Title	Studies on tropical plane curves in the symmetric truncated cubic forms
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Citation	大阪大学, 2024, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/98708
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Studies on tropical plane curves in the symmetric truncated cubic forms

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July 8, 2024

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Contents

Chapter 1

Introduction

1.1 Overview

The Min-Plus semiring is a semiring in which the sum of any two elements is defined as their minimum, and their product is defined as their sum. When the minimum operation is replaced by the maximum, the result is an isomorphic semiring known as the Max-Plus semiring. In 1978, Imre Simon introduced the semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ and later summarized several early results related to semirings in [IS88] where the Min-Plus semiring is characterized as the tropical semiring. J.-É.Pin discussed the development of problems related to tropical semirings in [P98]. In this framework, from well-known rings such as $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$, one can define their tropical counterparts–tropical reals, tropical rationals, and tropical integers–by augmenting the set with $\{\infty\}$.

J.Richter-Gebert, B.Sturmfels, and T.Theobald elaborated on the geometry over tropical semirings in [RST05]. They defined the tropical hypersurface of a tropical polynomial function. The tropical hypersurface of a two-variable tropical polynomial is referred to as a tropical curve. Furthermore, they also proved the tropical version of Bézout's Theorem. G. Mikhalkin, in [M05], showed a remarkable connection between tropical geometry and classical geometry, specifically related to the number of curves of a given degree and genus.

Valuation maps allow us to tropicalize algebraic geometry objects, such as functions and hypersurfaces. For the field of p-adic numbers, the p-adic valuation can serve as the valuation map. Meanwhile, when working with the field of formal Laurent series, the order of an element can be used as the valuation map. M.Einsiedler, M.Kapranov, and D.Lind, in [EKL06], considered the Log function as the valuation map for the base field and showed that the tropicalization of the hypersurface of f and the tropical hypersurface of the tropical f are equal after compactification.

By employing the field of Puiseux series as the base field, E.Katz, H.Markwig, and T.Markwig demonstrated the relationship between the j-invariant of a cubic polynomial and the cycle length of its tropical hypersurface in [KMM08]. In [M10], T.Markwig argued that the field of generalized Puiseux series serves as an ideal base field for concurrently working with algebraic and tropical geometry.

M.D.Vigeland established a connection to the group law on plane cubic curves by defining a group law on smooth plane tropical cubic curves in [V09]. Various works in tropical geometry have focused on specific two-variable polynomials and explored their tropical geometric properties. In [CS13], M.Chan and B.Sturmfels identified the plane cubic curves whose tropicalizations feature a hexagonal cycle and provided a detailed analysis of the tropical group law on these curves. Additionally, A.Nobe demonstrated in [N08] that the cycles of the tropical curves of a cubic share shapes with the uQRT maps, and further showed how point addition on a tropical curve can be understood through the uQRT map.

K.Kajiwara, M.Kaneko, A.Nobe, and T.Tsuda studied the tropicalization of the Hesse cubic curve and presented a duplication map of points on a specific nonsmooth tropical cubic curve in [KKNT09]. They constructed the level-three theta functions parametrization of the Hesse cubic curve and applied the ultradiscretization procedure to reveal its connection with the tropical duplication process. Furthermore, Nobe, in [N11], provided the addition formula for points on the tropical Hesse cubic curve, examining the tropical addition via intersection points. In [N16], Nobe investigated the tropical analogue of the group of linear automorphisms acting on the Hessian cubic curve.

This study concentrates on the tropical curves derived from symmetric Laurent polynomials of degree three, specifically those in which the terms x^3 and y^3 are truncated. We examine the conditions under which these tropical curves are either smooth or nonsmooth, as discussed in [T23]. As an application, we investigate the criteria for a symmetric truncated cubic curve to satisfy the honeycomb form and to align with the invariant curves of uQRT maps. In [NT23], H.Nakamura and R.S.Tarmidi showed that the symmetric truncated cubic is birationally equivalent to a form of elliptic curves introduced in [E07] by H.M.Edwards. Furthermore, we also delineate the addition-group structure on these tropical curves. The theta parametrization of the Edwards elliptic curves enables us to observe point addition through the ultradiscretization procedure. Lastly, we looked into a cryptographic aspect utilizing this cubic curve.

1.2 Main results

In this thesis, we consider a symmetric truncated cubic polynomial

$$f(x,y) = c_{12}(xy^2 + x^2y) + c_{34}(x^2 + y^2) + c_5(xy) + c_{67}(x + y) + c_8 \in \mathbb{K}[x,y] \tag{1.1}$$

and its tropical polynomial

$$\begin{aligned}\mathrm{trop}(f)(X,Y) = \min(&v_{12} + X + 2Y, v_{12} + 2X + Y, v_{34} + 2X, v_{34} + 2Y,\\ &v_5 + X + Y, v_{67} + X, v_{67} + Y, v_8)\end{aligned} \tag{1.2}$$

where $v_k = \mathrm{val}(c_k) \in \mathbb{Q}$ for $k \in \{12, 34, 5, 67, 8\}$.

After reviewing basic notions in Chapter 2, we have the connections between the tropical curve and the Newton polygon of a tropical polynomial. We let Δ_f shown in

Figure 1.1 be the Newton polygon of $\text{trop}(f)(X,Y)$. When Δ_f is divided into several cells, we call it as a subdivision of Δ_f. We can also map the value $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ to a subdivision of Δ_f. We call such subdivisions as regular subdivisions of Δ_f (see Definition 2.0.5 for the precise definition). A unimodular subdivision is one of the examples of a regular subdivision. It partitions Δ_f into a collection of the finest cells, i.e. triangular cells of area $\frac{1}{2}$.

In Chapter 3, we see that Proposition 3.1.1 enables us to characterize tropical smooth curves by unimodular subdivisions of the Newton polygon of $\text{trop}(f)(X,Y)$. Since tropical polynomial (1.2) is symmetric with respect to the interchange of its variables X and Y, any regular subdivisions of the Newton polygon Δ_f exhibits symmetry with respect to the dashed line on Figure 1.1. Thus, the unimodular subdivisions are limited to the five cases in Theorem 1.2.1.

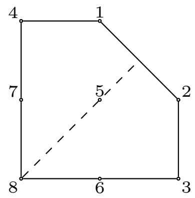


Figure 1.1: Newton polygon Δ_f.

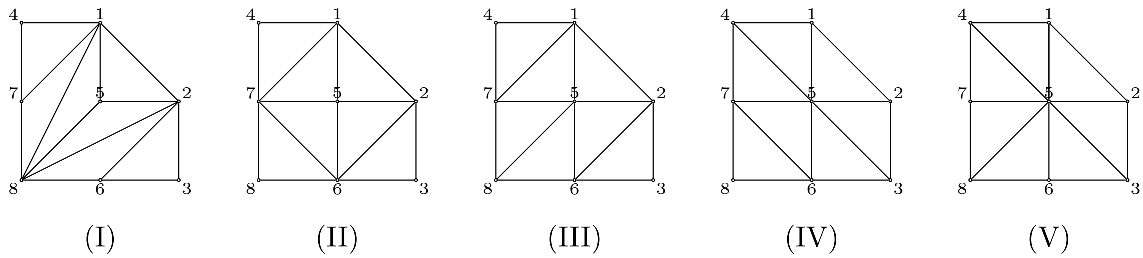


Figure 1.2: The unimodular subdivisions of Δ_f.

Theorem 1.2.1 (= Theorem 3.1.2). *Let $f(x,y)$ be the symmetric truncated cubic in equation* (1.1). *Then the possible cycles appearing in the tropical curves of* $\text{trop}(f)(X,Y)$ *are triangles, squares, pentagons, hexagons and heptagons. All possible unimodular subdivisions are as shown in Figure* 1.2, *respectively. Each of these cycles occurs if and only if* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ *satisfies inequalities listed in Table* 1.1.

We also discuss thoroughly about the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for the 17 non-smooth tropical curves of $\text{trop}(f)(X,Y)$. There are 23 other subdivisions of Δ_f as listed in Table 1.2. Among these subdivisions, 6 subdivisions are not related to the tropical curve $C(\text{trop}(f))$ for any $(v_{12}, v_{34}, v_5, v_{67}, v_8)$.

	Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$		
(I)	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - v_5 - v_{67} + v_8 < 0$	$-2v_{12} + 3v_5 - v_8 < 0$
(II)	$-v_5 + 2v_{67} - v_8 < 0$	$-v_{12} + 2v_5 - v_{67} < 0$	$v_{12} - v_{34} - v_5 + v_{67} < 0$
(III)	$v_5 - 2v_{67} + v_8 < 0$	$-v_{12} + v_5 + v_{67} - v_8 < 0$	$v_{12} - v_{34} - v_5 + v_{67} < 0$
(IV)	$-v_5 + 2v_{67} - v_8 < 0$	$-v_{34} + v_5 < 0$	$-v_{12} + v_{34} + v_5 - v_{67} < 0$
(V)	$v_5 - 2v_{67} + v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$-v_{12} + v_{34} + v_5 - v_{67} < 0$

Table 1.1: Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for all smooth tropical curves $C(\mathrm{trop}(f))$.

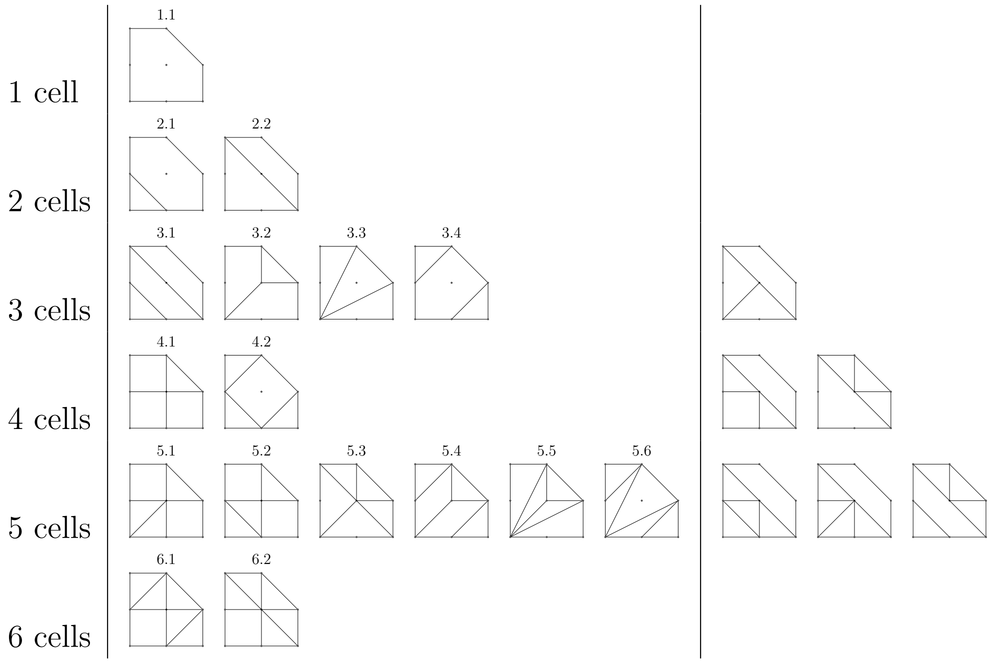


Table 1.2: Other subdivisions of Δ_f.

Theorem 1.2.2 (= Theorem 3.2.1)**.** *Let* $\mathrm{trop}(f)(X, Y)$ *be as defined in equation* (1.2) *and* Δ_f *be its Newton polygon. Then, the subdivisions on the right column of Table* 1.2 *never occur as the regular subdivisions of* Δ_f *for any* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$*.*

Theorem 1.2.3 (= Theorem 3.2.2)**.** *The conditions of* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ *for non-unimodular regular subdivisions in Table* 1.2 *are shown in Table* 1.3*.*

In Chapter 4, we provide some simple examples as applications of the characterization. We elaborated when truncated symmetric cubic forms satisfy the honeycomb form. We also analyzed the possible invariant curves of ultradiscrete QRT maps for various parameters. In Chapter 5, we observe the family of two-parameter Edwards curves $f_{r,s}$ by applying a unimodular transformation to the symmetric truncated cubic. In Chapter 6, we discussed the tropical group law of points on the curves of a symmetric truncated cubic polynomial.

	Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$		
1.1	$2v_{12} - 3v_{34} + v_8 = 0$	$-v_{12} + 2v_{34} - v_{67} \leq 0$	$v_{34} - v_5 \leq 0$
2.1	$v_{12} - 2v_{34} + v_{67} = 0$	$v_{34} - v_5 \leq 0$	$-v_{34} + 2v_{67} - v_8 < 0$
2.2	$v_{34} - 2v_{67} + v_8 \leq 0$	$v_{34} - v_5 \leq 0$	$-2v_{12} + 3v_{34} - v_8 < 0$
3.1	$v_{34} - v_5 \leq 0$	$-v_{12} + 2v_{34} - v_{67} < 0$	$-v_{34} + 2v_{67} - v_8 < 0$
3.2	$-2v_{12} + v_{34} + 2v_5 - v_8 = 0$	$v_{34} - 2v_{67} + v_8 \leq 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.3	$v_{34} - 2v_{67} + v_8 \leq 0$	$2v_{12} - 3v_5 + v_8 \leq 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.4	$v_{12} - 3v_{67} + 2v_8 = 0$	$v_{12} - 2v_5 + v_{67} \leq 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.1	$v_{12} - v_{34} - v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.2	$v_{12} - 2v_5 + v_{67} \leq 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$
5.1	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
5.2	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$
5.3	$v_{34} - 2v_{67} + v_8 \leq 0$	$-2v_{12} + v_{34} + 2v_5 - v_8 < 0$	$-v_{34} + v_5 < 0$
5.4	$-v_{12} + v_5 + v_{67} - v_8 = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
5.5	$v_{34} - 2v_{67} - v_8 \leq 0$	$2v_{12} - v_{34} - 2v_5 + v_8 < 0$	$-2v_{12} + 3v_5 - v_8 < 0$
5.6	$2v_{12} - 3v_5 + v_8 \leq 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
6.1	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
6.2	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$

Table 1.3: Conditions of v for subdivisions in Table 1.2.

Chapter 2

Tropical curves and symmetric truncated cubic forms

Let $\mathbb{K}$ be a field. A valuation map of $\mathbb{K}$ is a map $\text{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ that for $a, b \in \mathbb{K}$, it satisfies properties

1. $\text{val}(a) = \infty$ if and only if $a = 0$,
2. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$, and
3. $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$.

Remark 2.0.1. For $a, b \in \mathbb{K}$, a valuation map satisfies the following statements.

1. If $a^2 = a$, then $\text{val}(a) = 0$.
2. $\text{val}(a) = \text{val}(-a)$.
3. If $\text{val}(a) \neq \text{val}(b)$, then $\text{val}(a + b) = \min(\text{val}(a), \text{val}(b))$.

Proof. If $a^2 = a$, we have $\text{val}(a^2) = \text{val}(a)$ that implies $2\text{val}(a) = \text{val}(a)$ and the first statement follows. Moreover, since $a^2 = (-a)^2$, we have $2\text{val}(a) = 2\text{val}(-a)$ and the second statement follows.

Lastly, assume $\text{val}(a) > \text{val}(b)$. From the third property, we have $\text{val}(a + b) \geq \text{val}(b)$. Moreover,

$$\begin{aligned} \text{val}(b) =& \text{val}(a + b - a) \\ &\geq \min(\text{val}(a + b), \text{val}(-a)) = \min(\text{val}(a + b), \text{val}(a)) = \text{val}(a + b). \end{aligned}$$

The last line is an implication of the second statement and the assumption. Thus, the third claim follows. □

Set $\mathbb{K}^*$ denotes the nonzero elements of $\mathbb{K}$. In practical applications, as discussed in [M10], it is convenient to let $\mathbb{K}$ be the field of Puiseux series

$$\mathbb{C}\{\{t\}\} = \left\{ \sum_{k=m}^{\infty} a_k t^{\frac{k}{N}} : m \in \mathbb{Z}, N \in \mathbb{N}, a_k \in \mathbb{C} \right\}$$

with valuation

$$\begin{aligned} \mathrm{val}: &\mathbb{C}\{\{t\}\}^* \to \mathbb{Q} \\ &\sum_{k=m}^{\infty} a_k t^{\frac{k}{N}} \mapsto \min\left(\frac{k}{N} : a_k \neq 0\right). \end{aligned}$$

Let $\mathcal{I} \subset \mathbb{Z}^2$ be a non-empty subset and

$$f(x,y) = \sum_{(i,j)\in\mathcal{I}} a_{ij} x^i y^j \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$$

be a Laurent polynomial.

Definition 2.0.2. The tropical polynomial of $f(x,y)$ is the piecewise linear function

$$\mathrm{trop}(f)(X,Y) = \min(\mathrm{val}(a_{ij}) + i\cdot X + j\cdot Y : (i,j)\in\mathcal{I}).$$

Tropical curve $C(\mathrm{trop}(f))$ is the collection of coordinates $(X,Y)\in\mathbb{R}^2$ where $\mathrm{trop}(f)(X,Y)$ is nondifferentiable, i.e. the minimum value is attained at least twice.

A tropical curve forms a graph in $\mathbb{R}^2$. It contains bounded and unbounded edges. Some tropical curves have cycles formed by some bounded edges.

Remark 2.0.3. It is common to find other sources in the literature that express the tropicalization of a polynomial by using operations $(+, \max)$. The curve of a tropical polynomial in the form

$$\mathrm{trop}(f')(X,Y) = \max(-\mathrm{val}(a_{ij}) + i\cdot X + j\cdot Y)$$

and $C(\mathrm{trop}(f))$ are point-symmetric with respect to the origin O.

Proof. Let (X,Y) be a point on $C(\mathrm{trop}(f))$. Then there exist (i_1,j_1) and (i_2,j_2) such that

$$\mathrm{val}(a_{i_1 j_1}) + i_1 X + j_1 Y = \mathrm{val}(a_{i_2 j_2}) + i_2 X + j_2 Y$$

and less than other terms of $\mathrm{val}(a_{ij}) + iX + jY$. Thus, we have

$$-\mathrm{val}(a_{i_1 j_1}) + i_1(-X) + j_1(-Y) = -\mathrm{val}(a_{i_2 j_2}) + i_2(-X) + j_2(-Y)$$

and greater than other terms of $-\mathrm{val}(a_{ij}) + i(-X) + j(-Y)$. In other words, (X,Y) is a point on $C(\mathrm{trop}(f))$ if and only if $(-X,-Y)$ is a point on $C(\mathrm{trop}(f'))$. Thus, the tropical curves are point-symmetric with respect to the origin O. □

Expressions of the form $\mathrm{val}(a_{ij}) + i\cdot X + j\cdot Y$ can be disregarded when $\mathrm{val}(a_{ij})$ is ∞ or $-\infty$, depending on the operations we use for defining the tropical polynomial. Consequently, we define the following set for a tropical polynomial.

