(\widetilde{B}_2)]/(\widetilde{B}_2)\right)/(z_1 - z_2),$$

where z_1 and z_2 are new variables that do not belong to S and we let $\tilde{f}_i$ (resp. $\tilde{g}_i$) be the polynomial obtained by substituting z_1 (resp. z_2) for z appearing in f_i (resp. g_i) and let $\widetilde{B}_1 = \{\tilde{f}_1, \dots, \tilde{f}_n\}$ and $\widetilde{B}_2 = \{\tilde{g}_1, \dots, \tilde{g}_m\}$.

Proof. Since $\mathrm{var}(B_1 \cup B_2)$ contains all variables of S and I has the toric splitting $I = (B_1) + (B_2)$, we have $\Bbbk[\mathrm{var}(B_1 \cup B_2)]/((B_1) + (B_2)) = S/I$.

Moreover, it follows from $\mathrm{var}(\widetilde{B}_1) \cap \mathrm{var}(\widetilde{B}_2) = \emptyset$ that

$$\Bbbk[\mathrm{var}(\widetilde{B}_1)]/(\widetilde{B}_1) \otimes_{\Bbbk} \Bbbk[\mathrm{var}(\widetilde{B}_2)]/(\widetilde{B}_2) \cong \Bbbk[\mathrm{var}(\widetilde{B}_1 \cup \widetilde{B}_2)]/((\widetilde{B}_1) + (\widetilde{B}_2)),$$

and hence

$$\begin{aligned} S/I &= \Bbbk[\mathrm{var}(B_1 \cup B_2)]/((B_1) + (B_2)) \\ &\cong \Bbbk[\mathrm{var}(\widetilde{B}_1 \cup \widetilde{B}_2)]/((\widetilde{B}_1) + (\widetilde{B}_2) + (z_1 - z_2)) \\ &\cong \left(\Bbbk[\mathrm{var}(\widetilde{B}_1)]/(\widetilde{B}_1) \otimes_{\Bbbk} \Bbbk[\mathrm{var}(\widetilde{B}_2)]/(\widetilde{B}_2)\right)/(z_1 - z_2). \end{aligned}$$

□

Corollary 10.2.2. *Work with the notation and hypothesis of Proposition 10.2.1, and assume that* $A := \Bbbk[\mathrm{var}(\widetilde{B}_1)]/(\widetilde{B}_1)$, $B := \Bbbk[\mathrm{var}(\widetilde{B}_2)]/(\widetilde{B}_2)$ *and* $R := S/I$ *are semi-standard graded. Then the following conditions are equivalent:*

(1) R *is almost Gorenstein;*
(2) R *is Gorenstein;*
(3) A *and* B *are Gorenstein.*

Proof. It follows from Corollary 10.1.8 and Proposition 10.2.1 (even if $\mathrm{var}(B_1 \cup B_2)$ does not necessarily contain all variables of S, we can show this claim by applying polynomial extension and Corollary 10.1.9). □

We now give the proofs of our theorems and some examples. First, we provide an application to edge rings:

Theorem 10.2.3. *Let* G_1 *and* G_2 *be simple graphs. Suppose that* G_1 *is bipartite and* $\Bbbk[G_2]$ *is Cohen–Macaulay with* $e(\Bbbk[G_i]) > 1$ *for* $i = 1, 2$*. Then the following are equivalent:*

(1) $\Bbbk[G_1 \sharp G_2]$ *is almost Gorenstein;*
(2) $\Bbbk[G_1 \sharp G_2]$ *is Gorenstein;*
(3) $\Bbbk[G_1]$ *and* $\Bbbk[G_2]$ *are Gorenstein.*

Proof. It is known that the edge ring of a bipartite graph is normal ([64, Corollary 2.3]), and hence $\Bbbk[G_1]$ is Cohen–Macaulay. Note that $k := |V(G_1) \cap V(G_2)| \leq 2$ since G_1 is bipartite. If $k \leq 1$, then we have $\Bbbk[G_1 \sharp G_2] \cong \Bbbk[G_1] \otimes_{\Bbbk} \Bbbk[G_2]$ (cf. [84, Proposition 10.1.48]). Therefore, conditions (1), (2) and (3) are equivalent by Corollary 10.1.8.

Suppose that $k = 2$ and let $c = V(G_1) \cap V(G_2)$ (note that $c \in E(G)$). We can see that I_G has the toric splitting $I_G = I_{G_1} + I_{G_2}$ ([19, Corollary 4.8]) and the common variable that appears in generators of I_{G_1} and I_{G_2} is only x_c. Therefore, we get the desired result from Corollary 10.2.2. □

Next, we present an application to stable set rings:

Theorem 10.2.4. *Let* G_1 *and* G_2 *be simple graphs. Suppose that* $\Bbbk[\mathrm{Stab}_{G_i}]$ *is Cohen–Macaulay with* $e(\Bbbk[\mathrm{Stab}_{G_i}]) > 1$ *for* $i = 1, 2$*. Then the following are equivalent:*

(1) $\Bbbk[\mathrm{Stab}_{G_1+G_2}]$ *is almost Gorenstein;*
(2) $\Bbbk[\mathrm{Stab}_{G_1+G_2}]$ *is Gorenstein;*
(3) $\Bbbk[\mathrm{Stab}_{G_1}]$ *and* $\Bbbk[\mathrm{Stab}_{G_2}]$ *are Gorenstein.*

Proof. It follows immediately from Corollary 10.2.2 and Lemma 10.2.5 below. □

Lemma 10.2.5. *Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$ and let $B_1 = \{f_1, \ldots, f_n\}$ and $B_2 = \{g_1, \ldots, g_m\}$ be minimal generating systems of J_{G_1} and J_{G_2}, respectively. Then we have $\mathrm{var}(B_1) \cap \mathrm{var}(B_2) = \{x_\emptyset\}$. Moreover, $J_{G_1+G_2}$ has the toric splitting $J_{G_1+G_2} = J_{G_1} + J_{G_2}$.*

Proof. Notice that each non-empty stable set of $G_1 + G_2$ is that of only one of G_1 or G_2. The first assertion holds from this fact.

We can see that $J_{G_1} + J_{G_2} \subset J_G$ since $J_{G_i} \subset J_G$ for $i = 1, 2$. Thus it is enough to show that any binomial $f \in J_{G_1+G_2}$ belongs to $J_{G_1} + J_{G_2}$. We may assume that f can be written as

$$f = x_{S_1} \cdots x_{S_p} x_{T_1} \cdots x_{T_q} - x_{S'_1} \cdots x_{S'_r} x_{T'_1} \cdots x_{T'_u} x_\emptyset^a,$$

where S_i and S'_j (resp. T_k and T'_l) are non-empty stable sets of G_1 (resp. G_2), and $a = p + q - r - u \geq 0$. Moreover, we have $(S_i \cup S'_j) \cap (T_k \cup T'_l) = \emptyset$ for each i, j, k and l, which implies that $x_{S_1} \cdots x_{S_p} x_\emptyset^b - x_{S'_1} \cdots x_{S'_r} x_\emptyset^{b'}$ and $x_{T_1} \cdots x_{T_q} x_\emptyset^c - x_{T'_1} \cdots x_{T'_u} x_\emptyset^{c'}$ belong to J_{G_1} and J_{G_2} for some b, b', c and c', respectively.

Suppose that $a_1 = p - r \geq 0$. Then we can see that

$$f = \begin{cases} x_{T_1} \cdots x_{T_q} (x_{S_1} \cdots x_{S_p} - x_{S'_1} \cdots x_{S'_r} x_\emptyset^{a_1}) + \\ \qquad x_{S'_1} \cdots x_{S'_r} x_\emptyset^{a_1} (x_{T_1} \cdots x_{T_q} - x_{T'_1} \cdots x_{T'_u} x_\emptyset^{a-a_1}) & \text{if } a \geq a_1, \\ x_{T_1} \cdots x_{T_q} (x_{S_1} \cdots x_{S_p} - x_{S'_1} \cdots x_{S'_r} x_\emptyset^{a_1}) + \\ \qquad x_{S'_1} \cdots x_{S'_r} x_\emptyset^{a} (x_{T_1} \cdots x_{T_q} x_\emptyset^{a_1-a} - x_{T'_1} \cdots x_{T'_u}) & \text{if } a \leq a_1, \end{cases}$$

and hence $f \in J_{G_1} + J_{G_2}$.

If $a_1 < 0$, we have $a_2 = q - u \geq 0$. Otherwise, $a = (p - r) + (q - u) = a_1 + a_2 < 0$, contradicting the fact that $a \geq 0$. Therefore, we can show that $f \in J_{G_1} + J_{G_2}$ by the same argument and conclude that $J_{G_1+G_2} = J_{G_1} + J_{G_2}$.

□

Finally, we give two specific applications of our theorems.

Example 10.2.6. (a) For two integers $m \geq 1$ and $l \geq 0$, let $\mathfrak{g}^l_{r_1,\ldots,r_m}$ be the graph consisting of r_j cycles of length $2j + 1$ and l even cycles, such that all cycles share a single common vertex. If $l = 0$, then $\mathfrak{g}^l_{r_1,\ldots,r_m}$ consists of only odd cycles and we denote it by $\mathfrak{g}_{r_1,\ldots,r_m}$. This graph is the clique sum of $\mathfrak{g}_{r_1,\ldots,r_m}$ and l even cycles. It is known that the edge ring of an even cycle C is Gorenstein with $e(\Bbbk[C]) > 1$ (cf. [67, Theorem 2.1(b)]) and that almost Gorensteinness of the edge ring of $\mathfrak{g}_{r_1,\ldots,r_m}$ has been investigated in [1, Theorem 1.2].

From these facts and Theorem 10.2.3, we can characterize when $\Bbbk[\mathfrak{g}^l_{r_1,\ldots,r_m}]$ is (almost) Gorenstein:

- $\Bbbk[\mathfrak{g}^l_{r_1,\ldots,r_m}]$ is Gorenstein if and only if $r_1 + \cdots + r_m \leq 2$.

- $\Bbbk[\mathfrak{g}^l_{r_1,\dots,r_m}]$ is not Gorenstein but alomst Gorenstein if and only if $l = 0$, $m = 1$ and $r_1 \geq 3$.

(b) Let $G_1, \dots, G_k$ be simple connected graphs with at most one G_i not being bipartite and let C be an even cycle with at least k edges. In addition, let G be the graph obtained by identifying an edge of G_i with a distinct edge of C for each i (this graph appears in [19, Theorem 3.7]). If $e(\Bbbk[G_i]) > 1$ for all i, then we have the following equivalent conditions:

$$\Bbbk[G] \text{ is almost Gorenstein} \Leftrightarrow \Bbbk[G] \text{ is Gorenstein} \Leftrightarrow \Bbbk[G_i] \text{ is Gorenstein for all } i.$$

Example 10.2.7. For $m \geq 3$ and $n \geq 1$, we consider the *cone graph* $C_m + \overline{K}_n$, which is the join of the cycle graph C_m of length m and the empty graph $\overline{K}_n$ of order n. The following are known:

- The stable set ring of C_m is almost Gorenstein ([59, Theorem 4.1]). In particular, it is Gorenstein if and only if m is even, $m = 3$ or 5 ([67, Theorem 2.1(b)] and [37, Theorem 1]).
- The stable set ring of $\overline{K}_n$ is isomorphic to the Segre product of n polynomial rings in 2 variables and is Gorenstein. Note that $e(\Bbbk[\mathrm{Stab}_{\overline{K}_n}]) = 1$ if and only if $n = 1$.

Therefore, the following hold from Theorem 10.2.4:

- $\Bbbk[\mathrm{Stab}_{C_m+\overline{K}_n}]$ is Gorenstein if and only if m is even, $m = 3$ or 5.
- $\Bbbk[\mathrm{Stab}_{C_m+\overline{K}_n}]$ is not Gorenstein but almost Gorenstein if and only if $n = 1$, m is odd and $m \geq 7$.

Chapter 11

Nearly Gorenstein Ehrhart rings

In this chapter, we study nearly Gorensteinness of Ehrhart rings arising from integral polytopes. We give necessary conditions and sufficient conditions on integral polytopes for their Ehrhart rings to be nearly Gorenstein. Using this, we give an efficient method for constructing nearly Gorenstein polytopes. Moreover, we determine the structure of nearly Gorenstein $(0,1)$-polytopes and characterize nearly Gorensteinness of edge polytopes and graphic matroids. The contents of this chapter are contained in the author's paper [27] with T. Hall, M. Kölbl and S. Miyashita,

Throughout this chapter, let $P \subset \mathbb{R}^d$ be an integral polytope and we assume P is full-dimensional (i.e., $\dim P = d$) and has the facet presentation:

$$P = \bigcap_{F \in \Psi(P)} H^+(\mathbf{n}_F; h_F),$$

where each height h_F is an integer and each inner normal vector $\mathbf{n}_F \in \mathbb{Z}^d$ is primitive.

11.1 Necessary conditions

In this section, we will determine a necessary condition for P to be nearly Gorenstein, in terms of the polytope P itself. Before we proceed, let us first introduce some helpful notation. For a subset X of $\mathbb{R}^{d+1}$ and $k \in \mathbb{Z}$, let $X_k = \left\{x \in \mathbb{R}^d \ : \ (x,k) \in X\right\}$ be the k-th *piece* of X. Note the subtlety in our notation: while X is a subset of $\mathbb{R}^{d+1}$, its k-th piece X_k is a subset of $\mathbb{R}^d$. Moreover, for an integral polytope P, we denote its *codegree* by $a_P := \min\{n \in \mathbb{Z}_{>0} : \operatorname{int}(nP) \cap \mathbb{Z}^d \neq \emptyset\}$. When it is clear from context, we simply write a instead of a_P.

In order to describe the canonical module and the anti-canonical module of $A(P)$ in terms of P, we prepare some notation.

Note that

$$\operatorname{int}(C_P) = \left\{(x,k) \in \mathbb{R}^{d+1} \ : \ \langle \mathbf{n}_F, x \rangle + kh_F > 0 \text{ for all } F \in \Psi(P)\right\}.$$

Moreover, we define

$$\operatorname{ant}(C_P) := \left\{(x,k) \in \mathbb{R}^{d+1} \ : \ \langle \mathbf{n}_F, x \rangle + kh_F \geq -1 \text{ for all } F \in \Psi(P)\right\}.$$

Then the following is true.

Proposition 11.1.1 (see [31, Proposition 4.1 and Corollary 4.2])**.** *The canonical module of $A(P)$ and the anti-canonical module of $A(P)$ are given by the following, respectively:*

$$\omega_{A(P)} = \left\langle \mathbf{t}^x s^k : (x,k) \in \operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} \right\rangle \text{ and } \omega_{A(P)}^{-1} = \left\langle \mathbf{t}^x s^k : (x,k) \in \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1} \right\rangle.$$

Further, the negated a-invariant of $A(P)$ coincides with the codegree of P, i.e.

$$a(A(P)) = -\min \left\{ k \in \mathbb{Z}_{\geq 1} \ : \ \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \emptyset \right\}.$$

From this proposition, we can characterize nearly Gorensteinness in terms of polytopes.

Proposition 11.1.2. *Let $P \subset \mathbb{R}^d$ be an integral polytope with codegree a. Then P is nearly Gorenstein if and only if*

$$(C_P \cap \mathbb{Z}^{d+1}) \setminus \{0\} \subseteq \operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} + \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1}. \tag{11.1.1}$$

In particular, if P is nearly Gorenstein, then

$$P \cap \mathbb{Z}^d = \operatorname{int}(C_P)_a \cap \mathbb{Z}^d + \operatorname{ant}(C_P)_{1-a} \cap \mathbb{Z}^d. \tag{11.1.2}$$

The converse also holds if P has IDP.

Proof. By definition, P is nearly Gorenstein if and only if the trace $\operatorname{tr}(\omega_{A(P)})$ of the canonical ideal $\omega_{A(P)}$ of $A(P)$ contains the maximal ideal $\mathfrak{m}$ of $A(P)$. By Proposition 8.0.10, this trace is exactly the product $\omega_{A(P)} \cdot \omega_{A(P)}^{-1}$. Then, Proposition 11.1.1 tells us the monomial generators of $\omega_{A(P)}$ and $\omega_{A(P)}^{-1}$ in terms of the lattice points of $\operatorname{int}(C_P)$ and $\operatorname{ant}(C_P)$. We finally note that the maximal ideal $\mathfrak{m}$ can be generated by the monomials $\mathbf{t}^x s^k$, where (x,k) are lattice points in $C_P \setminus \{0\}$. From this, it is clear to see that P is nearly Gorenstein if and only if (11.1.1) holds.

We next prove that (11.1.2) follows from nearly Gorensteinness of P. First, note that the right hand side of (11.1.1) is contained in $C_P \cap \mathbb{Z}^{d+1}$ by definition. Therefore, when we take the 1-st piece of all three sets, we obtain the equality

$$P \cap \mathbb{Z}^d = (\operatorname{int}(C_P) \cap \mathbb{Z}^{d+1} + \operatorname{ant}(C_P) \cap \mathbb{Z}^{d+1})_1.$$

Note that when P is Gorenstein, $\operatorname{int}(C_P)_a \cap \mathbb{Z}^d$ and $\operatorname{ant}(C_P)_{-a} \cap \mathbb{Z}^d$ are singleton sets; therefore, the result easily follows. Otherwise, we claim that $\operatorname{ant}(C_P)_{1-b} \cap \mathbb{Z}^d$ is empty for all $b \geq a+1$. Since $\operatorname{int}(C_P)_b$ is empty for $b < a$, we obtain the desired result.

Finally, we show that the converse holds when P has IDP. Let $(x,k) \in C_P \cap \mathbb{Z}^d \setminus \{0\}$. Since P has IDP, there are $x_1, \ldots, x_k \in P \cap \mathbb{Z}^d$ such that $(x,k) = (x_1,1) + \cdots + (x_k,1)$. Further, each $x_i \in P \cap \mathbb{Z}^d$ can be written as the sum of lattice points in $\operatorname{int}(C_P)$ and $\operatorname{ant}(C_P)$. Therefore, (11.1.1) holds and so P is nearly Gorenstein. ☐

Definition 11.1.3. Let $P \subset \mathbb{R}^d$ be an integral polytope with codegree a. We define its *floor polytope* and *remainder polytopes* as

$$\lfloor P \rfloor := \operatorname{conv}(\operatorname{int}(P) \cap \mathbb{Z}^d) \qquad \text{and} \qquad \{P\} := \operatorname{conv}(\operatorname{ant}(C_P)_{1-a} \cap \mathbb{Z}^d),$$

respectively. Note that $\lfloor P \rfloor$ coincides with $\operatorname{conv}(\operatorname{int}(C_P)_1 \cap \mathbb{Z}^d)$.

We collate a couple of easy facts about these polytopes and reformulate part of Proposition 11.1.2 into the following statement.

Lemma 11.1.4. *Let $P \subset \mathbb{R}^d$ be an integral polytope with codegree a. Then:*

(1) $\lfloor aP \rfloor \subseteq \{x \in \mathbb{R}^d : \langle \mathbf{n}_F, x\rangle \geq 1 - ah_F \text{ for all } F \in \Psi(P)\}$;

(2) $\{P\} \subseteq \{x \in \mathbb{R}^d : \langle \mathbf{n}_F, x\rangle \geq (a-1)h_F - 1 \text{ for all } F \in \Psi(P)\}$;

(3) If P is nearly Gorenstein, then $P \cap \mathbb{Z}^d = \lfloor aP \rfloor \cap \mathbb{Z}^d + \{P\} \cap \mathbb{Z}^d$;

(4) If P has IDP and $P \cap \mathbb{Z}^d = \lfloor aP \rfloor \cap \mathbb{Z}^d + \{P\} \cap \mathbb{Z}^d$, then P is nearly Gorenstein.

Proof. Statements (1) and (2) follow immediately from the definition of the floor and remainder polytope. To prove statements (3) and (4), notice that the lattice points of $\mathrm{int}(C_P)_a$ coincide with those of $\lfloor aP \rfloor$ and the lattice points of $\mathrm{ant}(C_P)_{1-a}$ coincide with those of $\{P\}$. Then simply substitute this into Proposition 11.1.2. □

The following proposition is a necessary condition for an integral polytope to be nearly Gorenstein:

Proposition 11.1.5. *If P is nearly Gorenstein, then $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P.*

Proof. Let $x \in \lfloor aP \rfloor$ and $y \in \{P\}$. By statements (1) and (2) of Lemma 11.1.4, we have that, for all facets F of P, $n_F(x+y) \geq 1 - ah_F + (a-1)h_F - 1 = -h_F$. So, $x + y \in P$. Therefore, we obtain that $\lfloor aP \rfloor + \{P\} \subseteq P$.

On the other hand, let v be a vertex of P. Since P is an integral polytope, $v \in P \cap \mathbb{Z}^d$. Thus, by statement (3) of Lemma 11.1.4, can write v as the sum of an element of $\lfloor aP \rfloor \cap \mathbb{Z}^d$ and an element of $\{P\} \cap \mathbb{Z}^d$. This implies $P \subseteq \lfloor aP \rfloor + \{P\}$. □

Example 11.1.6. Consider the following integral polytope P:

$$P = \mathrm{conv}(\{(-1,0),(0,-1),(1,-1),(2,0),(2,1),(1,2),(0,2),(-1,1)\}).$$

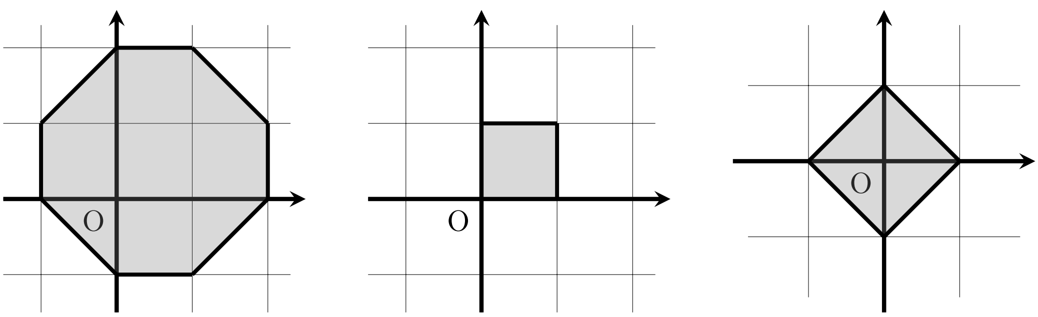

Figure 11.1: The polytope P (left) with its floor polytope $\lfloor P \rfloor$ (middle) and remainder polytope $\{P\}$ (right).

First, we note that $a_P = 1$. Next, we may compute the floor and remainder polytopes:

$$\lfloor P \rfloor = \mathrm{conv}(\{(0,0),(1,0),(1,1),(0,1)\}) \text{ and } \{P\} = \mathrm{conv}(\{(1,0),(0,1),(-1,0),(0,-1)\}).$$

By taking the Minkowski sum of these polytopes, we see that P satisfies the necessary condition to be Gorenstein given by Proposition 11.1.5, i.e. $P = \lfloor P \rfloor + \{P\}$. On the other hand, it is straightforward to verify that every lattice point of P can be written as the sum of a lattice point of $\lfloor P \rfloor$ and a lattice point of $\{P\}$. Since P has IDP (as is true for all polygons), statement (4) of Lemma 11.1.4 informs us that P is nearly Gorenstein.

Finally, we remark that the remainder polytope $\{P\}$ is reflexive. This is not a coincidence, as we will prove in Proposition 11.3.1.

11.2 A sufficient condition

In this section, we will explore sufficient conditions for an integral polytope to be nearly Gorenstein.

We first note that the converse of Proposition 11.1.5 does not hold in general.

