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SUPPLEMENTARY RESULTS ON COGENERATORS

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Introduction

We say that a ring Λ is a left *PF*-ring if every faithful left Λ-module is a completely faithful Λ-module, that is, a generator in ${}_{\Lambda}\mathfrak{M}$, the category of left Λ-modules. Right *PF*-ring is similarly defined, and a both left and right *PF*-ring is called a *PF*-ring. Several characterizations of *PF*-ring are researched by several authors (See [1], [8], [12], [13], [14], and [16]). Especially Λ is left *PF* if and only if Λ is left self injective and a cogenerator in ${}_{\Lambda}\mathfrak{M}$ (See [13]). A left self cogenerator ring is not always a left *PF*-ring, but a both left and right self cogenerator ring is always a *PF*-ring (See [8], [12], and [13]).

The aim of this paper is to study the relation between ring extension concerning the property of *PF* or cogenerator. In section 1 we give a generalization of T. Onodera's theorem; *Frobenius extension of a cogenerator is again a cogenerator* (See Satz 9 [12]). This theorem can be extended to the case of quasi-Frobenius estension (Theorem 1.1). Next we give a sufficient condition in order to reduce the property of *PF* or cogenerator of the over ring to the basic ring. In section 2 we study some structures of self cogenerator rings. And we give some necessary and sufficient conditions for cogenerators to be quasi-Frobenius rings. We assume all rings have units and all subrings have the common identities.

1. Quasi-Frobenius extension of cogenerator

Let Λ be a ring with 1 and Γ a subring of Λ having the same 1. Then we say that Λ is a left quasi-Frobenius extension of Γ, if Λ is left Γ-finitely generated projective and Λ is isomorphic to a direct summand of a finite direct sum of the copies of $\mathrm{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$ as Λ-Γ-module, i.e., ${}_{\Lambda}\Lambda_{\Gamma} \langle \oplus {}_{\Lambda}(\sum^{n}{}^{\oplus}\mathrm{Hom}({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma))_{\Gamma}$.

The following lemma is an immediate consequence of Satz 2 of [10].

Lemma 1.1. *Let $\Lambda \supset \Gamma$ be a ring extension. Then the following conditions are equivalent*:

1) *Λ is a left quasi-Frobenius extension of Γ.*

2) *Λ is left Γ-finitely generated projective and there exist Γ-Γ-homomorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$ from Λ to Γ and Λ-Γ-homomorphisms $\varphi_1, \varphi_2, \cdots, \varphi_n$ from* $\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ *to Λ such that $\sum \varphi_k(\alpha_k)=1$.*

3) *Λ is left Γ-finitely generated projective and there exist Γ-Γ-homomorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$ from Λ to Γ and casimir elements $\sum x_i^{(k)} \otimes y_i^{(k)}$ $k=1, 2, \cdots, n$ in $\Lambda\otimes_\Gamma\Lambda$ such that $\sum_k \sum_i x_i^{(k)}\alpha_k(y_i^{(k)})=1$.*

Proof. The equivalence 1)$\Leftrightarrow$2) have been proved by B. Müller in Satz 2 [10], so we need only to prove the equivalence 2)$\Leftrightarrow$3). Since ${}_\Gamma\Lambda$ is finitely generated projective, we have Λ-Λ-isomorphisms

$$\Lambda\otimes_\Gamma\Lambda \cong \mathrm{Hom}\,(\Gamma_\Gamma, \Lambda_\Gamma)\otimes_\Gamma\Lambda \cong \mathrm{Hom}\,(\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)$$

with $x\otimes y\to : f\to xf(y)$. Then the set of casimir elements of $\Lambda\otimes_\Gamma\Lambda$ is mapped onto $\mathrm{Hom}\,({}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, {}_\Lambda\Lambda_\Gamma)$ by this map. Hence the proof is straightforward.

