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ON THE Λ -MODULE STRUCTURES OF τ -HOMOTOPY GROUPS OF $X \wedge S_+^{1,0}$

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Introduction. Let X be a pointed τ -complex. The stable τ -homotopy group $\pi_{p,q}^S(X \wedge S_+^{1,0})$ is the E^1 -term of the forgetful spectral sequence associated with $\pi_{*,*}^S(X)$ [1], and is isomorphic to $\pi_{p+q}^S(X)$ additively since $S_+^{1,0}$ is an equivariant S -dual of itself [3]. Moreover, ρ acts as -1 on $\pi_{p,q}^S(X \wedge S_+^{1,0})$ [2]. (See [2], p. 365 for the definition of ρ). We define Λ to be the ring $\mathbb{Z}[\rho]/(1-\rho^2)$. The purpose of this paper is to prove the following theorem on the (unstable) τ -homotopy groups $\pi_{p,q}(X \wedge S_+^{1,0})$.

Theorem. Let $p \geq 1$ and $q \geq 2$. If $\pi_k(X \times X, X \vee X) = 0$ for each $k, q+2 \leq k \leq p+q+1$, then there exists an isomorphism of abelian groups

$$\phi_{p,q}: \pi_{p,q}(X \wedge S_+^{1,0}) \cong \pi_{p+q}(X) \oplus \pi_{q+1}(X).$$

Furthermore, the ρ -action on $\pi_{p,q}(X \wedge S_+^{1,0})$ is given by

$$\begin{aligned} \rho \cdot (\alpha, \beta) &= (-\alpha, \beta), & p \geq 2 \\ \rho \cdot (\alpha, \beta) &= (-\alpha, \alpha + \beta), & p = 1 \end{aligned}$$

where (α, β) is an element of $\pi_{p,q}(X \wedge S_+^{1,0})$ via $\phi_{p,q}$.

See §1 for the definition of $\phi_{p,q}$.

EXAMPLE. If $p+q+2 \leq 2n$, then we have

$$\pi_{p,q}(S^n X \wedge S_+^{1,0}) \cong \pi_{p+q}(S^n X) \oplus \pi_{q+1}(S^n X),$$

since $\pi_k(S^n X \times S^n X, S^n X \vee S^n X) = 0$ for $k \leq 2n-1$.

Corollary (cf. [2], Proposition 3.6). Let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. Then $\pi_{p,q}^S(X \wedge S_+^{1,0}) \cong \pi_{p+q}^S(X)$ and ρ acts as -1 on $\pi_{p,q}^S(X \wedge S_+^{1,0})$.

This follows from the above theorem, since $\pi_{q+n+1}(S^{2n} X) = 0$ for sufficiently large n .

Notations and elementary results in [2] are used freely.

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1. Let Y be a pointed τ -complex with involution τ . We recall first the forgetful exact sequence ([2], (10.5)),

$$(1) \quad \begin{array}{ccccccc} \cdots & \rightarrow & \pi_{r-1,s+1}(Y) & \xrightarrow{\psi_{r-1,s+1}} & \pi_{r+s}(Y) & \xrightarrow{\delta_{r,s}^*} & \pi_{r,s}(Y) \\ & & & & \searrow \chi_{r,s} & & \\ & & & & \pi_{r-1,s}(Y) & \xrightarrow{\psi_{r-1,s}} & \pi_{r+s-1}(Y) \rightarrow \cdots \end{array}$$

where $r \geq 1$. Moreover, by [2] Lemma 12.6, we have

$$(2) \quad \delta_{r,s}^* \psi_{r,s} = 1 - \rho \quad (\text{times}).$$

We denote by $\tau_*: \pi_{r+s}(Y) \rightarrow \pi_{r+s}(Y)$ the homomorphism induced by τ . Then we have

Proposition 1. $\psi_{r,s} \delta_{r,s}^* = 1 + (-1)^r \tau_*$.

Proof. Let $\alpha \in \pi_{r+s}(Y)$. Since $\psi_{r,s} \delta_{r,s}^*(\alpha)$ is an element of

$$\pi_{r+s}(Y) = [\Sigma^{r,s} \wedge S_+^{1,0}, Y]^r,$$

we obtain

$$\begin{aligned} & \psi_{r,s} \delta_{r,s}^*(\alpha)(s_1, \dots, s_r, t_1, \dots, t_s, -1) \\ &= \begin{cases} \alpha(s_1, \dots, s_{r-1}, 2s_r+1, t_1, \dots, t_s, -1), & -1 \leq s_r \leq 0, \\ \alpha(s_1, \dots, s_{r-1}, 1-2s_r, t_1, \dots, t_s, +1), & 0 \leq s_r \leq 1, \end{cases} \\ &= \begin{cases} \alpha(s_1, \dots, s_{r-1}, 2s_r+1, t_1, \dots, t_s), & -1 \leq s_r \leq 0, \\ \tau_*(\alpha)(-s_1, \dots, -s_{r-1}, 1-2s_r, t_1, \dots, t_s), & 0 \leq s_r \leq 1. \end{cases} \end{aligned}$$

This yields the result.