Example 11.2.1 (compare [60, Example 1.1])**.** Let $e_1, \ldots, e_6$ be the standard basis of the lattice $\mathbb{Z}^6$ and let $f = -e_1 - \cdots - e_5 + 3e_6$. Consider the integral polytope

$$Q \coloneqq \mathrm{conv}(\{e_1, \ldots, e_5, f, e_1 - e_6, \ldots, e_5 - e_6, f - e_6\})$$

and set $P \coloneqq 2Q$. Since $\lfloor P \rfloor = \{P\} = Q$, it's easy to see that $P = \lfloor P \rfloor + \{P\}$, meeting the necessary condition of Proposition 11.1.5 for nearly Gorensteinness.

On the other hand, Q is not IDP. In particular, $2Q \cap \mathbb{Z}^6 \neq (Q \cap \mathbb{Z}^6) + (Q \cap \mathbb{Z}^6)$. Thus, $P = 2Q$ fails the necessary condition of statement (3) in Lemma 11.1.4, and so P is not nearly Gorenstein.

So, we need to make more assumptions about P in order to be guaranteed nearly Gorensteinness. This brings us to the following result:

Theorem 11.2.2. *Let $P \subset \mathbb{R}^d$ be an integral polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then there exists some integer $K \geq 1$ (depending on P) such that for all $k \geq K$, the polytope kP is nearly Gorenstein.*

In order to prove the above, we rely on a few key ingredients. The first ingredient is an extension of known results from the reflexive case, which appear in [33].

Lemma 11.2.3. *Let $P \subset \mathbb{R}^d$ be an integral polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then the following statements hold:*

(1) $kP = \lfloor (k + a - 1)P \rfloor + \{P\}$, for all $k \geq 1$;

(2) $\lfloor k'P \rfloor = \lfloor aP \rfloor + (k' - a)P$, for all $k' \geq a$.

Before we give the proof, we will restrict these statements to the reflexive case for the sake of comparison. First, we have $a = 1$. Next, since $\lfloor P \rfloor$ is the origin, $P = \{P\}$. So, for reflexive polytopes, the statement (1) is equivalent to $kP = \lfloor kP \rfloor + P$. After cancellation by P, we obtain the reflexive version of statement (2): $\lfloor kP \rfloor = (k - 1)P$.

Proof of Lemma 11.2.3. Let $k \geq 1$ be an integer. Throughout this proof, we repeatedly use the two inequalities appearing in statements (1) and (2) of Lemma 11.1.4. We also use the inequalities appearing in the facet presentations for P and its dilates.

We first prove the "$\supseteq$" part of statement (1), i.e. that

$$kP \supseteq \lfloor (k+a-1)P \rfloor + \{P\}\,, \text{ for all } k \geq 1. \tag{11.2.1}$$

Let $x \in \lfloor (k+a-1)P \rfloor$ and $y \in \{P\}$. Then $n_F(x+y) \geq (1-(k+a-1)h_F)+((a-1)h_F-1) = -kh_F$, for all facets F of P. Thus, $x+y \in kP$.

Next, we note that $kP = (k-1)P + \lfloor aP \rfloor + \{P\}$. We substitute this into (11.2.1), then cancel $\{P\}$ from both sides to obtain $\lfloor (k+a-1)P \rfloor \subseteq (k-1)P + \lfloor aP \rfloor$.

We now prove the reverse inclusion of the above. Let $x \in (k-1)P$ and $y \in \lfloor aP \rfloor$. Then, $n_F(x+y) \geq -(k-1)h_F + (1-ah_F) = 1-(k+a-1)h_F$. Therefore, $x+y \in \lfloor (k+a-1)P \rfloor$. Thus, we obtain the equality $\lfloor (k+a-1)P \rfloor = (k-1)P + \lfloor aP \rfloor$. Setting $k' := k+a-1$ then gives us statement (2). Adding $\{P\}$ to both sides gives us statement (1). □

The main ingredient in proving Theorem 11.2.2 is a result of Haase and Hofmann, which allows us to guarantee that the second condition of statement (4) of Lemma 11.1.4 holds.

Theorem 11.2.4 ([26, Theorem 4.2])**.** *Let $P, Q \subset \mathbb{R}^d$ be integral polytopes such that the normal fan $\mathcal{N}(P)$ of P is a refinement of the normal fan $\mathcal{N}(Q)$ of Q. Suppose also that for each edge E of P, the corresponding face E' of Q has lattice length $\ell_{E'}$ satisfying $\ell_E \geq d\ell_{E'}$. Then $(P+Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d)$.*

In order to guarantee the first condition of statement (4) of Lemma 11.1.4, we need this next result:

Theorem 11.2.5 ([83, Theorem 1.3.3])**.** *Let $P \subset \mathbb{R}^d$ be an integral polytope. Then $(d-1)P$ has IDP.*

We are now ready to give the proof.

Proof of Theorem 11.2.2. We first wish to find a suitable K which satisfies

$$kP \cap \mathbb{Z}^d = \lfloor kP \rfloor \cap \mathbb{Z}^d + \{kP\} \cap \mathbb{Z}^d, \text{ for all } k \geq K.$$

Let a be the codegree of P. Looking at statement (2) of Lemma 11.2.3, we see that $(k-a)P$ is a Minkowski summand of $\lfloor kP \rfloor$; thus, we get a crude lower bound on the length of the edges of $\lfloor kP \rfloor$: for $k \geq a$, every edge E of $\lfloor kP \rfloor$ has lattice length $\ell_E \geq k-a$. Denote by L the maximum edge length of $\{aP\}$ and set $K := dL + a$. Note that for $k \geq a$, the polytopes $\{kP\}$ and $\{aP\}$ coincide. So, for all $k \geq K$, every edge E of $\lfloor kP \rfloor$ will have lattice length $\ell_E \geq k - a \geq dL$.

Further, statement (2) of Lemma 11.2.3 implies that, for $k \geq a+1$, the normal fan $\mathcal{N}(\lfloor kP \rfloor)$ coincides with $\mathcal{N}(P)$. Hence, $\mathcal{N}(\lfloor kP \rfloor)$ is a refinement of the normal fan of $\{kP\}$. Thus, we may apply Theorem 11.2.4, obtaining that $kP \cap \mathbb{Z}^d = \lfloor kP \rfloor \cap \mathbb{Z}^d + \{kP\} \cap \mathbb{Z}^d$.

Finally, since $a, L \geq 1$, we see that $K \geq d-1$. Thus, by Theorem 11.2.5, we have that kP has IDP. Therefore, by statement (4) of Lemma 11.1.4, we can conclude that kP is nearly Gorenstein for all $k \geq K$. □

Remark 11.2.6. We say that a graded ring R is *Gorenstein on the punctured spectrum* [30] if $\mathrm{tr}(\omega_R)$ contains $\mathfrak{m}^k$ for some integer $k \geq 0$. If $k = 0$, this is just the Gorenstein condition; if $k = 1$, it is the nearly Gorenstein condition. Now, for an integral polytope $P \subset \mathbb{R}^d$, it can be shown that its Ehrhart ring $A(P)$ is Gorenstein on the punctured spectrum if there exists a positive integer K such that $kP \cap \mathbb{Z}^d$ coincides with $(\mathrm{int}(C_P) \cap \mathbb{Z}^{d+1} + \mathrm{ant}(C_P) \cap \mathbb{Z}^{d+1})_k$, for all $k \geq K$. Therefore, using Theorem 11.2.2, it's straightforward to show that all integral polytopes P satisfying $P = \lfloor aP \rfloor + \{P\}$ are Gorenstein on the punctured spectrum.

11.3 Decompositions of nearly Gorenstein polytopes

In this section, we discuss decompositions of nearly Gorenstein polytopes. We consider whether nearly Gorenstein polytopes decompose into the Minkowski sum of Gorenstein polytopes (Questions 11.3.3 and 11.3.4). We give a way to systematically construct examples of nearly Gorenstein polytopes. This is then used to find a counterexample to Questions 11.3.3 and 11.3.4. Finally, we conclude the section with a result about indecomposable nearly Gorenstein polytopes.

Theorem 11.3.1. *Let $P \subset \mathbb{R}^d$ be an integral polytope which satisfies $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then we have*

$$\lfloor aP \rfloor = \left\{ x \in \mathbb{R}^d \ : \ \langle \mathbf{n}_F, x \rangle \geq 1 - ah_F \text{ for all } F \in \Psi(P) \right\} \text{ and}$$

$$\{P\} = \left\{ x \in \mathbb{R}^d \ : \ \langle \mathbf{n}_F, x \rangle \geq (a-1)h_F - 1 \text{ for all } F \in \Psi(P) \right\}.$$

In particular, the right hand sides of the equalities are integral polytopes. Furthermore, if $a = 1$, then $\{P\}$ is a reflexive polytope.

Proof. Label the two polytopes on the right-hand sides as Q_1 and Q_2, respectively. It's straightforward to see that $\lfloor aP \rfloor = \mathrm{conv}(Q_1 \cap \mathbb{Z}^d)$ and $\{P\} = \mathrm{conv}(Q_2 \cap \mathbb{Z}^d)$. Thus, $\lfloor aP \rfloor \subseteq Q_1$ and $\{P\} \subseteq Q_2$. Ultimately, we want to prove the reverse inclusions but first, we must show an intermediate equality: $P = Q_1 + Q_2$. Let $x \in Q_1$ and $y \in Q_2$. Then, for all facets F of P, we have $n_F(x + y) \geq 1 - ah_F + (a-1)h_F - 1 = -h_F$. Thus, $x + y \in P$ and so, $Q_1 + Q_2 \subseteq P$. Conversely, if we combine this with our assumption that $P = \lfloor aP \rfloor + \{P\}$, we obtain that, in fact, $P = Q_1 + Q_2$.

We now use the above equality to obtain that $\lfloor aP \rfloor = Q_1$ and $\{P\} = Q_2$, as follows. Assume towards a contradiction that $Q_1 \not\subseteq \lfloor aP \rfloor$, i.e. there exists a vertex v of Q_1 which doesn't belong to $\lfloor aP \rfloor$. Choose a normal vector $n \in (\mathbb{R}^d)^*$ which achieves its minimal value h_1 over Q_1 *only* at v (i.e. n lies in the interior of the cone σ_v in the (inner) normal fan $\mathcal{N}(Q_1)$ which corresponds to v). Denote by h_2 the minimal evaluation of n over Q_2 Then, the minimal evaluation of n over P is $h_1 + h_2$. However, for all $x \in \lfloor aP \rfloor$ and $y \in \{P\}$, we have that $n(x+y) > h_1 + h_2$. This contradicts the fact that $P = \lfloor aP \rfloor + \{P\}$. Therefore, the vertices of Q_1 coincide with the vertices of $\lfloor aP \rfloor$; in particular, $\lfloor aP \rfloor = Q_1$. We similarly obtain that $\{P\} = Q_2$.

Next, since $\lfloor aP \rfloor$ and $\{P\}$ are integral polytopes by definition, we note that Q_1 and Q_2 are integral polytopes in this situation.

Finally, suppose we are in the case when P has an interior lattice point, i.e. $a = 1$. By substituting this into the second equality, we see that the remainder polytope $\{P\}$ is indeed reflexive as all its facets lie at height 1.

□

In contrast, when P has no interior points, the remainder polytope $\{P\}$ is not necessarily even Gorenstein.

Example 11.3.2. Consider the polytope

$$P = \operatorname{conv}(\{(0,0,0),(2,0,0),(1,1,0),(0,1,0),(0,0,1),(2,0,1),(1,1,1),(0,1,1)\}).$$

We can verify that P is nearly Gorenstein and IDP, but the remainder polytope $\{P\}$ is not Gorenstein. However, $\{P\}$ can be written as the Minkowski sum of

$$\operatorname{conv}(\{(0,0,0),(1,0,0),(0,1,0)\}) \quad \text{and} \quad \operatorname{conv}(\{(-1,-1,-1),(-1,-1,0)\}),$$

which are both Gorenstein.

We see similar behavior when studying the nearly Gorensteinness for certain restricted classes of polytopes. This motivated us to pose the following question.

Question 11.3.3. *If P is nearly Gorenstein, then can we write $P = P_1 + \cdots + P_s$ for some Gorenstein integral polytopes $P_1, \ldots, P_s$?*

We recall that P is (*Minkowski*) *indecomposable* if P is not a singleton and if there exist integral polytopes P_1 and P_2 with $P = P_1 + P_2$, then either P_1 or P_2 is a singleton. Note that if P is not a singleton, then we can write $P = P_1 + \cdots + P_s$ for some indecomposable integral polytopes $P_1, \ldots, P_s$.

Then, Question 11.3.3 can be rephrased as:

Question 11.3.4. *If P has an indecomposable non-Gorenstein integral polytope as a Minkowski summand, then is P not nearly Gorenstein?*

This question has a positive answer for IDP $(0,1)$-polytopes, which is shown in Section 11.4. For the remainder of this section, we will build up some machinery which allows for the efficient construction of nearly Gorenstein polytopes. We then use this in Example 11.3.7 to give an answer to Questions 11.3.3 and 11.3.4.

Theorem 11.3.5. *Let $P \subset \mathbb{R}^d$ be a nearly Gorenstein polytope. Then there exists a reflexive polytope $Q \subset \mathbb{R}^d$ such that*

$$P = \left\{ x \in \mathbb{R}^d \ : \ \langle n, x \rangle \geq -h_n \text{ for all } n \in \partial Q^* \cap \mathbb{Z}^d \right\},$$

where h_n are integers and ∂Q^* denotes the boundary of Q^*. Moreover, the inequalities defined by $n \in \operatorname{vert}(Q^*)$ are irredundant, where $\operatorname{vert}(Q^*)$ denotes the set of vertices of Q^*. Furthermore, the number of facets of a nearly Gorenstein polytope is bounded by a constant depending on the dimension d.

Before we dive into the proof, it will be useful to have the following lemma.

Lemma 11.3.6. *Let P be an integral polytope satisfying $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Then $aP = \lfloor aP \rfloor + \{aP\}$. Moreover, $\{aP\} = (a-1)P + \{P\}$.*

Proof. We first wish to show that $(a-1)P + \{P\} \subseteq \{aP\}$. Let $x \in (a-1)P$ and $y \in \{P\}$. Then, by Lemma 11.1.4 (2), $n_F(x+y) \geq -(a-1)h_F + (a-1)h_F - 1 = -1$, for all facets F of P. So, $x + y \in \{aP\}$. Thus, $(a-1)P + \{P\} \subseteq \{aP\}$.

We can add $\lfloor aP \rfloor$ to both sides of the inclusion to get $aP \subseteq \lfloor aP \rfloor + \{aP\}$.

We next wish to show the reverse inclusion of the above. Let $z \in \lfloor aP \rfloor$ and $w \in \{aP\}$. Then $n_F(z+w) \geq (1 - ah_F) - 1 = -ah_F$, for all facets F of P. So, $z + w \in aP$. Therefore, $\lfloor aP \rfloor + \{aP\} \subseteq aP$. Combining the two inclusions gives the desired equality: $aP = \lfloor aP \rfloor + \{aP\}$.

Moreover, we obtain that $\lfloor aP \rfloor + \{P\} + (a-1)P = \lfloor aP \rfloor + \{aP\}$. Since Minkowski addition of convex sets satisfies the cancellation law, we may cancel both sides by $\lfloor aP \rfloor$ to obtain the equality $\{aP\} = (a-1)P + \{P\}$. □

Proof of Theorem 11.3.5. We wish to study the (inner) normal fan $\mathcal{N}(P)$ of P, as it's enough to show that its primitive ray generators all lie in $\partial Q^* \cap \mathbb{Z}^d$, for some reflexive polytope $Q \subset \mathbb{R}^d$. Let a be the codegree of P. Since dilation has no effect on the normal fan, we may pass to the normal fan of aP. Now, by Lemma 11.3.6, aP has a Minkowski decomposition into $\lfloor aP \rfloor$ and $\{aP\}$. Thus, $\mathcal{N}(aP)$ is the common refinement of $\mathcal{N}(\lfloor aP \rfloor)$ and $\mathcal{N}(\{aP\})$. By Proposition 11.3.1, we obtain that $Q \coloneqq \{aP\}$ is a reflexive polytope. Hence, the primitive ray generators of $\mathcal{N}(Q)$ are vertices of the reflexive polytope $Q^* \subset \mathbb{R}^d$; in particular, they are lattice points lying in the boundary of Q^*.

We next look at the contribution to $\mathcal{N}(aP)$ coming from $\lfloor aP \rfloor$. Let $n \in \mathbb{Z}^d$ be a primitive ray generator of $\mathcal{N}(\lfloor aP \rfloor)$. Then, by definition of the remainder polytope, $\langle n, x \rangle \geq -1$, for all $x \in Q$. But now, this means that n lies in Q^*. So, since $n \neq 0$ and Q is reflexive, we obtain that $n \in \partial Q^* \cap \mathbb{Z}^d$. Therefore, we have now shown that the primitive ray generators of $\mathcal{N}(P) = \mathcal{N}(aP)$ contain the vertices of Q^*, and that they all lie in $\partial Q^* \cap \mathbb{Z}^d$.

Finally, we note that the number of facets of a nearly Gorenstein polytope $P \subset \mathbb{R}^d$ is bounded by $c_d \coloneqq \sup_Q |\partial Q^* \cap \mathbb{Z}^d|$, where Q runs over all d-dimensional reflexive polytopes. Since there are only finitely reflexive polytopes in each dimension d, and all polytopes only have a finite number of boundary points, we see that c_d is a finite number. □

We will now detail how to construct nearly Gorenstein polytopes. First, choose a reflexive polytope $Q \subset \mathbb{R}^d$. Then, choose a (possibly empty) subset S' of the boundary lattice points of Q^* which are not vertices of Q^*. Now, for each $n \in S \coloneqq S' \cup \operatorname{vert}(Q^*)$, choose the height $h_n \in \mathbb{Z}$. Construct a polytope P' defined by $\langle n, x \rangle \geq -h_n$ for all $n \in S$, and assert that none of these inequalities are redundant. Next, we can dilate P' to rP' so that it's an integral polytope which contains an interior lattice point. By construction, its remainder polytope $\{rP'\}$ coincides with the reflexive polytope Q. In practice, rP' has a Minkowski decomposition into $\lfloor rP' \rfloor$ and $\{rP'\}$, but we don't yet have a proof that this always holds. Finally, we can use Theorem 11.2.2 to dilate rP' even further to $P \coloneqq krP'$ so that $P = \lfloor P \rfloor + \{P\}$ is nearly Gorenstein.

Example 11.3.7. Consider the polytope

$$P = \operatorname{conv}(\{(-4,-3,-4), (-3,-1,-3), (-2,-2,-3), (0,1,4), (0,4,1), (3,1,1)\}).$$

Note that P has many interior lattice points, it has codegree 1. Its floor polytope is

$$\lfloor P \rfloor = \operatorname{conv}(\{(-3,-2,-3),(0,3,1),(0,1,3),(2,1,1)\}).$$

This is an indecomposable simplex, which is not Gorenstein. Its remainder polytope is

$$\{P\} = \operatorname{conv}(\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\}),$$

which is clearly reflexive. We have $P = \lfloor P \rfloor + \{P\}$. We use `MAGMA` to verify that $P \cap \mathbb{Z}^3 = (\lfloor P \rfloor \cap \mathbb{Z}^3) + (\{P\} \cap \mathbb{Z}^3)$ and that P has IDP. Thus, we may conclude by Lemma 11.1.4 that P is a nearly Gorenstein polytope.

It can be shown that $P = \lfloor P \rfloor + \{P\}$ is the only non-trivial Minkowski decomposition of P. Thus, we may conclude that the nearly Gorenstein polytope P cannot be decomposed into Gorenstein polytopes. Therefore, we may answer Questions 11.3.3 and 11.3.4 in the negative.

We end this section by giving the following theorem about nearly Gorensteinness of indecomposable polytopes, which plays an important role in the characterisation of nearly Gorenstein $(0,1)$-polytopes in Section 11.4.

Theorem 11.3.8. *Let P be an indecomposable integral polytope. Then, P is nearly Gorenstein if and only if P is Gorenstein.*

Proof. It is already clear that Gorensteinness implies nearly Gorensteinness, so we just have to treat the converse implication. Suppose that P is nearly Gorenstein. By Proposition 11.1.5, we have that $P = \lfloor aP \rfloor + \{P\}$, where a is the codegree of P. Since P is indecomposable, either (i) $\lfloor aP \rfloor$ is a singleton or (ii) $\{P\}$ is a singleton.

We first deal with case (i). Consider aP. By Lemma 11.3.6, $aP = \lfloor aP \rfloor + \{aP\}$. Thus, aP is a translation of $\{aP\}$. By Proposition 11.3.1, $\{aP\}$ is reflexive. Thus, P is Gorenstein.

The argument for case (ii) is similar. We consider $\{aP\}$. By Lemma 11.3.6, $\{aP\} = (a-1)P + \{P\}$. Proposition 11.3.1 tells us that $\{aP\}$ is reflexive; therefore, $(a-1)P$ is a translation of a reflexive polytope. But this is an absurdity as it implies that $(a-1)P$ has an interior lattice point, contradicting that the codegree of P is a. Thus, this case cannot occur.

□

11.4 Nearly Gorenstein $(0,1)$-polytopes

In this section, we consider the case of $(0,1)$-polytopes. We provide the characterisation of nearly Gorenstein $(0,1)$-polytopes which have IDP. Moreover, we also characterise nearly Gorenstein edge polytopes of graphs satisfying the odd cycle condition and characterise nearly Gorenstein graphic matroid polytopes.

11.4.1 The characterisation of nearly Gorenstein $(0,1)$-polytopes

Lemma 11.4.1. *Let $P \subset \mathbb{R}^d$ be a $(0,1)$-polytope. Then, after a change of coordinates, we can write $P = P_1 \times \cdots \times P_s$ for some indecomposable $(0,1)$-polytopes $P_1, \dots, P_s$.*

Proof. As mentioned in the previous section, we can write $P = P_1' + \cdots + P_s'$ for some indecomposable integral polytopes $P_1', \dots, P_s'$.

First, we show that we can choose $P_1', \dots, P_s'$ so that these are $(0,1)$-polytopes. Suppose that we can write $P = P_1' + P_2'$ for some integral polytopes P_1' and P_2'. Then, for any $v \in P_1' \cap \mathbb{Z}^d$ and for any $u \in P_2' \cap \mathbb{Z}^d$, $v + u$ is a $(0,1)$-vector. Therefore, for any $i \in [d]$, $\pi_i(P_1' \cap \mathbb{Z}^d)$ can take one of the following forms: (i) $\{w_i\}$ or (ii) $\{w_i, w_i + 1\}$ for some $w_i \in \mathbb{Z}$. In case (i), $\pi_i(P_2' \cap \mathbb{Z}^d)$ is equal to $\{-w_i\}$, $\{-w_i + 1\}$ or $\{-w_i, -w_i + 1\}$. In case (ii), $\pi_i(P_2' \cap \mathbb{Z}^d)$ is equal to $\{-w_i\}$. Thus, in all cases, $P_1' - w$ and $P_2' + w$ are $(0,1)$-polytopes and we have $P = (P_1' - w) + (P_2' + w)$, where $w = (w_1, \dots, w_d)$.

Moreover, if we can write $P = P_1' + P_2'$ for some $(0,1)$-polytopes P_1' and P_2', then we can see that either $\pi_i(P_1')$ or $\pi_i(P_2')$ is equal to $\{0\}$ for any $i \in [d]$. Therefore, after a change of coordinates, we can write $P = P_1 \times P_2$ for some $(0,1)$-polytopes P_1 and P_2. □

Now, we provide the main theorem of this section.