Lemma 1.2. *If Γ is a left self cogenerator ring and Λ is a right quasi-Frobenius extension of Γ, then for any minimal left Λ-module I there exists a primitive idempotent e of Γ such that Γe is an injective hull of a minimal left ideal of Γ and Λe contains a copy of I.*

Proof. Let L be a maximal left ideal of Λ such that $\Lambda/L\cong I$. Then we can regard $\Gamma/\Gamma\cap L$ as a Γ-submodule of Λ/L. Since Γ is a left self cogenerator, there exists a minimal left ideal $\mathfrak{l}$ of Γ and an idempotent e in Γ such that $\mathfrak{l}$ is a homomorphic image of $\Gamma/\Gamma\cap L$ and Γe is an injective hull of $\mathfrak{l}$ by Lemma 1 [13]. Since Γe is Γ-injective there exists a Γ-homomorphism f of Λ/L to Γe such that $f(\Gamma/\Gamma\cap L)=\mathfrak{l}$ and $f(1+L)\neq 0$. On the other hand, since Λ is a right quasi-Frobenius extension of Γ, there exist $\sum x_i^{(k)}\otimes y_i^{(k)}$ $k=1, 2, \cdots, n$, in $\Lambda\otimes_\Gamma\Lambda$ such that $\sum x x_i^{(k)}\otimes y_i^{(k)}=\sum x_i^{(k)}\otimes y_i^{(k)}x$ for all k and x in Λ and $\alpha_k\in$ $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ $k=1, 2, \cdots, n$, such that $\sum_k\sum_i \alpha_k(x_i^{(k)})y_i^{(k)}=1$. If $\sum_i x_i^{(k)}f(y_i^{(k)}\bar{1})$ $=0$ for all k, then

$$\begin{aligned} f(\bar{1}) &= f(\sum_k\sum_i \alpha_k(x_i^{(k)})y_i^{(k)}\bar{1}) = \sum_k\sum_i \alpha_k(x_i^{(k)})f(y_i^{(k)}\bar{1}) \\ &= \sum_k \alpha_k(\sum_i x_i^{(k)}f(y_i^{(k)}\bar{1})) = \sum_k \alpha_k(0) = 0 \end{aligned}$$

a contradiction. Hence $\sum_i x_i^{(k)}f(y_i^{(k)}\bar{1})\neq 0$ for some k. Then the map: $\bar{\lambda}\to$ $\sum_i x_i^{(k)}f(y_i^{(k)}\bar{\lambda})$ is a non zero Λ-map form Λ/L to Λe, since $\sum_i x_i^{(k)}f(y_i^{(k)}\bar{\lambda})\in\Lambda\cdot\Gamma e$. Hence Λe contains a copy of $I=\Lambda/L$.

Corollary 1.1. *If Γ is a left self cogenerator ring and Λ is a right quasi-Frobenius extension of Γ, Λ is a right S-ring.*

Proof. A ring is a right S-ring if and only if it contains a copy of every minimal left module of it. Hence the proof is immediate by Lemma 1.2.

Now we come to our main purpose.

Theorem 1.1. *Let Λ be a quasi-Frobenius extension of Γ. Then if Λ is a left (or right) self cogenerator ring, Λ is so. And if Γ is left (or right) PF, Λ is so.*

Proof. Suppose Γ is a left self cogenerator ring. Let I be an arbitrary minimal left Λ-module. Then there exists an idempotent e of Γ such that Γe is Γ-injective and Λe contains a copy of I by Lemma 1.2. On the other hand, since Λ is a left quasi-Frobenius extension of Γ, we have ${}_\Lambda\Lambda_\Gamma < \oplus_\Lambda(\sum^{n}{}^{\oplus} \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma))_\Gamma$. Then, since $e\in\Gamma$,

$$ {}_\Lambda\Lambda e < \oplus\ {}_\Lambda(\sum^{n}{}^{\oplus} \mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)e) \cong {}_\Lambda(\sum^{n}{}^{\oplus}\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma e))\,. $$