Let X be a pointed τ -complex. Hereafter, we shall consider the case $Y = X \wedge S_+^{1,0}$. Let $X \vee X$ be a τ -complex with involution defined by $\tau(x, *) = (*, x)$. Since $X \wedge S_+^{1,0}$ is τ -homeomorphic to $X \vee X$ ([2], p. 370), $X \wedge S_+^{1,0}$ may be replaced by $X \vee X$ in the τ -homotopy groups. Thus we may assume that the involution on X is trivial.

As is well known, there exists an isomorphism

$$\pi_k(X \vee X) \cong \pi_k(X) \oplus \pi_k(X) \oplus \pi_{k+1}(X \times X, X \vee X).$$

Here we forget the involution of $X \vee X$, as usual.

If $\pi_{k+1}(X \times X, X \vee X) = 0$, then we have

$$\pi_k(X \vee X) \cong \pi_k(X) \oplus \pi_k(X).$$

By this isomorphism, we identify $\pi_k(X \vee X)$ with $\pi_k(X) \oplus \pi_k(X)$.

Let $p \geq 1$ and $q \geq 2$. In the following Lemmas 2 and 3, we assume $\pi_{p+q+1}(X \times X, X \vee X) = 0$.

Lemma 2. $\phi_{p,q} \delta_{p,q}^*(\alpha, \beta) = (\alpha + (-1)^p \beta, \beta + (-1)^p \alpha)$.

Proof. Let $\tau_*: \pi_{p+q}(X \vee X) \rightarrow \pi_{p+q}(X \vee X)$ be the map induced by τ . Then $\tau_*(\alpha, \beta) = (\beta, \alpha)$ where $(\alpha, \beta) \in \pi_{p+q}(X \vee X) \cong \pi_{p+q}(X) \oplus \pi_{p+q}(X)$. Thus, Lemma 2 follows from Proposition 1.

Lemma 3. Let $u \in \pi_{p,q}(X \vee X)$. Then $\phi_{p,q}(u)$ is of the form $(\alpha, (-1)^p \alpha)$ with $\alpha \in \pi_{p+q}(X)$.

Proof. Put $\psi_{p,q}(u) = (\alpha, \beta)$. It is sufficient to prove $\beta = (-1)^p \alpha$. Apply $\psi_{p,q}$ to $\delta_{p,q}^* \psi_{p,q}(u) = u - \rho \cdot u$ (2). Since $\psi_{p,q} \circ \rho = -\psi_{p,q}$ ([2], (9.9)), Lemma 2 shows that

$$(\alpha + (-1)^p \beta, \beta + (-1)^p \alpha) = (\alpha, \beta) + (\alpha, \beta),$$

and so $\beta = (-1)^p \alpha$ as required.

Since $\pi_{0,k}(X \vee X) = 0$, it follows from (1) with $Y = X \vee X$, $r = 1$ and $s = q$, that there exists an isomorphism

$$\delta_{1,q}^*: \pi_{q+1}(X \vee X) \xrightarrow{\cong} \pi_{1,q}(X \vee X).$$

Suppose that $\pi_{q+2}(X \times X, X \vee X) = 0$. We then define the homomorphism $\mathcal{X}_{p,q}: \pi_{p,q}(X \vee X) \rightarrow \pi_{q+1}(X)$ by the composition $p_2(\delta_{1,q}^*)^{-1} \mathcal{X}_{2,q} \cdots \mathcal{X}_{p,q}$ if $p \geq 2$, and define $\mathcal{X}_{1,q}$ by $p_2(\delta_{1,q}^*)^{-1}$, where p_2 denotes the projection to the second factor.

Moreover, suppose $\pi_{p+q+1}(X \times X, X \vee X) = 0$. Then we define the homomorphism

$$\phi_{p,q}: \pi_{p,q}(X \vee X) \rightarrow \pi_{p,q}(X) \oplus \pi_{q+1}(X)$$

by

$$\phi_{p,q}(u) = (p_1 \psi_{p,q}(u), \mathcal{X}_{p,q}(u)),$$

where p_1 denotes the projection to the first factor.