Theorem 11.4.2. *Let P be an IDP $(0,1)$-polytope. Then, P is nearly Gorenstein if and only if you can write $P = P_1 \times \cdots \times P_s$ for some Gorenstein $(0,1)$-polytopes $P_1, \dots, P_s$ with $|a_{P_i} - a_{P_j}| \leq 1$, where a_{P_i} and a_{P_j} are the respective codegrees of P_i and P_j, for $1 \leq i < j \leq s$.*

Proof. It follows from Lemma 11.4.1 that we can write $P = P_1 \times \cdots \times P_s$ for some indecomposable $(0,1)$-polytopes $P_1, \dots, P_s$. Thus, we have $\Bbbk[P] \cong \Bbbk[P_1] \# \cdots \# \Bbbk[P_s]$. Note that if P has IDP, then so is P_i for each $i \in [s]$, and $A(P)$ (resp. $A(P_i)$) coincides with $\Bbbk[P]$ (resp. $\Bbbk[P_i]$). Therefore, since P is nearly Gorenstein, $\Bbbk[P]$ is nearly Gorenstein, and hence $\Bbbk[P_i]$ is also nearly Gorenstein from Lemma 8.0.12 (1). Furthermore, P_i is nearly Gorenstein. Since P_i is indecomposable, P_i is Gorenstein by Theorem 11.3.8. Moreover, it follows from [31, Corollary 2.8] that $|a_{P_i} - a_{P_j}| \leq 1$ for $1 \leq i < j \leq s$.

The converse also holds from [31, Corollary 2.8]. □

From this theorem, we immediately obtain the following corollaries:

Corollary 11.4.3. *Question 11.3.3 is true for IDP $(0,1)$-polytopes.*

Corollary 11.4.4. *Let P be an IDP $(0,1)$-polytope. If $\Bbbk[P]$ is nearly Gorenstein, then $\Bbbk[P]$ is level.*

Proof. It follows immediately from Lemma 8.0.12 (2) and Theorem 11.4.2. □

The result of Theorem 11.4.2 can be applied to many classes of $(0,1)$-polytopes such as order polytopes and stable set polytopes. Nearly Gorensteinness of these polytopes has been studied in [30] and [36, 58], respectively. Theorem 11.4.2 enables us to recover [30, Theorem 5.4] and [36, Theorem B].

Furthermore, by using this theorem, we can study the nearly Gorensteinness of other classes of $(0,1)$-polytopes.

11.4.2 Nearly Gorenstein edge polytopes

We state the characterization of nearly Gorenstein edge polytopes.

Corollary 11.4.5. *Let G be a connected simple graph satisfying the odd cycle condition. Then, the edge polytope P_G of G is nearly Gorenstein if and only if P_G is Gorenstein or G is the complete bipartite graph $K_{n,n+1}$ for some $n \geq 2$.*

Proof. If P_G is nearly Gorenstein, then Theorem 11.4.2 allows us to write $P_G = P_1 \times \cdots \times P_s$ for some indecomposable Gorenstein $(0,1)$-polytopes $P_1, \ldots, P_s$. Then, we have $s \leq 2$ since $P_G \subset \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 + \cdots + x_d = 2\}$, where $d = |V(G)|$. In the case $s = 1$, P_G is Gorenstein. If $s = 2$, we can see that $P_1 = \text{conv}(\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}) \subset \mathbb{R}^n$ and $P_2 = \text{conv}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{d-n}\}) \subset \mathbb{R}^{d-n}$ for some $1 < n < d-1$. Therefore, we have $G = K_{n,d-n}$, and it is shown by [36, Proposition 1.5] that for any $1 < n < d-1$, $P_{K_{n,d-n}}$ is nearly Gorenstein if and only if $d - n \in \{n, n+1\}$. Since $P_{K_{n,n}}$ is Gorenstein, we get the desired result. □

Actually, Gorenstein edge polytopes have been investigated in [67].

11.4.3 Nearly Gorenstein graphic matroid polytopes

We start by giving one of several equivalent definitions of a matroid.

Definition 11.4.6. Let E be a finite set and let $\mathcal{B}$ be a subset of the power set of E satisfying the following properties:

1. $\mathcal{B} \neq \emptyset$.
2. If $A, B \in \mathcal{B}$ with $A \neq B$ and $a \in A \setminus B$, then there exists some $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Then the tuple $M = (E, \mathcal{B})$ is called a *matroid* with *ground set* E and *set of bases* $\mathcal{B}$.

Let now $G = (V, E)$ be a multigraph. The *graphic matroid* associated to G is the matroid M_G whose ground set is the set of edges E and whose bases are precisely the subsets of E which induce a spanning tree of G. Given two matroids $M_E = (E, \mathcal{B}_E)$ and $M_F = (F, \mathcal{B}_F)$, their *direct sum* $M_E \oplus M_F$ is the matroid with ground set $E \sqcup F$ such that for each basis B of $M_E \oplus M_F$, there exist bases $B_E \in \mathcal{B}_E$ and $B_F \in \mathcal{B}_F$ with $B = B_E \sqcup B_F$. If such a decomposition is not possible for a matroid M, we call it *irreducible*.

A graphic matroid with underlying multigraph G is irreducible if and only if its underlying graph is 2-connected. If it is not irreducible, its irreducible components correspond precisely to the 2-connected components of G.

For any matroid $M = (E, \mathcal{B})$, we can define its *matroid base polytope* (or simply *base polytope*) by

$$B_M = \text{conv}\left(\left\{\sum_{b \in B} e_b \colon B \in \mathcal{B}\right\}\right) \subset \mathbb{R}^{|E|}$$

where e_b is the unit vector in $\mathbb{R}^{|E|}$ corresponding to $b \in E$. If B_M comes from a graphic matroid M_G, we will call it B_G.

An alternative definition of matroid base polytopes is as follows.

Definition 11.4.7 ([20, Section 4]). A $(0,1)$-polytope $P \subset \mathbb{R}^d$ is called *(matroid) base polytope* if there is a positive integer h such that every vertex $v = (v_1, \dots, v_n)$ satisfies $\sum_{i=1}^d v_i = h$ and every edge (i.e. dimension 1 face) of P is a translation of a vector $e_i - e_j$ with $i \neq j$.

It is shown in [20, Theorem 4.1] that this definition is indeed equivalent to that of a base polytope as given above and that the underlying matroid is uniquely determined. This gives us the following two lemmas.

Lemma 11.4.8. *Let G be a multigraph and let $G_1, \dots G_n$ be its 2-connected components. Then B_G can be written as a direct product of the base polytopes $B_{G_1}, \dots, B_{G_n}$. Conversely, if B_G can be written as a direct product of polytopes $P_1, \dots, P_n$, where no P_i is itself a direct product, then these polytopes correspond to the base polytopes of the 2-connected components $G_1, \dots, G_n$ of G.*

Proof. The first statement is trivially satisfied.

The converse follows from two key insights. Firstly, the fact that if a base polytope B_M associated to a (not necessarily graphic) matroid M can be written as a direct product $P_1 \times P_2$, then P_1 and P_2 are again base polytopes. Secondly, if a graphic matroid M_G can be written as a direct sum $M_1 \oplus M_2$, then M_1 and M_2 are again graphic matroids corresponding to subgraphs of G which have at most one vertex in common.

The first insight follows from the alternative definition of a base polytope: Every edge of B_M is given by an edge in P_1 and a vertex of P_2, or vice versa. Hence, P_1 and P_2 must satisfy the definition as well, making them base polytopes with unique underlying matroids M_1 and M_2. The second insight is a classical result and can be found, among other places, in [80, Lemma 8.2.2]. □

The following proposition is the polytopal version of a classical result due to White.

Lemma 11.4.9 ([85, Theorem 1]). *Matroid base polytopes have IDP.*

We can now define Gorensteinness, nearly Gorensteinness, and levelness of a matroid by identifying it with its base polytope. In [34] and [48], a constructive, graph-theoretic criterion of Gorensteinness for graphic matroids was found. Since the direct product of two Gorenstein polytopes that have the same codegree is again Gorenstein, the characterisation is presented in terms of 2-connected graphs.

Proposition 11.4.10 ([48, Theorems 2.22 and 2.25]). *Let G be a 2-connected multigraph. Then the following are equivalent.*

1. *B_G is Gorenstein with codegree $a = 2$*
2. *Either G is the 2-cycle or G can be obtained from copies of the clique K_4 and Construction 2.15 in [48].*

The following are also equivalent.

1. *B_G is Gorenstein with codegree $a > 2$*
2. *G can be obtained from copies of the cycle C_a and Constructions 2.15, 2.17, 2.18 in [48] with $\delta = a$.*

The full characterisation of nearly Gorenstein graphic matroids is thus an immediate corollary of Theorem 11.4.2 and Proposition 11.4.10.

Corollary 11.4.11. *Let G be a multigraph with 2-connected components $G_1, \ldots, G_n$, then the following are equivalent.*

1. *B_G is nearly Gorenstein*
2. *$B_{G_1}, \ldots, B_{G_n}$ are Gorenstein with codegrees $a_1, \ldots, a_n$, where $|a_i - a_j| \leq 1$ for $1 \leq i < j \leq s$.*

Part IV

Conic divisorial ideals and non-commutative crepant resolutions of toric rings

Chapter 12

Descriptions of conic divisorial ideals

In this chapter, we give ways to describe the conic divisorial ideals of toric rings and determine the conic divisorial ideals of several classes of toric rings.

12.1 Preliminaries on conic divisorial ideals

First, we recall the definition of conic divisorial ideals of toric rings and review some basic facts about them.

Let $\tau^\vee \subset \mathbb{R}^d$ be a full-dimensional rational polyhedral cone defined by half-open spaces $H_i^+ \subset \mathbb{R}^d$ for $i = 1, \dots, n$, where $H_i^+ = \{\mathbf{x} \in \mathbb{R}^d : \langle \sigma_i, \mathbf{x} \rangle \geq 0\}$ for some $\sigma_i \in \mathbb{R}^d$. We set $\sigma(-) : \mathbb{R}^d \to \mathbb{R}^n$ by $\sigma(\mathbf{x}) := (\langle \sigma_1, \mathbf{x} \rangle, \dots, \langle \sigma_n, \mathbf{x} \rangle)$. Let R be the toric ring of $\tau^\vee$ with respect to $\mathbb{Z}^d$, that is,

$$R := \Bbbk[\tau^\vee \cap \mathbb{Z}^d] = \Bbbk[\mathbf{t}^\alpha : \alpha \in \mathbb{Z}^d \text{ and } \sigma(\alpha) \geq 0],$$

where $\geq$ stands for the component-wise inequality.

As mentioned in Section 2.1, there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^d \xrightarrow{\sigma(-)} \mathbb{Z}^n \xrightarrow{\pi} \mathrm{Cl}(R) \longrightarrow 0. \tag{12.1.1}$$

Definition 12.1.1 (see e.g., [9, Section 3]). A divisorial ideal $T(\mathbf{a})$ is said to be *conic* if there is $\mathbf{x} \in \mathbb{R}^d$ with $\mathbf{a} = \lceil \sigma(\mathbf{x}) \rceil$. In other words, there is $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{a} - \mathbf{1} < \sigma(\mathbf{x}) \leq \mathbf{a}$, where $\mathbf{1} = (1, 1, \dots, 1)$.

Note that a conic divisorial ideal is determined by the elements in $\mathbb{R}^d/\mathbb{Z}^d$ up to isomorphism since we see that $T(\sigma(\mathbf{x}')) \cong T(\sigma(\mathbf{x}))$ for $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$ with $\mathbf{x}' = \mathbf{x} + \mathbf{y}$ and $\mathbf{y} \in \mathbb{Z}^d$.

Let $\mathfrak{p}_i = T(\mathbf{e}_i)$, where $\mathbf{e}_i \in \mathbb{Z}^n$ denotes the i-th unit vector, and let us consider the prime divisor $\mathcal{D}_i := \mathrm{Spec}(R/\mathfrak{p}_i)$ on $\mathrm{Spec}\, R$. Then we see that the divisorial ideal $T(\mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_n)$ corresponds to the Weil divisor $-(a_1\mathcal{D}_1 + \cdots + a_n\mathcal{D}_n)$. By the fact that $\mathrm{Cl}(R) \cong \mathbb{Z}^n/\sigma(\mathbb{Z}^d)$, we obtain that

$$\sigma_1(\mathbf{e}_j)\mathcal{D}_1 + \cdots + \sigma_n(\mathbf{e}_j)\mathcal{D}_n = v_1^{(j)}\mathcal{D}_1 + \cdots + v_n^{(j)}\mathcal{D}_n = 0 \tag{12.1.2}$$

in $\mathrm{Cl}(R)$ for all $j \in [d]$, where for a vector $v \in \mathbb{R}^d$, $v^{(j)}$ denotes the j-th coordinate of v. Moreover, by using the exact sequence (12.1.1), we see that

$$\begin{aligned} \mathbb{Z}^d &\cong \left\{ (b_1, \dots, b_n) \in \mathbb{Z}^n : \sum_{i=1}^n b_i \mathcal{D}_i = 0 \text{ in } \mathrm{Cl}(R) \right\}, \text{ and} \\ \mathbb{R}^d &\cong \left\{ (b_1, \dots, b_n) \in \mathbb{R}^n : \sum_{i=1}^n b_i \mathcal{D}_i = 0 \text{ in } \mathrm{Cl}(R) \otimes_{\mathbb{Z}} \mathbb{R} \right\}. \end{aligned} \tag{12.1.3}$$

Remark that $\sum_{i=1}^n b_i \mathcal{D}_i = 0$ holds if and only if $(b_1, \dots, b_n) = \sigma(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{Z}^d$.

By using those descriptions, we can characterize what kinds of elements in $\mathrm{Cl}(R)$ correspond to conic divisorial ideals as follows (see [43, Subsection 2.1]):

Lemma 12.1.2 (see [7, Corollary 1.2] and [68, Proposition 3.2.3])**.** *There exists a one-to-one correspondence among the following objects:*

(1) *a conic divisorial ideal $T(a_1, \dots, a_n)$;*

(2) *an $\mathbb{R}$-divisor $\sum_{i=1}^n \delta_i \mathcal{D}_i$ with $(\delta_1, \dots, \delta_n) \in (-1, 0]^n$ up to equivalence, where we say that two $\mathbb{R}$-divisors are equivalent if their difference is in (12.1.3);*

(3) *a full-dimensional cell of the decomposition of the semi-open cube $(-1, 0]^d$ by hyperplanes $H_{i,q} := H(\sigma_i, -q) = \{\mathbf{x} \in \mathbb{R}^d : \sigma_i(\mathbf{x}) = q\}$ for some $q \in \mathbb{Z}$ and $i = 1, \dots, n$.*

We identify the cell $\bigcap_{i=1}^n L_{i,a_i}$ with $T(a_1, \dots, a_n)$, where

$$L_{i,a_i} = \{\mathbf{x} \in \mathbb{R}^d : a_i - 1 < \sigma_i(\mathbf{x}) \le a_i\}.$$

Next, we discuss the representation of toric rings as the rings of invariants. In what follows, we assume that $\mathrm{Cl}(R) \cong \mathbb{Z}^r$. Then, we can rewrite R as the ring of invariants under the action of $G = \mathrm{Hom}(\mathrm{Cl}(R), \Bbbk^\times) \cong (\Bbbk^\times)^r$ on $S = \Bbbk[x_1, \dots, x_n]$. Let $X(G)$ be the character group of G. We can see that $X(G) \cong \mathrm{Cl}(R)$, and hence we can use the same symbol for both of a character and the corresponding weight. When we consider the prime divisor $\mathcal{D}_i$ on $\operatorname{Spec} R$ as the element in $X(G) \cong \mathbb{Z}^r$ via the surjection π in (12.1.1), we denote it by β_i. Note that if we can write $R = \Bbbk[P]$ for an appropriate integral polytope P, then β_i's coincide with the weights of $\Bbbk[P]$ defined in Section 4.1. For a character $\chi \in X(G)$, we also denote by V_χ the irreducible representation corresponding to χ, and we let $W = \bigoplus_{i=1}^n V_{\beta_i}$. Then, the symmetric algebra $\operatorname{Sym} W$ of the G-representation W is isomorphic to S, and the algebraic torus G acts on S, which is the action induced by $g \cdot x_i = \beta_i(g) x_i$ for $g \in G$. This action gives the $\mathrm{Cl}(R)$-grading on S, and the degree zero part coincides with the G-invariant components. In particular, we have $R = S^G$ (see e.g., [8, Theorem 2.1]). In addition, for a character χ, we define $M_\chi = (S \otimes_\Bbbk V_\chi)^G$. This is an R-module called the *module of covariants* associated to V_χ and is generated by $f \in S$ with $g \cdot f = \chi(g) f$ for any $g \in G$. Note that for $\chi = \sum_i a_i \beta_i \in X(G)$, we have $T(a_1, \dots, a_n) = M_{-\chi}$.

We introduce notions of quasi-symmetric and weakly-symmetric:

Definition 12.1.3 ([73, Definition 2.2])**.** A G-representation W is *quasi-symmetric* if for every line $l \subset X(G)_{\mathbb{R}} = X(G) \otimes_{\mathbb{Z}} \mathbb{R}$ passing through the origin, we have $\sum_{\beta_i \in l} \beta_i = 0$.

It is *weakly-symmetric* if for every line l, the cone spanned by $\beta_i \in l$ is either zero or l. We say that a toric ring R is quasi-symmetric (resp. weakly-symmetric) if $R \cong S^G$ with $S = \operatorname{Sym} W$ and W is a quasi-symmetric (resp. weakly-symmetric) representation.

Note that quasi-symmetric representations are weakly-symmetric. If W is quasi-symmetric, then the top exterior power of W is the trivial representation, and hence $R = S^G$ is Gorenstein.

Now, we discuss the description of conic divisorial ideals of toric rings by using weights $\beta_1, \dots, \beta_n$. By the arguments in [70, Section 10.6], we can get the following proposition.

Proposition 12.1.4. *Let $\chi \in X(G)$. Then, $M_{-\chi}$ is conic if and only if one can write $-\chi = \sum_i a_i\beta_i$ with $a_i \in [0, 1)$ for all $i \in [n]$.*

Let $\bar{\beta}_1, \dots, \bar{\beta}_{n'}$ be weights of R such that n' is the minimal number with $\{\bar{\beta}_1, \dots, \bar{\beta}_{n'}\} = \{\beta_1, \dots, \beta_n\}$, and let m_i be the multiplicity of $\bar{\beta}_i$ for $i \in [n']$, that is, $m_i = |\{j \in [n] : \beta_j = \bar{\beta}_i\}|$. By Proposition 12.1.4, each element of $\mathcal{W}(R) \cap \mathbb{Z}^r$ one-to-one corresponds to a conic divisorial ideal of R, where

$$\mathcal{W}(R) = \Big\{ \sum_{i=1}^{n} a_i\beta_i \in \mathbb{R}^r : a_i \in [0, 1) \Big\} = \Big\{ \sum_{i=1}^{n'} \bar{a}_i\bar{\beta}_i \in \mathbb{R}^r : \bar{a}_i \in [0, m_i) \Big\}.$$

On the other hand, we define

$$\mathcal{W}'(R) = \Big\{ \sum_{i=1}^{n} a_i\beta_i \in \mathbb{R}^r : a_i \in [0, 1] \Big\} = \Big\{ \sum_{i=1}^{n'} \bar{a}_i\bar{\beta}_i \in \mathbb{R}^r : \bar{a}_i \in [0, m_i] \Big\}.$$

Note that $\mathcal{W}'(R)$ is an integral polytope since it is the Minkowski sum of lattice segments $\{\bar{a}_i\bar{\beta}_i : \bar{a}_i \in [0, m_i]\}$.

According to oriented matroid theory, we can determine the faces of $\mathcal{W}'(R)$ as follows. We define the sign function $\operatorname{sign} : \mathbb{R} \to \{+, -, 0\}$ by setting

$$\operatorname{sign}(x) = \begin{cases} + & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ - & \text{if } x < 0, \end{cases}$$

and define partial order on $\{+, -, 0\}$ by setting $0 \prec +$ and $0 \prec -$, while $+$ and $-$ are incomparable. We consider the subset of $\{+, -, 0\}^{n'}$:

$$S = \{(\operatorname{sign}(\langle \bar{\beta}_1, \mathbf{n} \rangle), \dots, \operatorname{sign}(\langle \bar{\beta}_{n'}, \mathbf{n} \rangle)) : \mathbf{n} \in \mathbb{R}^{n'} \setminus \{0\}\}.$$

Note that S can be regarded as a poset by using componentwise partial ordering: for $s, s' \in S$, $s \preceq s'$ if and only if $s^{(i)} \preceq s'^{(i)}$ for all $i \in [n']$. By [3, Proposition 2.2.2], there is an order-reversing bijection between S and the set of faces of $\mathcal{W}'(R)$ (except for the empty set and $\mathcal{W}'(R)$ itself), partially ordered by inclusion. In particular, by considering the correspondence between the facets of $\mathcal{W}'(R)$ and the minimal elements of S, the following lemma holds:

Lemma 12.1.5. *If there exist* $\mathbf{n} \in \mathbb{Z}^r \setminus \{0\}$ *and* $\bar{\beta}_{i_1}, \ldots, \bar{\beta}_{i_{r-1}}$ *such that* $\bar{\beta}_{i_1}, \ldots, \bar{\beta}_{i_{r-1}}$ *are linearly independent and* $\langle \mathbf{n}, \bar{\beta}_{i_j} \rangle = 0$ *for all* $j \in [r-1]$, *then*

$$F = \Big\{ \sum_{\langle \mathbf{n}, \beta_i \rangle > 0} \beta_i + \sum_{\langle \mathbf{n}, \beta_i \rangle = 0} a_i \beta_i \in \mathbb{R}^r : a_i \in [0,1] \Big\}$$

is a facet of $\mathcal{W}'(R)$. *Conversely, all facets of* $\mathcal{W}'(R)$ *are obtained in this way.*

Our goal is to determine the facet defining inequalities of a convex polytope representing conic classes. Let $m \in \mathbb{Z}_{>0}$ and let $p_i, q_i \in \mathbb{Z}_{>0}$ for $i \in [m]$. Moreover, for $i \in [m]$ and $j \in [r]$, let c_{ij} be an integer such that the greatest common divisor of $c_{i1}, \ldots, c_{ir}$ is equal to 1 for all $i \in [m]$. We define two convex polytopes:

$$\mathcal{C} = \{(z_1, \ldots, z_r) \in \mathbb{R}^r : -q_i \le \sum_{j=1}^{r} c_{ij} z_j \le p_i \text{ for all } i \in [m]\} \text{ and}$$

$$\mathcal{C}' = \{(z_1, \ldots, z_r) \in \mathbb{R}^r : -q_i - 1 \le \sum_{j=1}^{r} c_{ij} z_j \le p_i + 1 \text{ for all } i \in [m]\}.$$

Note that if $\mathcal{C}'$ is an integral polytope, then we have $\operatorname{int}(\mathcal{C}') \cap \mathbb{Z}^r = \mathcal{C} \cap \mathbb{Z}^r$.

The following lemma is useful for describing the conic divisorial ideals of toric rings.

Lemma 12.1.6. *(i) If* $\mathcal{W}'(R) = \mathcal{C}'$, *then one has* $\mathcal{W}(R) \cap \mathbb{Z}^r = \mathcal{C} \cap \mathbb{Z}^r$.

(ii) Suppose that $\mathcal{W}'(R) \subset \mathcal{C}'$. *If all vertices of* $\mathcal{C}'$ *are in* $\mathcal{W}'(R)$, *then* $\mathcal{W}'(R) = \mathcal{C}'$.

Proof. (i) We show that $\operatorname{int}(\mathcal{W}'(R)) = \mathcal{W}(R)$. This implies

$$\mathcal{W}(R) \cap \mathbb{Z}^r = \operatorname{int}(\mathcal{W}'(R)) \cap \mathbb{Z}^r = \operatorname{int}(\mathcal{C}') \cap \mathbb{Z}^r = \mathcal{C} \cap \mathbb{Z}^r. \tag{12.1.4}$$

Note that $\dim \mathcal{W}'(R) = \dim \mathcal{C}' = r$. For any $\beta \in \operatorname{int}(\mathcal{W}'(R))$, there exists $k > 1$ such that $k\beta \in \mathcal{W}'(R)$. Thus, we have $\beta \in \mathcal{W}(R)$ and hence $\operatorname{int}(\mathcal{W}'(R)) \subset \mathcal{W}(R)$.