Definition 2.0.4. The support of $f(x, y)$, or alternatively the support of $\text{trop}(f)(X, Y)$, is the set

$$\text{Supp}(f) = \{(i, j) \in \mathbb{Z}^2 : a_{ij} \neq 0\}$$

and the Newton polygon of $\text{trop}(f)(X, Y)$, denoted by Δ_f, is the convex hull of $\text{Supp}(f)$. Let Γ_d be the triangle with vertices $(0, 0), (0, d), (d, 0)$. If Δ_f fits inside Γ_d but not inside Γ_{d+1}, then we say $C(\text{trop}(f))$ has degree d. If $\Delta_f = \Gamma_d$, we say $C(\text{trop}(f))$ has a full support.

The structure of tropical curve $C(\text{trop}(f))$, including its vertex count and the presence of cycles, exhibits a connection with the regular subdivision of Δ_f as defined below. The definition of subdivisions can be summarized in the diagram in Figure 2.1.

Definition 2.0.5. Let $v = (\text{val}(a_{ij}) : a_{ij} \neq 0) \in \mathbb{Q}^{\text{Supp}(f)}$. Furthermore, let Δ_f be the Newton polygon of $\text{trop}(f)(X, Y)$ and $\overline{\Delta_f}$ be the convex hull of

$$\{(i, j, \text{val}(a_{ij})) : (i, j) \in \text{Supp}(f)\} \subseteq \mathbb{Z}^2 \times \mathbb{R}.$$

The regular subdivision Subdiv_v is the image of corner edges of the upper part of $\overline{\Delta_f}$ under the projection to $\mathbb{Z}^2$ that subdivide Δ_f into smaller polygons. Each small polygon is called a cell. A cell is primitive when all of its lattice points are its vertices. It is unimodular if it is a triangle of area half. A regular subdivision is primitive (resp. unimodular) when all of its cells are primitive (resp. unimodular).

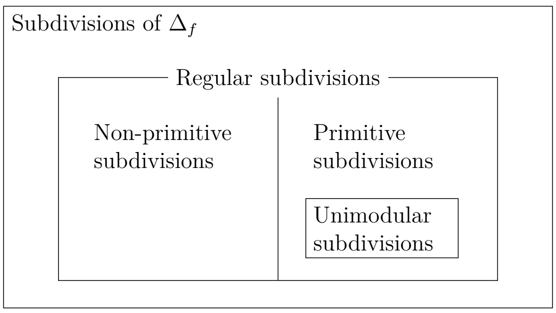


Figure 2.1: Subdivisions of a Newton polygon.

In this thesis, the points on Δ_f are numbered according to the position of the term corresponding to it in the tropical polynomial $\text{trop}(f)(X, Y)$. Furthermore, we will name the cells of a regular subdivision of Δ_f by using these numbers.

Example 2.0.6. Assume tropical polynomial

$$\text{trop}(f)(X, Y) = \min(v_1 + X + 2Y, v_2 + 2X + Y, v_3 + 2X, v_4 + 2Y, \\ v_5 + X + Y, v_6 + X, v_7 + Y, v_8).$$

For vector $(v_1, \ldots, v_8) = (0, 0, 0, 0, -1, 0, 0, 0)$, the regular subdivision Subdiv_v is shown in Figure 2.2. We can write this subdivision as $[[1, 2, 5], [1, 4, 5], [2, 3, 5], [4, 5, 8], [3, 5, 8]]$.

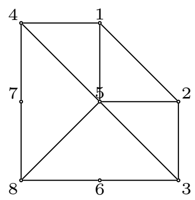


Figure 2.2: The regular subdivision that is dual to
$\text{trop}(f)(X,Y) = \min(X+2Y, 2X+Y, 2X, 2Y, -1+X+Y, X, Y, 0)$.

Additionally, we may notice that we have the same regular subdivision for a different vector $(v_1, \ldots, v_8) = (0,0,0,0,-2,0,0,0)$.

Definition 2.0.7. Let $v \in \mathbb{Q}^{\text{Supp}(f)}$. The collection of vectors v that yield the same regular subdivision forms a polyhedral cone in $\mathbb{R}^{\text{Supp}(f)}$. The collection of these cones defines the secondary fan of the Newton polygon Δ_f.

Furthermore, the polyhedral cones corresponding to unimodular subdivisions are top-dimensional cones, see [KMM08].

Definition 2.0.8. Let $i \in \mathbb{N}$ and p be a cell of a regular subdivision. The i^{th} term of tropical polynomial $\text{trop}(f)(X,Y)$ is denoted by $\text{trop}(f)(X,Y)_i$. If the name of cell p contains i, then we write $i \in p$. If $i \in p$ but i is contained inside cell p, we say p covers i.

Remark 2.0.9. [V10, Lemma 3.2] Let $f = \sum_{(i,j)\in\mathcal{I}} a_{ij}x^iy^j$ be a polynomial and let $v = (\text{val}(a_{ij}) : a_{ij} \neq 0)$. For any edge E that is pointing outward from vertex V on the tropical curve $C(\text{trop}(f))$, there is a cell p in the subdivision Subdiv_v bounded by edge D where E is the inward normal vector of D. The opposite holds as well. This dualism is illustrated in Figure 2.3.

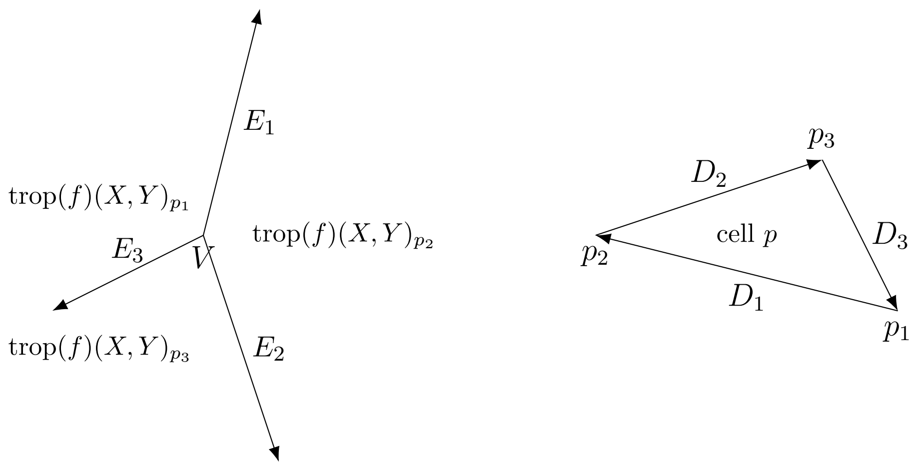


Figure 2.3: The dualism of vertex V of a tropical curve and cell p of a subdivision.

Remark 2.0.10. Let $\text{trop}(f)(X,Y)$ be a tropical polynomial and p be a cell of its regular subdivision. Let (X_p, Y_p) be a vertex on a tropical curve $C(\text{trop}(f))$ that corresponds to cell p. Then (X_p, Y_p) is the solution of the system of linear equations

$$\{\text{trop}(f)(X,Y)_i = \text{trop}(f)(X,Y)_j : i,j \in p \text{ and } i \neq j\}.$$

Let $i \in p$ and we have cell p corresponds to inequalities

$$\text{trop}(f)(X_p,Y_p)_i \begin{cases} \leq \text{trop}(f)(X_p,Y_p)_j : j \notin p, p \text{ covers } j \\ < \text{trop}(f)(X_p,Y_p)_j : j \notin p, p \text{ doesn't cover } j \end{cases} . \tag{2.1}$$

Proof. Let $p = [p_1, \ldots, p_t]$ where for $i = 1, \ldots, t$, $p_i \in \mathbb{N}$ are points on the regular subdivision that form the vertices of cell p. Cell p is a t sides polygon. It is dual to vertex (X_p, Y_p) whose t emerging edges separate t regions (see Figure 2.3). Each region is the collection of (X,Y) where $\text{trop}(f)(X,Y) = \text{trop}(f)(X,Y)_{p_i}$, $(i = 1, \ldots, t)$. Thus, (X_p, Y_p) is the intersection point between t terms $\text{trop}(f)(X,Y)_{p_i}$, $(i = 1, \ldots, t)$.

Furthermore, from the definition of a tropical curve, we know that vertex (X_p, Y_p) is a point where the minimum value $\text{trop}(f)(X,Y)$ is attained t times by

$$\text{trop}(f)(X,Y)_{p_1} = \cdots = \text{trop}(f)(X,Y)_{p_t}.$$

Hence, for $j \neq p_1, \ldots, p_t$, we have $\text{trop}(f)(X,Y)_{p_1} \leq \text{trop}(f)(X,Y)_j$ when j is covered by cell p and $\text{trop}(f)(X,Y)_{p_1} < \text{trop}(f)(X,Y)_j$ for other j. □

The edges of a tropical curve connect vertices that lie on $\mathbb{Q}^2$. Meanwhile, the edges of a regular subdivision connect lattice vertices $\mathbb{Z}^2$.

Definition 2.0.11. Let E be an edge in a tropical curve and D be the corresponding edge in its dual subdivision. If n is the number of lattice points on D, then the weight of E, denoted by ω_E, is defined to be $n-1$. If E is a bounded edge, let $|E|$ and $|D|$ be the Euclidean lengths of E and D, respectively. The lattice length of E, denoted by l_E, is given by $l_E = \frac{|E|}{|D|}$. If P and Q are two points on E, the lattice length between P and Q is

$$l_{PQ} = \frac{|PQ|}{|E|} l_E.$$

The relationship between tropical curves and regular subdivisions establishes a coherent mapping between various elements of the two entities. Specifically, there is a one-to-one correspondence between edges of a tropical curve and edges of a regular subdivision, between vertices of a tropical curve and cells of a regular subdivision, and between regions of a tropical curve and vertices of a subdivision.

Remark 2.0.12. For a polynomial $f(x,y) = \sum_{(i,j)\in\mathcal{I}} a_{ij}x^i y^j$, let $v = (\text{val}(a_{ij}) : a_{ij} \neq 0)$. Let $u = \begin{pmatrix} \text{u}_1 \\ \text{u}_2 \end{pmatrix}$ and $u' = \begin{pmatrix} \text{u}'_1 \\ \text{u}'_2 \end{pmatrix}$ be primitive integer vectors that satisfy

$$\text{u}_1 - \text{u}'_2 = 0 \qquad \text{u}_2 + \text{u}'_1 = 0. \tag{2.2}$$

Suppose V is a vertex on $C(\text{trop}(f))$ and edges $E_1, \ldots, E_t$ emerge from it following a clockwise orientation. Let $u_1, \ldots, u_t$ be their primitive integer directions, respectively. We have $u_1 + \cdots + u_t = 0$. Furthermore, vertex V is dual to cell p in Subdiv_v that is bounded by edges $D_1, \ldots, D_t$ that can be written as vectors $\omega_1 u'_1, \ldots, \omega_t u'_t$, following a clockwise orientation.

Proof. The first claim that says $u_1 + \cdots + u_t = 0$ is the direct corollary of Remark 2.0.9. If cell p of Subdiv_v is dual to vertex V, it is bounded by t edges $D_1, \ldots, D_t$. For $i = 1, \ldots, t$, we can assume the primitive integer directions of D_i is u'_i since vectors u_i and u'_i are perpendicular to each other. Moreover, from the definition of the weight of E_i, the Euclidean length of edges D_i is $|D_i| = \omega_i |u'_i|$. Thus, the result follows. □

Triangular cells have a special role because it allows us to define the multiplicity of trivalent vertices that are dual to it.

Definition 2.0.13. Let V be a trivalent vertex on a tropical curve with edges E_1, E_2, E_3 emerging from it. Let $\omega_1, \omega_2, \omega_3$ be the weights and v_1, v_2, v_3 be the primitive integer directions of these edges, respectively. The multiplicity of V, denoted by mult_V, is defined by the absolute values of

$$\omega_1\omega_2 \begin{vmatrix} v_1 & v_2 \end{vmatrix} = \omega_1\omega_3 \begin{vmatrix} v_1 & v_3 \end{vmatrix} = \omega_2\omega_3 \begin{vmatrix} v_2 & v_3 \end{vmatrix}.$$

In [V09], Vigeland focused on a tropical curve whose vertices are all trivalent and of multiplicity one.

Definition 2.0.14. A tropical curve is defined to be smooth if each vertex is trivalent and has multiplicity one.

We can also notice that a tropical curve whose vertices are all trivalent of multiplicity one is dual to a unimodular subdivision. Step by step proof is is given in Proposition 3.1.1.

Chapter 3

Explicit criteria for various types of subdivisions

The dualism between a regular subdivision and a tropical curve tells that we can identify all tropical curves from the families of regular subdivisions. In this chapter, we will describe all combinatorial possibilities of the subdivisions of Δ_f and determine the values of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for each subdivision, [T23].

3.1 Smooth tropical curves

First, we want to identify the subdivisions that are dual to smooth tropical curves.

Proposition 3.1.1. *Let $C(\mathrm{trop}(f))$ be a tropical curve and Subdiv_v be a subdivision that are dual. Tropical curve $C(\mathrm{trop}(f))$ is smooth if and only if Subdiv_v is a unimodular triangulation.*

Proof. Suppose Subdiv_v is a unimodular triangulation. Let p be a triangular cell in Subdiv_v and its borders be vectors a', b', c'. Since the area of p is half, there is no lattice point contained in p except its vertices, and vectors a', b', c' are primitive integer directions. It implies $\omega_a = \omega_b = \omega_c = 1$. Furthermore,

$$\frac{1}{2} = \text{area of } p = \frac{1}{2}(a'_1 b'_2 - a'_2 b'_1) = \frac{1}{2}(b'_1 c'_2 - b'_2 c'_1) = \frac{1}{2}(c'_1 a'_2 - c'_2 a'_1).$$

Let V be the vertex on $C(\mathrm{trop}(f))$ that is dual to cell p. The direction of three emerging edges from V will have primitive integer vectors a, b, c satisfying Remark 2.0.12. Due to the equations (2.2), the absolute values of determinants $\begin{vmatrix} a & b \end{vmatrix}, \begin{vmatrix} a & c \end{vmatrix}, \begin{vmatrix} b & c \end{vmatrix}$ are one. Thus we have the absolute value of $\omega_a \omega_b \begin{vmatrix} a & b \end{vmatrix}$ is one, which establishes $\mathrm{mult}_V = 1$. For a treatment of general tropical curves, we refer the reader to [MS15].

Conversely, suppose that $C(\mathrm{trop}(f))$ is smooth. Let V be a vertex and the three emerging edges have primitive integer directions a, b, c. Let p be the triangular cell in the subdivision that is dual to V. If a', b', c' are primitive integer vectors as mentioned in

Remark 2.0.12, the boundaries of p are vectors $\omega_a a', \omega_b b', \omega_c c'$. Since we have $\text{mult}_V = 1$, the weights of a, b, c are $\omega_a = \omega_b = \omega_c = 1$ and the absolute values of $\begin{vmatrix} a & b \end{vmatrix}$, $\begin{vmatrix} a & c \end{vmatrix}$, and $\begin{vmatrix} b & c \end{vmatrix}$ are one. Furthermore, applying the relation (2.2) of Remark 2.0.12 yields

$$\text{area of } p = \frac{1}{2}\begin{vmatrix} a & b \end{vmatrix} = \frac{1}{2}\begin{vmatrix} b & c \end{vmatrix} = \frac{1}{2}\begin{vmatrix} c & a \end{vmatrix}.$$

Therefore, we can assert that the subdivision Subdiv_v is a unimodular triangulation. □

Theorem 3.1.2 (= Theorem 1.2.1)**.** *Let $f(x,y)$ be the symmetric truncated cubic in equation* (1.1)*. Then the possible cycles appearing in the tropical curves of* $\text{trop}(f)(X,Y)$ *are triangles, squares, pentagons, hexagons and heptagons. All possible unimodular subdivisions are as shown in Figure* 1.2*, respectively. Each of these cycles occurs if and only if* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ *satisfies inequalities listed in Table* 3.1*.*

	Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$		
(I)	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - v_5 - v_{67} + v_8 < 0$	$-2v_{12} + 3v_5 - v_8 < 0$
(II)	$-v_5 + 2v_{67} - v_8 < 0$	$-v_{12} + 2v_5 - v_{67} < 0$	$v_{12} - v_{34} - v_5 + v_{67} < 0$
(III)	$v_5 - 2v_{67} + v_8 < 0$	$-v_{12} + v_5 + v_{67} - v_8 < 0$	$v_{12} - v_{34} - v_5 + v_{67} < 0$
(IV)	$-v_5 + 2v_{67} - v_8 < 0$	$-v_{34} + v_5 < 0$	$-v_{12} + v_{34} + v_5 - v_{67} < 0$
(V)	$v_5 - 2v_{67} + v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$-v_{12} + v_{34} + v_5 - v_{67} < 0$

Table 3.1: Conditions of $v_{12}, v_{34}, v_5, v_{67}, v_8$ for all smooth tropical curves $C(\text{trop}(f))$.

Proof. We will simultaneously prove the five cases since their proofs share similarities. There are three main steps involved in determining the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for each subdivision in Figure 1.2. We will apply Remark 2.0.10. First, we calculate the coordinates of the vertices of a smooth tropical curve. Since a vertex in a tropical curve corresponds to a cell in a subdivision, each smooth tropical curve has seven vertices. The second step involves finding the inequalities that determine each cell of the subdivision. Lastly, we assemble a collection of inequalities that is equivalent to the union of all inequalities obtained in the second step.

The five subdivisions on Figure 1.2 can be written as

1. $\mathcal{S}_I = [[1,2,5],[1,5,8],[2,5,8],[1,7,8],[2,6,8],[1,4,7],[2,3,6]]$,
2. $\mathcal{S}_{II} = [[1,2,5],[1,5,7],[2,5,6],[1,4,7],[2,3,6],[5,6,7],[6,7,8]]$,
3. $\mathcal{S}_{III} = [[1,2,5],[1,5,7],[2,5,6],[1,4,7],[2,3,6],[5,7,8],[5,6,8]]$,
4. $\mathcal{S}_{IV} = [[1,2,5],[1,4,5],[2,3,5],[4,5,7],[3,5,6],[5,6,7],[6,7,8]]$, and
5. $\mathcal{S}_V = [[1,2,5],[1,4,5],[2,3,5],[4,5,7],[3,5,6],[5,7,8],[5,6,8]]$.