Since Γe is Γ-injective, Hom $({}_\Gamma\Lambda, {}_\Gamma\Gamma e)$ is Λ-injective by Prop. 1.4 Chap. VI [3]. Therefore Λe is Λ-injective, and Λ contains a copy of the injective hull of I. Thus we see that Λ is a left self cogenerator. Next suppose Γ is left *PF*. By Remark 2 on Page 349 [10], Λ is left (Λ, Γ)-injective. While, Λ is left Γ-injective, as ${}_\Gamma\Lambda < \oplus\ {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)$ and Γ is left self injective. Then Λ is left self injective, since ${}_\Lambda\Lambda < \oplus\ {}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ and the latter is left Λ-injective by Prop. 1.4, Chap. VI [3]. Thus Λ is a left self injective self cogenerator, and Λ is left *PF*.

REMARK. In the proof of Theorem 1.1, in case Γ is left *PF*, we need only the condition that Λ is left (Λ, Γ)-injective and Λ is left Γ-finitely generated projective. But if Λ is a right quasi-Frobenius extension of Γ, Λ is also left Γ-finitely generated projective. Hence we get the following fact;

In case Λ is a left semi-simple and right quasi-Frobenius extension of Γ, if Γ is left PF, Λ is so.

Now we consider the converse situation.

Proposition 1.1. *Let $\Lambda\supset\Gamma$ be a ring extension such that $\Gamma_\Gamma < \oplus\Lambda_\Gamma$, and Λ is left Γ-torsionless and right Γ-flat. Then if Λ is a left self cogenerator, Γ is so, and if Λ is left PF, Γ is so.*

Proof. Suppose Λ is a left self cogenerator. Let $\Lambda=\Gamma\oplus M$ with M a right Γ-submodule and $\mathfrak{l}$ and arbitrary maximal left ideal of Γ. Then $\Lambda\mathfrak{l}=\Gamma\mathfrak{l}\oplus M\mathfrak{l}=\mathfrak{l}\oplus M\mathfrak{l}\neq\Lambda$. Let L be a maximal left ideal of Λ which contains $\Lambda\mathfrak{l}$. Then $L\cap\Gamma=\mathfrak{l}$, and we can regard $\Gamma/\mathfrak{l}$ as a Γ-submodule of Λ/L. Since Λ is a cogenerator in ${}_\Lambda\mathfrak{M}$, it contains a copy of the injective hull E of Λ/L. Since Λ is right Γ-flat, E is left Γ-injective. Therefore Λ contains a copy of the injective hull E' ($\subset E$) of $\Gamma/\mathfrak{l}$ as left Γ-module. Hence Λ is a left Γ-cogenerator by Lemma 1 [13]. Since ${}_\Gamma\Lambda$ is a submodule of ${}_\Gamma(\Pi\Gamma)$, ${}_\Gamma(\Pi\Gamma)$ is, consequently, ${}_\Gamma\Gamma$ is a cogenerator by Lemma 1 [14]. Next suppose Λ is left *PF*. Then Γ is a left self cogenerator by the above proof. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$ be distinct maximal left ideal of Γ and L_i be a maximal left ideal of Λ containing $\Lambda\mathfrak{l}_i$ for $i=1, 2$. Then $L_1 \neq L_2$, since $L_i \cap \Gamma = \mathfrak{l}_i$. Since $\Lambda/\mathrm{Rad}\,\Lambda$ is semi-simple artinean by Th. 7 [1], Λ has but finitely many maximal left ideals. Hence Γ has but finitely many maximal left ideals. Therefore Γ is left *PF* by Th. 1 [14].