To prove the reverse inclusion, we need only show that if $\beta \in \mathcal{W}'(R)$ is in the boundary $\partial\mathcal{W}'(R)$ of $\mathcal{W}'(R)$, then $\beta \notin \mathcal{W}(R)$. Let $\mathcal{S} = \sigma(\tau^\vee \cap \mathbb{Z}^d)$ and let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the i-th projection for each $i \in [n]$. Note that the group of differences $\langle \mathcal{S} \rangle$ of $\mathcal{S}$ coincide with $\sigma(\mathbb{Z}^d)$ and $\pi_i(s) = \sigma_i(\alpha)$ for $s = \sigma(\alpha) \in \mathcal{S}$. Since $\pi_i(\langle \mathcal{S} \rangle) = \sigma_i(\mathbb{Z}^d) = \mathbb{Z}$ for all $i \in [n]$ and for all $i, j \in [n]$ with $i \ne j$, there exists $s = \sigma(\alpha) \in \sigma(\tau^\vee \cap \mathbb{Z}^d)$ such that $\pi_j(s) = \sigma_j(\alpha) > 0 = \sigma_i(\alpha) = \pi_i(s)$, the set $T_i = \{\mathbf{e}_j + \langle \mathcal{S} \rangle : i \ne j \in [n]\}$ generates $\mathbb{Z}^n/\langle \mathcal{S} \rangle$ as a semigroup for every $i \in [n]$ ([11, Theorem 2]). This implies

$$\mathbb{Z}_{\ge 0}\beta_1 + \cdots + \widehat{\mathbb{Z}_{\ge 0}\beta_i} + \cdots + \mathbb{Z}_{\ge 0}\beta_n = \mathbb{Z}^r$$

for all $i \in [n]$, where $\widehat{\ }$ indicates an element to be omitted. Therefore, for any $\mathbf{n} \in \mathbb{Z}^r$, there exists $j \in [n]$ such that $\langle \mathbf{n}, \beta_j \rangle > 0$. Since $\beta \in \partial\mathcal{W}'(R)$, there is a facet F of $\mathcal{W}'(R)$ with $\beta \in F$. From Lemma 12.1.5, β_j with $\langle \mathbf{n}_F, \beta_j \rangle > 0$ must appear as a summand in $\beta = \sum_{i \in [n]} a_i \beta_i$, where $\mathbf{n}_F \in \mathbb{Z}^r$ is an outer normal vector of the supporting hyperplane defining F. Thus, we have $\beta \notin \mathcal{W}(R)$.

(ii) Let $v_1, \ldots, v_s$ be the vertices of $\mathcal{C}'$. Since $v_k \in \mathcal{W}'(R)$ for each $k \in [s]$, we can write $v_k = \sum_{i=1}^{n} a_{ki} \beta_i$ for some $a_{ki} \in [0,1]$. On the other hand, for any $z \in \mathcal{C}'$, we can also write $z = \sum_{k=1}^s t_k v_k$ with $t_k \in [0,1]$ and $\sum_{k=1}^s t_k = 1$. Thus, $z = \sum_{k=1}^s t_k(\sum_{i=1}^n a_{ki}\beta_i) = \sum_{i=1}^n(\sum_{k=1}^s t_k a_{ki})\beta_i$. Since $\sum_{k=1}^s t_k a_{ki} \in [0,1]$ for all $i \in [n]$, we have $z \in \mathcal{W}'(R)$, and hence $\mathcal{W}'(R) = \mathcal{C}'$. □

12.2 Conic divisorial ideals of toric rings

In this section, we determine conic divisorial ideals of the edge rings of complete multipartite graphs, Hibi rings and stable set rings. The contents of Subsection 12.2.1 (resp. Subsections 12.2.2 and 12.2.3) are contained in the author's paper [40] with A. Higashitani (resp. the author's paper [51]).

12.2.1 Conic divisorial ideals of edge rings of complete multipartite graphs

Given integers $1 \leq r_1 \leq \cdots \leq r_n$, let $\mathcal{C}(r_1, \dots, r_n)$ be a convex polytope defined as follows:

$$\begin{aligned}\mathcal{C}(r_1,\dots,r_n) =\Bigg\{&(z_1,\dots,z_n) \in \mathbb{R}^n : -r_i \leq z_j - z_i \leq r_j \text{ for } 1 \leq i,j \leq n,\\ &-|J| - \sum_{i\in[n-1]\setminus I} r_i - \sum_{j\in J} r_j + 1 \leq \sum_{i\in I} z_i - \sum_{j\in J} z_j \leq |J|+1\\ &\text{for } I, J \subset [n-1] \text{ with } |I| = |J|+1 \text{ and } I \cap J = \emptyset,\\ &-|J| - \sum_{i\in[n-1]\setminus I} r_i - \sum_{j\in J} r_j + 2 \leq \sum_{i\in I} z_i - \sum_{j\in J} z_j \leq |J|\\ &\text{for } I \subset [n-1] \text{ and } J \subset [n] \text{ with } |I| = |J|+1,\ n \in J \text{ and } I \cap J = \emptyset\Bigg\},\end{aligned} \tag{12.2.1}$$

where J is regarded as a multi-set and $J = \emptyset$ might happen, while I is a usual non-empty set. For the explicit descriptions in the cases where $n = 3$ and $n = 4$, see Example 12.2.2.

Theorem 12.2.1. *Let $K_{r_1,\dots,r_n}$ be the complete multipartite graph with $1 \leq r_1 \leq \cdots \leq r_n$. Then the conic divisorial ideals of $\Bbbk[K_{r_1,\dots,r_n}]$ one-to-one correspond to the points in $\mathcal{C}(r_1,\dots,r_n) \cap \mathbb{Z}^n$ if $n \leq 4$.*

In what follows, we consider $G = K_{r_1,\dots,r_n}$ with $1 \leq r_1 \leq \cdots \leq r_n$, and assume that $n = 3$ with $r_1 \geq 2$ or $n \geq 4$. Note that G is non-bipartite. Let $V(G) = \bigsqcup_{i=1}^n V_i$ with $|V_i| = r_i$, let $V_i = \{v_{i1}, \dots, v_{ir_i}\}$ and let $E(G) = \{\{a,b\} : a \in V_i, b \in V_j \text{ for } 1 \leq i \neq j \leq n\}$.

In the sequel, we identify the entry of $\mathbb{R}^d$ with the vertex of G and assume that v_{n,r_n} corresponds to the last (d-th) coordinate of $\mathbb{R}^d$.

Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ with $\pi(x_1,\dots,x_d) = (x_1,\dots,x_{d-1})$. For the proof of Theorem 12.2.1, we replace P_G by the projected polytope $\pi(P_G)$ and we write $C_{\pi(P_G)}$ as C_G.

First, we observe how the supporting hyperplanes of C_G look like. We see that the variable "x_d" in P_G changes into "$2x_d - \sum_{i=1}^{d-1} x_i$" since $\sum_{i=1}^d x_i = 2$ holds. Hence, by

(9.3.1), the system of supporting hyperplanes of C_G becomes as follows:

$$\begin{aligned} &x_i \geq 0 \ \ (i = 1, \ldots, d-1), \quad 2x_d - \sum_{i=1}^{d-1} x_i \geq 0, \\ &x_d - \sum_{j \in V_i} x_j \geq 0 \ \ (i = 1, \ldots, n-1), \qquad \sum_{k \in V(G) \setminus V_n} x_k - x_d \geq 0. \end{aligned} \tag{12.2.2}$$

Apply the following unimodular transformation:

$$x_i \mapsto y_i \ \text{ for } i = 1, \ldots, d-1, \ \text{ and } \quad \sum_{k \in V(G) \setminus V_n} x_k - x_d \mapsto y_d.$$

Then (12.2.2) changes as follows:

$$\begin{aligned} &y_i \geq 0 \ \ (i = 1, \ldots, d), \\ &-y_d + \sum_{k \in V \setminus (V_i \sqcup V_n)} y_k \geq 0 \ \ (i = 1, \ldots, n-1), \\ &-2y_d + \sum_{k \in V \setminus V_n} y_k - \sum_{u \in V_n \setminus \{v_{n, r_n}\}} y_u \geq 0. \end{aligned} \tag{12.2.3}$$

Let

$$C'_G = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \text{ satisfies all inequalities in (12.2.3)}\}.$$

Since the edge ring $\Bbbk[G]$ is isomorphic to the $\Bbbk$-algebra $\Bbbk[C'_G \cap \mathbb{Z}^d]$, we consider C'_G.

In what follows, let

$$\mathbb{R}^d \ni \sigma_i = \begin{cases} \mathbf{e}_i & \text{for } i = 1, \ldots, d, \\ \sum_{k \in V \setminus V_n} \mathbf{e}_k - \sum_{\ell \in V_{i-d}} \mathbf{e}_\ell - \mathbf{e}_d & \text{for } i = d+1, \ldots, d+n. \end{cases} \tag{12.2.4}$$

Then each inequality in (12.2.3) corresponds to $\langle \sigma_i, \mathbf{y} \rangle \geq 0$, where $\mathbf{y} = (y_1, \ldots, y_d)$.

Before proving Theorem 12.2.1, we describe $\mathcal{C}(r_1, \ldots, r_n)$ more explicitly for small n's.

Example 12.2.2. Let $n = 3$. Then

$$\begin{aligned} \mathcal{C}(r_1, r_2, r_3) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : &-r_2 \leq z_1 - z_2 \leq r_1, \ -r_3 \leq z_1 - z_3 \leq r_1, \\ &-r_3 \leq z_2 - z_3 \leq r_2, \ -r_2 + 1 \leq z_1 \leq 1, \\ &-r_1 + 1 \leq z_2 \leq 1, \ -r_3 + 1 \leq z_1 + z_2 - z_3 \leq 1\}. \end{aligned} \tag{12.2.5}$$

Note that the inequality $-r_2 + 1 \leq z_1 \leq 1$ (resp. $-r_1 + 1 \leq z_2 \leq 1$) comes from the second family in (12.2.1) with $I = \{1\}$ (resp. $I = \{2\}$) and $J = \emptyset$ and the inequality $-r_3 + 1 \leq z_1 + z_2 - z_3 \leq 1$ comes from the third family in (12.2.1) with $I = \{1, 2\}$ and $J = \{3\}$.

Let $n = 4$. Then

$$\begin{aligned} \mathcal{C}(r_1, r_2, r_3, r_4) = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : & -r_j \leq z_i - z_j \leq r_i \text{ for } 1 \leq i < j \leq 4, \\ & -\sum_{j \in \{1,2,3\} \setminus \{i\}} r_j + 1 \leq z_i \leq 1 \text{ for } i = 1, 2, 3, \\ & -2r_k \leq z_i + z_j - z_k \leq 2 \text{ for } \{i, j, k\} = \{1, 2, 3\}, \\ & -r_k - r_4 + 1 \leq z_i + z_j - z_4 \leq 1 \text{ for } \{i, j, k\} = \{1, 2, 3\}, \\ & -2r_4 \leq z_1 + z_2 + z_3 - 2z_4 \leq 2\}. \end{aligned} \tag{12.2.6}$$

Note that the third family of the inequalities $-2r_k \leq z_i + z_j - z_k \leq 2$ (as well as the fourth one) for $\{i, j, k\} = \{1, 2, 3\}$ is regarded as three inequalities.

Theorem 12.2.1 directly follows from Lemma 12.2.3, Proposition 12.2.4 and Lemma 12.2.5 below.

Lemma 12.2.3. *Let* $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{Z}^n$. *Assume that the following conditions (a) and (b) are equivalent:*

(a) there exists $\mathbf{x} \in (-1, 0]^d$ *such that* $c_j - 1 < \langle \sigma_{d+j}, \mathbf{x} \rangle \leq c_j$ *holds for* $j = 1, \dots, n$*;*

(b) $\mathbf{c} \in \mathcal{C}(r_1, \dots, r_n) \cap \mathbb{Z}^n$.

Then the conic divisorial ideals one-to-one correspond to the points in $\mathcal{C}(r_1, \dots, r_n) \cap \mathbb{Z}^n$.

Proof. Let $m = d + n$.
Conic $\Rightarrow \mathcal{C}(r_1, \dots, r_n)$: Take any $\mathbf{a} = (a_1, \dots, a_m) \in \mathrm{Cl}(R) \subset \mathbb{Z}^m$ (cf. (12.1.3)) corresponding to a conic divisorial ideal $T(\mathbf{a})$.

We consider the decomposition of the semi-open cube $(-1, 0]^d$ cut by the hyperplanes defined from σ_i $(i = 1, \dots, m)$ in (12.2.4). More precisely, by identifying a conic divisorial ideal $T(a_1, \dots, a_m)$ with a full-dimensional cell of the decomposition $\bigcap_{i=1}^m L_{i,a_i}$, where $L_{i,a_i} = \{\mathbf{x} \in \mathbb{R}^d : a_i - 1 < \langle \sigma_i, \mathbf{x} \rangle \leq a_i\}$ for $i = 1, \dots, m$, we analyze which $(a_1, \dots, a_m) \in \mathbb{Z}^m$ defines a conic divisorial ideal.

Here, we notice that $\bigcap_{i=1}^d L_{i,a_i} \subset (-1, 0]^d$ holds if and only if $a_1 = \cdots = a_d = 0$, and in this case, we have $\bigcap_{i=1}^d L_{i,a_i} = (-1, 0]^d$. Hence, we see that $a_1 = \cdots = a_d = 0$ and we may discuss the remaining linear forms $\sigma_{d+1}, \dots, \sigma_{d+n}$.

In what follows, we show that $(a_{d+1}, \dots, a_{d+n}) \in \mathcal{C}(r_1, \dots, r_n) \cap \mathbb{Z}^n$. By definition of $\bigcap_{i=1}^m L_{i,a_i}$ and since it becomes full-dimensional, we see that

$$a_{d+i} - 1 < \langle \sigma_{d+i}, \mathbf{y} \rangle \leq a_{d+i} \text{ for } i = 1, \dots, n,$$

where $\mathbf{y} = (y_1, \dots, y_d)$ and $-1 < y_i \leq 0$ for each $i = 1, \dots, d$. Therefore, we conclude that $(a_{d+1}, \dots, a_{d+n}) \in \mathcal{C}(r_1, \dots, r_n) \cap \mathbb{Z}^n$ since (a) implies (b).

$\mathcal{C}(r_1, \dots, r_n) \Rightarrow$ conic: Take any $\mathbf{a} \in \mathcal{C}(r_1, \dots, r_n) \cap \mathbb{Z}^n$. We show that a divisorial ideal $T(0, \dots, 0, a_1, \dots, a_n)$ is conic. For this purpose, we prove that an $\mathbb{R}$-divisor $\sum_{i=1}^m b_i \mathcal{D}_i$ which is equivalent to $-\sum_{i=1}^n a_i \mathcal{D}_{d+i}$ satisfies that $-\mathbf{1} < \mathbf{b} \leq \mathbf{0}$ (see Lemma 12.1.2). By definition, we have $b_i + a_{i-d} = \langle \sigma_i, \mathbf{x} \rangle$ for each $i = 1, \dots, m$ for some $\mathbf{x} \in \mathbb{R}^d$, where we let $a_j = 0$ if $j \leq 0$. Here, we have $b_i = \langle \sigma_i, \mathbf{x} \rangle = x_i$ for $i = 1, \dots, d$. Since we can choose $\mathbf{x} \in \mathbb{R}^d$ up to $\mathbb{R}^d/\mathbb{Z}^d$, we may assume that $-1 < x_i \leq 0$ for $i = 1, \ldots, d$. Hence, $-1 < b_i \leq 0$ holds for $i = 1, \ldots, d$.

By $\mathbf{a} \in \mathcal{C}(r_1, \ldots, r_n) \cap \mathbb{Z}^n$, since (b) implies (a), we see that $a_j - 1 < \langle \sigma_{d+j}, \mathbf{x} \rangle \leq a_j$ holds for each $j = 1, \ldots, n$. Hence,

$$-1 < b_j = \langle \sigma_{d+j}, \mathbf{x} \rangle - a_j \leq 0 \text{ for } j = 1, \ldots, n,$$

as desired. □

Proposition 12.2.4. *The implication (a) ⇒ (b) in Lemma 12.2.3 holds for any n.*

Proof. On the first family of the inequalities in (12.2.1), since we have $\langle \sigma_{d+i}, \mathbf{x} \rangle \leq c_i < \langle \sigma_{d+i}, \mathbf{x} \rangle + 1$ for each $i = 1, \ldots, n$ by our assumption, we see that

$$\langle \sigma_{d+j} - \sigma_{d+i}, \mathbf{x} \rangle - 1 < c_j - c_i < \langle \sigma_{d+j} - \sigma_{d+i}, \mathbf{x} \rangle + 1$$

for each $1 \leq i, j \leq n$. Here, we observe that

$$\sigma_{d+j} - \sigma_{d+i} = \sum_{k \in V_i} \mathbf{e}_k - \sum_{\ell \in V_j} \mathbf{e}_\ell \text{ for each } 1 \leq i, j \leq n. \tag{12.2.7}$$

Hence, we obtain that

$$\begin{aligned} -r_i - 1 < \sum_{k \in V_i} x_k - \sum_{\ell \in V_j} x_\ell - 1 = \langle \sigma_{d+j} - \sigma_{d+i}, \mathbf{x} \rangle - 1 \\ < c_j - c_i < \langle \sigma_{d+j} - \sigma_{d+i}, \mathbf{x} \rangle + 1 = \sum_{k \in V_i} x_k - \sum_{\ell \in V_j} x_\ell + 1 < r_j + 1 \end{aligned}$$

for each $1 \leq i, j \leq n$.

On the second family, for $I, J \subset [n-1]$ with $|I| = |J| + 1$ and $I \cap J = \emptyset$, where J is regarded as a multi-set, it follows from (12.2.7) that

$$\begin{aligned} \sum_{i \in I} \sigma_{d+i} - \sum_{j \in J} \sigma_{d+j} &= \sigma_{d+i_0} + \sum_{\ell \in \bigcup_{j \in J} V_j} \mathbf{e}_\ell - \sum_{k \in (\bigcup_{i \in I} V_i) \setminus V_{i_0}} \mathbf{e}_k \\ &= \sum_{k \in V \setminus (V_n \cup \bigcup_{i \in I} V_i)} \mathbf{e}_k + \sum_{\ell \in \bigcup_{j \in J} V_j} \mathbf{e}_\ell - \mathbf{e}_d, \end{aligned}$$

where $i_0 \in I$ and $\bigcup_{j \in J} V_j$ is regarded as a multi-set. Similarly to the above discussions, we obtain that

$$\begin{aligned} &\sum_{i \in I} c_i - \sum_{j \in J} c_j < \left\langle \sum_{i \in I} \sigma_{d+i} - \sum_{j \in J} \sigma_{d+j}, \mathbf{x} \right\rangle + |I| < 1 + |I| = |J| + 2, \quad \text{and} \\ &\sum_{i \in I} c_i - \sum_{j \in J} c_j > \left\langle \sum_{i \in I} \sigma_{d+i} - \sum_{j \in J} \sigma_{d+j}, \mathbf{x} \right\rangle - |J| > -|J| - \sum_{i \in [n-1] \setminus I} r_i - \sum_{j \in J} r_j. \end{aligned} \tag{12.2.8}$$

On the third family, for $I \subset [n-1]$ and $J \subset [n]$ with $|I| = |J| + 1$, $n \in J$ and $I \cap J = \emptyset$, since we see from $n \in J$ that

$$\sum_{i \in I} \sigma_{d+i} - \sum_{j \in J} \sigma_{d+j} = \sum_{k \in V \setminus (V_n \cup \bigcup_{i \in I} V_i)} \mathbf{e}_k + \sum_{\ell \in (\bigcup_{j \in J} V_j) \setminus \{v_{n, r_n}\}} \mathbf{e}_\ell,$$

we obtain the conclusion by slightly modifying the estimation from (12.2.8). □

Lemma 12.2.5. *The implication (b) ⇒ (a) in Lemma 12.2.3 holds if $n = 3$ or $n = 4$.*

Proof. Let $n = 3$. Then $\mathcal{C}(r_1, r_2, r_3)$ is explicitly described as in (12.2.5). By the direct computation, we can list the vertices of $\mathcal{C}(r_1, r_2, r_3)$ as follows:

$$\begin{aligned}&(1,1,1),\ (1,1,r_3+1),\ (1,-r_1+1,-r_1+r_3+1),\ (-r_2+1,1,-r_2+r_3+1),\\&(-r_2+1,-r_1+1,-r_1-r_2+1),\ (-r_2+1,-r_1+1,-r_1-r_2+r_3+1),\\&(1,-r_1+1,-r_1+1),\ (-r_2+1,1,-r_2+1).\end{aligned}$$

From the same argument as in Lemma 12.1.6 (ii), it suffices to show the existence of $\mathbf{x}$ satisfying (a) for those vertices. Given $\mathbf{x} \in (-1, 0]^d$, let

$$y_1 = \sum_{k\in V_1} x_k,\ \ y_2 = \sum_{k\in V_2} x_k,\ \ y_3 = \sum_{k\in V_3} x_k - x_d, \text{ and } y_d = x_d.$$

(Remark that $d = |V| = |V_1| + |V_2| + |V_3| = r_1 + r_2 + r_3$.) In our case, it suffices to show that for each vertex $(c_1, c_2, c_3) \in \mathcal{C}(r_1, r_2, r_3)$, there is $\mathbf{y} = (y_1, y_2, y_3, y_d) \in (-r_1, 0] \times (-r_2, 0] \times (-r_3 + 1, 0]^d \times (-1, 0]$ such that

$$c_1 - 1 < y_2 - y_d \le c_1,\ \ c_2 - 1 < y_1 - y_d \le c_2 \ \text{ and } \ c_3 - 1 < y_1 + y_2 - y_3 - 2y_d \le c_3.$$

We list how to choose such $\mathbf{y}$'s for each vertex as follows:

$$\begin{aligned}&(1,1,1) : \mathbf{y} = (0,0,0,-\epsilon), \quad (1,1,r_3+1) : \mathbf{y} = (0,0,-r_3+1+\epsilon,-1+\epsilon),\\&(1,-r_1+1,-r_1+r_3+1) : \mathbf{y} = (-r_1+\epsilon,0,-r_3+1+\epsilon,-1+\epsilon),\\&(-r_2+1,1,-r_2+r_3+1) : \mathbf{y} = (0,-r_2+\epsilon,-r_3+1+\epsilon,-1+\epsilon),\\&(-r_2+1,-r_1+1,-r_1-r_2+1) : \mathbf{y} = (-r_1+\epsilon,-r_2+\epsilon,0,-\epsilon),\\&(-r_2+1,-r_1+1,-r_1-r_2+r_3+1) : \mathbf{y} = (-r_1+\epsilon,-r_2+\epsilon,-r_3+1+\epsilon,-1+\epsilon),\\&(1,-r_1+1,-r_1+1) : \mathbf{y} = (-r_1+\epsilon,0,0,-\epsilon),\\&(-r_2+1,1,-r_2+1) : \mathbf{y} = (0,-r_2+\epsilon,0,-\epsilon),\end{aligned}$$

where $\epsilon > 0$ is sufficiently small.

For the case $n = 4$, we may apply the same discussions as above by using (12.2.6), although the computations become much more complicated. □

12.2.2 Conic divisorial ideals of Hibi rings

In this subsection, we consider conic divisorial ideals of Hibi rings.