By using Remark 2.0.10, the coordinates (X,Y) corresponding to all cells of the subdivisions are

1. $\mathcal{S}_{I,(X,Y)} = [(-v_{12} + v_5, -v_{12} + v_5), (-2v_5 + v_{12} + v_8, -v_{12} + v_5),$
   $(-v_{12} + v_5, -2v_5 + v_{12} + v_8), (-v_{12} + 2v_{67} - v_8, -v_{67} + v_8),$
   $(-v_{67} + v_8, -v_{12} + 2v_{67} - v_8), (-v_{12} + v_{34}, -v_{34} + v_{67}), (-v_{34} + v_{67}, -v_{12} + v_{34})],$

2. $\mathcal{S}_{II,(X,Y)} = [(-v_{12} + v_5, -v_{12} + v_5), (-v_5 + v_{67}, -v_{12} + v_5), (-v_{12} + v_5, -v_5 + v_{67}),$
   $(-v_{12} + v_{34}, -v_{34} + v_{67}), (-v_{34} + v_{67}, -v_{12} + v_{34}), (-v_5 + v_{67}, -v_5 + v_{67}),$
   $(-v_{67} + v_8, -v_{67} + v_8)],$

3. $\mathcal{S}_{III,(X,Y)} = [(-v_{12} + v_5, -v_{12} + v_5), (-v_5 + v_{67}, -v_{12} + v_5), (-v_{12} + v_5, -v_5 + v_{67}),$
   $(-v_{12} + v_{34}, -v_{34} + v_{67}), (-v_{34} + v_{67}, -v_{12} + v_{34}), (-v_{67} + v_8, -v_5 + v_{67}),$
   $(-v_5 + v_{67}, -v_{67} + v_8)],$

4. $\mathcal{S}_{IV,(X,Y)} = [(-v_{12} + v_5, -v_{12} + v_5), (-v_{12} + v_{34}, -v_{12} + v_5), (-v_{12} + v_5, -v_{12} + v_{34}),$
   $(-v_{34} + v_{67}, -v_5 + v_{67}), (-v_5 + v_{67}, -v_{34} + v_{67}), (-v_5 + v_{67}, -v_5 + v_{67}),$
   $(-v_{67} + v_8, -v_{67} + v_8)]$, and

5. $\mathcal{S}_{V,(X,Y)} = [(-v_{12} + v_5, -v_{12} + v_5), (-v_{12} + v_{34}, -v_{12} + v_5), (-v_{12} + v_5, -v_{12} + v_{34}),$
   $(-v_{34} + v_{67}, -v_5 + v_{67}), (-v_5 + v_{67}, -v_{34} + v_{67}), (-v_5 + v_{67}, -v_{67} + v_8),$
   $(-v_{67} + v_8, -v_5 + v_{67})].$

Meanwhile, the union of inequalities that determine all cells are

1. $\mathcal{S}_{I,\text{inequalities}} = \{v_5 < v_{34}, 2v_5 < v_{12} + v_{67}, 3v_5 < 2v_{12} + v_8, 4v_5 < v_8 + v_{34} + 2v_{12},$
   $2v_8 < -v_{12} + (3v_{67}), 3v_8 < v_{34} - (2v_{12}) + (4v_{67}), v_{12} < 2v_{34} - v_{67}, -v_{34} + 2v_{67} < v_8,$
   $v_8 + v_{12} - v_{67} < v_5, -v_{34} + v_{12} + v_{67} < v_5, -v_{34} + 2v_{12} + v_8 < 2v_5\},$

2. $\mathcal{S}_{II,\text{inequalities}} = \{v_5 < v_{34}, 2v_5 < v_{12} + v_{67}, 3v_5 < 2v_{12} + v_8, v_5 < v_8 + v_{12} - v_{67},$
   $3v_5 < v_{67} + v_{12} + v_{34}, v_{12} < 2v_{34} - v_{67}, -v_8 + 2v_{67} < v_5, -v_{34} + 2v_{67} < v_8,$
   $- v_{12} + 3v_{67} < 2v_8, -v_{34} + v_{12} + v_{67} < v_5\},$

3. $\mathcal{S}_{III,\text{inequalities}} = \{v_5 < v_{34}, v_5 < -v_8 + 2v_{67}, 2v_5 < v_{12} + v_{67}, 3v_5 < 2v_{12} + v_8,$
   $v_5 < v_8 + v_{12} - v_{67}, 2v_5 < -v_8 + v_{34} + 2v_{67}, 3v_5 < v_{67} + v_{12} + v_{34}, v_{12} < 2v_{34} - v_{67},$
   $- v_{34} + 2v_{67} < v_8, -v_{34} + v_{12} + v_{67} < v_5\},$

4. $\mathcal{S}_{IV,\text{inequalities}} = \{v_5 < v_{34}, 2v_5 < v_{12} + v_{67}, 3v_5 < 2v_{12} + v_8, v_5 < -v_{34} + v_{12} + v_{67},$
   $2v_5 < -v_{34} + 2v_{12} + v_8, -v_8 + 2v_{67} < v_5,$
   $- v_{34} + 2v_{67} < v_8, -v_{12} + 3v_{67} < 2v_8\}$, and

5. $\mathcal{S}_{V,\text{inequalities}} = \{v_5 < v_{34}, v_5 < -v_8 + 2v_{67}, 2v_5 < v_{12} + v_{67}, 3v_5 < 2v_{12} + v_8,$
   $v_5 < v_8 + v_{12} - v_{67}, v_5 < -v_{34} + v_{12} + v_{67}, 2v_5 < -v_8 + v_{34} + 2v_{67},$
   $2v_5 < -v_{34} + 2v_{12} + v_8, -v_{34} + 2v_{67} < v_8\}.$

Finally, we will proceed to simplify the five sets of inequalities that are mentioned above. This reduction can be accomplished through Maple calculations, as demonstrated in Section 3.3. As a result, we obtain the final conditions for $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ that determine each subdivision, which are as follows.

1. $\mathcal{S}_{I,v} = \{-v_8 - v_{34} + 2v_{67} < 0, -v_5 + v_8 + v_{12} - v_{67} < 0, 3v_5 - v_8 - 2v_{12} < 0\}$,
2. $\mathcal{S}_{II,v} = \{-v_5 - v_8 + 2v_{67} < 0, -v_5 + v_{12} - v_{34} + v_{67} < 0, 2v_5 - v_{12} - v_{67} < 0\}$,
3. $\mathcal{S}_{III,v} = \{-v_5 + v_{12} - v_{34} + v_{67} < 0, v_5 - v_8 - v_{12} + v_{67} < 0, v_5 + v_8 - 2v_{67} < 0\}$,
4. $\mathcal{S}_{IV,v} = \{-v_5 - v_8 + 2v_{67} < 0, v_5 - v_{12} + v_{34} - v_{67} < 0, v_5 - v_{34} < 0\}$, and
5. $\mathcal{S}_{V,v} = \{-v_8 - v_{34} + 2v_{67} < 0, v_5 - v_{12} + v_{34} - v_{67} < 0, v_5 + v_8 - 2v_{67} < 0\}$.

□

3.2 Non-smooth tropical curves

Due to Proposition 3.1.1, we know that all regular subdivisions that are non-unimodular correspond to nonsmooth tropical curves of cubic polynomial $f(x, y)$ in equation (1.1). Table 1.2 shows all subdivisions of Newton polygon Δ_f that have some non-unimodular cells. Now, we will determine which subdivisions in Table 1.2 are regular subdivisions of Δ_f.

Theorem 3.2.1 (= Theorem 1.2.2)**.** *Let* $\operatorname{trop}(f)(X, Y)$ *be as defined in equation* (1.2) *and* Δ_f *be its Newton polygon. Then, the subdivisions in the right column of Table* 1.2 *never occur as the regular subdivisions of* Δ_f *for any* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$*.*

Proof. The proof can be accomplished by examining the shape of the subdivision. By contradiction, assume that the subdivisions are viable. In doing so, we observe that the interior point $(1, 1)$ forms a vertex of the Newton polygon Δ_f. However, it is evident that its dual does not form a closed cycle in a tropical curve. □

We can see that the regular subdivisions in Table 1.2 can be derived through a coarsening process from the unimodular subdivisions shown in Figure 1.2. This implies that the linear expressions representing the polyhedral cones for subdivisions in Table 1.2 may not all be strict inequalities, unlike those in Table 3.1. Moreover, we can arrange all of the non-unimodular and unimodular subdivisions as shown in Figure 3.1.

Theorem 3.2.2 (= Theorem 1.2.3)**.** *The conditions of* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ *for non-unimodular regular subdivisions in Table* 1.2 *are shown in Table* 3.2*.*

Proof. We will do the proof in three separate steps. The first step is replicating the three steps in the proof of Theorem 3.1.2 and achieving the sets of linear expressions in $(v_{12}, v_{34}, v_5, v_{67}, v_8)$. Unlike the proof of Theorem 3.1.2, we may notice linear equality due

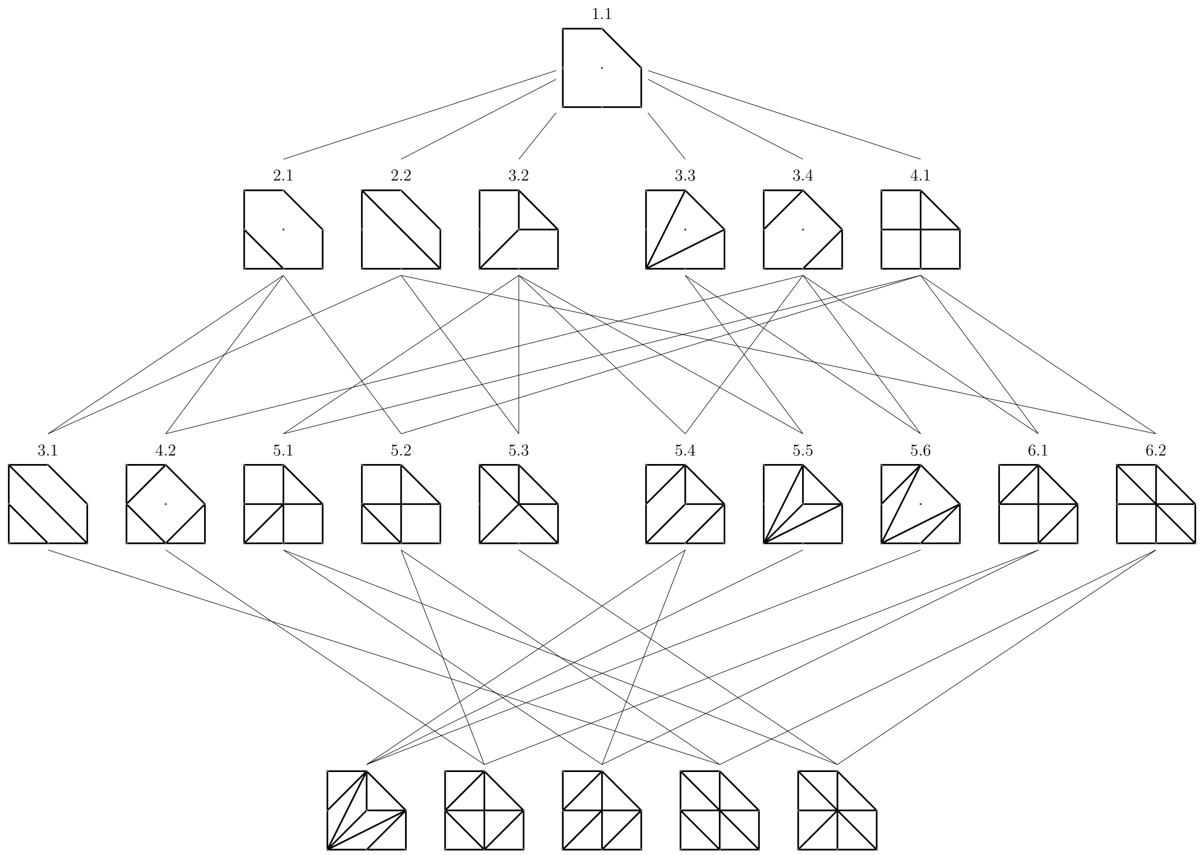


Figure 3.1: A poset of all regular subdivisions of Δ_f from the coarsest to the finest.

to the cells that are formed by four vertices or more. At this step, we let all inequalities (3.2) produced by the Maple calculation in Section 3.3 be strict inequalities to create similarity with the proof of Theorem 3.1.2. The linear expressions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for each non-unimodular subdivision obtained by this first step are written in Table 3.3.

In the second step, we separate the non-unimodular subdivisions into the primitive and the non-primitive. Since a primitive cell does not cover any point of Δ_f, the inequality expressions in Table 3.3 for the primitive subdivisions will remain strict inequalities according to Remark 2.0.10. Thus, the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for the subdivisions 4.1, 5.1, 5.2, 5.4, 6.1, 6.2 are as shown in Table 3.3.

The third step is changing the sign of some strict inequalities to non-strict inequalities on the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for the non-primitive subdivisions. We do this by applying Remark 2.0.10. In subdivision 1.1, all inequalities are non-strict because all points are covered by the single cell $[1, 2, 3, 4, 8]$.

Lastly, we see the finer regular subdivisions on the second and third lines of Figure 3.1. We want to see which inequalities are non-strict by evaluating the non-primitive cells. Due to the symmetrical property of $\mathrm{trop}(f)(X, Y)$, we can reduce the observation to Table 3.4. Furthermore, we change the inequalities of subdivisions containing the non-primitive cells accordingly. Thus, the result follows. $\square$

	Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$		
1.1	$2v_{12} - 3v_{34} + v_8 = 0$	$-v_{12} + 2v_{34} - v_{67} \leq 0$	$v_{34} - v_5 \leq 0$
2.1	$v_{12} - 2v_{34} + v_{67} = 0$	$v_{34} - v_5 \leq 0$	$-v_{34} + 2v_{67} - v_8 < 0$
2.2	$v_{34} - 2v_{67} + v_8 \leq 0$	$v_{34} - v_5 \leq 0$	$-2v_{12} + 3v_{34} - v_8 < 0$
3.1	$v_{34} - v_5 \leq 0$	$-v_{12} + 2v_{34} - v_{67} < 0$	$-v_{34} + 2v_{67} - v_8 < 0$
3.2	$-2v_{12} + v_{34} + 2v_5 - v_8 = 0$	$v_{34} - 2v_{67} + v_8 \leq 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.3	$v_{34} - 2v_{67} + v_8 \leq 0$	$2v_{12} - 3v_5 + v_8 \leq 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.4	$v_{12} - 3v_{67} + 2v_8 = 0$	$v_{12} - 2v_5 + v_{67} \leq 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.1	$v_{12} - v_{34} - v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.2	$v_{12} - 2v_5 + v_{67} \leq 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$
5.1	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
5.2	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$
5.3	$v_{34} - 2v_{67} + v_8 \leq 0$	$-2v_{12} + v_{34} + 2v_5 - v_8 < 0$	$-v_{34} + v_5 < 0$
5.4	$-v_{12} + v_5 + v_{67} - v_8 = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
5.5	$v_{34} - 2v_{67} - v_8 \leq 0$	$2v_{12} - v_{34} - 2v_5 + v_8 < 0$	$-2v_{12} + 3v_5 - v_8 < 0$
5.6	$2v_{12} - 3v_5 + v_8 \leq 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
6.1	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
6.2	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$

Table 3.2: Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for subdivisions in Table 1.2.

	Conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$		
1.1	$2v_{12} - 3v_{34} + v_8 = 0$	$-v_{12} + 2v_{34} - v_{67} < 0$	$v_{34} - v_5 < 0$
2.1	$v_{12} - 2v_{34} + v_{67} = 0$	$v_{34} - v_5 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$
2.2	$v_{34} - 2v_{67} + v_8 < 0$	$v_{34} - v_5 < 0$	$-2v_{12} + 3v_{34} - v_8 < 0$
3.1	$v_{34} - v_5 < 0$	$-v_{12} + 2v_{34} - v_{67} < 0$	$-v_{34} + 2v_{67} - v_8 < 0$
3.2	$-2v_{12} + v_{34} + 2v_5 - v_8 = 0$	$v_{34} - 2v_{67} + v_8 < 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.3	$v_{34} - 2v_{67} + v_8 < 0$	$2v_{12} - 3v_5 + v_8 < 0$	$2v_{12} - 3v_{34} + v_8 < 0$
3.4	$v_{12} - 3v_{67} + 2v_8 = 0$	$v_{12} - 2v_5 + v_{67} < 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.1	$v_{12} - v_{34} - v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$
4.2	$v_{12} - 2v_5 + v_{67} < 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$
5.1	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
5.2	$-v_{12} + v_{34} + v_5 - v_{67} = 0$	$v_{12} - 2v_{34} + v_{67} < 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$
5.3	$v_{34} - 2v_{67} + v_8 < 0$	$-2v_{12} + v_{34} + 2v_5 - v_8 < 0$	$-v_{34} + v_5 < 0$
5.4	$-v_{12} + v_5 + v_{67} - v_8 = 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
5.5	$v_{34} - 2v_{67} - v_8 < 0$	$2v_{12} - v_{34} - 2v_5 + v_8 < 0$	$-2v_{12} + 3v_5 - v_8 < 0$
5.6	$2v_{12} - 3v_5 + v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$	$v_{12} - 3v_{67} + 2v_8 < 0$
6.1	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + 3v_{67} - 2v_8 < 0$	$v_{12} - v_{34} - v_{67} + v_8 < 0$
6.2	$v_5 - 2v_{67} + v_8 = 0$	$-v_{12} + v_{34} + v_{67} - v_8 < 0$	$-v_{34} + 2v_{67} - v_8 < 0$

Table 3.3: The linear expressions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ after the first step.