If $\Lambda \supset \Gamma$ is a ring extension such that ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, then ${}_\Gamma\Gamma_\Gamma \langle \oplus\, {}_\Gamma\Lambda_\Gamma$ by Prop. 5.6 [7]. Hence this type of extension satisfies the condition of Proposition 1.1. Hence we have

Proposition 1.2. *Let $\Lambda \supset \Gamma$ be a ring extension such that ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, and suppose* 1) *$\Lambda$ is a quasi-Frobenius extension of Γ, or* 2) *Λ is a left semi-simple and right quasi-Frobenius extension of Γ. Then Λ is a left self cogenerator* (*or left PF*) *if and only if Γ is so.*

Proof. All we have to do is to prove that if Γ is a left self cogenerator, Λ is a left self cogenerator in case 2). Let I be an arbitrary minimal left Λ-module. Then there exists an idempotent e of Γ such that Γe is Γ-injective and Λe contains a copy of I by Lemma 1.2. Since ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, ${}_\Gamma\Lambda e \langle \oplus\, {}_\Gamma(\Gamma e\oplus\cdots\oplus\Gamma e)$. So, Λe is Γ-injective, and we see that Λe is Λ-injective since Λ is left semi-simple over Γ. Hence Λ contains a copy of injective hull of every minimal left Λ-module, and Λ is a left self cogenerator.

Corollary 1.2. *Let $\Lambda \supset \Gamma$ be a separable extension such that ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$. Then Λ is left* (*or right*) *self cogenerator if and only if Γ is so, and Λ is left* (*or righ*) *PF if and only if Γ is so.*

Proof. It is already known that a separable extension of this type becomes a quasi-Frobenius extension. Therefore the proof is direct from Proposition 1.2.

REMARK. The fact that a separable extension $\Lambda \supset \Gamma$ such that ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$ becomes a quasi-Frobenius extension was informed to the author at the seminar of ring theory on Norikura in August 1968. As the proof is not so difficult we shall state here for the convenience to readers.
Since ${}_\Gamma\Lambda_\Gamma \langle \oplus\, {}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, we have ${}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Gamma \langle \oplus\, {}_\Gamma(\sum^\oplus \mathrm{Hom}\,(\Gamma_\Gamma, \Gamma_\Gamma))_\Gamma$. Then ${}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma) \otimes {}_\Gamma\Lambda_\Lambda \langle \oplus\, {}_\Gamma(\sum^\oplus \Gamma \otimes_\Gamma \Lambda)_\Lambda \cong {}_\Gamma(\sum^\oplus \Lambda)_\Lambda$. Since Λ is a separable extension of Γ, there exists $\sum x_i \otimes y_i$ in $\Lambda \otimes_\Gamma \Lambda$ such that $\sum x x_i \otimes y_i = \sum x_i \otimes y_i x$ for all $x \in \Lambda$ and $\sum x_i y_i = 1$. Then the map

$$\mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma) \otimes {}_\Gamma\Lambda \to \mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma): \qquad f \otimes \lambda \to f \circ \lambda$$

(Γ, Λ)-splits by the map

$$\mathrm{Hom}\,(\Lambda_\Gamma,\ \Gamma_\Gamma) \to \mathrm{Hom}\,(\Lambda_\Gamma,\ \Gamma_\Gamma)\otimes{}_\Gamma\Lambda: \qquad f \to \sum f\circ x_i \otimes y_i$$

Hence we see ${}_\Gamma\mathrm{Hom}(\Lambda_\Gamma,\ \Gamma_\Gamma)_\Lambda \langle \oplus\, {}_\Gamma(\sum^\oplus \Lambda)_\Lambda$, and Λ is a left quasi-Frobenius extension of Γ by Satz 2 [10]. Similarly it is a right quasi-Frobenius extension.