Let Π be a poset such that the Hasse diagram $\mathcal{H}(\widehat{\Pi})$ of $\widehat{\Pi}$ has $d+1$ vertices and n edges. For $p \in \widehat{\Pi} \setminus \{\hat{1}\}$, let $U(p) = \{\{p, q\} \in E(\mathcal{H}(\widehat{\Pi})) : q \text{ covers } p\}$. Similarly, for $p \in \widehat{\Pi} \setminus \{\hat{0}\}$, let $D(p) = \{\{p, q\} \in E(\mathcal{H}(\widehat{\Pi})) : q \text{ is covered by } p\}$.

By definition of $\sigma_\Pi(-)$ (see (3.2.1)) and (12.1.2), we see that the prime divisors $\mathcal{D}_e$ indexed by the edges e of $\mathcal{H}(\widehat{\Pi})$ satisfy the relations:

$$\sum_{e\in U(p)} \mathcal{D}_e = \sum_{e'\in D(p)} \mathcal{D}_{e'} \text{ for } p \in \widehat{\Pi} \setminus \{\hat{0}, \hat{1}\},\ \sum_{e\in U(\hat{0})} \mathcal{D}_e = 0 \ \text{ and } \ \sum_{e\in D(\hat{1})} \mathcal{D}_e = 0. \qquad (12.2.9)$$

In particular, we can take prime divisors corresponding to edges not contained in a spanning tree as generators of $\mathrm{Cl}(\Bbbk[\Pi])$, thus we obtain that $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^n/\sigma_\Pi(\mathbb{Z}^d) \cong \mathbb{Z}^{n-d}$ ([28, Theorem]). Moreover, any Weil divisor can be described as $\sum_{i=1}^{n-d} a_i \mathcal{D}_{e_{d+i}}$ and we identify this with $(a_1, \cdots, a_{n-d}) \in \mathbb{Z}^{n-d}$.

Note that $\mathcal{H}(\widehat{\Pi})$ can be regarded as a directed graph by orienting the edge $\{p,q\} \in E(\mathcal{H}(\widehat{\Pi}))$ from p to q if q covers p. In what follows, we fix a spanning tree T of $\mathcal{H}(\widehat{\Pi})$ and let $e_1, \cdots, e_d$ be its edges. Thus, let $e_{d+1}, \cdots, e_n$ be the remaining edges of $\mathcal{H}(\widehat{\Pi})$. In addition, for $i \in [n-d]$, let F_i be the fundamental cycle of e_{d+i} with respect to T and we assume that $e_{d+i} \in \mathrm{supp}^+(F_i)$. For $e \in E(\mathcal{H}(\widehat{\Pi}))$, let β_e be the weight corresponding to the prime divisor $\mathcal{D}_e$. Then, $\beta_{e_{d+i}} = \mathbf{e}_i$ for $i \in [n-d]$ and other weights β_{e_j} for $j \in [d]$ are determined uniquely by the relation (12.2.9).

Proposition 12.2.6. *Work with the same notation as above. Then $\beta_e = \sum_{i\in[n-d]} \mathbf{v}(F_i)^{(e)}\mathbf{e}_i$. Moreover, for $w = \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e\beta_e$ and a cycle C of $\mathcal{H}(\widehat{\Pi})$, we have*

$$\sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} w^{(i)} = \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e \mathbf{v}(C)^{(e)}.$$

Proof. Let $\gamma_e = \sum_{i\in[n-d]} \mathbf{v}(F_i)^{(e)}\mathbf{e}_i$ for $e \in E(\mathcal{H}(\widehat{\Pi}))$ and we show that $\gamma_e = \beta_e$. We see that $\gamma_{e_{d+j}} = \sum_{i\in[n-d]} \mathbf{v}(F_i)^{(e_{d+j})}\mathbf{e}_i = \mathbf{e}_j = \beta_{e_{d+j}}$ for $j \in [n-d]$. Thus, it is enough to show that γ_e's satisfy the relation (12.2.9). For $p \in \widehat{\Pi} \setminus \{\hat{0}, \hat{1}\}$, let $u(p) = \sum_{e\in U(p)} \gamma_e$ and $d(p) = \sum_{e'\in D(p)} \gamma_{e'}$. We fix $i \in [n-d]$. If $p \notin V(F_i)$, then $U(p) \cap E(F_i) = \emptyset$ and $D(p) \cap E(F_i) = \emptyset$. Thus, $u(p)^{(i)} = d(p)^{(i)} = 0$. Suppose that $p \in V(F_i)$. Then, there are exactly two edges e_1, e_2 with $e_1, e_2 \in E(F_i)$, and only the following two situations may happen: (a) $e_1, e_2 \in U(p)$ or $e_1, e_2 \in D(p)$, or (b) $e_1 \in U(p)$, $e_2 \in D(p)$ or $e_2 \in U(p)$, $e_1 \in D(p)$. In case (a), we can see that $e_1 \in \mathrm{supp}^+(F_i), e_2 \in \mathrm{supp}^-(F_i)$ or $e_2 \in \mathrm{supp}^+(F_i), e_1 \in \mathrm{supp}^-(F_i)$, and hence $u(p)^{(i)} = d(p)^{(i)} = 0$. In case (b), we can see that $e_1 \in \mathrm{supp}^+(F_i), e_2 \in \mathrm{supp}^+(F_i)$ or $e_1 \in \mathrm{supp}^-(F_i), e_2 \in \mathrm{supp}^-(F_i)$, and hence $u(p)^{(i)} = d(p)^{(i)} = 1$ or -1. Therefore, $u(p) = d(p)$ holds. Similarly, we have $u(\hat{0}) = d(\hat{1}) = 0$.

Moreover, for $w = \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e\beta_e$, we have $w^{(i)} = \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e \mathbf{v}(F_i)^{(e)}$. Furthermore, we see that $\mathbf{v}(C) = \sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})}\mathbf{v}(F_i)$ for a cycle C in $\mathcal{H}(\widehat{\Pi})$ since $\mathbf{v}(F_1), \dots, \mathbf{v}(F_{n-d})$ form bases of the flow space of $\mathcal{H}(\widehat{\Pi})$. Therefore, we obtain that

$$\begin{aligned}\sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} w^{(i)} &= \sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} \Big(\sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e \mathbf{v}(F_i)^{(e)} \Big) \\ &= \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e \Big(\sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} \mathbf{v}(F_i)^{(e)} \Big) = \sum_{e\in E(\mathcal{H}(\widehat{\Pi}))} a_e \mathbf{v}(C)^{(e)}.\end{aligned}$$

□

In what follows, we will give a correspondence of which elements in $\mathrm{Cl}(\Bbbk[\Pi]) \cong \mathbb{Z}^{n-d}$ describe the conic divisorial ideals.

Let $\mathcal{C}(\Pi)$ be a convex polytope defined by

$$\mathcal{C}(\Pi) = \Bigg\{ (z_1, \cdots, z_{n-d}) \in \mathbb{R}^{n-d} : \\ -|\operatorname{supp}^-(C)| + 1 \leq \sum_{i \in [n-d]} \mathbf{v}(C)^{(e_{d+i})} z_i \leq |\operatorname{supp}^+(C)| - 1 \Bigg\}, \tag{12.2.10}$$

where C runs over all circuits in $\mathcal{H}(\widehat{\Pi})$. Moreover, let $\mathcal{C}'(\Pi)$ be a convex polytope defined by

$$\mathcal{C}'(\Pi) = \Bigg\{ (z_1, \cdots, z_{n-d}) \in \mathbb{R}^{n-d} : \\ -|\operatorname{supp}^-(C')| \leq \sum_{i \in [n-d]} \mathbf{v}(C')^{(e_{d+i})} z_i \leq |\operatorname{supp}^+(C')| \Bigg\},$$

where C' runs over all cycles in $\mathcal{H}(\widehat{\Pi})$.

Theorem 12.2.7. *Let the notation be the same as above. Then, each point in $\mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$ one-to-one corresponds to the conic divisorial ideal of $\Bbbk[\Pi]$.*

Remark 12.2.8. This result has already been given in [43, Theorem 2.4]. It was proved that each conic divisorial ideal of $\Bbbk[\Pi]$ corresponds to a point in $\mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$ (the set $\mathcal{C}(\Pi)$ defined in [43] coincides with (12.2.10)). In the fifth step of the proof, it seems to prove the converse, that is, for each point $(m_{d+1}, \dots, m_n) \in \mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$, there exist $\mathbf{x} \in (-1, 0]^d$ and $(\overline{m}_1, \dots, \overline{m}_n) \in \sigma_P(\mathbb{Z}^d)$ such that $m_i - \overline{m}_i = \ulcorner \sigma_{e_i}(\mathbf{x}) \urcorner$ for all $i \in [n]$, where $m_1 = \cdots = m_d = 0$. But in fact, they have proven that for each point $(m_{d+1}, \dots, m_n) \in \mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$ and for each $j \in \{d+1, \dots, n\}$, there exist $\mathbf{x} \in (-1, 0]^d$ and $(\overline{m}_1, \dots, \overline{m}_n) \in \sigma_\Pi(\mathbb{Z}^d)$ such that $m_i - \overline{m}_i = \ulcorner \sigma_{e_i}(\mathbf{x}) \urcorner$ for $i \in [d]$ and $i = j$, which does not guarantee that $m_i - \overline{m}_i = \ulcorner \sigma_{e_i}(\mathbf{x}) \urcorner$ for $i \in \{d+1, \dots, n\} \setminus \{j\}$.

We re-prove this theorem by a different technique. We derive it by showing $\mathcal{W}'(\Bbbk[\Pi]) = \mathcal{C}'(\Pi)$ and using Lemma 12.1.6 (i), rather than showing $\mathcal{W}(\Bbbk[\Pi]) \cap \mathbb{Z}^{n-d} = \mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$ directly.

Proof of Theorem 12.2.7. It is enough to show that $\mathcal{W}(\Bbbk[\Pi]) \cap \mathbb{Z}^{n-d} = \mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$. We first show that $\mathcal{W}'(\Bbbk[\Pi]) \subset \mathcal{C}'(\Pi)$. Take $w = \sum_{e \in E(\mathcal{H}(\widehat{\Pi}))} a_e \beta_e \in \mathcal{W}'(\Bbbk[\Pi])$. For each cycle C of $\mathcal{H}(\widehat{\Pi})$, it follows from Proposition 12.2.6 that

$$\sum_{i \in [n-d]} \mathbf{v}(C)^{(e_{d+i})} w^{(i)} = \sum_{e \in E(\mathcal{H}(\widehat{\Pi}))} a_e \mathbf{v}(C)^{(e)} = \sum_{e \in \operatorname{supp}^+(C)} a_e - \sum_{e' \in \operatorname{supp}^-(C)} a_{e'}. \tag{12.2.11}$$

Since $a_e \in [0, 1]$, we have

$$-|\operatorname{supp}^-(C)| \leq \sum_{i \in [n-d]} \mathbf{v}(C)^{(e_{d+i})} w^{(i)} \leq |\operatorname{supp}^+(C)|. \tag{12.2.12}$$

Thus, $w \in \mathcal{C}'(\Pi)$, and hence $\mathcal{W}'(\Bbbk[\Pi]) \subset \mathcal{C}'(\Pi)$. Moreover, the hyperplanes

$$\sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} z_i = |\operatorname{supp}^+(C)| \text{ and } \sum_{i\in[n-d]} \mathbf{v}(C)^{(e_{d+i})} z_i = -|\operatorname{supp}^-(C)| \qquad (12.2.13)$$

are supporting hyperplanes of $\mathcal{C}'(\Pi)$ because there exist elements in $\mathcal{W}'(\Bbbk[\Pi]) \subset \mathcal{C}'(\Pi)$ such that the equality of each side of (12.2.12) holds respectively.

To show that $\mathcal{C}'(\Pi) \subset \mathcal{W}'(\Bbbk[\Pi])$, we prove that any vertex of $\mathcal{C}'(\Pi)$ is in $\mathcal{W}'$. Since the hyperplanes (12.2.13) support $\mathcal{C}'(\Pi)$, any vertex v of $\mathcal{C}'(\Pi)$ can be represented as the intersection of $n-d$ hyperplanes of them. By reversing the direction of cycles $C_1, \dots, C_{n-d}$ of $\mathcal{H}(\widehat{\Pi})$, we may assume that these hyperplanes have the following forms:

$$\sum_{i\in[n-d]} \mathbf{v}(C_k)^{(e_{d+i})} z_i = |\operatorname{supp}^+(C_k)| \quad \text{for } k \in [n-d]. \qquad (12.2.14)$$

From Lemma 12.2.9 below, we have $v \in \mathcal{W}'$, and hence $\mathcal{W}'(\Bbbk[\Pi]) = \mathcal{C}'(\Pi)$ by Lemma 12.1.6 (ii).

Moreover, from Lemma 12.1.6 (i), we have

$$\mathcal{W}(\Bbbk[\Pi]) \cap \mathbb{Z}^{n-d} = \Bigg\{ (z_1, \cdots, z_{n-d}) \in \mathbb{Z}^{n-d} : \\ -|\operatorname{supp}^-(C')| + 1 \le \sum_{i\in[n-d]} \mathbf{v}(C')^{(e_{d+i})} z_i \le |\operatorname{supp}^+(C')| - 1 \Bigg\},$$

where C' runs over all cycles in $\mathcal{H}(\widehat{\Pi})$. By the same argument as in the fourth step of the proof of [43, Theorem 2.4], the inequalities arising from a cycle having a chord can be omitted. Therefore, we obtain that $\mathcal{W}(\Bbbk[\Pi]) \cap \mathbb{Z}^{n-d} = \mathcal{C}(\Pi) \cap \mathbb{Z}^{n-d}$. $\square$

Lemma 12.2.9. *Let $C^+ = \bigcup_{k\in[n-d]} \operatorname{supp}^+(C_k)$ and $C^- = \bigcup_{k\in[n-d]} \operatorname{supp}^-(C_k)$. Suppose that the intersection of (12.2.14) is a unique point $v = (v_1, \dots, v_{n-d}) \in \mathbb{R}^{n-d}$. Then, v is a vertex of $\mathcal{C}'(\Pi)$ if and only if $C^+ \cap C^- = \emptyset$, in which case v is in $\mathcal{W}'$.*

Proof. It is enough to show that $v \in \mathcal{W}' \subset \mathcal{C}'(\Pi)$ (resp. $v \notin \mathcal{C}'(\Pi)$) if $C^+ \cap C^- = \emptyset$ (resp. $C^+ \cap C^- \neq \emptyset$). Suppose that $C^+ \cap C^- = \emptyset$. Then, $\sum_{e\in C^+} \beta_e \in \mathcal{W}'$ satisfies (12.2.14) for all $k \in [n-d]$. In fact, by (12.2.11) and $C^+ \cap C^- = \emptyset$, we have

$$\sum_{i\in[n-d]} \mathbf{v}(C_k)^{(e_{d+i})} \Big(\sum_{e\in C^+} \beta_e \Big)^{(i)} = \sum_{e\in \operatorname{supp}^+(C_k)} 1 = |\operatorname{supp}^+(C_k)|.$$

Therefore, $v = \sum_{e\in C^+} \beta_e \in \mathcal{W}' \subset \mathcal{C}'(\Pi)$.

If $C^+ \cap C^- \neq \emptyset$, there exist $s, t \in [n-d]$ with $\operatorname{supp}^+(C_s) \cap \operatorname{supp}^-(C_t) \neq \emptyset$. We may assume that $s = 1$ and $t = 2$. We set $\mathbf{u} = \mathbf{v}(C_1) + \mathbf{v}(C_2)$ and

$$C^* = (\operatorname{supp}^+(C_1) \cap \operatorname{supp}^-(C_2)) \cup (\operatorname{supp}^+(C_2) \cap \operatorname{supp}^-(C_1)).$$

Note that $C^* \neq \emptyset$. Since v satisfies (12.2.14), we obtain that

$$\begin{aligned}\sum_{i\in[n-d]} \mathbf{u}^{(e_{d+i})} v_i &= \sum_{i\in[n-d]} \left(\mathbf{v}(C_1)^{(e_{d+i})} + \mathbf{v}(C_2)^{(e_{d+i})}\right) v_i \\ &= |\operatorname{supp}^+(C_1)| + |\operatorname{supp}^+(C_2)| \\ &= |\operatorname{supp}^+(C_1) \setminus C^*| + |\operatorname{supp}^+(C_2) \setminus C^*| + |C^*|. \end{aligned} \tag{12.2.15}$$

On the other hand, since $\mathbf{u}$ is in the flow space of $\mathcal{H}(\widehat{\Pi})$, we can write $\mathbf{u} = \sum_{i=1}^m \mathbf{v}(D_i)$, where $D_1, \ldots, D_m$ are cycles of $\mathcal{H}(\widehat{\Pi})$ with $\operatorname{supp}^+(D_k) \subset (\operatorname{supp}^+(C_1) \cup \operatorname{supp}^+(C_2)) \setminus C^*$ and $\operatorname{supp}^-(D_k) \subset (\operatorname{supp}^-(C_1) \cup \operatorname{supp}^-(C_2)) \setminus C^*$ for all $k \in [m]$. This fact follows from a similar argument as in the proof of [21, Theorem 14.2.2]. If $v \in \mathcal{C}'(\Pi)$, then

$$\sum_{i\in[n-d]} \mathbf{v}(D_k)^{(e_{d+i})} v_i \leq |\operatorname{supp}^+(D_k)| \quad \text{for all } k \in [m].$$

Thus, we have

$$\begin{aligned}\sum_{i\in[n-d]} \mathbf{u}^{(e_{d+i})} v_i &= \sum_{k\in[m]} \Big(\sum_{i\in[n-d]} \mathbf{v}(D_k)^{(e_{d+i})} v_i \Big) \\ &\leq \sum_{k\in[m]} |\operatorname{supp}^+(D_k)| = |\operatorname{supp}^+(C_1) \setminus C^*| + |\operatorname{supp}^+(C_2) \setminus C^*|, \end{aligned}$$

a contradiction to (12.2.15). Hence, we obtain that $v \notin \mathcal{C}'(\Pi)$. □

12.2.3 Conic divisorial ideals of stable set rings

In this subsection, we consider conic divisorial ideals of stable set rings of perfect graphs.

In what follows, let G be a perfect graph on the vertex set $V(G) = [d]$ which has maximal cliques $Q_0, Q_1, \ldots, Q_n$. By definition of $\sigma_G(-)$ (see (3.2.3)) and (12.1.2), we see that the prime divisor $\mathcal{D}_i$ for $i \in \{0, 1, \ldots, n+d\}$ satisfies the relations:

$$\sum_{j=0}^n \mathcal{D}_j = 0 \quad \text{ and } \quad \mathcal{D}_{n+k} = \sum_{j=0}^n \chi_j(k) \mathcal{D}_j \ \text{ for } k \in [d], \tag{12.2.16}$$

where

$$\chi_j(k) = \begin{cases} 1 & \text{if } k \in Q_j, \\ 0 & \text{if } k \notin Q_j. \end{cases}$$

In particular, we can see that prime divisors $\mathcal{D}_1, \ldots, \mathcal{D}_n$ generate $\operatorname{Cl}(\Bbbk[\operatorname{Stab}_G])$, thus we have that $\operatorname{Cl}(\Bbbk[\operatorname{Stab}_G]) \cong \mathbb{Z}^{n+d+1}/\sigma_G(\mathbb{Z}^{d+1}) \cong \mathbb{Z}^n$ (see [41]). Furthermore, let β_i be the weight corresponding to the prime divisor $\mathcal{D}_i$. Then, we can determine the weights β_i ($i \in \{0, 1, \ldots, n+d\}$) by the relation (12.2.16):

$$\beta_i = \begin{cases} \mathbf{e}_i & \text{if } i \in \{0, 1, \ldots, n\}, \\ \sum_{j=0}^n \chi_j(k) \mathbf{e}_j & \text{if } i \in \{n+1, \ldots, n+d\}, \end{cases} \tag{12.2.17}$$

where we let $\mathbf{e}_0 = -\mathbf{e}_1 - \cdots - \mathbf{e}_n$. For $v \in V(G)$ and a multiset $L \subset \{0, 1, \dots, n\}$, let $m_L(v) = |\{l \in L : v \in Q_l\}|$. Moreover, for multisets $I, J \subset \{0, 1, \dots, n\}$, we set

$$X_{IJ}^+ = \{v \in V(G) : m_{IJ}(v) > 0\} \text{ and } X_{IJ}^- = \{v \in V(G) : m_{IJ}(v) < 0\}, \tag{12.2.18}$$

where $m_{IJ}(v) = m_I(v) - m_J(v)$.

Let $\mathcal{C}(G)$ and $\mathcal{C}'(G)$ be two convex polytopes defined by

$$\mathcal{C}(G) = \Bigg\{ (z_1, \cdots, z_n) \in \mathbb{R}^n : \\ -|J| + \sum_{v \in X_{IJ}^-} m_{IJ}(v) + 1 \le \sum_{i \in I} z_i - \sum_{j \in J} z_j \le |I| + \sum_{v \in X_{IJ}^+} m_{IJ}(v) - 1 \\ \text{for multisets } I, J \subset \{0, 1, \dots, n\} \text{ with } |I| = |J| \text{ and } I \cap J = \emptyset \Bigg\} \text{ and} \tag{12.2.19}$$

$$\mathcal{C}'(G) = \Bigg\{ (z_1, \cdots, z_n) \in \mathbb{R}^n : \\ -|J| + \sum_{v \in X_{IJ}^-} m_{IJ}(v) \le \sum_{i \in I} z_i - \sum_{j \in J} z_j \le |I| + \sum_{v \in X_{IJ}^+} m_{IJ}(v) \\ \text{for multisets } I, J \subset \{0, 1, \dots, n\} \text{ with } |I| = |J| \text{ and } I \cap J = \emptyset \Bigg\}, \tag{12.2.20}$$

where we let $z_0 = 0$.

Remark 12.2.10. Note that an infinite number of inequalities appear in (12.2.19) and (12.2.20). But in fact, only finitely many inequalities are needed since it follows from Theorem 12.2.12 below that $\mathcal{C}'(G)$ coincides with $\mathcal{W}'(\Bbbk[\mathrm{Stab}_G])$. Therefore, $\mathcal{C}(G)$ and $\mathcal{C}'(G)$ are polytopes. On the other hand, by using Lemma 12.1.5, we can determine the facet defining inequalities of $\mathcal{C}'(G)$. For example, since $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ must appear as a weight of $\Bbbk[\mathrm{Stab}_G]$, for each $i, j \in \{0, 1, \dots, n\}$ with $i \neq j$, the inequality

$$-1 - |X_{\{i\}\{j\}}^-| \le z_i - z_j \le 1 + |X_{\{i\}\{j\}}^+|$$

defines a facet of $\mathcal{C}'(G)$.

Example 12.2.11. Let Γ be the graph on the vertex set $\{1, \dots, 7\}$ with the edge set

$$E(\Gamma) = \{12, 13, 23, 24, 25, 34, 36, 45, 46, 56, 57, 67\}.$$

See Figure 12.1.

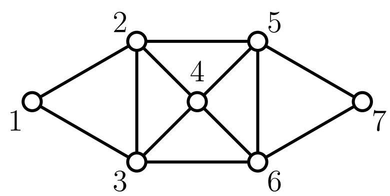

Figure 12.1: The graph Γ

Then, Γ is a perfect graph and has 6 maximal cliques:

$Q_0 = \{1,2,3\}, Q_1 = \{2,3,4\}, Q_2 = \{2,4,5\}, Q_3 = \{3,4,6\}, Q_4 = \{4,5,6\}$ and $Q_5 = \{5,6,7\}$.