Cell p	Coordinate (X_p, Y_p)	Inequalities	Subdivisions
$[1,2,3,4,6,7]$	$(-v_{34}+v_{67}, -v_{34}+v_{67})$	$v_{12}-v_{34}-v_5+v_{67} \leq 0$ $v_{12} = 2v_{34}-v_{67}$	2.1
$[1,2,3,4]$	$(-v_{12}+v_{34}, -v_{12}+v_{34})$	$v_{34}-v_5 \leq 0$	2.2, 3.1
$[3,4,8]$	$(-\frac{v_{34}}{2}+\frac{v_8}{2}, -\frac{v_{34}}{2}+\frac{v_8}{2})$	$v_{34}-v_5 \leq 0$ $v_{34}-2v_{67}+v_8 \leq 0$	2.2
$[3,4,6,7]$	$(-v_{34}+v_{67}, -v_{34}+v_{67})$	$v_{34}-v_5 \leq 0$	3.1
$[1,4,5,8]$	$(-v_{12}+v_{34}, -v_{12}+v_5)$	$-v_{12}+v_{34}+v_5-v_{67} \leq 0$	3.2
$[1,4,8]$	$(-v_{12}+v_{34}, -\frac{v_{34}}{2}+\frac{v_8}{2})$	$v_{34}-2v_{67}+v_8 \leq 0$	3.3
$[1,2,8]$	$(-\frac{v_{12}}{3}+\frac{v_8}{3}, -\frac{v_{12}}{3}+\frac{v_8}{3})$	$2v_{12}-3v_5+v_8 \leq 0$	3.3, 5.6
$[1,2,6,7,8]$	$(-v_{67}+v_8, -v_{67}+v_8)$	$v_{12}-2v_5+v_{67} \leq 0$ $v_{12}-3v_{67}+2v_8 = 0$	3.4
$[1,2,6,7]$	$(-\frac{v_{12}}{2}+\frac{v_{67}}{2}, -\frac{v_{12}}{2}+\frac{v_{67}}{2})$	$v_{12}-2v_5+v_{67} \leq 0$	4.2
$[3,5,8]$	$(-\frac{v_{34}}{2}+\frac{v_8}{2}, -v_5+\frac{v_8}{2}+\frac{v_{34}}{2})$	$v_{34}-2v_{67}+v_8 \leq 0$	5.3

Table 3.4: Non-strict inequalities correspond to non-primitive cells.

3.3 Maple code

In the proof of Theorem 3.1.2, we can narrow down several inequalities to three by checking the equivalence of the two sets of inequalities. We can also do this calculation in Maple by using its `PolyhedralSets` package. We employ the following commands to calculate the conditions for $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ in the case of smooth tropical curves in the proof of Theorem 3.1.2.

```
I_ineqs:= {v[5] <= v[34], v[5] <= v[12]/2 + v[67]/2, v[5] <= (2*v[12])/3
   + v[8]/3, v[5] <= v[8]/4 + v[34]/4 + v[12]/2, v[8] <= -v[12]/2 + (3*v
   [67])/2, v[8] <= v[34]/3 - (2*v[12])/3 + (4*v[67])/3, v[12] <= 2*v[34]
    - v[67], -v[34] + 2*v[67] <= v[8], v[8] + v[12] - v[67] <= v[5], -v
   [34] + v[12] + v[67] <= v[5], -v[34]/2 + v[12] + v[8]/2 <= v[5]}:
I_pseudo_polyhedral:= PolyhedralSets:-PolyhedralSet(I_ineqs);

II_ineqs:= {v[5] <= v[34], v[5] <= v[12]/2 + v[67]/2, v[5] <= (2*v[12])/3
    + v[8]/3, v[5] <= v[8] + v[12] - v[67], v[5] <= v[67]/3 + v[12]/3 + v
   [34]/3, v[12] <= 2*v[34] - v[67], -v[8] + 2*v[67] <= v[5], -v[34] + 2*
   v[67] <= v[8], -v[12]/2 + (3*v[67])/2 <= v[8], -v[34] + v[12] + v[67]
   <= v[5]}:
II_pseudo_polyhedral:= PolyhedralSets:-PolyhedralSet(II_ineqs);

III_ineqs:= {v[5] <= v[34], v[5] <= -v[8] + 2*v[67], v[5] <= v[12]/2 + v
   [67]/2, v[5] <= (2*v[12])/3 + v[8]/3, v[5] <= v[8] + v[12] - v[67], v
   [5] <= -v[8]/2 + v[34]/2 + v[67], v[5] <= v[67]/3 + v[12]/3 + v[34]/3,
    v[12] <= 2*v[34] - v[67], -v[34] + 2*v[67] <= v[8], -v[34] + v[12] +
```

```
   v[67] <= v[5]}:
III_pseudo_polyhedral:= PolyhedralSets:-PolyhedralSet(III_ineqs);

IV_ineqs:= {v[5] <= v[34], v[5] <= v[12]/2 + v[67]/2, v[5] <= (2*v[12])/3
    + v[8]/3, v[5] <= -v[34] + v[12] + v[67], v[5] <= -v[34]/2 + v[12] +
   v[8]/2, -v[8] + 2*v[67] <= v[5], -v[34] + 2*v[67] <= v[8], -v[12]/2 +
   (3*v[67])/2 <= v[8]}:
IV_pseudo_polyhedral:= PolyhedralSets:-PolyhedralSet(IV_ineqs);

V_ineqs:= {v[5] <= v[34], v[5] <= -v[8] + 2*v[67], v[5] <= v[12]/2 + v
   [67]/2, v[5] <= (2*v[12])/3 + v[8]/3, v[5] <= v[8] + v[12] - v[67], v
   [5] <= -v[34] + v[12] + v[67], v[5] <= -v[8]/2 + v[34]/2 + v[67], v[5]
    <= -v[34]/2 + v[12] + v[8]/2, -v[34] + 2*v[67] <= v[8]}:
V_pseudo_polyhedral:= PolyhedralSets:-PolyhedralSet(V_ineqs);
```

This gives polyhedra (3.1).

$$\begin{aligned}
I_pseudo_polyhedral &:= \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_8 - v_{34} + 2v_{67} \leq 0, \\ -v_5 + v_8 + v_{12} - v_{67} \leq 0, v_5 - \frac{v_8}{3} - \frac{2v_{12}}{3} \leq 0] \end{cases} \\
II_pseudo_polyhedral &:= \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_5 - v_8 + 2v_{67} \leq 0, \\ -v_5 + v_{12} - v_{34} + v_{67} \leq 0, v_5 - \frac{v_{12}}{2} - \frac{v_{67}}{2} \leq 0] \end{cases} \\
III_pseudo_polyhedral &:= \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_5 + v_{12} - v_{34} + v_{67} \leq 0, \\ v_5 - v_8 - v_{12} + v_{67} \leq 0, v_5 + v_8 - 2v_{67} \leq 0] \end{cases} \\
IV_pseudo_polyhedral &:= \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_5 - v_8 + 2v_{67} \leq 0, \\ v_5 - v_{12} + v_{34} - v_{67} \leq 0, v_5 - v_{34} \leq 0] \end{cases} \\
V_pseudo_polyhedral &:= \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_8 - v_{34} + 2v_{67} \leq 0, \\ v_5 - v_{12} + v_{34} - v_{67} \leq 0, v_5 + v_8 - 2v_{67} \leq 0] \end{cases}
\end{aligned} \tag{3.1}$$

The command `PolyhedralSets:-PolyhedralSet()` calculates a set of non-strict inequalities. However, we know that a unimodular subdivision corresponds to a top-dimensional cone of the secondary fan of Δ_f. Thus, we simply change all inequality expressions in (3.1) to strict inequalities to achieve a five-dimensional polyhedral cone.

For non-unimodular regular subdivisions, we employ the following commands to get preliminary inequality expressions (3.2) that correspond to the non-unimodular subdivisions.

```
ineqs_11:= {u[8] = -2*u[12] + 3*u[34], u[34] <= u[5], 2*u[34] - u[67] <=
   u[12]}:
pseudo_polyhedral_11:= PolyhedralSets:-PolyhedralSet(ineqs_11);

ineqs_21:= {u[12] = 2*u[34] - u[67], -u[8] + 2*u[67] <= u[5], -u[34] + 2*
   u[67] <= u[8], -u[12]/2 + (3*u[67])/2 <= u[8], u[12] - 3*u[34] + 3*u
   [67] <= u[8], u[12] - u[34] + u[67] <= u[5]}:
pseudo_polyhedral_21:= PolyhedralSets:-PolyhedralSet(ineqs_21);

ineqs_22:= {u[8] <= -u[34] + 2*u[67], u[34] <= u[5], -2*u[12] + 3*u[34]
   <= u[8], 2*u[34] - u[67] <= u[12]}:
pseudo_polyhedral_22:= PolyhedralSets:-PolyhedralSet(ineqs_22);

ineqs_31:= {u[34] <= u[5], -u[8] + 2*u[67] <= u[5], -2*u[12] + 3*u[34] <=
    u[8], -u[34] + 2*u[67] <= u[8], 2*u[34] - u[67] <= u[12], -u[12]/2 +
   (3*u[67])/2 <= u[8]}:
pseudo_polyhedral_31:= PolyhedralSets:-PolyhedralSet(ineqs_31);

ineqs_32:= {u[8] = u[34] - 2*u[12] + 2*u[5], u[5] <= u[34], u[5] <= u
   [12]/2 + u[67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[5] <= u[12] - u[34]
   + u[67]}:
pseudo_polyhedral_32:= PolyhedralSets:-PolyhedralSet(ineqs_32);

ineqs_33:= {u[8] <= -2*u[12] + 3*u[34], u[8] <= -u[34] + 2*u[67], u[8] <=
    -u[12]/2 + (3*u[67])/2, u[8] <= u[67] - u[12] + u[34], (2*u[12])/3 +
   u[8]/3 <= u[5], u[8]/2 - u[34]/2 + u[12] <= u[5]}:
pseudo_polyhedral_33:= PolyhedralSets:-PolyhedralSet(ineqs_33);

ineqs_34:= {u[12] = 3*u[67] - 2*u[8], u[8] <= u[67] - u[12] + u[34], u
   [12] <= 2*u[34] - u[67], -u[34] + 2*u[67] <= u[8], u[12] - u[34] + u
   [67] <= u[5], u[12] - u[67] + u[8] <= u[5]}:
pseudo_polyhedral_34:= PolyhedralSets:-PolyhedralSet(ineqs_34);

ineqs_41:= {u[5] = -u[8] + 2*u[67], u[5] = u[12] - u[34] + u[67], u[5] <=
    u[34], u[5] <= u[12]/2 + u[67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[5]
   <= u[12] - u[67] + u[8], u[12] <= 2*u[34] - u[67], -u[34] + 2*u[67] <=
    u[8]}:
pseudo_polyhedral_41:= PolyhedralSets:-PolyhedralSet(ineqs_41);

ineqs_42:= {u[12] <= 2*u[34] - u[67], -u[8] + 2*u[67] <= u[5], -u[34] +
   2*u[67] <= u[8], -u[12]/2 + (3*u[67])/2 <= u[8], u[12]/2 + u[67]/2 <=
   u[5], u[12] - u[34] + u[67] <= u[5]}:
pseudo_polyhedral_42:= PolyhedralSets:-PolyhedralSet(ineqs_42);
```

```
ineqs_51:= {u[5] = u[12] - u[34] + u[67], u[5] <= u[34], u[5] <= -u[8] +
   2*u[67], u[5] <= u[12]/2 + u[67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[5]
    <= u[12] - u[67] + u[8], u[5] <= -u[8]/2 + u[34]/2 + u[67], u[12] <=
   2*u[34] - u[67], -u[34] + 2*u[67] <= u[8]}:
pseudo_polyhedral_51:= PolyhedralSets:-PolyhedralSet(ineqs_51);

ineqs_52:= {u[5] = u[12] - u[34] + u[67], u[5] <= u[34], u[5] <= u[12]/2
   + u[67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[12] <= 2*u[34] - u[67], -u
   [8] + 2*u[67] <= u[5], -u[34] + 2*u[67] <= u[8], -u[12]/2 + (3*u[67])
   /2 <= u[8]}:
pseudo_polyhedral_52:= PolyhedralSets:-PolyhedralSet(ineqs_52);

ineqs_53:= {u[5] <= u[34], u[5] <= u[12]/2 + u[67]/2, u[5] <= (2*u[12])/3
    + u[8]/3, u[5] <= u[12] - u[34] + u[67], u[5] <= -u[8]/2 + u[34]/2 +
   u[67], u[5] <= u[8]/2 - u[34]/2 + u[12], u[5] <= u[8]/4 + u[12]/2 + u
   [34]/4, u[8] <= -u[34] + 2*u[67]}:
pseudo_polyhedral_53:= PolyhedralSets:-PolyhedralSet(ineqs_53);

ineqs_54:= {u[5] = u[12] - u[67] + u[8], u[5] <= u[34], u[5] <= u[12]/2 +
    u[67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[8] <= -u[12]/2 + (3*u[67])
   /2, u[8] <= u[34]/3 + (4*u[67])/3 - (2*u[12])/3, u[12] <= 2*u[34] - u
   [67], -u[34] + 2*u[67] <= u[8], u[12] - u[34] + u[67] <= u[5]}:
pseudo_polyhedral_54:= PolyhedralSets:-PolyhedralSet(ineqs_54);

ineqs_55:= {u[5] <= u[34], u[5] <= u[12]/2 + u[67]/2, u[5] <= (2*u[12])/3
    + u[8]/3, u[5] <= u[8]/4 + u[12]/2 + u[34]/4, u[8] <= -2*u[12] + 3*u
   [34], u[8] <= -u[34] + 2*u[67], u[8] <= u[67] - u[12] + u[34], u[12] -
    u[67] + u[8] <= u[5], u[8]/2 - u[34]/2 + u[12] <= u[5]}:
pseudo_polyhedral_55:= PolyhedralSets:-PolyhedralSet(ineqs_55);

ineqs_56:= {u[8] <= -2*u[12] + 3*u[34], u[8] <= -u[12]/2 + (3*u[67])/2, u
   [8] <= u[34]/3 + (4*u[67])/3 - (2*u[12])/3, u[12] <= 2*u[34] - u[67],
   -u[34] + 2*u[67] <= u[8], (2*u[12])/3 + u[8]/3 <= u[5], u[12] - u[34]
   + u[67] <= u[5], u[12] - u[67] + u[8] <= u[5]}:
pseudo_polyhedral_56:= PolyhedralSets:-PolyhedralSet(ineqs_56);

ineqs_61:= {u[5] = -u[8] + 2*u[67], u[5] <= u[34], u[5] <= u[12]/2 + u
   [67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[5] <= u[12] - u[67] + u[8], u
   [5] <= u[67]/3 + u[12]/3 + u[34]/3, u[12] <= 2*u[34] - u[67], -u[34] +
    2*u[67] <= u[8], u[12] - u[34] + u[67] <= u[5]}:
pseudo_polyhedral_61:= PolyhedralSets:-PolyhedralSet(ineqs_61);
```

```
ineqs_62:= {u[5] = -u[8] + 2*u[67], u[5] <= u[34], u[5] <= u[12]/2 + u
    [67]/2, u[5] <= (2*u[12])/3 + u[8]/3, u[5] <= u[12] - u[34] + u[67], u
    [5] <= u[12] - u[67] + u[8], u[5] <= u[8]/2 - u[34]/2 + u[12], -u[34]
    + 2*u[67] <= u[8]}:
pseudo_polyhedral_62:= PolyhedralSets:-PolyhedralSet(ineqs_62);
```

$$
\begin{aligned}
&pseudo\_polyhedral\_11 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [2v_{12} - 3v_{34} + v_8 = 0, \\ -v_{12} + 2v_{34} - v_{67} \leq 0, v_{34} - v_5 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_21 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{12} - 2v_{34} + v_{67} = 0, \\ v_{34} - v_5 \leq 0, -v_{34} + 2v_{67} - v_8 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_22 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{34} - 2v_{67} + v_8 \leq 0, \\ v_{34} - v_5 \leq 0, -2v_{12} + 3v_{34} - v_8 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_31 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{34} - v_5 \leq 0, \\ -v_{12} + 2v_{34} - v_{67} \leq 0, -v_{34} + 2v_{67} - v_8 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_32 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-2v_{12} + v_{34} + 2v_5 - v_8 = 0, \\ v_{34} - 2v_{67} + v_8 \leq 0, 2v_{12} - 3v_{34} + v_8 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_33 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{34} - 2v_{67} + v_8 \leq 0, \\ 2v_{12} - 3v_5 + v_8 \leq 0, 2v_{12} - 3v_{34} + v_8 \leq 0] \end{cases} \\
&pseudo\_polyhedral\_34 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{12} - 3v_{67} + 2v_8 = 0, \\ v_{12} - 2v_5 + v_{67} \leq 0, v_{12} - 2v_{34} + v_{67} \leq 0] \end{cases} \\
&pseudo\_polyhedral\_41 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{12} - v_{34} - v_{67} + v_8 = 0, \\ -v_{12} + v_{34} + v_5 - v_{67} = 0, v_{12} - 2v_{34} + v_{67} \leq 0] \end{cases} \\
&pseudo\_polyhedral\_42 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{12} - 2v_5 + v_{67} \leq 0, \\ v_{12} - 2v_{34} + v_{67} \leq 0, -v_{12} + 3v_{67} - 2v_8 \leq 0] \end{cases}
\end{aligned} \tag{3.2}
$$

$$pseudo\_polyhedral\_51 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_{12} + v_{34} + v_5 - v_{67} = 0, \\ -v_{34} + 2v_{67} - v_8 \leq 0, v_{12} - v_{34} - v_{67} + v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_52 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_{12} + v_{34} + v_5 - v_{67} = 0, \\ v_{12} - 2v_{34} + v_{67} \leq 0, -v_{12} + v_{34} + v_{67} - v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_53 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{34} - 2v_{67} + v_8 \leq 0, \\ -2v_{12} + v_{34} + 2v_5 - v_8 \leq 0, -v_{34} + v_5 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_54 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [-v_{12} + v_5 + v_{67} - v_8 = 0, \\ -v_{34} + 2v_{67} - v_8 \leq 0, vv_{12} - 3v_{67} + 2v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_55 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_{34} - 2v_{67} - v_8 \leq 0, \\ 2v_{12} - v_{34} - 2v_5 + v_8 \leq 0, -2v_{12} + 3v_5 - v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_56 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [2v_{12} - 3v_5 + v_8 \leq 0, \\ -v_{34} + 2v_{67} - v_8 \leq 0, v_{12} - 3v_{67} + 2v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_61 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_5 - 2v_{67} + v_8 = 0, \\ -v_{12} + 3v_{67} - 2v_8 \leq 0, v_{12} - v_{34} - v_{67} + v_8 \leq 0] \end{cases}$$

$$pseudo\_polyhedral\_62 := \begin{cases} Coordinates : [v_5, v_8, v_{12}, v_{34}, v_{67}] \\ Relations : [v_5 - 2v_{67} + v_8 = 0, \\ -v_{12} + v_{34} + v_{67} - v_8 \leq 0, -v_{34} + 2v_{67} - v_8 \leq 0] \end{cases}$$

We simply change the non-strict inequalities in (3.2) that are related to the primitive subdivisions. On the other hand, for a non-primitive subdivision, we observed the non-primitive cells one by one to determine the non-strict inequalities in the proof of Proposition 3.2.2.