2. We shall prove the theorem by the induction on p . Let $p = 1$. Put $(\delta_{1,q}^*)^{-1}(u) = (\alpha, \beta)$, and we obtain

$$\begin{aligned} \phi_{1,q}(u) &= (p_1 \psi_{1,q}(u), \mathcal{X}_{1,q}(u)) && \text{by definition} \\ &= (p_1 \psi_{1,q} \delta_{1,q}^*(\alpha, \beta), p_2(\alpha, \beta)) \\ &= (\alpha - \beta, \beta) && \text{by Lemma 2.} \end{aligned}$$

Therefore, $\phi_{1,q}$ is an isomorphism. When $p \geq 2$, by the induction hypothesis, we get an isomorphism

$$\phi_{p-1,q}: \pi_{p-1,q}(X \vee X) \cong \pi_{p+q-1}(X) \oplus \pi_{q+1}(X).$$

Consider the forgetful exact sequence (1) with $Y = X \vee X$, $r = p$ and $s = q$. Let $u \in \pi_{p,q}(X \vee X)$. Then $\phi_{p-1,q}(\mathcal{X}_{p,q}(u)) = (0, \mathcal{X}_{p,q}(u))$ since $\phi_{p-1,q}\mathcal{X}_{p,q} = 0$.

We assume $\phi_{p,q}(u) = (0, 0)$. Then $p_1\phi_{p,q}(u) = 0$ and $\mathcal{X}_{p,q}(u) = 0$ by definition. Since $\phi_{p-1,q}$ is an isomorphism, we get $\mathcal{X}_{p,q}(u) = 0$. Therefore, there exists an element $(\alpha, \beta) \in \pi_{p+q}(X \vee X)$ such that $\delta_{p,q}^*(\alpha, \beta) = u$. It follows from Lemma 2 that (α, β) is congruent to $(\alpha + (-1)^p\beta, 0) \bmod \text{Im } \phi_{p-1,q+1}$. Thus $\delta_{p,q}^*(\alpha + (-1)^p\beta, 0) = u$. Applying $p_1\phi_{p,q}$ to this, we get an equality

$$\alpha + (-1)^p\beta = p_1\phi_{p,q}(u) = 0$$

which implies $u = 0$. Hence $\phi_{p,q}$ is a monomorphism.

We show that $\phi_{p,q}$ is an epimorphism. Let $(\alpha, \beta) \in \pi_{p+q}(X) \oplus \pi_{q+1}(X)$. By Lemma 3, $\phi_{p-1,q}(\phi_{p-1,q})^{-1}(0, \beta) = (0, 0)$. Therefore, there exists an element $v \in \pi_{p,q}(X \vee X)$ such that $\mathcal{X}_{p,q}(v) = (\phi_{p-1,q})^{-1}(0, \beta)$. This implies $\mathcal{X}_{p,q}(v) = \beta$. Then we have

$$\phi_{p,q}(\delta_{p,q}^*(\alpha, 0) + v - \delta_{p,q}^*(p_1\phi_{p,q}(v), 0)) = (\alpha, \beta)$$

as can be easily checked.

We now turn to the ρ -action on $\pi_{p,q}(X \vee X)$. Let $u \in \pi_{p,q}(X \vee X)$ and $\phi_{p,q}(u) = (\alpha, \beta)$. Then $\psi_{p,q}(u) = (\alpha, (-1)^p\alpha)$ by Lemma 3. From (2), we obtain $\delta_{p,q}^*(\alpha, (-1)^p\alpha) = u - \rho \cdot u$. Recall that $(\alpha, 0)$ is congruent to $(0, (-1)^p\alpha) \bmod \text{Im } \phi_{p-1,q+1}$ if $p \geq 2$. Thus $2 \cdot \delta_{p,q}^*(\alpha, 0) = u - \rho \cdot u$. Applying $\phi_{p,q}$, we obtain

$$2 \cdot (\alpha, 0) = \phi_{p,q}(u) - \phi_{p,q}(\rho \cdot u).$$

This shows that $\phi_{p,q}(\rho \cdot u) = (-\alpha, \beta)$ for $p \geq 2$. Let $u \in \pi_{1,q}(X \vee X)$ and $\phi_{1,q}(u) = (\alpha, \beta)$. Then $\psi_{1,q}(u) = (\alpha, -\alpha)$ by Lemma 3. The same method gives rise to

$$\phi_{1,q}\delta_{1,q}^*(\alpha, -\alpha) = (\alpha, \beta) - \phi_{1,q}(\rho \cdot u).$$

By the definition of $\phi_{1,q}$, the left hand side coincides with $(2\alpha, -\alpha)$. Hence $\phi_{1,q}(\rho \cdot u) = (-\alpha, \alpha + \beta)$ as required. This completes the proof of the theorem stated in the introduction.

REMARK. Let $X \times X$ be a τ -space with involution defined by $\tau(x_1, x_2) = (x_2, x_1)$. Then we have an isomorphism of abelian groups

$$\xi_{p,q}: \pi_{p,q}(X \times X) \cong \pi_{p+q}(X) \quad \text{for } p \geq 1, q \geq 2.$$

The correspondence is given by $\xi_{p,q}(u) = p_1\psi_{p,q}(u)$. Moreover, ρ acts as -1 on $\pi_{p,q}(X \times X)$.

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