Let $I = \{1,1,5\}$ and $J = \{0,2,3\}$. Then, we have

$$m_{IJ}(1) = -1,\ m_{IJ}(2) = m_{IJ}(3) = m_{IJ}(4) = m_{IJ}(5) = m_{IJ}(6) = 0 \ \text{ and } \ m_{IJ}(7) = 1.$$

Thus, we obtain that $X_{IJ}^{+} = \{7\}$ and $X_{IJ}^{-} = \{1\}$. Therefore, we get the inequality

$$-4 \le 2z_1 + z_5 - z_2 - z_3 \le 4.$$

Indeed, this inequality is a facet defining inequality of $\mathcal{C}'(\Gamma)$.

Theorem 12.2.12. *Let the notation be the same as above. Then, each point in $\mathcal{C}(G) \cap \mathbb{Z}^n$ one-to-one corresponds to the conic divisorial ideal of $\Bbbk[\mathrm{Stab}_G]$.*

Proof. We prove that $\mathcal{W}(\Bbbk[\mathrm{Stab}_G]) \cap \mathbb{Z}^n = \mathcal{C}(G) \cap \mathbb{Z}^n$ by the same discussion as in the case of Hibi rings. We take $w = \sum_{i=0}^{n+d} a_i \beta_i \in \mathcal{W}'(\Bbbk[\mathrm{Stab}_G])$. It follows from (12.2.17) that

$$w^{(i)} = a_i + \sum_{v \in Q_i} a_{v+n} - \sum_{u \in Q_0} a_{u+n} - a_0.$$

Therefore, for multisets $I, J \subset \{0,1,\dots,n\}$ with $|I| = |J|$ and $I \cap J = \emptyset$,

$$\begin{aligned}
\sum_{i\in I} w^{(i)} - \sum_{j \in J} w^{(j)} &= \sum_{i \in I} \Big(a_i + \sum_{v \in Q_i} a_{v+n} - \sum_{u \in Q_0} a_{u+n} - a_0 \Big) \\
&\qquad - \sum_{j \in J} \Big(a_j + \sum_{v \in Q_j} a_{v+n} - \sum_{u \in Q_0} a_{u+n} - a_0 \Big) \\
&= \sum_{i \in I} a_i + \sum_{v=1}^{d} m_I(v) a_{v+n} - \sum_{u=1}^{d} m_J(u) a_{u+n} - \sum_{j \in J} a_j \\
&= \sum_{i \in I} a_i - \sum_{j \in J} a_j + \sum_{v=1}^{d} m_{IJ}(v) a_{v+n}. \qquad (12.2.21)
\end{aligned}$$

Since $a_i \in [0,1]$, we have

$$-|J| + \sum_{v \in X_{IJ}^{-}} m_{IJ}(v) \le \sum_{i \in I} w^{(i)} - \sum_{j \in J} w^{(j)} \le |I| + \sum_{v \in X_{IJ}^{+}} m_{IJ}(v).$$

Thus, $w \in \mathcal{C}'(G)$, and hence $\mathcal{W}'(\Bbbk[\mathrm{Stab}_G]) \subset \mathcal{C}'(G)$. Furthermore, the hyperplanes

$$\sum_{i \in I} z_i - \sum_{j \in J} z_j = |I| + \sum_{v \in X_{IJ}^{+}} m_{IJ}(v) \ \text{and} \ \sum_{i \in I} z_i - \sum_{j \in J} z_j = -|J| + \sum_{v \in X_{IJ}^{-}} m_{IJ}(v). \qquad (12.2.22)$$

are supporting hyperplanes of $\mathcal{C}'(G)$.

Next, we prove that any vertex of $\mathcal{C}'(G)$ is in $\mathcal{W}'(\Bbbk[\mathrm{Stab}_G])$. We consider n supporting hyperplanes (12.2.22) whose intersection is a unique point u. By alternating I_k and J_k, we may assume that these hyperplanes have the following forms:

$$\sum_{i\in I_k} w^{(i)} - \sum_{j\in J_k} w^{(j)} = |I_k| + \sum_{v\in X^+_{I_kJ_k}} m_{I_kJ_k}(v) \quad \text{for } k\in[n]. \tag{12.2.23}$$

From the following lemma, we have $u \in \mathcal{W}'(\Bbbk[\mathrm{Stab}_G])$, and hence $\mathcal{W}(\Bbbk[\mathrm{Stab}_G]) \cap \mathbb{Z}^n = \mathcal{C}(G) \cap \mathbb{Z}^n$. □

Lemma 12.2.13. *Let $X^+ = \bigcup_{k\in[n]} X^+_{I_kJ_k}$ and $X^- = \bigcup_{k\in[n]} X^-_{I_kJ_k}$. Suppose that the intersection of (12.2.23) is a unique point $u = (u_1,\dots,u_n) \in \mathbb{R}^n$. Then, u is a vertex of $\mathcal{C}'(G)$ if and only if $X^+ \cap X^- = \emptyset$, in which case u is in $\mathcal{W}'(\Bbbk[\mathrm{Stab}_G])$.*

Proof. It is enough to show that $u \in \mathcal{W}' \subset \mathcal{C}'(G)$ (resp. $u \notin \mathcal{C}'(G)$) if $X^+ \cap X^- = \emptyset$ (resp. $X^+ \cap X^- \neq \emptyset$). Suppose that $X^+ \cap X^- = \emptyset$. From (12.2.21), we can see that $\sum_{l=0}^{n+d} \alpha_l\beta_l \in \mathcal{W}'$ satisfies (12.2.23) for all $k\in[n]$, where

$$\alpha_l = \begin{cases} 1 & \text{if } l \in \bigcup_{k=1}^n I_k \text{ or } l-n \in X^+,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, we have $u = \sum_{l=0}^{n+d} \alpha_l\beta_l \in \mathcal{W}' \subset \mathcal{C}'(P)$.

If $X^+ \cap X^- \neq \emptyset$, there exist $s,t\in[n]$ with $X^+_{I_sJ_s} \cap X^-_{I_tJ_t} \neq \emptyset$. We may assume that $s=1$ and $t=2$. We set $I' = (I_1\cup I_2)\setminus(J_1\cup J_2)$, $J' = (J_1\cup J_2)\setminus(I_1\cup I_2)$,

$$X_1^* = X^+_{I_1J_1} \cap X^-_{I_2J_2} \text{ and } X_2^* = X^+_{I_2J_2} \cap X^-_{I_1J_1}.$$

Here, I' and J' are regarded as multisets. Note that $X_1^* \neq \emptyset$, $X^+_{I'J'} \cup X_1^* \cup X_2^* = X_1^+ \cup X_2^+$ and $m_{I'J'}(v) = m_{I_1J_1}(v) + m_{I_2J_2}(v)$ for all $v \in V(G)$. Since u satisfies (12.2.14), we obtain that

$$\begin{aligned}\sum_{i\in I'} u_i - \sum_{j\in J'} u_j &= \Big(\sum_{i\in I_1} u_i - \sum_{j\in J_1} u_j\Big) + \Big(\sum_{i'\in I_2} u_{i'} - \sum_{j\in J_2} u_{j'}\Big)\\ &= |I_1| + \sum_{v\in X^+_{I_1J_1}} m_{I_1J_1}(v) + |I_2| + \sum_{v'\in X^+_{I_2J_2}} m_{I_2J_2}(v')\\ &= |I_1| + |I_2| + \sum_{v\in X^+_{I'J'}\setminus(X_1^*\cup X_2^*)} m_{I'J'}(v) + \sum_{v\in X_1^*} m_{I_1J_1}(v) + \sum_{v\in X_2^*} m_{I_2J_2}(v).\end{aligned} \tag{12.2.24}$$

On the other hand, since $|I'| = |J'|$ and $I'\cap J' = \emptyset$, we have

$$\sum_{i\in I'} u_i - \sum_{j\in J'} u_j \le |I'| + \sum_{v\in X^+_{I'J'}} m_{I'J'}(v)$$

if $u \in \mathcal{C}'(G)$, a contradiction to (12.2.24) because $|I'| \le |I_1| + |I_2|$ and $m_{I'J'}(v) < m_{I_1J_1}(v)$ (resp. $m_{I'J'}(v) < m_{I_2J_2}(v)$) for $v \in X_1^*$ (resp. $v \in X_2^*$). Thus, we obtain that $v \notin \mathcal{C}'(G)$. □

We finally state the description of conic divisorial ideals of the stable set ring arising from the comparability graph of a poset Π. In this case, we expect to be able to describe them in terms of Π as follows.

Let Π be a poset and let $\mathcal{Q}(\widehat{\Pi})$ denote the set of maximal chains of $\widehat{\Pi}$. Moreover, let $C = (p_1, \dots, p_s)$ be a cycle of $\mathcal{H}(\widehat{\Pi})$ and we may assume that

$$p_{m_1} = p_1 \prec p_2 \prec \cdots \prec p_{M_1} \succ \cdots \succ p_{m_2} \prec \quad \cdots \quad \prec p_{M_k} \succ \cdots \succ p_s \succ p_{m_{k+1}} = p_1.$$

We set $\mathbf{U}_C = \mathcal{Q}(\widehat{\Pi}_{\succeq p_{M_1}}) \times \cdots \times \mathcal{Q}(\widehat{\Pi}_{\succeq p_{M_k}})$ and $\mathbf{D}_C = \mathcal{Q}(\widehat{\Pi}_{\preceq p_{m_1}}) \times \cdots \times \mathcal{Q}(\widehat{\Pi}_{\preceq p_{m_k}})$, where $\widehat{\Pi}_{\succeq p} = \{q \in \widehat{\Pi} : q \succeq p\}$ for $p \in \widehat{\Pi}$ (we define $\widehat{\Pi}_{\preceq p}$ analogously). Furthermore, for $U = (U_1, \dots, U_k) \in \mathbf{U}_C$ and $D = (D_1, \dots, D_k) \in \mathbf{D}_C$, the sets

$$Q_i^{\uparrow} = D_i \cup \{p_{m_i}, p_{m_i+1} \dots, p_{M_i}\} \cup U_i \quad \text{and} \quad Q_i^{\downarrow} = U_i \cup \{p_{M_i}, p_{M_i+1} \dots, p_{m_{i+1}}\} \cup D_{i+1}$$

are maximal chains of $\widehat{\Pi}$ for each $i \in [k]$, where $U_{k+1} = U_1$. Fix $Q_0 \in \mathcal{Q}(\widehat{\Pi})$ and we define

$$\mathfrak{C}_\Pi = \Bigg\{ z \in \mathbb{R}^{\mathcal{Q}(\widehat{\Pi}) \setminus \{Q_0\}} : \\ -k - \sum_{l=1}^{k} (m_{l+1} - M_l - 1) + 1 \leq \sum_{i \in 1}^{k} z^{(Q_i^{\uparrow})} - \sum_{j \in 1}^{k} z^{(Q_j^{\downarrow})} \leq k + \sum_{l=1}^{k} (M_l - m_l - 1) - 1 \\ \text{for } U \in \mathbf{U}_C \text{ and } D \in \mathbf{D}_C \Bigg\},$$

where C runs over all circuits in $\mathcal{H}(\widehat{\Pi})$ and we let $m_{k+1} = s+1$ and $z^{(Q_0)} = 0$.

We call a poset $\mathcal{X}$ *general X-shape* if $\mathcal{X}$ is the ordinal sum of some chains and some disjoint unions of two chains. The Hasse diagram of a general X-shape poset looks like the one shown in Figure 12.2.

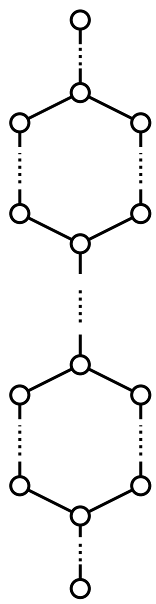

Figure 12.2: The general X-shape poset

Suppose that $\widehat{\Pi}$ contains a general X-shape subposet $\mathcal{X}$ with $\mathcal{Q}(\mathcal{X}) \subset \mathcal{Q}(\widehat{\Pi})$ and $|\mathcal{Q}(\mathcal{X})| \geq 4$. For $Q \in \mathcal{Q}(\mathcal{X})$, there exists $\overline{Q} \in \mathcal{Q}(\mathcal{X})$ with $Q \cup \overline{Q} = \mathcal{X}$, which is uniquely determined. We set

$$\mathfrak{X}_\Pi = \left\{ z \in \mathbb{R}^{\mathcal{Q}(\widehat{\Pi})\setminus\{Q_0\}} : -1 \leq z^{(Q)} + z^{(\overline{Q})} - z^{(Q')} - z^{(\overline{Q'})} \leq 1 \text{ for } Q, Q' \in \mathcal{Q}(\mathcal{X}) \right\},$$

where $\mathcal{X}$ runs over all general X-shape subposet of $\widehat{\Pi}$ with $\mathcal{Q}(\mathcal{X}) \subset \mathcal{Q}(\widehat{\Pi})$ and $|\mathcal{Q}(\mathcal{X})| \geq 4$.

Conjecture 12.2.14. *Let* Π *be a poset. Then, the conic divisorial ideals of* $\Bbbk[\mathcal{C}_\Pi]$ *one-to-one correspond to the points in* $\mathfrak{C}_\Pi \cap \mathfrak{X}_\Pi \cap \mathbb{Z}^{\mathcal{Q}(\widehat{\Pi})\setminus\{Q_0\}}$.

Example 12.2.15. Let $\Pi = \{p_1, \ldots, p_6\}$ be the poset which is the ordinal sum of the disjoint union of two elements and the disjoint union of three elements. The Hasse diagram of $\widehat{\Pi}$ is shown in Figure 12.3.

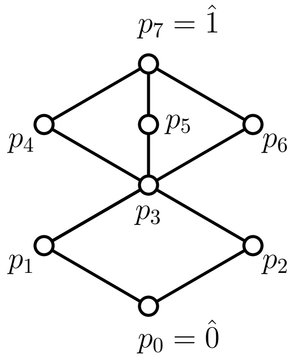

Figure 12.3: The Hasse diagram of $\widehat{\Pi}$

It has 6 maximal chains, 4 circuits and 3 general X-shape subposets satisfying the appropriate conditions. Let $C = (p_3, p_4, p_7, p_5)$ be a circuit of $\mathcal{H}(\widehat{\Pi})$. In this case, we can see that

$$\mathbf{U}_C = \mathcal{Q}(\widehat{\Pi}_{\succeq p_7}) = \{\{p_7\}\} \quad \text{and} \quad \mathbf{D}_C = \mathcal{Q}(\widehat{\Pi}_{\preceq p_3}) = \{\{p_0, p_1, p_3\}, \{p_0, p_2, p_3\}\}.$$

Let $U = U_1 = \{p_7\} \in \mathbf{U}_C$ and $D = D_1 = \{p_0, p_1, p_3\} \in \mathbf{D}_C$. Then, we have

$$Q_1^{\uparrow} = \{p_0, p_1, p_3, p_4, p_7\} \quad \text{and} \quad Q_1^{\downarrow} = \{p_7, p_5, p_3, p_1, p_0\},$$

and hence we get the inequality

$$-1 \leq z^{(Q_1^{\uparrow})} - z^{(Q_1^{\downarrow})} \leq 1.$$

Next, we consider the general X-shape subposet $\mathcal{X} = \widehat{\Pi}\setminus\{p_5\}$. Let $Q = \{p_0, p_1, p_3, p_4, p_7\}$ and $Q' = \{p_0, p_2, p_3, p_4, p_7\}$. Then, we have

$$\overline{Q} = \{p_0, p_2, p_3, p_6, p_7\} \quad \text{and} \quad \overline{Q'} = \{p_0, p_1, p_3, p_6, p_7\},$$

and hence we get the inequality

$$-1 \leq z^{(Q)} + z^{(\overline{Q})} - z^{(Q')} - z^{(\overline{Q'})} \leq 1.$$

We can see that the inequalities appearing in $\mathfrak{C}_\Pi$ and $\mathfrak{X}_\Pi$ are special forms of those appearing in $\mathcal{C}(G(\Pi))$. Therefore, we have $\mathcal{C}(G(\Pi)) \subset \mathfrak{C}_\Pi \cap \mathfrak{X}_\Pi$, that is, each conic divisorial ideal of $\Bbbk[\mathcal{C}_\Pi]$ corresponds to a point in $\mathfrak{C}_\Pi \cap \mathfrak{X}_\Pi \cap \mathbb{Z}^{\mathcal{Q}(\widehat{\Pi}) \setminus \{Q_0\}}$. We expect that the converse is true.

12.3 Quasi-symmetric or weakly-symmetric toric rings

In this section, we characterize when Hibi rings and stable set rings are quasi-symmetric or weakly-symmetric. The contents of this section are contained in the author's paper [51].

We prove the following theorems:

Theorem 12.3.1. *Let Π be a poset. We consider the following conditions:*

(i) Π *is a general X-shape poset;*

(ii) $\Bbbk[\Pi]$ *is isomorphic to the tensor product of a polynomial ring and some Segre products of two polynomial rings;*

(iii) $\Bbbk[\Pi]$ *is weakly-symmetric;*

(iv) $\Bbbk[\Pi]$ *is quasi-symmetric.*

Then, (i), (ii) and (iii) are equivalent. Furthermore, if $\Bbbk[\Pi]$ *is Gorenstein, then the above four conditions are equivalent.*

Theorem 12.3.2. *Let* $\Bbbk[\mathrm{Stab}_G]$ *be the stable set ring of a perfect graph G. We consider the following conditions:*

(i) G has at most 2 maximal cliques;

(ii) $\Bbbk[\mathrm{Stab}_G]$ *is isomorphic to the tensor product of a polynomial ring and the Segre products of two polynomial rings;*

(iii) $\Bbbk[\mathrm{Stab}_G]$ *is weakly-symmetric;*

(iv) $\Bbbk[\mathrm{Stab}_G]$ *is quasi-symmetric.*

Then, (i), (ii) and (iii) are equivalent. Furthermore, if $\Bbbk[\mathrm{Stab}_G]$ *is Gorenstein, then the above four conditions are equivalent.*

12.3.1 Proof of Theorem 12.3.1

Proof. Let Π be a poset such that $\mathcal{H}(\widehat{\Pi})$ has $d+1$ vertices and n edges.

(i) $\Rightarrow$ (ii): This follows immediately from Proposition 3.2.1 (i) and (ii).

(ii) $\Rightarrow$ (iii): If $\Bbbk[\Pi]$ is isomorphic to the tensor product of a polynomial ring and some Segre products of two polynomial rings, then it is also isomorphic to the Hibi ring of a general X-shape poset $\mathcal{X}$. We can easily compute the weights of $\Bbbk[\Pi] \cong \Bbbk[\mathcal{X}]$ by using

Proposition 12.2.6. In fact, since the circuits of $\mathcal{H}(\widehat{\mathcal{X}})$ are precisely the fundamental cycles of $\mathcal{H}(\widehat{\mathcal{X}})$ and their edge sets are disjoint, the weights of $\Bbbk[\Pi]$ is $\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_{n-d}$ or 0, and $\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_{n-d}$ must appear. Therefore, $\Bbbk[\Pi]$ is weakly-symmetric.

Before proving (iii) $\Rightarrow$ (i), we give an easy observation: Π has an element which is comparable with any other element of Π if and only if $\mathcal{H}(\widehat{\Pi})$ is not 2-connected, i.e., there exists an element p in Π such that $\mathcal{H}(\widehat{\Pi}) \setminus p$ is not connected. In this case, we can see that $\Pi = \Pi_{\prec p} \oplus \Pi_{\succ p}$. Therefore, we may assume that $\mathcal{H}(\widehat{\Pi})$ is 2-connected by Proposition 3.2.1 (ii).

(iii) $\Rightarrow$ (i): Since $\mathcal{H}(\widehat{\Pi})$ is 2-connected, it can be constructed from the Hasse diagram H (see Figure 12.4) by successively adding paths to graphs already constructed (see e.g., [17, Proposition 3.1.1]). In this case, we can replace "paths" with "chains". Moreover, by removing an edge from H and each added chain, we get a spanning tree T of $\mathcal{H}(\widehat{\Pi})$. We denote those edges by $e_1, \dots, e_{n-d}$ and assume that $e_1 \in E(H)$ and e_2 is in the added chain to the first. For $i \in [n-d]$, let F_i be the fundamental cycle of e_i with respect to T. Note that $F_1 = H$.

Since $\Bbbk[\Pi]$ is weakly-symmetric and any weight of $\Bbbk[\Pi]$ is in $\{0, 1, -1\}^{n-d}$, there is the weight $-\mathbf{e}_2$. However, it is impossible because any edge of F_2 is contained in $E(F_1)$ or the edge set of the added chain to the first, in particular, $\mathrm{supp}^-(F_2) \subset E(F_1)$. Thus, $\mathcal{H}(\widehat{\Pi}) = H$.

Suppose that $\Bbbk[\Pi]$ is Gorenstein. If the condition (i) is satisfied, then we can compute the weights of $\Bbbk[\Pi]$ as in (ii) $\Rightarrow$ (iii) and check that $\Bbbk[\Pi]$ is quasi-symmetric since Π is pure. Clearly, (iv) $\Rightarrow$ (iii) holds, and hence those four conditions are equivalent. □

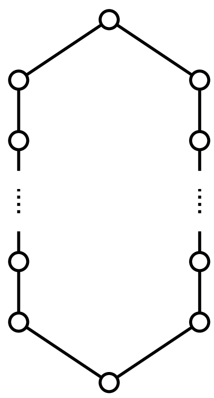

Figure 12.4: The Hasse diagram H

12.3.2 Proof of Theorem 12.3.2

Proof. Let G be a perfect graph with maximal cliques $Q_0, Q_1, \dots, Q_n$.

(i) $\Rightarrow$ (ii): In the case $n = 0$, $\Bbbk[\mathrm{Stab}_G]$ is the polynomial ring with $|Q_0| + 1$ variables over $\Bbbk$. Suppose that $n = 1$ and let $Q = Q_0 \cap Q_1$. Note that for each $v \in Q$, we have $\{v, w\} \in E(G)$ for all $w \in V(G)$. By observing stable sets of G, we can see that $\Bbbk[\mathrm{Stab}_G] \cong \Bbbk[\mathrm{Stab}_{G \setminus Q}] \otimes_\Bbbk \Bbbk[\mathrm{Stab}_{G_Q}]$ and $\Bbbk[\mathrm{Stab}_{G \setminus Q}]$ is isomorphic to the Segre product of $\Bbbk[\mathrm{Stab}_{G_{Q_0} \setminus Q}]$ and $\Bbbk[\mathrm{Stab}_{G_{Q_1} \setminus Q}]$. Furthermore, $\Bbbk[\mathrm{Stab}_{G_Q}]$, $\Bbbk[\mathrm{Stab}_{G_{Q_0} \setminus Q}]$ and $\Bbbk[\mathrm{Stab}_{G_{Q_1} \setminus Q}]$

are polynomial rings. Thus, $\Bbbk[\mathrm{Stab}_G]$ is isomorphic to the tensor product of a polynomial ring and the Segre products of two polynomial rings

(ii) $\Rightarrow$ (iii): This is the same as in the case of Hibi rings.

(iii) $\Rightarrow$ (i): Since $\beta_i \in \{0,1,-1\}^n$ for any $i \in \{0,1,\dots,n+d\}$ and $\Bbbk[\mathrm{Stab}_G]$ is weakly-symmetric, the weights $-\mathbf{e}_k$ ($k \in \{0,1,\dots,n\}$) must appear. Equivalently, for each $j \in \{0,1,\dots,n\}$, there exists $v_j \in V(G)$ such that $v_j \notin Q_j$ and $v_j \in Q_l$ for any $l \in \{0,1,\dots,n\} \setminus \{j\}$.

If $n \geq 2$, then $\{v_s, v_t\} \in E(G)$ for any $s,t \in \{0,1,\dots,n\}$ because there exists $u \in \{0,1,\dots,n\}$ with $v_s, v_t \in Q_u$. Therefore, $\{v_0, v_1, \dots, v_n\}$ is a clique of G, and a maximal clique containing it is different from $Q_0, Q_1, \dots, Q_n$. Hence, we have $n \leq 1$.