Chapter 4

Examples

4.1 Symmetric honeycomb

For a cubic polynomial $g(x, y)$, Chan-Sturmfels [CS13] defined the tropical curve $C(\text{trop}(g))$ to be in *honeycomb* form if it contains a trivalent hexagonal cycle. Furthermore, a honeycomb tropical curve is called *symmetric* when the hexagonal cycle has six edges with the same lattice length and the three bounded edges emerging from the cycle also have the same lattice length. It is mentioned that a cubic in the form

$$g(x, y) = a(x^3 + y^3 + 1) + b(x^2y + x^2 + xy^2 + x + y^2 + y) + xy$$

is a symmetric honeycomb if and only if $\text{val}(a) > 2\text{val}(b) > 0$. In this section, we want to examine the conditions for our truncated symmetric cubic

$$f(x, y) = c_{12}(xy^2 + x^2y) + c_{34}(x^2 + y^2) + c_5xy + c_{67}(x + y) + c_8 \tag{4.1}$$

to be in honeycomb form.

Corollary 4.1.1. *Let $f(x, y)$ be the cubic in equation* (4.1)*. The tropical curve of* $\text{trop}(f)(X, Y)$ *is in honeycomb form if and only if*

$$-v_5 + 2v_{67} - v_8 < 0 \qquad -v_{34} + v_5 < 0 \qquad -v_{12} + v_{34} + v_5 - v_{67} < 0.$$

Proof. Tropical curve $C(\text{trop}(f))$ contains a trivalent hexagonal cycle if and only if it is dual to the regular subdivision in Figure 4.1. Thus, this is the case (IV) of Table 3.1. □

Proposition 4.1.2 (Two types of truncated honeycomb)**.** *Let $f(x, y)$ be as defined in equation* (4.1)*, and suppose tropical polynomial* $\text{trop}(f)(X, Y)$ *satisfies the conditions in Corollary* 4.1.1*. In this case, the six edges emanating from the hexagonal cycle can be classified as either:*

(a) five rays and one bounded edge (called the tail), or

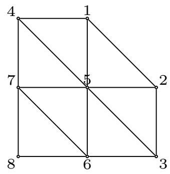


Figure 4.1: The regular subdivision of Δ_f corresponding to $C(\text{trop}(f))$ with a trivalent hexagonal cycle.

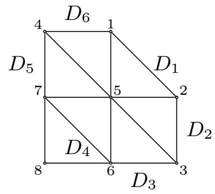
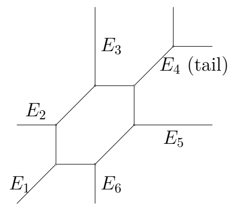
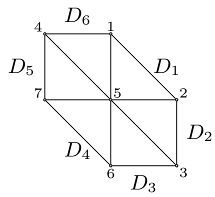
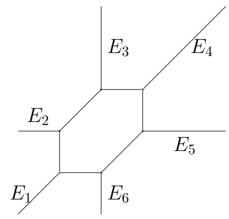
	Edges emerging from the cycle	Subdivision	Tropical curve
(a)	Five rays and one bounded edge		
(b)	Six rays		

Table 4.1: Two types of truncated honeycomb.

(b) six rays,

as illustrated in Table 4.1*. The cases (a),(b) occur according to whether* $c_8 \neq 0$, $c_8 = 0$, *respectively.*

Proof. For $i = 1, 2, 3, 5, 6$, edges E_i of the tropical curves correspond to edges D_i of the subdivisions. When $c_8 \neq 0$, the Newton polygon Δ_f takes the form shown in case (a). In this scenario, edge D_4 does not lie on the border of Δ_f, resulting in its correspondence edge, E_4, being a bounded edge. If $c_8 = 0$, Δ_f exhibits the shape depicted in case (b). In this case, edge D_4 takes part as the border of Δ_f, causing E_4 to form a ray. □

We shall say a truncated honeycomb $C(\text{trop}(f))$ is *quasi-symmetric* if the six sides of the hexagon have the same lattice length. A quasi-symmetric truncated honeycomb is *symmetric* (following the definition in [CS13]) if and only if the hexagon has six emanating rays and does not possess a tail, that is of type (b) of Proposition 4.1.2.

Proposition 4.1.3 (Quasi-symmetric truncated honeycombs)**.** *Let $f(x, y)$ be as in equation* (4.1) *and suppose $C(\text{trop}(f))$ is a truncated honeycomb. Then $C(\text{trop}(f))$ is quasi-symmetric if and only if*

$$2v_{34} = v_{12} + v_{67} \text{ and } -v_5 + 2v_{67} < v_8.$$

The lattice length of the hexagon's side is $|v_{34} - v_5|$ *and the tail is equal to* $|v_5 - 2v_{67} + v_8|$*. Additionally,* $C(\mathrm{trop}(f))$ *is symmetric if and only if*

$$2v_{34} = v_{12} + v_{67} \text{ and } v_8 = \infty.$$

Proof. A truncated honeycomb tropical curve is illustrated in Figure 4.2. The lattice

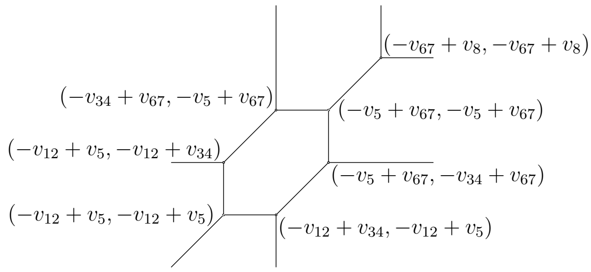


Figure 4.2: A quasi-symmetric truncated honeycomb tropical curve.

lengths of the bounded edges can be determined by the differences of coordinates X or Y in Figure 4.2. Thus, a truncated honeycomb tropical curve is quasi-symmetric if and only if $|v_{34} - v_5| = |v_{12} - v_{34} - v_5 + v_{67}|$. From the last two inequalities of Corollary 4.1.1, we have $v_{34} - v_5 = v_{12} - v_{34} - v_5 + v_{67}$, thus $2v_{34} = v_{12} + v_{67}$. Together with the first inequality of Corollary 4.1.1, the result follows. Hence, the lattice lengths of the edges on the hexagonal cycle are $|v_{34} - v_5|$, while the tail has lattice length $|v_5 - 2v_{67} + v_8|$.

Meanwhile, truncated honeycomb $C(\mathrm{trop}(f))$ is symmetric if and only if the tail has infinite lattice length. That is $|v_5 - 2v_{67} + v_8| = v_5 - 2v_{67} + v_8 = \infty$. Hence, $v_5 = \infty$ or $v_8 = \infty$. If $v_5 = \infty$, the edges $[5, i]$, where $i = 1, 2, 3, 4, 6, 7$, of the regular subdivisions on Table 4.1 do not exist. Thus, $v_8 = \infty$. □

Example 4.1.4. Let $(v_{12}, v_{34}, v_5, v_{67}) = (3, 2, 0, 1)$. If $v_8 = 3$, $C(\mathrm{trop}(f))$ is a quasi-symmetric truncated honeycomb where the hexagon's sides have length 2 and the tail has length 1 as shown in Figure 4.3(I). If $v_8 = \infty$, tropical curve $C(\mathrm{trop}(f))$ is a symmetric truncated honeycomb as illustrated in Figure 4.3(II).

(I) $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (3, 2, 0, 1, 3)$ (II) $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (3, 2, 0, 1, \infty)$

Figure 4.3: Quasi-symmetric and symmetric truncated honeycombs in Example 4.1.4.

4.2 Nobe's one-parameter family f_k

Nobe studied the relation between the invariant curves of a certain piecewise linear dynamical system called the ultradiscrete QRT map and the cycle of a tropical elliptic curve in [N08]. For a fixed $(v_{12}, v_{34}, v_{67}, v_8) \in \mathbb{R}^4$, we modify our tropical polynomial $\mathrm{trop}(f)(X, Y)$ and consider a one-parameter family of tropical curves $\{C(\mathrm{trop}(f_k))\}_{k\in\mathbb{R}}$ of tropical polynomial

$$\mathrm{trop}(f_k)(X, Y) = \min(v_{12} + X + 2Y, v_{12} + 2X + Y, v_{34} + 2X, v_{34} + 2Y, \\ k + X + Y, v_{67} + X, v_{67} + Y, v_8).$$

According to [N08, Lemma 1], there is a one-parameter family of ultradiscrete QRT maps whose invariant curve I_k coincides with the cycle part of $C(\mathrm{trop}(f_k))$ for each $k \in \mathbb{R}$.

Example 4.2.1 ([N08, Example 1])**.** Since we are dealing with operations $(+, \min)$ instead of $(+, \max)$ like [N08], we apply Remark 2.0.3. Therefore, we substitute the following negative values of Nobe's parameters,

$$v_{12} = -10 \quad v_{34} = 0 \quad v_{67} = -5 \qquad v_8 = 0,$$

to Table 3.1 and we obtain Table 4.2. We see that the invariant curves $I_k (k \in \mathbb{R})$ are

	Cycle shape	Conditions of k
(I)	Triangle	$-10 < 0$ $-5 < k$ $k < -\frac{20}{3}$
(II)	Square	$-10 < k$ $k < -7.5$ $-15 < k$
(III)	Pentagon	$k < -10$ $k < -5$ $-15 < k$
(IV)	Hexagon	$-10 < k$ $k < 0$ $k < -15$
(V)	Heptagon	$k < -10$ $-10 < 0$ $k < -15$

Table 4.2: The conditions of $C(\mathrm{trop}(f_k))$ for $(v_{12}, v_{34}, v_{67}, v_8) = (-10, 0, -5, 0)$.

classified into heptagon for $k \in (-\infty, -15)$, pentagon for $k \in [-15, -10)$, and square for $k \in [-10, -7.5)$. The values $(-10, 0, k, -5, 0)$ when $k = -15$, $k = -10$, and $k \geq -7.5$ lie in the polyhedral cones of cases (5.1), (6.1), (4.2) of Proposition 3.2.2, respectively.

We want to give one more example of a family of invariant curves I_k that contains a triangular shape. Furthermore, it seems natural to ask if there exist a fixed $(v_{12}, v_{34}, v_{67}, v_8)$ so the family $\{C(\mathrm{trop}(f_k))\}_{k\in\mathbb{R}}$ varies from a triangular shape to a heptagonal shape. We will discuss this in Proposition 4.2.3.

Example 4.2.2. Let us present the case $(v_{12}, v_{34}, v_{67}, v_8) = (0, 14, 4, 0)$ and substitute it to the Table 3.1 in Theorem 3.1.2. Thus, we have Table 4.3. Then we have the cycle

	Cycle shape	Conditions of k
(I)	Triangle	$-6 < 0$ $-4 < k$ $k < 0$
(II)	Square	$8 < k$ $k < 2$ $-10 < k$
(III)	Pentagon	$k < 8$ $k < -4$ $-10 < k$
(IV)	Hexagon	$8 < k$ $k < 14$ $k < -10$
(V)	Heptagon	$k < 8$ $-6 < 0$ $k < -10$

Table 4.3: Conditions of $C(\mathrm{trop}(f_k))$ for $(v_{12}, v_{34}, v_{67}, v_8) = (0, 14, 4, 0)$.

part of $C(\mathrm{trop}(f_k))$ forms a heptagon for $k < -10$, a pentagon for $-10 \le k < -4$, and a triangle for $-4 \le k < 0$. The nonsmooth tropical curves for $k = -10$, $k = -4$, and $k \ge 0$ are the cases (5.1), (5.4), and (5.6) of Proposition 3.2.2, respectively. Figure 4.4 illustrates the family $\{C(\mathrm{trop}(f_k))\}_{k\in\mathbb{R}}$ for the given $(v_{12}, v_{34}, v_{67}, v_8)$.

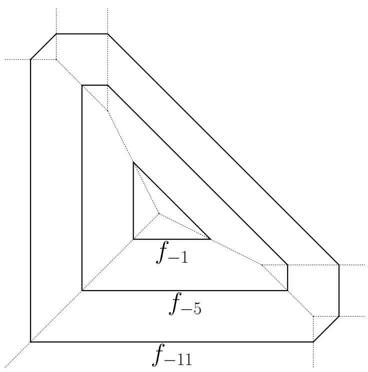


Figure 4.4: $C(\mathrm{trop}(f_k))$ for $k = -1, -5, -11$ in Example 4.2.2.

Proposition 4.2.3. *For any fixed* $(v_{12}, v_{34}, v_{67}, v_8)$*, invariant curves* I_k *with triangle and square shapes are not possible to coexist in the family* $\{C(\text{trop}(f_k))\}_{k\in\mathbb{R}}$*.*

Proof. For a fixed $(v_{12}, v_{34}, v_{67}, v_8)$, the invariant curve I_k is the cycle of a smooth tropical curve $C(\text{trop}(f_k))$ for most of $k \in \mathbb{R}$. Suppose we have $(v_{12}, v_{34}, v_{67}, v_8) = (a, b, c, d)$ such that I_{k_1} is a triangle and I_{k_2} is a square for some $k_1, k_2 \in \mathbb{R}$. Thus, from Table 3.1, we have $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (a, b, k_1, c, d)$ satisfies case (I) and $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (a, b, k_2, c, d)$ satisfies case (II). In other words, we have

$$a - c + d < k_1 < \frac{2a + d}{3} \quad \text{and} \quad 2c - d < k_2 < \frac{a + c}{2}.$$

This implies $a - 3c + 2d < 0$ and $-a + 3c - 2d < 0$, which are contradiction. □

Chapter 5

Two-parameter family of Edwards curves $f_{r,s}$

5.1 Unimodular transformation

Let ϕ be an integral unimodular affine transformation

$$\begin{aligned} \phi :& \mathbb{Z}^2 \to \mathbb{Z}^2 \\ & (i,j) \mapsto (i,j)A + \tau \end{aligned}$$

where $A \in \mathrm{GL}_2(\mathbb{Z})$ and $\tau \in \mathbb{Z}^2$. For $(i,j) \in \mathbb{Z}^2$, let $(xy)^{(i,j)}$ denote monomial $x^i y^j$ and $(i,j) \cdot (X,Y)$ denote $iX + jY$.

Definition 5.1.1. For a non-empty set $\mathcal{I} \subset \mathbb{Z}^2$, let

$$f(x,y) = \sum_{(i,j)\in\mathcal{I}} a_{ij}(xy)^{(i,j)} \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$$

be a Laurent polynomial and

$$\mathrm{trop}(f)(X,Y) = \min(\mathrm{val}(a_{ij}) + (i,j)\cdot(X,Y) : (i,j) \in \mathcal{I})$$

be its tropicalization. Map ϕ acts on $f(x,y)$ to form

$$f^{\phi}(x,y) = \sum_{(i,j)\in\mathcal{I}} a_{ij}(xy)^{\phi(i,j)}$$

and acts on $\mathrm{trop}(f)(X,Y)$ to form

$$\mathrm{trop}(f)^{\phi}(X,Y) = \min(\mathrm{val}(a_{ij}) + \phi(i,j)\cdot(X,Y) : (i,j) \in \mathcal{I}).$$

This definition implies that we have

$$\mathrm{trop}(f)^{\phi}(X,Y) = \mathrm{trop}(f^{\phi})(X,Y).$$

In [KMM09], Katz-Markwig-Markwig mentioned that integral unimodular affine transformations preserve the lattice length of edges on a tropical curve.

In this section, we consider a specific transformation by letting

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tau = (2,2)$$

and let $g(x,y) = f^{\phi}(x,y)$ where $f(x,y)$ is the symmetric truncated cubic polynomial in (1.1). Thus, we have

$$\begin{aligned} g(x,y) =& c_{12}(x+y) + c_{34}(x^2+y^2) + c_5xy + c_{67}(x^2y+y^2x) + c_8x^2y^2 \\ =& f\left(\frac{1}{x}, \frac{1}{y}\right) \cdot x^2y^2 \end{aligned}$$

The tropicalization of g is

$$\begin{aligned} \operatorname{trop}(g)(X,Y) = \min(& v_{12}+X, v_{12}+Y, v_{34}+2X, v_{34}+2Y, v_5+X+Y, \\ & v_{67}+2X+Y, v_{67}+X+2Y, v_8+2X+2Y). \end{aligned}$$

Next we want to show the exact relation between the tropical curves of $\operatorname{trop}(g)(X,Y)$ and $\operatorname{trop}(f)(X,Y)$.

Lemma 5.1.2. *Let $f(x,y)$ and $g(x,y)$ be the Laurent polynomials that satisfy*

$$g(x,y) = f\left(\frac{1}{x}, \frac{1}{y}\right) \cdot x^2y^2.$$

Then we have $C(\operatorname{trop}(f)) = -1 \cdot C(\operatorname{trop}(g))$ holds. In other words, these two tropical curves are symmetric about the center O.

Proof. From the tropicalization of

$$g(x,y) = f\left(\frac{1}{x}, \frac{1}{y}\right) \cdot x^2y^2,$$

we have

$$\operatorname{trop}(g)(X,Y) = \operatorname{trop}(f)(-X,-Y) + 2X + 2Y.$$

Note that $2X+2Y$ does not exhibit any singularities for any (X,Y). Therefore, we have (X,Y) is a point on $C(\operatorname{trop}(g))$ if and only if $(-X,-Y)$ is a point on $C(\operatorname{trop}(f))$. In other words,

$$C(\operatorname{trop}(g)) = -1 \cdot C(\operatorname{trop}(f))$$

holds. □

Furthermore, Lemma 5.1.2 implies that the two tropical curves share the same structure for the same $(v_{12}, v_{34}, v_5, v_{67}, v_8)$.