If $\Bbbk[\mathrm{Stab}_G]$ is Gorenstein, then the conditions (i), (ii), (iii), and (iv) are equivalent by the same argument as in the case of Hibi rings. □

Chapter 13

Constructions of NCCRs

In this chapter, we recall the definition of NCCRs and construct an NCCR for certain toric rings; Gorenstein edge rings of complete multipartite graphs and a special family of stable set rings.

13.1 Preliminaries on NCCRs

In this section, we recall the definition of N(C)CRs and prepare some notation and lemmas.

Definition 13.1.1. Let R be a Cohen–Macaulay normal domain, let $M \neq 0$ be a reflexive R-module, and let $E = \mathrm{End}_R(M)$. Moreover, let $\mathrm{gldim}\, E$ denote the global dimension of E.

- We call E a *non-commutative resolution* (*NCR*, for short) of R if $\mathrm{gldim}\, E < \infty$ ([14]).

In addition, suppose that R is Gorenstein.

- We call E a *non-commutative crepant resolution* (*NCCR*, for short) of R if E is an NCR and is an MCM R-module ([82]).
- Moreover, we say that an NCCR E is *splitting* if M is a finite direct sum of rank one reflexive R-modules. A splitting NCCR is also called "toric NCCR" when R is a toric ring (see [4]).

Since conic divisorial ideals of a toric ring R are rank one reflexive R-modules, the endomorphism ring E of the finite direct sum of some of them is a toric NCCR if E is an MCM R-module and $\mathrm{gldim}\, E < \infty$.

We use the same notation as in Section 12.1. Let $\mathcal{A} = \mathrm{mod}(G, S)$ be the category of finitely generated G-equivariant S-modules. Given $\chi \in X(G)$, let $P_\chi = V_\chi \otimes_{\Bbbk} S$. Note that $P_\chi \in \mathcal{A}$ and $M_\chi = P_\chi^G$. For a subset $\mathcal{L} \subset X(G)$, we set

$$P_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} P_\chi \quad \text{and} \quad \Lambda_{\mathcal{L}} = \mathrm{End}_{\mathcal{A}}(P_{\mathcal{L}}).$$

Moreover, for $\chi \in X(G)$, let $P_{\mathcal{L},\chi} = \mathrm{Hom}_{\mathcal{A}}(P_{\mathcal{L}}, P_\chi)$.

Let $Y(G)$ denote the group of one-parameter subgroups of G and let $Y(G)_{\mathbb{R}} = Y(G) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that $Y(G) \cong \mathbb{Z}^r$ and $Y(G)_{\mathbb{R}} \cong \mathbb{R}^r$. We say that $\chi \in X(G)$ is *separated from* $\mathcal{L}$ *by* $\lambda \in Y(G)_{\mathbb{R}}$ if it holds that $\langle \lambda, \chi \rangle < \langle \lambda, \chi' \rangle$ for each $\chi' \in \mathcal{L}$.

Our goal is to choose $\mathcal{L} \subset \mathcal{W}(R) \cap X(G)$ such that $E = \mathrm{End}_R(M_{\mathcal{L}})$ becomes an NCCR, where $R = S^G$ and $M_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} M_\chi$. To show $\mathrm{gldim}\, E < \infty$, we use the following facts:

- We note that $\Lambda_{\mathcal{L}}^G = \mathrm{End}_R(M_{\mathcal{L}}) = E$ and that $\mathrm{gldim}\, \Lambda_{\mathcal{L}} < \infty$ implies $\mathrm{gldim}\, E < \infty$ (see [43, Section 3]).
- If $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for all $\chi \in X(G)$, where $\mathrm{pdim}\, P_{\mathcal{L},\chi}$ denotes the projective dimension of $P_{\mathcal{L},\chi}$ over $\Lambda_{\mathcal{L}}$, then $\mathrm{gldim}\, \Lambda_{\mathcal{L}} < \infty$ ([70, Lemma 10.1]).
- By the same argument as in [70, Section 10.3], we can see that if $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for each $\chi \in \widetilde{\mathcal{L}}$, where $\mathcal{L} \subset \mathcal{W}(R) \cap X(G) \subset \widetilde{\mathcal{L}} \subset X(G)$, then $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for all $\chi \in X(G)$.

Moreover, we note the following fact to show that E is an MCM R-module:

- Since $\mathrm{End}_R(M_{\mathcal{L}}) \cong \bigoplus_{\chi,\chi' \in \mathcal{L}} M_{\chi-\chi'}$, the endomorphism ring E is an MCM R-module if $M_{\chi-\chi'}$ is an MCM R-module for any $\chi, \chi' \in \widetilde{\mathcal{L}}$.

By summarizing those facts, we obtain the following lemma:

Lemma 13.1.2. *Let the notation be the same as above. Then, E is an NCCR of R if there exists $\widetilde{\mathcal{L}} \subset X(G)$ satisfying the following two conditions:*

(a) $\chi - \chi' \in \widetilde{\mathcal{L}}$ for any $\chi, \chi' \in \mathcal{L}$, and M_χ is an MCM R-module for any $\chi \in \widetilde{\mathcal{L}}$.

(b) $\mathcal{W}(R) \cap X(G) \subset \widetilde{\mathcal{L}}$ and $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for each $\chi \in \widetilde{\mathcal{L}}$.

In addition, we give another lemma to verify the condition (b) in Lemma 13.1.2.

Lemma 13.1.3. *Let the notation be the same as above.*

(i) ([70, Section 10.1]) *If χ is in $\mathcal{L}$, then $P_{\mathcal{L},\chi}$ is a right projective $\Lambda_{\mathcal{L}}$-module, and hence $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$.*

(ii) ([70, Lemma 10.2]) *Let $\chi \in X(G)$ be separated from $\mathcal{L}$ by $\lambda \in Y(G)_{\mathbb{R}}$. Then, we obtain the acyclic complex*

$$0 \to \bigoplus_{\mu_{d_\lambda}} P_{\mathcal{L},\mu_{d_\lambda}} \to \cdots \to \bigoplus_{\mu_1} P_{\mathcal{L},\mu_1} \to P_{\mathcal{L},\chi} \to 0,$$

where for each $p \in [d_\lambda]$ with $d_\lambda = |\{i \in [m] : \langle \beta_i, \lambda \rangle > 0\}|$, we let $\mu_p = \chi + \beta_{i_1} + \cdots + \beta_{i_p}$ with $\{i_1, \dots, i_p\} \subset [m]$ and $\langle \beta_{i_j}, \lambda \rangle > 0$ for all $j \in [p]$.
This implies that $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ if $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\mu_p} < \infty$ for each μ_p.

13.2 NCCR of Gorenstein edge rings of complete multipartite graphs

In this section, we provide an NCCR of Gorenstein edge rings of complete multipartite graphs $K_{r_1,\dots,r_n}$. The contents of this section are contained in the author's paper [40] with A. Higashitani.

It is known by [15, Theorem 1.1] that if a ring admits an NCCR, then it should be $\mathbb{Q}$-Gorenstein. This means that if the divisor class group of a normal Cohen–Macaulay domain R is torsionfree and R admits an NCCR, then R is Gorenstein.

From Theorem 5.1.7, $\Bbbk[K_{r_1,\dots,r_n}]$ is Gorenstein if $\Bbbk[K_{r_1,\dots,r_n}]$ has an NCCR and we know when $\Bbbk[K_{r_1,\dots,r_n}]$ is Gorenstein (Proposition 9.1.1). Moreover, it follows from Propositions 6.4.1 and 6.4.2 that $\Bbbk[K_{r_1,r_2}]$ and $\Bbbk[K_{1,r_2,r_3}]$ are isomorphic to Hibi rings and the ranks of their divisor class groups are at most 2. Furthermore, it is shown in [61] that Hibi rings whose divisor class groups have at most rank 2 have an NCCR. Therefore, our remaining tasks are the study of the edge rings of $K_{2,2,2}$ and $K_{1,1,1,1} = K_4$. We note that $\mathrm{Cl}(\Bbbk[K_{2,2,2}]) \cong \mathbb{Z}^3$ and $\mathrm{Cl}(\Bbbk[K_4]) \cong \mathbb{Z}^4$.

The following is the main theorem of this section:

Theorem 13.2.1. *Let R be the edge ring of $G = K_{2,2,2}$ or $G = K_4$. Let*

$$\mathcal{L} = \{(0,0,-1), (0,0,0), (1,0,0), (1,-1,0), (1,0,1), (0,-1,-1), (1,-1,-1), (0,-1,-2)\} \subset \mathcal{C}(2,2,2)$$

if $G = K_{2,2,2}$, and let

$$\mathcal{L} = \{(0,0,0,0), (1,0,0,0), (1,0,0,1), (1,1,0,1), (1,1,1,2)\} \subset \mathcal{C}(1,1,1,1)$$

if $G = K_4$. Then $\mathrm{End}_R\left(\bigoplus_{\chi\in\mathcal{L}} M_\chi\right)$ is an NCCR of R, respectively.

Finally, in this section, we give an NCCR for $\Bbbk[K_{2,2,2}]$ and $\Bbbk[K_4]$, i.e., we prove Theorem 13.2.1.

This subsection is devoted to giving a proof of Theorem 13.2.1.

First, let us describe the set of conic divisorial ideals in the cases $K_{2,2,2}$ and K_4. The direct computations imply that those correspond to the following set of the lattice points by Theorem 12.2.1:

$$\begin{aligned}\mathcal{C}(2,2,2) \cap \mathbb{Z}^3 &= \{\pm(1,1,a) : a = 1,2,3\} \cup \{\pm(1,0,a), \pm(0,1,a) : a = 0,1,2\} \\ &\quad \cup \{\pm(1,-1,a), (0,0,a) : a = -1,0,1\}, \\ \mathcal{C}(1,1,1,1) \cap \mathbb{Z}^4 &= \{\pm(1,1,1,2)\} \cup \{\pm(\alpha,1) : \alpha \in \{0,1\}^3\} \\ &\quad \cup \{\pm(1,0,0,0), \pm(0,1,0,0), \pm(0,0,1,0), (0,0,0,0)\}.\end{aligned}$$

As in Theorem 13.2.1, let

$$\mathcal{L} = \{(0,0,-1), (0,0,0), (1,0,0), (1,-1,0), (1,0,1), (0,-1,-1), (1,-1,-1), (0,-1,-2)\}$$

if $G = K_{2,2,2}$, and

$$\mathcal{L} = \{(0,0,0,0), (1,0,0,0), (1,0,0,1), (1,1,0,1), (1,1,1,2)\}$$

if $G = K_4$. We set

$$\widetilde{\mathcal{L}} = \mathcal{C}(2,2,2) \cap \mathbb{Z}^3 \text{ if } G = K_{2,2,2}, \text{ and}$$
$$\widetilde{\mathcal{L}} = \mathcal{C}(1,1,1,1) \cap \mathbb{Z}^3 \cup \{\pm(0,1,1,2)\} \text{ if } G = K_4.$$

One can check by `Macaulay2` ([24]) that the divisorial ideals corresponding to $\pm(0,1,1,2)$ are MCM $\Bbbk[K_4]$-modules. Moreover, one can also verify that $\chi - \chi' \in \widetilde{\mathcal{L}}$ holds for $\chi, \chi' \in \mathcal{L}$ in both cases. Note that all conic divisorial ideals are rank one MCMs. Hence, E is an MCM by Lemma 13.1.2 (a).

Our remaining task is to show that $\operatorname{gldim} E < \infty$. By Lemmas 13.1.2 and 13.1.3, we can conclude this if the following procedures terminate:

1. Choose $\chi \in \widetilde{\mathcal{L}} \setminus \mathcal{L}$. (Note that $\operatorname{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} = 0$ if $\chi \in \mathcal{L}$.)
2. Find $\lambda \in Y(G)_{\mathbb{R}}$ such that χ is separated from $\mathcal{L}$ by λ and $\chi + \beta_{i_1} + \cdots + \beta_{i_p} \in \mathcal{L}$ for any $\{i_1, \ldots, i_p\} \subset [m]$ with $\langle \beta_{i_j}, \lambda \rangle > 0$ for each j.
3. If $\mathcal{L} \cup \{\chi\} = \widetilde{\mathcal{L}}$, then terminate the procedure. Otherwise, replace $\mathcal{L}$ by $\mathcal{L} \cup \{\chi\}$ and go back to (1).

For a while, we consider the case $K_{2,2,2}$. Then one has $m = d + n = 6 + 3 = 9$. For the computations of $\beta_1, \ldots, \beta_9$, since we have

$$\sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_1 \rangle \mathcal{D}_j = \mathcal{D}_1 + \mathcal{D}_8 + \mathcal{D}_9 = 0, \quad \sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_2 \rangle \mathcal{D}_j = \mathcal{D}_2 + \mathcal{D}_8 + \mathcal{D}_9 = 0,$$
$$\sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_3 \rangle \mathcal{D}_j = \mathcal{D}_3 + \mathcal{D}_7 + \mathcal{D}_9 = 0, \quad \sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_4 \rangle \mathcal{D}_j = \mathcal{D}_4 + \mathcal{D}_7 + \mathcal{D}_9 = 0,$$
$$\sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_5 \rangle \mathcal{D}_j = \mathcal{D}_5 - \mathcal{D}_9 = 0, \text{ and } \sum_{j=1}^{9} \langle \sigma_j, \mathbf{e}_6 \rangle \mathcal{D}_j = \mathcal{D}_6 - \mathcal{D}_7 - \mathcal{D}_8 - 2\mathcal{D}_9 = 0,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_6 \in \mathbb{R}^6$ denote the unit vectors of $\mathbb{R}^6$ and σ_i's are as in (12.2.4), by letting $\mathcal{D}_7 = (1,0,0)$, $\mathcal{D}_8 = (0,1,0)$ and $\mathcal{D}_9 = (0,0,1)$, we obtain that

$$\mathcal{D}_1 = \mathcal{D}_2 = (0,-1,-1), \; \mathcal{D}_3 = \mathcal{D}_4 = (-1,0,-1), \; \mathcal{D}_5 = (0,0,1) \text{ and } \mathcal{D}_6 = (1,1,2).$$

Hence, we can choose $\beta_1, \ldots, \beta_9$ as the following multi-set:

$$\{\beta_1, \ldots, \beta_9\} = \{(1,0,0), (0,1,0), (0,0,1) \times 2, (0,-1,-1) \times 2, (-1,0,-1) \times 2, (1,1,2)\},$$

where $\times 2$ stands for the duplicate.

We list the ordering of choices of $\chi \in \widetilde{\mathcal{L}} \setminus \mathcal{L}$ and the corresponding λ as follows:

χ	λ
(0,-1,0)	(1,1,-1)
(1,1,1)	(0,-1,0)
(-1,-1,-2)	(1,0,0)
(0,1,1)	(0,-1,0)
(0,0,1)	(1,1,-1)
(0,1,2)	(0,-1,0)

χ	λ
(-1,-1,-1)	(1,0,0)
(-1,0,-1)	(1,0,0)
(-1,1,0)	(0,-1,0)
(0,1,0)	(0,-1,0)
(-1,0,0)	(1,1,-1)
(1,1,2)	(0,0,-1)

χ	λ
(-1,1,1)	(0,-1,0)
(-1,0,-2)	(0,-1,1)
(-1,-1,-3)	(0,0,1)
(1,0,2)	(0,1,-1)
(1,-1,1)	(0,1,-1)
(-1,1,-1)	(0,-1,0)
(1,1,3)	(0,0,-1)

We read off the lines from top to bottom of the left-most table at first and go to the right. By this ordering, we can directly check that the procedure terminates.

For example, at first, for $(0,-1,0) \in \widetilde{\mathcal{L}} \setminus \mathcal{L}$, we let $\lambda = (1,1,-1)$, and we see that $-1 = \langle \lambda, (0,-1,0) \rangle < \langle \lambda, \chi' \rangle$ for each $\chi' \in \mathcal{L}$ since $\langle \lambda, \chi' \rangle \in \{0,1\}$ for each χ'. Moreover, we also have $\langle \beta, \lambda \rangle > 0$ for $\beta \in \{\beta_1, \dots, \beta_9\}$ if and only if $\beta \in \{(1,0,0),(0,1,0)\}$ and we can check that all of $\chi + (1,0,0)$, $\chi + (0,1,0)$ and $\chi + (1,1,0)$ belong to $\mathcal{L}$. Thus, we add $(0,-1,0)$ to $\mathcal{L}$. Next, take $\chi = (1,1,1) \in \widetilde{\mathcal{L}} \setminus \mathcal{L}$ and let $\lambda = (0,-1,0)$. Then we see that $-1 = \langle \lambda, \chi \rangle < \langle \lambda, \chi' \rangle$ for each $\chi' \in \mathcal{L}$ since $\langle \lambda, \chi' \rangle \in \{0,1\}$ for each χ'. Moreover, we also have $\langle \beta, \lambda \rangle > 0$ if and only if $\beta \in \{(0,-1,-1) \times 2\}$ and $\chi + (0,-1,-1)$ and $\chi + (0,-2,-2)$ belong to $\mathcal{L}$. Thus, we add $(1,1,1)$ to $\mathcal{L}$. We repeat this procedure until $\mathcal{L}$ coincides with $\widetilde{\mathcal{L}}$.

In the case K_4, one has $m = 4 + 4 = 8$. Since the method for the proof is completely the same as the case of $K_{2,2,2}$, we just list $\{\beta_1, \dots, \beta_8\}$, the ordering of choices of $\chi \in \widetilde{\mathcal{L}} \setminus \mathcal{L}$ and the corresponding λ below:

$$\begin{aligned}\{\beta_1, \dots, \beta_8\} = \{&(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),\\ &(1,1,1,2),(0,-1,-1,-1),(-1,0,-1,-1),(-1,-1,0,-1)\};\end{aligned}$$

χ	λ
(1,1,1,1)	(0,-1,-1,1)
(0,1,1,1)	(2,0,0,-1)
(0,1,0,1)	(1,0,1,-1)
(0,0,0,1)	(1,1,0,-1)
(0,0,0,-1)	(0,0,0,1)
(0,0,-1,-1)	(0,0,1,0)
(0,-1,-1,-1)	(0,1,0,0)

χ	λ
(-1,-1,-1,-1)	(1,0,0,0)
(-1,0,0,-1)	(1,0,0,0)
(-1,-1,-1,-2)	(0,0,0,1)
(-1,0,0,0)	(1,-1,0,0)
(-1,0,-1,-1)	(0,0,1,0)
(0,-1,0,0)	(0,2,0,-1)
(0,1,0,0)	(0,-2,0,1)

χ	λ
(1,0,1,1)	(-1,1,-1,0)
(0,0,1,1)	(1,1,0,-1)
(0,-1,0,-1)	(0,1,0,0)
(-1,-1,0,-1)	(1,1,0,-1)
(0,0,-1,0)	(0,0,2,-1)
(0,0,1,0)	(0,0,-2,1)
(0,1,1,2)	(1,-1,0,0)
(0,-1,-1,-2)	(0,0,0,1)

13.3 NCCR of a special family of stable set rings

Finally, in this section, we introduce a perfect graph $G_{r_1,\dots,r_n}$ and give an NCCR for its stable set ring. The contents of this section are contained in the author's paper [51].

13.3.1 Perfect graphs $G_{r_1,\dots,r_n}$

For an integer $n \geq 3$ and positive integers $r_1, \dots, r_n$, let $G_{r_1,\dots,r_n}$ be the graph on the vertex set $V(G_{r_1,\dots,r_n}) = [2d]$ with the edge set $E(G_{r_1,\dots,r_n}) = \bigcup_{i=0}^{n} \{\{v,u\} : v, u \in Q_i\}$, where $d = \sum_{k=1}^{n} r_k$, $Q_0 = \{d+1, \dots, 2d\}$ and for $i \in [n]$, we let

$$Q_i^+ = \Big\{r_1 + \cdots + r_{i-1} + 1, \dots, r_1 + \cdots + r_{i-1} + r_i\Big\},$$
$$Q_i^- = \Big\{d + r_1 + \cdots + r_{i-1} + 1, \dots, d + r_1 + \cdots + r_{i-1} + r_i\Big\} \text{ and}$$
$$Q_i = Q_i^+ \cup (Q_0 \setminus Q_i^-).$$

Note that $Q_i^+ = Q_i \setminus Q_0$ and $Q_i^- = Q_0 \setminus Q_i$.

Example 13.3.1. We give drawings of $G_{1,1,1}$ and $G_{1,1,1,1}$ in Figures 13.1 and 13.2, respectively.

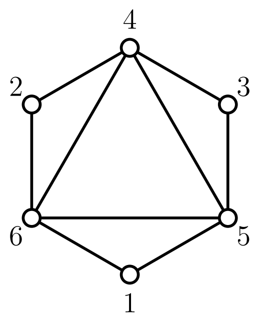

Figure 13.1: The graph $G_{1,1,1}$

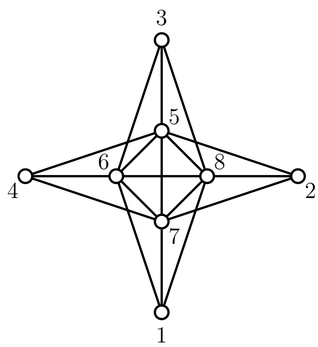

Figure 13.2: The graph $G_{1,1,1,1}$

We can see that $G_{1,1,1}$ has maximal stable sets $\{1,2,3\}$, $\{1,4\}$, $\{2,5\}$ and $\{3,6\}$.

The graph $G_{r_1,\dots,r_n}$ has the following properties.

Proposition 13.3.2. *(i) The maximal cliques of $G_{r_1,\dots,r_n}$ are precisely $Q_0, Q_1, \dots, Q_n$.*

(ii) A subset $S \subset V(G_{r_1,\dots,r_n})$ is a maximal stable set of $G_{r_1,\dots,r_n}$ if and only if $S = \{v_i, v_i'\}$ or $\{v_1, \dots, v_n\}$ for some $i \in [n]$, $v_i \in Q_i^+$ and $v_i' \in Q_i^-$.

(iii) The graph $G_{r_1,\dots,r_n}$ is chordal (and hence perfect), but is not a comparability graph.

(iv) The stable set ring $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ is Gorenstein, and $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]) \cong \mathbb{Z}^n$.

(v) One has

$$\mathcal{C}(G_{r_1,\dots,r_n}) = \{(z_1, \cdots, z_n) \in \mathbb{R}^n : -r_i \leq z_i \leq r_i \text{ for } i \in [n]\}.$$

Proof. (i) By the definition of $G_{r_1,\dots,r_n}$, we see that $Q_0, Q_1, \dots, Q_n$ are cliques of $G_{r_1,\dots,r_n}$ and we can easily check that these are maximal. If there exists a maximal clique Q of $G_{r_1,\dots,r_n}$ which is different from $Q_0, Q_1, \dots, Q_n$, then there is an element $u_i \in Q \setminus Q_i$ for each $i \in \{0, 1, \dots, n\}$. We may assume that $u_0 \in Q_1$. Then, since $u_0 \in Q_1^+$ and $u_1 \notin Q_1$, we have $\{u_0, u_1\} \notin E(G_{r_1,\dots,r_n})$, a contradiction to $u_0, u_1 \in Q$.

(ii) It follows from the definition of $G_{r_1,\dots,r_n}$ that $\{v_i, v_i'\}$ and $\{v_1,\dots,v_n\}$ are maximal stable sets of $G_{r_1,\dots,r_n}$. Suppose that S is a maximal stable set of $G_{r_1,\dots,r_n}$. If there exists a vertex $v_0' \in S \cap Q_0$, then $i \in [n]$ with $v_0' \notin Q_i$ is uniquely determined and we can see that $S = \{v_0', v_i\}$ for some $v_i \in Q_i^+$ since $\{v_0', v\} \in E(G_{r_1,\dots,r_n})$ for any $v \in V(G_{r_1,\dots,r_n}) \setminus Q_i^+$. If $S \cap Q_0 = \emptyset$, then we have $S = \{v_1,\dots,v_n\}$ for some $v_i \in Q_i^+$ ($i \in [n]$) since $S \subset V(G_{r_1,\dots,r_n}) \setminus Q_0 = Q_1^+ \cup \dots \cup Q_n^+$.