5.2 Tropical curves of $f_{r,s}$

Let $\mathbb{K}$ be a valuated field and $q \in \mathbb{K}$ such that $\mathrm{val}(q) > 0$. For Euler functions

$$\epsilon = \prod_{n=1}^{\infty}(1+q^n) = 1 + q + q^2 + 2q^3 + \dots,$$
$$\bar{\epsilon} = \prod_{n=1}^{\infty}(1+(-q)^n) = 1 - q + q^2 - 2q^3 + \dots, \text{ and}$$

$r, s \in \mathbb{K}$ such that $\epsilon r \neq \bar{\epsilon} s$, define coefficients

$$\begin{aligned}
d_{12} &= 2\epsilon\bar{\epsilon}(\epsilon^4 - \bar{\epsilon}^4)(\bar{\epsilon}s - \epsilon r),\\
d_{34} &= (\epsilon^4 - \bar{\epsilon}^4)(\bar{\epsilon}^2 s^2 - \epsilon^2 r^2),\\
d_5 &= 8\epsilon\bar{\epsilon}(\epsilon r - \bar{\epsilon}s)(\bar{\epsilon}^3 r - \epsilon^3 s),\\
d_{67} &= 2(\epsilon r - \bar{\epsilon}s)\{(\bar{\epsilon}^4 - \epsilon^4)rs + 2\epsilon\bar{\epsilon}(\bar{\epsilon}^2 r^2 - \epsilon^2 s^2)\}, \text{ and}\\
d_8 &= 2(\epsilon^2 s^2 - \bar{\epsilon}^2 r^2)(\bar{\epsilon}^2 s^2 - \epsilon^2 r^2).
\end{aligned} \tag{5.1}$$

For $i \in \{12, 34, 5, 67, 8\}$, let $u_i = \mathrm{val}(d_i)$. In [NT23], it is shown that

$$f_{r,s}(x,y) = d_{12}(x+y) + d_{34}(x^2+y^2) + d_5xy + d_{67}(x^2y + y^2x) + d_8x^2y^2$$

is birationally equivalent to an Edwards elliptic curve that can be parameterized by theta functions, [E07]. We will discuss it in the next section. We can see that Laurent polynomial

$$\begin{aligned}
g_{r,s}(x,y) &:= f_{r,s}\left(\frac{1}{x}, \frac{1}{y}\right) \cdot x^2y^2\\
&= d_{12}(xy^2 + x^2y) + d_{34}(x^2+y^2) + d_5xy + d_{67}(x+y) + d_8
\end{aligned}$$

shares the same Newton polygon with our symmetric truncated cubic in (1.1). By Lemma 5.1.2, the tropical curve of tropical polynomial

$$\begin{aligned}
\mathrm{trop}(f_{r,s})(X,Y) = \min(&u_{12} + X, u_{12} + Y, u_{34} + 2X, u_{34} + 2Y, u_5 + X + Y,\\
&u_{67} + 2X + Y, u_{67} + 2Y + X, u_8 + 2X + 2Y)
\end{aligned}$$

and the tropical curve $C(\mathrm{trop}(g_{r,s}))$ are point-symmetric with respect to the origin O.

In this section, we want to discuss the possible tropical curves of $\mathrm{trop}(f_{r,s})$. We have

$$\begin{aligned}\mathrm{val}(\epsilon) &= \sum_{n=1}^{\infty}\mathrm{val}(1+q^n)\\ &= 0\end{aligned} \quad \text{and} \quad \begin{aligned}\mathrm{val}(\bar{\epsilon}) &= \sum_{n=1}^{\infty}\mathrm{val}(1+(-q)^n)\\ &= 0\end{aligned}$$

due to Remark 2.0.1. Furthermore, we have $\mathrm{val}(\epsilon^4 - \bar{\epsilon}^4) = 1$. For a big $N \in \mathbb{N}$, let

$$r = r_0 + r_1q^{\frac{1}{N}} + \dots + r_Nq + r_{N+1}q^{1+\frac{1}{N}} + \dots \text{ and}$$

$$s = s_0 + s_1 q^{\frac{1}{N}} + \cdots + s_N q + s_{N+1} q^{1+\frac{1}{N}} + \ldots$$

where $r_i, s_i \in \mathbb{K}$ and r_0, s_0 are nonzero. Assume

$$f_1 = \bar{\epsilon}s - \epsilon r, f_2 = \bar{\epsilon}^2 s^2 - \epsilon^2 r^2, f_3 = \epsilon^2 s^2 - \bar{\epsilon}^2 r^2, f_4 = \bar{\epsilon}^3 r - \epsilon^3 s, f_5 = (\bar{\epsilon}^4 - \epsilon^4) rs - 2\epsilon\bar{\epsilon} f_3,$$

and $F_i = \mathrm{val}(f_i)$ for $i = 1, \ldots, 5$. We can know the values of F_i by expanding f_i. Furthermore, we have

$$u_{12} = 1 + F_1, u_{34} = 1 + F_2, u_5 = F_1 + F_4, u_{67} = F_1 + F_5, \text{ and } u_8 = F_2 + F_3. \tag{5.2}$$

Proposition 5.2.1. *Table* 5.1 *shows the conditions for* $(F_1, F_2, F_3, F_4, F_5)$ *of the regular subdivisions related to* $\mathrm{trop}(f_{r,s})(X, Y)$.

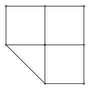
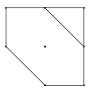
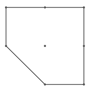
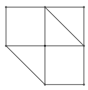
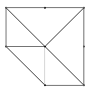
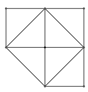
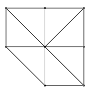
(a)		$2F_1 - 2F_2 + F_5 - 1 < 0$ $F_3 - F_5 = 0$ $F_1 - F_2 - F_4 + F_5 = 0$
(b)		$2F_1 - 2F_2 - F_3 + 2F_5 - 1 < 0$ $-F_1 + F_2 - F_4 + 1 \leq 0$ $2F_1 - 2F_2 + F_5 - 1 = 0$
(c)		$2F_1 - 2F_2 + F_3 - 1 = 0$ $-2F_1 + 2F_2 - F_5 + 1 \leq 0$ $-F_1 + F_2 - F_4 + 1 \leq 0$
(d)		$2F_1 - 2F_2 + F_5 - 1 < 0$ $-F_3 + F_5 < 0$ $-F_1 + F_2 + F_4 - F_5 = 0$
(e)		$F_1 - F_2 + F_4 - 1 < 0$ $-F_3 + 2F_4 - 1 < 0$ $-2F_1 + 2F_2 + F_3 - 2F_5 + 1 \leq 0$
(f)		$F_1 - F_2 - F_3 - F_4 + 2F_5 < 0$ $F_4 - F_5 < 0$ $F_1 - F_2 - F_4 + F_5 < 0$
(g)		$-F_1 + F_2 + F_3 + F_4 - 2F_5 < 0$ $2F_1 - 2F_2 - F_3 + 2F_5 - 1 < 0$ $-F_1 + F_2 + F_4 - F_5 < 0$

Table 5.1: Summaries of the regular subdivisions related to $\mathrm{trop}(f_{r,s})(X, Y)$.

Proof. Table 5.1 can be obtained by substituting equations (5.2) to Table 3.2 of Proposition 3.2.2 and Table 3.1 of Theorem 3.1.2. Cases (a) - (e) are cases (4.1), (2.1), (1.1), (5.2), (5.3) of Proposition 3.2.2, respectively. Meanwhile, cases (f) and (g) are the square and heptagon cases of Theorem 3.1.2. □

Proposition 5.2.2. *The tropical curves* $\mathrm{trop}(f_{r,s})(X, Y)$ *are limited to the cases in Table* 5.2.

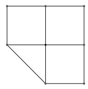
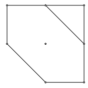
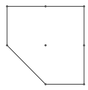
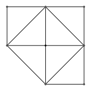
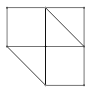
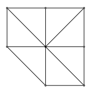
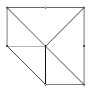
Cases	Regular subdivisions
1. $s_0^2 - r_0^2 \neq 0$. 2. $s_0 - r_0 = \cdots = s_{t-1} - r_{t-1} = 0$ and $s_t - r_t \neq 0$ for some $1 \leq t \leq N-1$. 3. $s_0 + r_0 = \cdots = s_{t-1} + r_{t-1} = 0$ and $s_t + r_t \neq 0$ for some $1 \leq t \leq N-1$. 4. $s_0 + r_0 = \cdots = s_{N-1} + r_{N-1} = 0$ and $s_N + r_N \neq \pm 2r_0, 0$.	
5. $s_0 - r_0 = \cdots = s_{N-1} - r_{N-1} = 0$ and $s_N - r_N = -2r_0$	
6. $s_0 - r_0 = \cdots = s_{N-1} - r_{N-1} = 0$ and $s_N - r_N \neq -2r_0$	
7. $s_0 + r_0 = \cdots = s_{N-1} + r_{N-1} = 0$ and $s_N + r_N = -2r_0$	
8. $s_0 + r_0 = \cdots = s_{N-1} + r_{N-1} = 0$ and $s_N + r_N = 2r_0$	
9. $s_0 + r_0 = \cdots = s_N + r_N = 0$ and $s_t + r_t \neq 0$ for some $N+1 \leq t \leq 2N-1$.	
10. $s_0 + r_0 = \cdots = s_N + r_N = \cdots = s_{2N-1} + r_{2N-1} = 0$	

Table 5.2: The regular subdivisions that are dual to $C(\text{trop}(f_{r,s})(X,Y))$.

Proof. For $i = 1, \ldots, 5$, the value of F_i is determined by the leading term of f_i. Expressions f_i can be seen as functions of q with coefficients $r_0, r_1, \ldots, s_0, s_1, \ldots$ that follow certain patterns. Their expansions are

$$\begin{aligned}
f_1 =& (s_0 - r_0) + (s_1 - r_1)q^{\frac{1}{N}} + \cdots + (s_{N-1} - r_{N-1})q^{\frac{N-1}{N}} + (s_N - r_N - s_0 - r_0)q + \ldots, \\
f_2 =& (s_0^2 - r_0^2) + (2s_0s_1 - 2r_0r_1)q^{\frac{1}{N}} + (2s_0s_2 + s_1^2 - 2r_0r_2 - r_1^2)q^{\frac{2}{N}} + \ldots \\
&+ (2s_0s_N + 2s_1s_{N-1} + \cdots - 2r_0r_N - 2r_1r_{N-1} - \cdots - 2s_0^2 - 2r_0^2)q \\
&+ (2s_0s_{N+1} + 2s_1s_N + \cdots - 2r_0r_{N+1} - 2r_1r_N - \cdots - 4s_0s_1 - 4r_0r_1)q^{1+\frac{1}{N}} \\
&+ \ldots, \\
f_3 =& (s_0^2 - r_0^2) + (2s_0s_1 - 2r_0r_1)q^{\frac{1}{N}} + (2s_0s_2 + s_1^2 - 2r_0r_2 - r_1^2)q^{\frac{2}{N}} + \ldots \\
&+ (2s_0s_N + 2s_1s_{N-1} + \cdots - 2r_0r_N - 2r_1r_{N-1} - \cdots + 2s_0^2 + 2r_0^2)q
\end{aligned}$$

$$
\begin{aligned}
&+ (2s_0 s_{N+1} + 2s_1 s_N + \cdots - 2r_0 r_{N+1} - 2r_1 r_N - \cdots + 4s_0 s_1 + 4r_0 r_1) q^{1+\frac{1}{N}} \\
&+ \ldots, \\
f_4 =& (r_0 - s_0) + (r_1 - s_1) q^{\frac{1}{N}} + \cdots + (r_{N-1} - s_{N_1}) q^{\frac{N-1}{N}} + (r_N - s_N - 3r_0 - 3s_0) q + \ldots, \\
f_5 =& (2r_0^2 - 2s_0^2) + (4r_0 r_1 - 4s_0 s_1) q^{\frac{1}{N}} + (4r_0 r_2 + 2r_1^2 - 4s_0 s_2 - 2s_1^2) q^{\frac{2}{N}} + \ldots \\
&+ (4r_0 r_N + 4r_1 r_{N-1} + \cdots - 4s_0 s_N - 4s_1 s_{N-1} - \cdots - 4r_0^2 - 4s_0^2 - 8r_0 s_0) q \\
&+ (4r_0 r_{N+1} + 4r_1 r_N + \cdots - 4s_0 s_{N+1} - 4s_1 s_N - \cdots - 8r_0 r_1 - 8s_0 s_1 - 8r_0 s_1 - 8s_0 r_1) q^{1+\frac{1}{N}} \\
&+ \ldots.
\end{aligned}
$$

Table 5.3 shows the values of $(F_1, \ldots, F_5)$ for all of the ten cases. We can check that each

	F_1	F_2	F_3	F_4	F_5
Case 1	0	0	0	0	0
Case 2	$\frac{t}{N}$	$\frac{t}{N}$	$\frac{t}{N}$	$\frac{t}{N}$	$\frac{t}{N}$
Case 3	0	$\frac{t}{N}$	$\frac{t}{N}$	0	$\frac{t}{N}$
Case 4	0	1	1	0	1
Case 5	1	1	$F_3 > 1$	1	1
Case 6	$F_1 \geq 1$	$F_2 = F_1$	1	$F_4 \geq 1$	$F_5 \geq 1$
Case 7	0	$F_2 > 1$	1	0	1
Case 8	0	1	$F_3 > 1$	0	1
Case 9	0	1	1	0	$1 < F_5 < 2$
Case 10	0	1	1	0	$F_5 \geq 2$

Table 5.3: The values of $(F_1, \ldots, F_5)$ for the ten cases.

value of $(F_1, \ldots, F_5)$ satisfies the condition of the corresponding regular subdivision as mentioned in Proposition 5.2.1. □

5.3 The parametrization of the cycle by ultradiscrete theta functions

Edwards in [E07] showed that the elliptic curves of form

$$
E_a := x^2 + y^2 = a^2(1 + x^2 y^2)
$$

can be parameterized by theta functions

$$
\begin{aligned}
\theta_1(z|\tau) =& -i \sum_{n \in \mathbb{Z}} (-1)^n \exp\left(\pi i \tau \left(\frac{1}{2} + n\right)^2\right) \exp(\pi i z(2n+1)), \\
\theta_2(z|\tau) =& \sum_{n \in Z} \exp\left(\pi i \tau \left(\frac{1}{2} + n\right)^2\right) \exp(\pi i z(2n+1)),
\end{aligned}
$$

$$\theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i \tau n^2)\exp(\pi i z 2n),$$
$$\theta_4(z|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n \exp(\pi i \tau n^2)\exp(\pi i z 2n)$$

where $z, \tau \in \mathbb{C}$ and $\mathrm{im}(\tau) > 0$.

Remark 5.3.1. [E07, Theorem 15.1] Let $\tau \in \mathbb{C}$ where $\mathrm{im}(\tau) > 0$ and

$$a = \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)}.$$

Then elliptic curves E_a can be parameterized by

$$x(z) = \frac{\theta_1(z|2\tau)}{\theta_4(z|2\tau)}, \quad y(z) = \frac{\theta_2(z|2\tau)}{\theta_3(z|2\tau)}$$

for $z \in \mathbb{C}$.

Through rational substitutions

$$x = \frac{r\mathrm{x} + \bar{\epsilon}}{s\mathrm{x} + \epsilon}, \quad y = \frac{r\mathrm{y} + \bar{\epsilon}}{s\mathrm{y} + \epsilon}$$

we see that $f_{r,s}(\mathrm{x}, \mathrm{y})$ is the numerator of the rational function. Thus, we can parametrized $f_{r,s}(\mathrm{x}, \mathrm{y})$ by utilizing the theta parametrization of E_a as demonstrated in [NT23]. The ultradiscretization of these theta parametrizations allows us to express the cycle part of $C(\mathrm{trop}(f_{r,s}))$ as two periodic functions. For $t \in \mathbb{R}$, let

$$\Theta^{\mathrm{odd}}(t) = -2(2\left\lfloor \frac{t}{2} \right\rfloor + 1 - t)^2$$
$$\Theta^{\mathrm{even}}(t) = -2(2\left\lfloor \frac{t+1}{2} \right\rfloor - t)^2.$$

Remark 5.3.2. [NT23, Theorem 1.2] For $t \in \mathbb{R}$, expressions

$$X(t) = Y(t - \frac{1}{2})$$
$$Y(t) = \max(\Theta^{\mathrm{odd}}(t), -1 + \Theta^{\mathrm{even}}(t)) - \max(\mathrm{val}_{\mathbb{K}}(r-s) + \Theta^{\mathrm{even}}(t), -\mathrm{val}_{\mathbb{K}}(r+s) + \Theta^{\mathrm{odd}}(t))$$

trace the cycle part of $C(\mathrm{trop}(f_{r,s}))$ as $\{(-X(t), -Y(t)) : t \in \mathbb{R}\}$.

By assuming $\delta = \mathrm{val}_{\mathbb{K}}(r+s) - \mathrm{val}_{\mathbb{K}}(r-s)$ and

$$Y_\delta(u) = \max(\Theta^{\mathrm{odd}}(t), -1 + \Theta^{\mathrm{even}}(t)) - \max(\delta + \Theta^{\mathrm{even}}(t), \Theta^{\mathrm{odd}}(t)), \tag{5.3}$$

we have

$$\begin{pmatrix} -X(t) \\ -Y(t) \end{pmatrix} = \begin{pmatrix} -Y_\delta(t - \frac{1}{2}) \\ -Y_\delta(t) \end{pmatrix} + \mathrm{val}_{\mathbb{K}}(r+s) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.4}$$

Since the last term does not play a role in determining the shape of $(-X(t), -Y(t))$, we have that the value of δ determines the shape of the cycle part of $C(\mathrm{trop}(f_{r,s}))$.

Remark 5.3.3. [NT23, Corollary 1.3] The cycle part of $C(\mathrm{trop}(f_{r,s}))$ is determined by the value δ with the following rules.

1. Curve $C(\mathrm{trop}(f_{r,s}))$ has no cycle if and only if $\delta \leq -1$.
2. Curve $C(\mathrm{trop}(f_{r,s}))$ has a square cycle if and only if $-1 < \delta \leq 1$.
3. Curve $C(\mathrm{trop}(f_{r,s}))$ has a heptagonal cycle if and only if $1 < \delta < 2$.
4. Curve $C(\mathrm{trop}(f_{r,s}))$ has a pentagonal cycle if and only if $2 \leq \delta$.

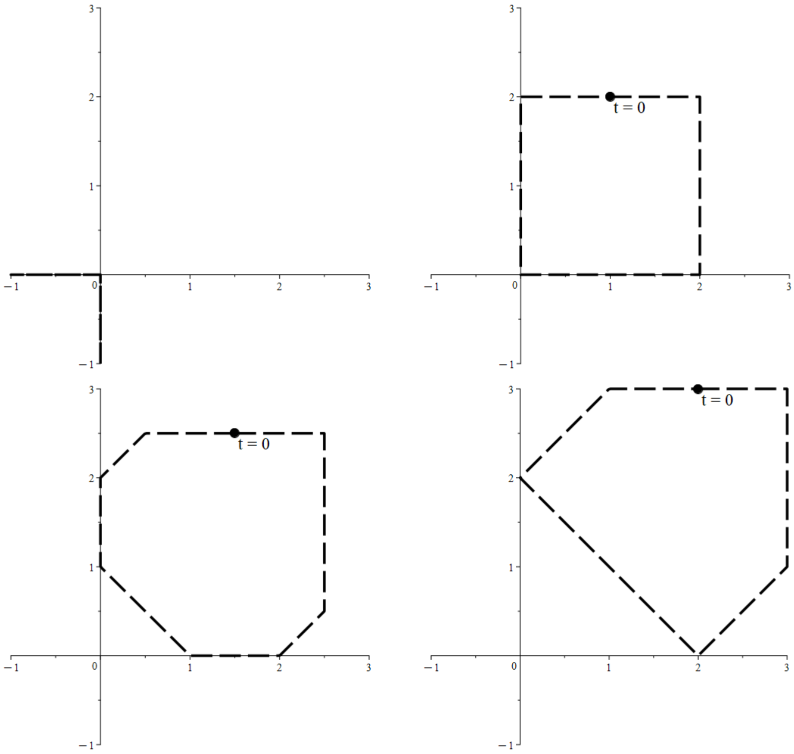


Figure 5.1: From left top: Curves $(-Y_\delta(t-\frac{1}{2}), -Y_\delta(t))$ for $\delta = -1$, $\delta = 1$, $\delta = 1.5$, and $\delta = 2$.

Chapter 6

Tropical group of points

6.1 Intersection points on tropical curves

A group of points has been defined on various tropical curves in [V10], [N16], and [N11]. In this section, we adopt the definitions related to the group of points on smooth tropical curves of degree three, also known as tropical elliptic curves, [V10]. We particularly emphasize the tropical elliptic curve associated with the tropical polynomial (1.2). In this section, we will introduce two kinds of intersections involving a tropical curve. The first one is the intersection between two tropical curves. The second one is the intersection between a tropical curve and another piecewise linear expression.

Definition 6.1.1. Let C_1 and C_2 be two tropical curves. The two curves intersect transversally when they do not intersect at any of the vertices of the two curves. The set of intersection points are denoted as $C_1 \cap C_2$. When the two curves do not intersect transversally, we wiggle the two curves as far as ϵ to C_1^ϵ and C_2^ϵ until they intersect transversally. Then we define the stable intersection

$$C_1 \cap_{\mathrm{st}} C_2 = \lim_{\epsilon \to 0} C_1^\epsilon \cap C_2^\epsilon.$$

For $i = 1, 2$, let $P \in C_1 \cap_{\mathrm{st}} C_2$ be a point on edges E_i of C_i. Let u_i be the primitive integer directions and ω_i be the weights of E_i. The multiplicity of P, denoted by $\mathrm{mult}(P)$, is the absolute value of

$$\omega_1 \omega_2 \left| u_1 u_2 \right|.$$

In classical algebra, we know that two curves of degree d_1 and d_2 intersect at $d_1 d_2$ points in projective space by counting multiplicities. Similarly, we encounter a similar number of intersection points of two tropical curves.

Theorem 6.1.2 (Tropical Bezout-Bernstein)**.** *Let C_1 and C_2 be two tropical curves of degree d_1 and d_2, respectively. Assume one of the tropical curves have full support. By counting multiplicities, the stable intersection contains $d_1 d_2$ points.*

Next, we will consider the intersection of a tropical curve with the variety of a tropical rational function.

Definition 6.1.3. A tropical rational function is a piecewise linear function of the form

$$h(X,Y) = \text{trop}(f)(X,Y) - \text{trop}(g)(X,Y)$$

where $f(x,y)$ and $g(x,y)$ are polynomials with $\Delta_f = \Delta_g$. The variety $V(h)$ of $h(X,Y)$ is the collection of coordinates (X,Y) where the function is not linear.

When dealing with the intersection of a tropical curve C and $V(h)$, we observe the restriction of $h(X,Y)$ to C that is piecewise linear on each edge of C with integer slopes.

Definition 6.1.4. Let $\bar{h}$ be the restriction of $h(X,Y)$ to C. Note that for any point P on a tropical curve C, we have P is either a vertex or a 2-valent vertex. The order of a point P with respect to $\bar{h}$, denoted by $\text{ord}_{\bar{h}}(P)$, is the sum of the outgoing slopes of $\bar{h}$ along the edges adjacent to P.

Example 6.1.5. Let $\text{trop}(f)(X,Y) = \min(1+X, Y, 4)$, $\text{trop}(g)(X,Y) = \min(X, 1+Y, 4)$, and $h(X,Y) = \text{trop}(f)(X,Y) - \text{trop}(g)(X,Y)$. Let C be a tropical curve that is situated with $V(h)$ as shown in Figure 6.1. The orders of some points $P_1, P_2 \in C$ with respect to $\bar{h}$ are

$$\begin{aligned}
\text{ord}_{\bar{h}}(P_1) =& (0,1)\cdot(-1,1) + (1,0)\cdot(-1,1) + (-1,-1)\cdot(-1,1) = 0, \\
\text{ord}_{\bar{h}}(P_2) =& (-1,0)\cdot(-1,1) + (1,0)\cdot(0,0) = 1.
\end{aligned}$$

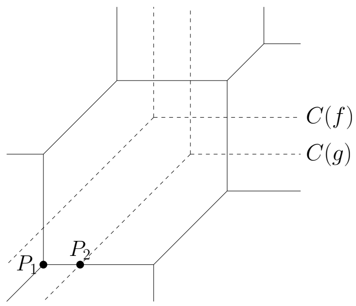


Figure 6.1: Tropical curve C and variety $V(h)$ of Example 6.1.5.

It is worth noting that for points P on a tropical curve that are not contained in $V(h)$, their order is zero.

The collection of points on a tropical curve C forms a group that is called the group of divisors. The intersection points play an important role in this group.

Definition 6.1.6. A divisor $\mathcal{D}$ on C is a finite sum $\mathcal{D} = \sum \mu_P P$ where $\mu_P \in \mathbb{Z}$ and P are points on C. The collection of these divisors forms the group of divisors $\text{Div}(C)$. The degree of $\mathcal{D}$ is $\sum \mu_P$. An important subgroup of $\text{Div}(C)$ is the group of degree-zero divisors that is denoted by $\text{Div}^0(C)$.

The most important divisors on a tropical curve C are ones that are related to a polynomial $f(x,y)$ or to a tropical rational polynomial $h(X,Y)$.

Definition 6.1.7. Let C be a tropical curve. For a polynomial $f(x,y)$, let $\mathcal{P} = C \cap_{\mathrm{st}} C(\mathrm{trop}(f))$. The divisor related to $f(x,y)$ is

$$\mathrm{div}(f) = \sum_{P \in \mathcal{P}} \mathrm{mult}(P)P.$$

For a tropical rational polynomial $h(X,Y)$, the divisor related to $h(X,Y)$ is

$$\mathrm{div}(h) = \sum_{P \in C} \mathrm{ord}_{\bar{h}}(P)P$$

and is called a principal divisor. When $h(X,Y) = \mathrm{trop}(f)(X,Y) - \mathrm{trop}(g)(X,Y)$, [V09] shows that $\mathrm{div}(h) = \mathrm{div}(f) - \mathrm{div}(g)$. Two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent, denoted by $\mathcal{D}_1 \sim \mathcal{D}_2$, if $\mathcal{D}_1 - \mathcal{D}_2$ is a principal divisor. The group $\mathrm{Div}^0(C)/\sim$ is called the Jacobian of C, denoted by $\mathrm{Jac}(C)$.

Example 6.1.8. Consider a tropical curve C, tropical line L_1, and tropical line L_2 in Figure 6.2. We have $\mathcal{D}(L_1) = P_1 + P_2 + P_3$ and $\mathcal{D}(L_2) = Q_1 + Q_2 + Q_3$. Thus, we can say $P_1 + P_2 + P_3 \sim Q_1 + Q_2 + Q_3$.

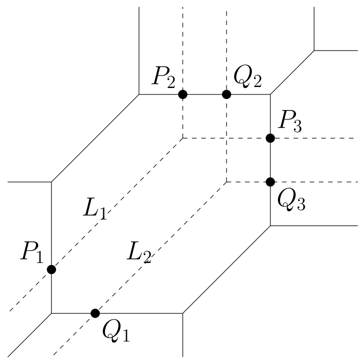


Figure 6.2: Equivalent divisors.

6.2 Point addition by intersection points

From this point, we let C be a tropical elliptic curve. Thus, it has a unique cycle that is denoted by $\overline{C}$. We can attach a metric on $\overline{C}$ as follows.

Definition 6.2.1. Fix a point $\mathcal{O}$ on $\overline{C}$ and name the vertices of $\overline{C}$ with $V_1, \ldots, V_n$ in counter-clockwise direction so that either $\mathcal{O} = V_1$ or $\mathcal{O}$ is between V_1 and V_n. For $i = 1, \ldots, n$, let E_i be the edge connecting V_i and V_{i+1} where $V_1 = V_{n+1}$. Let $\mathcal{L}$ be the sum of the lattice length (see Definition 2.0.11) of the edges of cycle $\overline{C}$ and we define the bijection map $\alpha : \overline{C} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ as follows.

1. $\alpha(\mathcal{O}) = 0$,
2. $\alpha(V_1) = l_{\mathcal{O}V_1}$,
3. $\alpha(V_i) = \alpha(V_{i-1}) + l_{E_{i-1}}$, and
4. for P on E_i, $\alpha(P) = \alpha(V_i) + l_{V_iP}$.

Furthermore, for any two points P and Q on $\overline{C}$, the sign displacement of these two points is

$$d_C(P, Q) = \alpha(Q) - \alpha(P).$$

The sign displacement has several properties, discussed next.

Remark 6.2.2 ([V09, Lemma 6.3])**.** For any three points P, Q, and R on $\overline{C}$, the sign displacement satisfies the following properties.

1. $d_C(P, Q) + d_C(Q, P) = 0$.
2. $d_C(P, Q) + d_C(Q, R) = d_C(P, R)$.
3. For points P' and Q', divisors $P+Q \sim P'+Q'$ if and only if $d_C(P, P') = -d_C(Q, Q')$.

The group of point addition on $\overline{C}$ is algebraically defined as follows.

Definition 6.2.3. Due to the properties of d_C, for a fixed $\mathcal{O} \in \overline{C}$, the map

$$\begin{aligned} \tau_{\mathcal{O}} : \overline{C} &\to \mathrm{Jac}(C) \\ P &\mapsto P - \mathcal{O} \end{aligned}$$

is bijection. For points $P, Q \in \overline{C}$, the point $P + Q$ is the preimage $\tau_{\mathcal{O}}^{-1}(P + Q - 2\mathcal{O})$.

We can identify the point $P + Q$ by using the sign displacement.

Proposition 6.2.4. *For $P, Q \in \overline{C}$, a point $T \in \overline{C}$ is the point $P + Q$ if and only if*

$$d_C(\mathcal{O}, T) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q). \tag{6.1}$$

Proof. Let $T \in \overline{C}$ satisfy equation (6.1). By adding $d_C(Q, \mathcal{O})$ to both sides, we have $d_C(Q, T) = d_C(\mathcal{O}, P)$. Remark 6.2.2 implies

$$\begin{aligned} T + \mathcal{O} &\sim P + Q \\ T - \mathcal{O} &\sim P + Q - 2\mathcal{O}. \end{aligned}$$

From the definition of $\tau_{\mathcal{O}}$, we have $T\tau_{\mathcal{O}}^{-1}(T - \mathcal{O}) = \tau_{\mathcal{O}}^{-1}(P + Q - 2\mathcal{O}) = P + Q$. Conversely, it is shown in [V09, Theorem 6.6] that the point $P + Q$ satisfies (6.1). □

For some pairs of points on $\overline{C}$, the addition can be achieved geometrically in the following manner.

Remark 6.2.5. Let (P, Q) be a pair of points on $\overline{C}$. It is defined as a good pair when a tropical line L_{PQ} such that $P, Q \in L_{PQ} \cap_{\text{st}} \overline{C}$ is present. For a good pair (P, Q), let R be the third intersection point in $L_{PQ} \cap_{\text{st}} \overline{C}$. If $(\mathcal{O}, R)$ is a good pair, let $L_{\mathcal{O}R}$ be the tropical line such that $\mathcal{O}, R \in L_{\mathcal{O}R} \cap_{\text{st}} \overline{C}$. The point $P + Q$ is the third intersection point in $L_{\mathcal{O}R} \cap_{\text{st}} \overline{C}$.

Next, we are going to provide an addition calculation related to the smooth tropical curves mentioned in Theorem 3.1.2. Let $\overline{C}$ be the cycle of the smooth tropical curves of

$$\begin{aligned}\text{trop}(f)(X, Y) = \min(&v_{12} + X + 2Y, v_{12} + 2X + Y, v_{34} + 2X, v_{34} + 2Y \\ &v_5 + X + Y, v_{67} + X, v_{67} + Y, v_8)\end{aligned}$$

that is a symmetrical triangle, square, pentagon, hexagon, or heptagon. Let $P(X_1, Y_1)$ and $Q(X_2, Y_2)$ be two points on $\overline{C}$. Occasions may arise wherein numerous tropical lines passing through points P and Q are discernible, such as when $X_1 = X_2$. In these occurrences, a singular tropical line is rigorously defined, originating from the tropical determinant of a matrix.

Definition 6.2.6. For any two points $P(X_1, Y_1)$ and $Q(X_2, Y_2)$ on $\overline{C}$, let

$$L_{PQ}(X, Y) = \begin{vmatrix} 0 & X & Y \\ 0 & X_1 & Y_1 \\ 0 & X_2 & Y_2 \end{vmatrix}_{\text{trop}} = \min(X_1 + Y_2, X_2 + Y_1, X + \min(Y_1, Y_2), Y + \min(X_1, X_2))$$

be the unique tropical line passing P and Q.

Line L_{PQ} has one vertex (V_X, V_Y) that is always within the cycle $\overline{C}$. Furthermore, we have

$$\begin{aligned} V_X &= \min(X_1 + Y_2, X_2 + Y_1) - \min(Y_1, Y_2), \\ V_Y &= \min(X_1 + Y_2, X_2 + Y_1) - \min(X_1, X_2). \end{aligned}$$

Inspired by some calculations in [O06], we have the following coordinate-wise expression of point addition.

Proposition 6.2.7. *The intersection points of $L_{PQ} \cap_{st} \overline{C}$ are*

$$\begin{aligned} P_1 =& (v_5 - v_{12} + \max(0, V_X - V_Y), v_5 - v_{12} + \max(-(V_X - V_Y), 0)), \\ P_2 =& (V_X, \min(v_8 - V_X, v_{67}, v_{34} + V_X) - v_5), \\ P_3 =& (\min(v_8 - V_Y, v_{67}, v_{34} + V_Y) - v_5, V_Y). \end{aligned}$$

The pair of points (P, Q) is a good pair if and only if $\{P, Q\} \subseteq \{P_1, P_2, P_3\}$. Furthermore, when (P, Q) is a good pair of points, the coordinate of the third intersection point R in $L_{PQ} \cap_{\text{st}} \overline{C}$ is

$$R = P_1 + P_2 + P_3 - P - Q,$$

that is the sum of coordinates in $\mathbb{R}^2$.

Proof. Assume the intersection points $L_{PQ} \cap_{\text{st}} \overline{C}$ are situated as shown in Figure 6.3. When $V_Y > V_X$, we have coordinate P_1 is the solution of $X - Y = V_X - V_Y$ and $X = v_5 - v_{12}$. Meanwhile, when $V_X > V_Y$, coordinate P_1 is the solution of $X - Y = V_X - V_Y$ and $Y = v_5 - v_{12}$. The coordinate of P_2 is the solution of $X = V_X$ and the smallest value of $Y = v_8 - v_5 - V_X$, $Y = v_{67} - v_5$, or $Y = v_{34} - v_5 + V_X$. Similarly, coordinate P_3 is the solution of $Y = V_Y$ and the smallest value of $X = v_8 - v_5 - V_Y$, $X = v_{67} - v_5$, or $X = v_{34} - v_5 + V_Y$.

The second claim follows from the definition of a good pair of points. Furthermore, it is clear that the coordinate of the third intersection of $L_{PQ} \cap_{\text{st}} \overline{C}$ is the sum of all the three intersection points minus the pair of good points. □

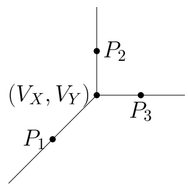


Figure 6.3: Intersection points $L_{PQ} \cap_{\text{st}} \overline{C}$.

When working with the symmetric truncated cubic, we can choose a point $\mathcal{O}$ so that line $L_{\mathcal{O}R}$ in Remark 6.2.5 is not needed.

Proposition 6.2.8. *Let $\overline{C}$ be the cycle of the tropical curves mentioned in Theorem 3.1.2 and $\mathcal{O}$ be the vertex that is dual to cell $[1, 2, 5]$. For any tropical line L such that $L \cap_{\text{st}} \overline{C} = \{P_1, P_2, P_3\}$, we have*

$$d_C(\mathcal{O}, P_1) + d_C(\mathcal{O}, P_2) + d_C(\mathcal{O}, P_3) = 0.$$

Proof. Assume the intersection points $L \cap_{\text{st}} \overline{C}$ are situated as shown in Figure 6.3. If $P_1 = \mathcal{O}$, the result follows due to the symmetric property of $\overline{C}$. If P_1 is somewhere on the horizontal (or vertical) edge emerging from $\mathcal{O}$, we translate line L horizontally (or vertically) until P_1 becomes $P_1' = \mathcal{O}$ and P_2 becomes P_2' (or P_3 becomes P_3') and the sum of the sign displacement from $\mathcal{O}$ to the new points is zero.

In counter-clockwise direction from point $\mathcal{O}$, the edges of $\overline{C}$ have the integer primitive directions limited to $(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1)$, or $(0, -1)$. Thus, we have $d_C(P_1, P_1') = d_C(P_2', P_2)$ (or $d_C(P_1, P_1') = d_C(P_3', P_3)$) holds. By considering

$$d_C(\mathcal{O}, P_1) = d_C(\mathcal{O}, P_1') - d_C(P_1, P_1') \text{ and } d_C(\mathcal{O}, P_2) = d_C(\mathcal{O}, P_2') + d_C(P_2', P_2)$$

(or $d_C(\mathcal{O}, P_3) = d_C(\mathcal{O}, P_3') + d_C(P_3', P_3)$), we have

$$d_C(\mathcal{O}, P_1) + d_C(\mathcal{O}, P_2) + d_C(\mathcal{O}, P_3) = 0.$$

□

Corollary 6.2.9. *Let $\overline{C}$ be the cycle of the tropical curves mentioned in Theorem 3.1.2. Fix the point that is dual to cell $[1, 2, 5]$ as the point $\mathcal{O}$. Let (P, Q) be a good pair of points on $\overline{C}$ and let R be the third intersection point in $L_{PQ} \cap_{\text{st}} \overline{C}$. Then the coordinate of $P + Q$ can be obtained by flipping the coordinate of R.*

Proof. From Proposition 6.2.8, we have the points P, Q, R satisfy

$$d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q) + d_C(\mathcal{O}, R) = 0$$

and Proposition 6.2.4 implies

$$d_C(\mathcal{O}, P + Q) + d_C(\mathcal{O}, R) = 0.$$

It tells us that the displacements of point $P+Q$ and point R from point $\mathcal{O}$ have the same magnitude but opposite direction. Hence, the point $P + Q$ is the image of point R with respect to $X = Y$. □

Next, we want to show how to deal with a bad pair (P, Q). It is shown in [V09] that we can transform a bad pair (P, Q) to a good pair (P', Q') as follows. Choose tropical lines L_P and L_Q such that $L_P \cap_{\text{st}} \overline{C} = \{P, P_1, P_2\}$, $L_Q \cap_{\text{st}} \overline{C} = \{Q, Q_1, Q_2\}$, and we have two good pairs (P_1, Q_1) and (P_2, Q_2). Next, we can have two tropical lines L_1 and L_2 such that $L_1 \cap_{\text{st}} \overline{C} = \{P_1, Q_1, P'\}$ and $L_2 \cap_{\text{st}} \overline{C} = \{P_2, Q_2, Q'\}$. Under this procedure, we have $P + Q \sim P' + Q'$. Figure 6.4 illustrates this procedure for a bad pair on various cycles $\overline{C}$.