(iii) We can see that $G_{r_1,\dots,r_n}$ arises from $n+1$ complete graphs with d vertices by pasting them. Thus, by Proposition 2.3.1, $G_{r_1,\dots,r_n}$ is chordal.

If $G_{r_1,\dots,r_n}$ is a comparability graph, then so is any induced subgraph of $G_{r_1,\dots,r_n}$. However, for any $n \geq 3$ and $r_1,\dots,r_n$, the graph $G_{r_1,\dots,r_n}$ contains $G_{1,1,1}$ as an induced subgraph and we can check that $G_{1,1,1}$ is not a comparability graph, a contradiction.

(iv) Since $G_{r_1,\dots,r_n}$ is perfect and maximal cliques $Q_0, Q_1, \dots, Q_n$ of $G_{r_1,\dots,r_n}$ have the same cardinality d, the stable set ring $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ is Gorenstein. Moreover, we have $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]) \cong \mathbb{Z}^n$ because $G_{r_1,\dots,r_n}$ has $n+1$ maximal cliques.

(v) From (12.2.17), we can see that for $j \in \{0, 1, \dots, 2d+n\}$,

$$\beta_i = \begin{cases} \mathbf{e}_j & \text{if } i = j \text{ or } i - n \in Q_j^+; \\ -\mathbf{e}_j & \text{if } i - n \in Q_j^-. \end{cases} \tag{13.3.1}$$

Therefore, $\{\bar{\beta}_1, \dots, \bar{\beta}_{n'}\} = \{\mathbf{e}_0, \pm\mathbf{e}_1, \dots, \pm\mathbf{e}_n\}$ with $n' = 2n+1$. Let $\mathbf{n} \in \mathbb{Z}^n$. If there are $i_1, \dots, i_{n-1} \subset [2n+1]$ such that $\bar{\beta}_{i_k}$'s are linearly independent and $\langle \mathbf{n}, \bar{\beta}_{i_k} \rangle = 0$ for all $k \in [n-1]$, then $\mathbf{n}$ must be the following form:

$$\mathbf{n} = m\mathbf{e}_i \quad \text{or} \quad \mathbf{n} = m(\mathbf{e}_i - \mathbf{e}_j) \quad (m \in \mathbb{Z} \setminus \{0\})$$

for some $i, j \in [n]$. By Lemma 12.1.5 (the observation mentioned in Remark 12.2.10), we have

$$\begin{aligned} \mathcal{C}'(G_{r_1,\dots,r_n}) &= \mathcal{W}'(\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]) \\ &= \Big\{(z_1, \cdots, z_n) \in \mathbb{R}^n : -1 - |X^-_{\{i\}\{j\}}| \leq z_i - z_j \leq 1 + |X^+_{\{i\}\{j\}}| \\ &\qquad\qquad \text{for } i, j \in \{0, 1, \dots, n\}\Big\}. \end{aligned}$$

Moreover, we can determine $X^{\pm}_{\{i\}\{j\}}$ defined in (12.2.18) as follows: for $i, j \in [n]$,

$$X^+_{\{i\}\{0\}} = Q_i^+, \quad X^-_{\{i\}\{0\}} = Q_i^-, \quad X^+_{\{i\}\{j\}} = Q_i^+ \cup Q_j^- \ \text{ and } \ X^-_{\{i\}\{j\}} = Q_j^+ \cup Q_i^-.$$

Hence, we get

$$\begin{aligned} \mathcal{C}(G_{r_1,\dots,r_n}) = \{(z_1, \cdots, z_n) \in \mathbb{R}^n : & -r_i \leq z_i \leq r_i \text{ for } i \in [n], \\ & -r_i - r_j \leq z_i - z_j \leq r_i + r_j \text{ for } i, j \in [n]\}. \end{aligned}$$

Clearly, the inequality $-r_i - r_j \leq z_i - z_j \leq r_i + r_j$ can be omitted, and hence we obtain the desired result. $\square$

13.3.2 Construction of an NCCR for $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$

This subsection is devoted to giving a proof of the following theorem:

Theorem 13.3.3. *Let*

$$\mathcal{L} = \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : 0 \le z_i \le r_i \text{ for } i \in [n]\} \subset \mathcal{C}(G_{r_1,\dots,r_n}) \cap \mathbb{Z}^n.$$

Then, $E = \mathrm{End}_R(M_{\mathcal{L}})$ is an NCCR of $R = \Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$. In particular, $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ has a toric NCCR.

We set $\widetilde{\mathcal{L}} = \mathcal{C}(G_{r_1,\dots,r_n}) \cap \mathbb{Z}^n$. Note that $\widetilde{\mathcal{L}} = \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : -r_i \le z_i \le r_i \text{ for } i \in [n]\}$ from Proposition 13.3.2 (v).

In [43, Theorem 3.6], a toric NCCR is given for the Segre product of polynomial rings by taking $\mathcal{L}, \widetilde{\mathcal{L}} \subset X(G)$ (these are different from $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ defined above, but very similar) and using the same arguments as in Lemma 13.1.3. Indeed, the proof of Theorem 13.3.3 can be obtained by the same procedure as that of [43, Theorem 3.6]. However, we give a self-contained proof since $\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ is not the Segre product of polynomial rings.

Proof of Theorem 13.3.3. It is enough to show that $\widetilde{\mathcal{L}}$ satisfies the conditions (a) and (b) in Lemma 13.1.2. First, we can check (a) since

$$\{\chi - \chi' : \chi, \chi' \in \mathcal{L}\} = \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : -r_i \le z_i \le r_i \text{ for } i \in [n]\} = \widetilde{\mathcal{L}},$$

and M_χ is a conic divisorial ideal of R for any $\chi \in \widetilde{\mathcal{L}}$.

Next, we show (b). We see that $\mathcal{L} \subset \mathcal{W}(R) \cap X(G) = \mathcal{C}(G_{r_1,\dots,r_n}) \cap X(G) = \widetilde{\mathcal{L}}$. We have already computed the weights of R in the proof of Proposition 13.3.2 (v). We set $\bar{\beta}_i = \mathbf{e}_i$, $\bar{\beta}_{n+i} = -\mathbf{e}_i$ for $i \in [n]$ and $\bar{\beta}_{2n+1} = \mathbf{e}_0$. Note that for $i \in [n]$, the multiplicity m_i of $\bar{\beta}_i$ is equal to $r_i + 1$. We also set

$$\widetilde{\mathcal{L}}_j = \{(z_1, \cdots, z_n) \in \widetilde{\mathcal{L}} : z_j \ge 0, \cdots, z_n \ge 0\}$$

for $j \in [n]$ and

$$\widetilde{\mathcal{L}}_j(k) = \{(z_1, \cdots, z_n) \in \widetilde{\mathcal{L}}_j : z_j \ge -k\}.$$

for $0 \le k \le r_j$. Note that $\mathcal{L} = \widetilde{\mathcal{L}}_1 \subset \widetilde{\mathcal{L}}_2 \subset \cdots \subset \widetilde{\mathcal{L}}_n \subset \widetilde{\mathcal{L}}$, $\widetilde{\mathcal{L}}_{j+1}(0) = \widetilde{\mathcal{L}}_j(r_j)$ for $j \in [n-1]$ and $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_n(r_n)$. Moreover, for any $j \in [n]$, $k \in [r_j]$ and any $\chi \in \widetilde{\mathcal{L}}_j(k) \setminus \widetilde{\mathcal{L}}_j(k-1)$, we can see that

$$\langle \mathbf{e}_j, \chi \rangle < \langle \mathbf{e}_j, \chi' \rangle \text{ for any } \chi' \in \widetilde{\mathcal{L}}_j(k-1).$$

Hence, χ is separated from $\widetilde{\mathcal{L}}_j(k-1)$ by $\mathbf{e}_j$, and we have that $\langle \mathbf{e}_j, \bar{\beta}_j \rangle > 0$ and $\langle \mathbf{e}_j, \bar{\beta}_i \rangle \le 0$ for any $i \ne j$.

We prove that $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $j \in [n]$, $k \in \{0, 1, \dots, r_j\}$ and $\chi \in \widetilde{\mathcal{L}}_j(k)$ by the induction on j and k.

($j = 1$ and $k = 0$) In this case, we have $\widetilde{\mathcal{L}}_1(0) = \mathcal{L}$. Hence, $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $\chi \in \mathcal{L}$ by Lemma 13.1.3 (i).

($j = 1$ and $k > 0$) Assume that $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $\chi \in \widetilde{\mathcal{L}}_1(k-1)$. Then, for any $\chi' \in \widetilde{\mathcal{L}}_1(k) \setminus \widetilde{\mathcal{L}}_1(k-1)$, we see that $\chi' + \beta_{i_1} + \cdots + \beta_{i_p} \in \widetilde{\mathcal{L}}_1(k-1)$, where $\beta_{i_1} = \cdots = \beta_{i_p} = \bar{\beta}_1$ and $p \in [r_1 + 1]$. Hence, $\mathrm{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi'} < \infty$ by Lemma 13.1.3 (ii).

($j > 1$ and $k \geq 0$) Assume that $\operatorname{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $\chi \in \widetilde{\mathcal{L}}_{j-1}(r_{j-1})$. The case $k = 0$ is trivial since $\widetilde{\mathcal{L}}_j(0) = \widetilde{\mathcal{L}}_{j-1}(r_{j-1})$. Suppose that $k > 0$ and $\operatorname{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $\chi \in \widetilde{\mathcal{L}}_j(k-1)$. Then, for any $\chi' \in \widetilde{\mathcal{L}}_j(k) \backslash \widetilde{\mathcal{L}}_j(k-1)$, we see that $\chi' + \beta_{i_1} + \cdots + \beta_{i_p} \in \widetilde{\mathcal{L}}_j(k-1)$, where $\beta_{i_1} = \cdots = \beta_{i_p} = \bar{\beta}_j$ and $p \in [r_j + 1]$. Hence, $\operatorname{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi'} < \infty$ by Lemma 13.1.3 (ii).

Consequently, we obtain that $\operatorname{pdim}_{\Lambda_{\mathcal{L}}} P_{\mathcal{L},\chi} < \infty$ for any $\chi \in \widetilde{\mathcal{L}}_n(r_n) = \widetilde{\mathcal{L}}$.

□

Bibliography

[1] K. Bhaskara, A. Higashitani, and N. Shibu Deepthi, *h-vectors of edge rings of odd-cycle compositions*,arXiv:2311.13573.

[2] M. Beck and S. Robins, "Computing the continuous discretely", Undergraduate Texts in Mathematics. Springer, New York, second edition, 2015.

[3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. M. Ziegler, Oriented Matroids. Second edition. Encyclopedia of Mathematics and its Applications, 46. Cambridge University Press, Cambridge, 1999.

[4] R. Bocklandt, Generating toric noncommutative crepant resolutions, *J. Algebra*, **364** (2012), 119–147.

[5] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, (English summary) Computational algebra and number theory (London, 1993), *J. Symbolic Comput*, **24** (1997), no. 3–4, 235-–265.

[6] N. Broomhead, Dimer model and Calabi-Yau algebras, Mem. Amer. Math. Soc., **215** no. 1011, (2012).

[7] W. Bruns, Conic divisor classes over a normal monoid algebra, *Commutative algebra and algebraic geometry, Contemp. Math.*, **390**, Amer. Math. Soc., (2005), 63–71.

[8] W. Bruns and J. Gubeladze, Divisorial linear algebra of normal semigroup rings, *Algebra and Represent. Theory* **6** (2003), 139–168.

[9] W. Bruns and J. Gubeladze, Polytopes, rings and K-theory, Springer Monographs in Mathematics. Springer, Dordrecht, (2009).

[10] W. Bruns and J. Herzog, "Cohen-Macaulay rings, revised edition", Cambridge University Press, 1998.

[11] L. Chouinard, Krull semigroups and divisor class groups, *Canad. J. Math.* **33** (1981), 1459–1468.

[12] V. Chvátal, On certain polytopes associated with graphs, *J. Combin. Theory, Ser.* (B) **18** (1975), 138–154.

[13] D. A. Cox, J. B. Little and H. K. Schenck, Toric varieties. Graduate Studies in Mathematics, **124**, American Mathematical Society, Providence, RI, 2011.

[14] H. Dao, O. Iyama, R. Takahashi and C. Vial, *Non-commutative resolutions and Grothendieck groups*, J. Noncommut. Geom., **9** (2015) no. 1, 21–34.

[15] H. Dao, O. Iyama, R. Takahashi and M. Wemyss, Gorenstein modifications and $\mathbb{Q}$-Gorenstein rings, *J. Algebraic Geom.* **29** (2020), 729–751.

[16] E. De Negri and T. Hibi, Gorenstein algebras of Veronese type, *J. Algebra* **193** (1997), 629–639.

[17] R. Diestel, "Graph theory" Fifth edition, GTM **173**, Springer, Berlin, 2017.

[18] E. Faber, G. Muller and K. E. Smith, Non-commutative resolutions of toric varieties, *Adv. Math.* **351** (2019), 236–274.

[19] G. Favacchio, J. Hofscheier, G. Keiper, and A. Van Tuyl, Splittings of toric ideals, *J. Algebra*, 574:409–433, 2021.

[20] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, *Advances in Mathematics*, 63(3):301–316, 1987.

[21] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[22] S. Goto, R. Takahashi, N. Taniguchi, Almost Gorenstein rings – towards a theory of higher dimension, *J. Pure Appl. Algebra* **219**, (2015), 2666–2712.

[23] J. Gouveia, P. A. Parrilo and R. R. Thomas, Theta bodies for polynomial ideals, *SIAM J. Optim.* **20** (2010), 2097–2118.

[24] D. Grayson and M. Stillman. Macaulay2, a software system for research in algebraic geometry, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[25] B. Grünbaum, "Convex Polytopes" Second edition, Graduate Texts in Mathematics, **221**, Springer, New York, (2003).

[26] C. Haase and J. Hofmann, Convex-normal (pairs of) polytopes, *Canadian Mathematical Bulletin*, 60(3):510–521, 2017.

[27] T. Hall, M. Kölbl, K. Matsushita and S. Miyashita, Nearly Gorenstein Polytopes, *Electron. J. Combin.*, **30** (2023), no. 4, Paper No. 4.42.

[28] M. Hashimoto, T. Hibi and A. Noma, Divisor class groups of affine semigroup rings associated with distributive lattices, *J. Algebra* **149** (2), (1992), 352–357.

[29] J. Herzog, T. Hibi and H. Ohsugi, Binomial ideals, Graduate Texts in Mathematics, **279**, Springer, Cham, (2018).

[30] J. Herzog, T. Hibi, and D. I. Stamate, The trace of the canonical module, *Israel J. Mathematics*, 233:133–165, (2019).

[31] J. Herzog, F. Mohammadi and J. Page, Measuring the non-Gorenstein locus of Hibi rings and normal affine semigroup rings, *J. Algebra*, 540:78–99, 2019.

[32] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws. In: Nagata, M., Matsumura, H. (eds.) Commutative Algebra and Combinatorics. Advanced Studies in Pure Mathematics, vol. 11, pp. 93–109. North-Holland, Amsterdam (1987).

[33] T. Hibi, Dual polytopes of rational convex polytopes, *Combinatorica*, 12:237–240, 1992.

[34] T. Hibi, M. Lasoń, K. Matsuda, M. Michałek and M. Vodička, Gorenstein graphic matroids, *Israel J. Mathematics*, 243(1):1–26, 2021.

[35] T. Hibi and N. Li, Unimodular Equivalence of Order and Chain polytopes, *Math. Scand.* **118**, No. 1 (2016), 5–12.

[36] T. Hibi and D. I. Stamate, Nearly Gorenstein rings arising from finite graphs, *Electron. J. Combin.* **28**(3), (2021), 3.28.

[37] T. Hibi and A. Tsuchiya, Odd cycles and Hilbert functions of their toric rings, *Mathematics*, 8(1):22, 2019.

[38] A. Higashitani, Almost Gorenstein homogeneous rings and their h-vectors, *J. Algebra* **456** (2016), 190–206.

[39] A. Higashitani and N. Shibu Deepthi, The h-vectors of the edge rings of a special family of graphs, *Communications in Algebra*, 51(12):5287–5296, 2023.

[40] A. Higashitani and K. Matsushita, Conic divisorial ideals and non-commutative crepant resolutions of edge rings of complete multipartite graphs, *J. Algebra* **594** (2022), 685–711.

[41] A. Higashitani and K. Matsushita, Three families of toric rings arising from posets or graphs with small class groups, *J. Pure Appl. Algebra*, **226** no. 10, (2022), 107079.

[42] A. Higashitani and K. Matsushita, Levelness versus almost Gorensteinness of edge rings of complete multipartite graphs, *Communications in Algebra*, **50** (6):2637–2652, 2022.

[43] A. Higashitani and Y. Nakajima, Conic divisorial ideals of Hibi rings and their applications to non-commutative crepant resolutions, *Selecta Math.* **25** (2019), 25pp.

[44] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *Ann. Math.* **96** (1972), 318–337.

[45] A. Ishii and K. Ueda, Dimer models and the special McKay correspondence, *Geom. Topol.* **19**, (2015), 3405–3466.

[46] O. Iyama and M. Wemyss, Maximal modifications and Auslander–Reiten duality for non-isolated singularities, *Invent. Math.* **197**(3), (2014), 521–586.

[47] V. Kaibel and M. Wolff, Simple 0/1-polytopes, European *J. Combinatorics* **21** (2000), 139-144.

[48] M. Kölbl, Gorenstein graphic matroids from multigraphs, *Annals of Combinatorics*, 24(2):395–403, 2020.

[49] K. Matsuda, H. Ohsugi and K. Shibata, Toric Rings and Ideals of Stable Set Polytopes, *Mathematics* **7** (2019), 7(7), 613.

[50] K. Matsushita, Torsionfreeness for divisor class groups of toric rings of integral polytopes, *J. Algebra*, **644** (2024), 749–760.

[51] K. Matsushita, Conic divisorial ideals of toric rings and applications to Hibi rings and stable set rings, arXiv:2210.02031.

[52] K. Matsushita, Toric rings of $(0,1)$-polytopes with small rank, *Illinois J. Math.*, to appear.

[53] K. Matsushita, S. Miyashita, Conditions of multiplicity and applications for almost Gorenstein graded rings, arXiv:2311.17387.

[54] S. Miyashita, Levelness versus nearly Gorensteinness of homogeneous rings, *J. Pure Appl. Algebra*, **228** (2024), no. 4, 107553.

[55] S. Miyashita, Comparing generalized Gorenstein properties in semi-standard graded rings, *J. Algebra*, **647** (2024), 823-–843.

[56] M. Miyazaki, On the generators of the canonical module of a Hibi ring: A criterion of level property and the degrees of generators, *J. Algebra* **480** (2017), 215–236.

[57] M. Miyazaki, Almost Gorenstein Hibi rings, *J. Algebra* **493** (2018), 135–149.

[58] M. Miyazaki, Gorenstein on the punctured spectrum and nearly Gorenstein property of the Ehrhart ring of the stable set polytope of an h-perfect graph, *arXiv:2201.02957.*

[59] M. Miyazaki, Non-Gorenstein locus and almost Gorenstein property of the Ehrhart ring of the stable set polytope of a cycle graph, *Taiwanese J. Mathematics*, 27(3):441–459, 2023.

[60] M. Mustaţa and S. Payne, Ehrhart polynomials and stringy Betti numbers, *Math. Ann.*, 333:787–795, 2005.

[61] Y. Nakajima, Non-commutative crepant resolutions of Hibi rings with small class groups, *J. Pure Appl. Algebra* **223** (2019), 3461–3484.

[62] M. Newman, Integral Matrices, Pure and Applied Mathematics **45**, Academic Press, New York, 1972.

[63] J. Oxley, Matroid theory, Oxford Graduate Texts in Mathematics, Vol. 21, Oxford University Press, Oxford, second edition, 2011.

[64] H. Ohsugi and T. Hibi, Normal polytopes arising from finite graphs, *J. Algebra* **207** (1998), 409–426.

[65] H. Ohsugi and T. Hibi, Compressed polytopes, initial ideals and complete multipartite graphs, *Illinois J. Math.* **44**, No. 2 (2000), 391–406.

[66] H. Ohsugi and T. Hibi, Convex polytopes all of whose reverse lexicographic initial ideals are squarefree, *Proc. Amer. Math. Soc.*, **129** (2001), No.9, 2541–2546.

[67] H. Ohsugi and T. Hibi, Special simplices and Gorenstein toric rings, *J. Combin. Theory Ser.* A **113** (2006), no. 4, 718—725.

[68] K. E. Smith and M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, *Proc. London Math. Soc.* (3) **75** (1997), no. 1, 32–62.

[69] A. Simis, W. V. Vasconcelos and R. H. Villarreal, The integral closure of subrings associated to graphs, *J. Algebra* **199** (1998), 281–289.

[70] Š. Špenko and M. Van den Bergh, Non-commutative resolutions of quotient singularities for reductive groups, *Invent. Math.* **210** (2017), no. 1, 3–67.

[71] Š. Špenko and M. Van den Bergh, Non-commutative crepant resolutions for some toric singularities I, *Int. Math. Res. Not. IMRN* (2020), no. 21, 8120—8138.

[72] Š. Špenko and M. Van den Bergh, *Non-commutative crepant resolutions for some toric singularities II*, J. Noncommut. Geom. **14** (2020), no. 1, 73-–103.

[73] Š. Špenko and M. Van den Bergh, J. P. Bell, On the noncommutative Bondal-Orlov conjecture for some toric varieties. *Math. Z.* **300** (2022), no. 1, 1055–1068.

[74] R. P. Stanley, Cohen–Macaulay complexes, in: M. Aigner (Ed.), Higher Combinatorics, Reidel, Dor-drecht and Boston, (1977), 51–62.

[75] R.P. Stanley. Hilbert functions of graded algebras. *Adv. Math.* **28** (1978), 57–83.

[76] R. P. Stanley, Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333–342. MR **82a:**52007.

[77] R. P. Stanley, Linear diophantine equations and local cohomology, *Invent. Math.* **68** (1982), 175–193.

[78] R. P. Stanley, Two Poset Polytopes, *Discrete Comput. Geom.* **1** (1986), 9–23.

[79] S. Sullivant, Compressed polytopes and statistical disclosure limitation, *Tohoku Math. J.* **58** (2006), 433–445.

[80] K. Truemper, Matroid decomposition, volume 6, Citeseer, 1992.

[81] M. Van den Bergh, Cohen-Macaulayness of semi-invariants for tori, *Trans. Amer. Math. Soc.* **336** (1993), no. 2, 557–580.

[82] M. Van den Bergh, Non-Commutative Crepant Resolutions, *The Legacy of Niels Henrik Abel*, Springer-Verlag, Berlin, (2004), 749–770.

[83] N. Viêt, J. Gubeladze and W. Bruns, Normal polytopes, triangulations, and Koszul algebras, *Journal für die reine und angewandte Mathematik*, 485:123–160, 1997.

[84] R. H. Villarreal, "Monomial algebras", Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.

[85] N. L. White, The basis monomial ring of a matroid, *Advances in Mathematics*, 24(2):292–297, 1977.

[86] K. Yanagawa, Castelnuovo's Lemma and h-vectors of Cohen–Macaulay homogeneous domains, *J. Pure Appl. Algebra* **105**, (1995) 107–116.

[87] G. M. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, **152**, Springer-Verlag, New York, 1995.