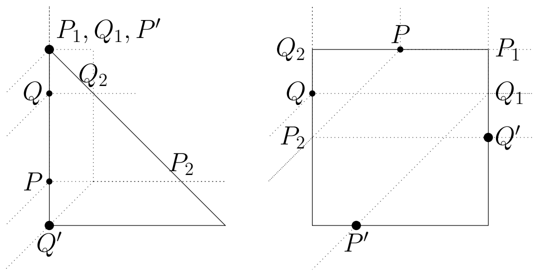


Figure 6.4: Moving a bad pair of points.

6.3 Edwards case by ultradiscrete functions parametrization

Now we want to discuss the possibility of performing point addition through theta parametrization of a curve. This technique is simpler because we do not need to check if the given points are a good pair. However, [CM17] argued that point addition on tropical Hesse curves is better done through intersection points because it involves less calculation.

In order to make a parallel connection with point addition, we modify the parametrization in Remark 5.3.3 so its period is the cycle length of the cycle.

Corollary 6.3.1. *Let $Y_\delta(t)$ be as defined at (5.3) and assume $1 \leq \delta$. Let the points (X, Y) of the cycle of $C(\text{trop}(f_{r,s}))$ be parameterized by*

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} -Y_\delta(\frac{t}{4} - \frac{1}{2}) \\ -Y_\delta(\frac{t}{4}) \end{pmatrix} + \text{val}_{\mathbb{K}}(r+s) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, for two points $A = (X(a), Y(a))$ and $B = (X(b), Y(b))$ where $a, b \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, we have

$$A + B = (X(a+b), Y(a+b)).$$

Proof. Notice that for $1 \leq \delta$, we have constant cycle length $\mathcal{L} = 8$. Since the period of the cycle parametrization in (5.4) is 2, substituting $t \mapsto \frac{t}{4}$ to equation (5.4) makes $(X(t), Y(t)) : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \bar{C}$ a bijection map. Thus, we can apply Proposition 6.2.4. □

6.4 Cryptographic applications

In this section we apply a cipher construction that is a modification of the one proposed in [CM17]. Assume Alice and Bob share the tropical polynomial function $\text{trop}(f)(X, Y)$ where its unique cycle $\bar{C}$ is of length $\mathcal{L} \in \mathbb{R}$ and is parametrized by $(X(t), Y(t) : t \in \mathbb{R}/\mathcal{L}\mathbb{Z})$. Let $\mathcal{O} = (X(0), Y(0))$ be the identity of the point addition on $\bar{C}$.

Suppose Alice wants to send a message of 500 alphanumeric characters to Bob. We assume that each character is represented by 7-bit data. So we have that Alice's message is an element of $\{0,1\}^{3500}$. Assume we partition the message into 350 blocks where each block is an element of $\{0,1\}^{10}$. Let $m_0 \in \{0,1\}^{10}$ be a part of the message that Alice wants to send to Bob. Alice maps m_0 to point $M_0 \in \bar{C}$ through the following procedure. First, Alice converts m_0 to $\bar{m_0} \in \mathbb{N}$ that will be in the interval $[0, 2^{10}-1]$. Let $t_0 = \bar{m_0}\frac{\mathcal{L}}{2^{10}} \in \frac{\mathcal{L}}{2^{10}}\mathbb{Z}$. Lastly, Alice defines $M_0 = (X(t_0), Y(t_0))$.

Let $r \in \mathbb{N}$ be a public property denoting the number of iterations during the ciphering process. Through the Diffie-Hellman Key Exchange procedure, Alice and Bob share $u_0 = \frac{\mathcal{L}}{2^{10}}n_0 \in \frac{\mathcal{L}}{2^{10}}\mathbb{Z}$ such that $0 < n_0 < 2^{10} - 1$ and $\bar{k} \in \mathbb{N}$ such that $0 < \bar{k} < 2^r - 1$. Let $S_0 = (X(u_0), Y(u_0)) \in \bar{C}$ be a secret point and $k \in \{0,1\}^r$–that is the binary form of $\bar{k}$–be a secret key.

Define the two halving functions

$$h_0(X(t), Y(t)) = \left(X\left(\frac{t}{2}\right), Y\left(\frac{t}{2}\right)\right) \text{ and}$$
$$h_1(X(t), Y(t)) = \left(X\left(\frac{t+\mathcal{L}}{2}\right), Y\left(\frac{t+\mathcal{L}}{2}\right)\right).$$

During the ciphering process, Alice obtains the points $M_i = (X(t_i), Y(t_i))$ with $t_i \in \frac{\mathcal{L}}{2^{10+i}}\mathbb{Z}$ $(i = 1, \ldots, r)$ in Table 6.1. Then Alice eventually obtains the ciphered coordinate $M_r = (X(t_r), Y(t_r)) \in \bar{C}$. Alice calculates $\bar{m_r} = t_r\frac{2^{10+r}}{\mathcal{L}}$ and convert it into the ciphered binary message $m_r \in \{0,1\}^{10+r}$ to be sent to Bob.

Meanwhile, Bob receives the ciphered binary message m_r. By using the secret point S_0 and the secret key k, Bob can determine secret points $S_1, S_2, \ldots, S_r$. Now, Bob deciphers

Iteration	Ciphered points	Secret points
1	$M_1 = h_{k_1}(M_0 + S_0)$	$S_1 = h_{k_1}(S_0)$
2	$M_2 = h_{k_2}(M_1 + S_1)$	$S_2 = h_{k_2}(S_1)$
$\vdots$	$\vdots$	$\vdots$
r	$M_r = h_{k_r}(M_{r-1} + S_{r-1})$	$S_r = h_{k_r}(S_{r-1})$

Table 6.1: Alice's encryption process.

the message by converting m_r to $\bar{m_r} \in \mathbb{N}$. Then, Bob calculates $t_r = \bar{m_r}\frac{\mathcal{L}}{2^{10+r}}$ and obtains $M_r = (X(t_r), Y(t_r)) \in \bar{C}$. During the r iterations, Bob obtains the following points on $\bar{C}$, see Table 6.2. Now Bob has $M_0 = (X(t_0), Y(t_0))$ where $t_0 \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. Thus, Bob can get

Iteration	Secret points	Points on $\bar{C}$
1	S_{r-1}	$M_{r-1} = 2M_r - S_{r-1}$
2	S_{r-2}	$M_{r-2} = 2M_{r-1} - S_{r-2}$
$\vdots$	$\vdots$	$\vdots$
r	S_0	$M_0 = 2M_1 - S_0$

Table 6.2: Bob's decryption process.

$\bar{m_0} = t_0\frac{2^{10}}{\mathcal{L}} \in \mathbb{N}$ that is in the interval $[0, 2^{10} - 1]$. Lastly, Bob can obtain the decrypted message $m_0 \in \{0,1\}^{10}$ that is the binary form of $\bar{m_0}$.

Below is an example when working with a nonsmooth pentagonal cycle of

$$\begin{aligned}\text{trop}(f)(X,Y) = \min(&5 + X + 2Y, 5 + 2X + Y, 3 + 2X, 3 + 2Y,\\ &-5 + X + Y, 5 + X, 5 + Y, -1).\end{aligned}$$

In this case, we have $\mathcal{L} = 40$. Let $r = 10$, $u_0 = \frac{625}{128}$, and $k = [1,0,1,1,1,1,1,0,0,0]$. Assume $m_0 = [1,1,0,0,1,1,1,1,0,0]$ is the message Alice wants to send. We have $\bar{m_0} = 3^5$. Figure 6.5 shows the plots of $(i, X(t_i))$ and $(i, Y(t_i))$ for $i = 0,\ldots,10$. Additionally, we can see the randomness of points $M_0,\ldots,M_{10}$ in Figure 6.6. The ciphered coordinate is

$$M_{10} = \left(\frac{539945}{131072}, -\frac{508631}{131072}\right) = \left(X\left(\frac{1850665}{131072}\right), Y\left(\frac{1850665}{131072}\right)\right) \tag{6.2}$$

and the ciphered binary message is

$$m_{10} = [1,0,1,0,1,0,1,1,1,0,1,0,0,1,0,1,1,0,1,0].$$

Furthermore, we want to measure decorrelation and diffusion properties of this ciphering procedure. For observing the decorrelation property, pick 400 random messages $m_{0,1}, m_{0,2},\ldots,m_{0,400}$. We compare them with the first 10 bits of ciphered messages $m_{r,1}, m_{r,2},\ldots,m_{r,400}$ and see how many bits are changed. For the same $\text{trop}(f)(X,Y)$,

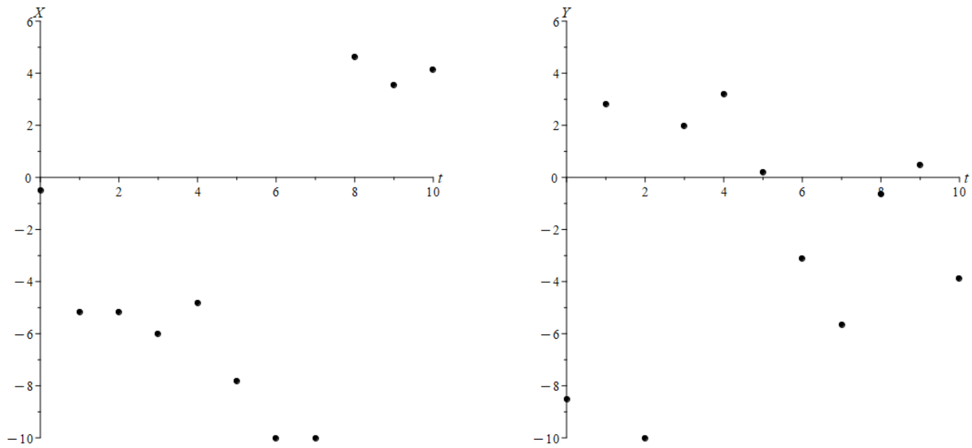


Figure 6.5: The X and Y coordinates during the 10 iterations.

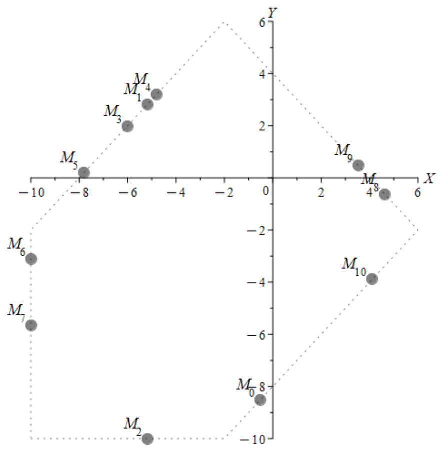


Figure 6.6: The coordinates $(X(t_i), Y(t_i))$ for $i = 1, \ldots, 10$.

r, u, and k, Figure 6.7 shows the percentage of bit-changes. Even though the first bit typically stays the same and the second bit tends to change, the average probability of a bit change is approximately 47.9%. This indicates there is a good decorrelation between the original message m_0 and its encrypted counterpart m_r.

For observing the diffusion property, we pick 40 random messages $m_{0,1}, m_{0,2}, \ldots, m_{0,40}$. For each sample $m_{0,i}$, where $i = 1, \ldots, 40$, let $m_{0,i}^1, m_{0,i}^2, \ldots, m_{0,i}^{10}$ be the messages that differ from $m_{0,i}$ only on its $1^{\text{st}}, 2^{\text{nd}}, \ldots, 10^{\text{th}}$ bits, respectively. For $h = 1, \ldots, 10$, let H^h be the set of 40 Hamming distances between $m_{r,i}$ and $m_{r,i}^h$ (for $i = 1, \ldots, 40$). The horizontal axis of Figure 6.8 indicates the location of the bit that is changed. The vertical axis shows the possible Hamming distance between two ciphered messages. For our settings of $\text{trop}(f)(X, Y)$, r, u, and k, the length of the ciphered message is $10 + r = 20$ bits. Each column of H^h in Figure 6.8 (for $h = 1, \ldots, 10$) has 40 points of Hamming distances between $m_{r,i}$ and $m_{r,i}^h$ (for $i = 1, \ldots, 40$). The average of each column shows a stability of the Hamming distances regardless the location of the bit that is changed.

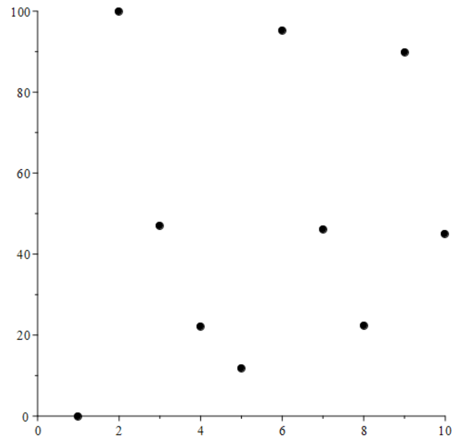


Figure 6.7: Percentages of bit-changes on 400 random messages.

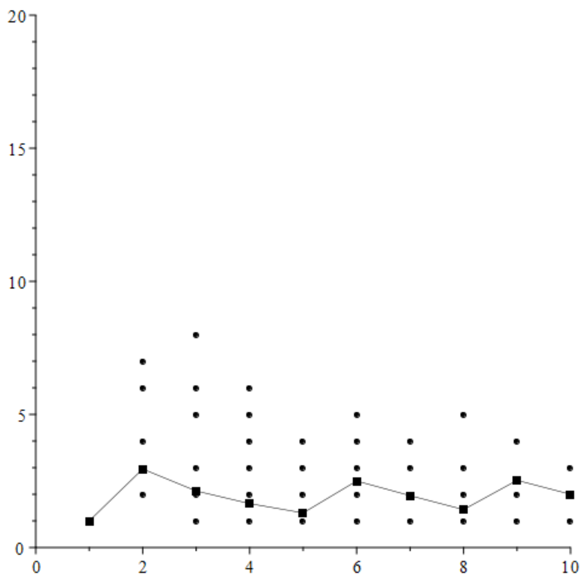


Figure 6.8: The graph of Hamming distances $H^1, H^2, \ldots, H^{10}$.

Bibliography

[CM17] J.M.Chauvet and E.Mahé. *Cryptography from the tropical Hessian pencil.* Groups Complexity Cryptology, De Gruyter **9(1)** (2017), 19–29.

[CS13] M.Chan and B.Sturmfels. *Elliptic curves in honeycomb form.* Contemp. Math., Amer. Math. Soc. **589** (2013), 87–107.

[E07] H.M.Edwards. *A normal form for elliptic curves.* Bull. Amer. Math. Soc., Amer. Math. Soc. **44(3)** (2007), 393–422.

[EKL06] M.Einsiedler, M.Kapranov, and D.Lind. *Non-archimedean amoebas and tropical varieties.* J. Reine Angew. Math., De Gruyter **601** (2006), 139–157.

[IS88] I.Simon *Recognizable sets with multiplicities in the tropical semiring.* Mathematical Foundations of Computer Science, Lecture Notes in Computer Science, Springer **324** (1988), 107–120 J. Reine Angew. Math., De Gruyter **601** (2006), 139–157.

[KKNT09] K.Kajiwara, M.Kaneko, A.Nobe, and T.Tsuda. *Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve.* Kyushu J. Math. **63** (2009), 315–338.

[KMM08] E.Katz, H.Markwig, and T.Markwig. *The j-invariant of a plane tropical cubic.* Journal of Algebra **320** (2008), 3832–3848.

[KMM09] E.Katz, H.Markwig, and T.Markwig. *The tropical j-invariant*, LMS J. Comput. Math. **12** (2009), 275–294.

[M05] G.Mikhalkin *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc., **18(2)** (2005), 313–377.

[M10] T.Markwig. *A field of generalised puiseux series for tropical geometry.* Rend. Sem. Mat. Univ. Politec. Torino **1** (2010), 79–92.

[MS15] D.Maclagan and B.Sturmfels. *Introduction to Tropical Geometry.* American Mathematical Society, 2015.

[N08] A.Nobe. *Ultradiscrete QRT maps and tropical elliptic curves.* J. Phys. A: Math. Theor. **41** (2008) 125205 (12pp).

[N11] A.Nobe. *On the addition formula for the tropical Hesse pencil*. RIMS Kokyuroku **1765** (2011), 188–208.

[N16] A.Nobe. *Group actions on the tropical Hesse pencil*. Japan J. Indust. Appl. Math. **33** (2016), 537-–556.

[NT23] H.Nakamura and R.S.Tarmidi. *On a two-parameter family of tropical Edwards curves*. Kyushu J. Math. (to appear).

[O06] C.Ormerod. *An ultradiscrete QRT mapping from tropical elliptic curves*, arXiv:math-ph/0609060.

[P98] J.-É.Pin. *Tropical semirings*. Idempotency (Publications of the Newton Institute) **11** (1998), 50–69.

[RST05] J.Richter-Gebert, B.Sturmfels and T.Theobald. *First steps in tropical geomtery*. Idempotent mathematics and mathematical physics, Contemp. Math., Amer. Math. Soc **377** (2005), 289–317.

[T23] R.S.Tarmidi. *Note on smoothness condition on tropical elliptic curves of symmetric truncated cubic forms*. Math. J. Okayama Univ. **66** (2024), 171–187.

[V09] M.D.Vigeland. *The group law on a tropical, elliptic curve*. Math. Scand. **104** (2009), 188–204.

[V10] M.D.Vigeland. *Smooth tropical surfaces with infinitely many tropical lines*. Ark. Mat. **48** (2010), 177—206.