There have been a problem whether a left *PF*-ring is always a right *PF*-ring. It is very likely that the answer is no. But in a special case we have

Theorem 1.2. *In case a ring Λ is a quasi-Frobenius algebra over a commutative ring R, the following conditions are equivalent*;

1) *R is a self cogenerator ring.*
2) *R is a PF-ring.*
3) *Λ is a left self cogenerator.*
4) *Λ is a left PF-ring.*
5) *Λ is a right self cogenerator.*
6) *Λ is a right PF-ring.*

Proof. Since every commutative self cogenerator ring is *PF*, by Prop. 2 [14] and since a quasi-Frobenius R-algebra is R-finitely generated projective and ${}_R R \langle \oplus\, {}_R\Lambda$, the proof is immediate by Proposition 1.2.

2. Remark on the structure of self cogenerator rings

We denote the left (resp. right) socle of a ring Λ by S_l (resp. S_r) and the socle of Λ as Λ-Λ-module by S_t.

Theorem 2.1. *If Λ is a left self cogenerator ring, then $S_t \subset S_r \subset S_l$. If furthermore Λ contains the injective hull of every minimal left ideal of it, $S_t = S_r = S_l$.*

Proof. Let $\mathfrak{a}$ be an arbitrary minimal two sided ideal of Λ. $\mathfrak{a}$ contains a minimal right ideal $\mathfrak{r}$ by Lemma 3 [14]. Then $\Lambda\mathfrak{r} = \mathfrak{a}$ since $\mathfrak{a}$ is minimal. Since $\Lambda\mathfrak{r}$ is a sum of minimal right ideals of Λ which are isomorphic to $\mathfrak{r}$, $\mathfrak{a} \subset S_r$, and consequently $S_t \subset S_r$. Suppose $x\Lambda$ is a minimal right ideal, and let $\mathfrak{l}$ be a maximal left ideal of Λ which contains $l(x)$, the left annihilators of x. Then $\Lambda/\mathfrak{l}$ is isomorphic to some minimal left ideal Λy such that $E(\Lambda y) \subset \Lambda$, where by $E(\Lambda y)$ we mean the injective hull of Λy and we have a homomorphism $\Lambda x \to \Lambda y$ which maps x to y. Since $E(\Lambda y)$ is contained in Λ, we see $x\lambda = y$ for some $\lambda \in \Lambda$, and $y\Lambda \subset x\Lambda$. This implies $x\Lambda = y\Lambda$, for $x\Lambda$ is minimal, and $x = yz$ for some $z \in \Lambda$. Then we see Λx is minimal, since it is a homomorphic image of Λy which is minimal. Therefore $S_r \subset S_l$. Next, suppose that Λ contains the injective hull of every minimal left ideal of it. Let $\mathfrak{l}$ be an arbitrary minimal left ideal of Λ, and $\mathfrak{a}$ be the sum of all minimal left ideals which are isomorphic to $\mathfrak{l}$. Clearly $\mathfrak{a}$ is a two sided ideal. Let $\mathfrak{b}$ be a non zero two sided subideal of $\mathfrak{a}$, $\mathfrak{l}_1$ a minimal left ideal contained in $\mathfrak{b}$, and $\mathfrak{l}_2$ an arbitrary minimal left ideal contained in $\mathfrak{a}$. Evidently we have $\mathfrak{l}\cong\mathfrak{l}_1\cong\mathfrak{l}_2$. Since there exist minimal injective ideals E_i such that $\mathfrak{l}_i\subset E_i\subset\Lambda$ $i=1,2$ by assumption, there exist idempotents e_i such that $E_i=\Lambda e_i$ $i=1,2$, and the isomorphism $\mathfrak{l}_1\cong\mathfrak{l}_2$ is extended to $\Lambda e_1\cong\Lambda e_2$. The latter isomorphism is given by the right multiplication of some $e_1\lambda e_2$. Hence $\mathfrak{l}_2=\mathfrak{l}_1e_1\lambda e_2\subset\mathfrak{b}$. This shows $\mathfrak{a}=\mathfrak{b}$, and $\mathfrak{a}$ is a minimal two sided ideal. Hence $\mathfrak{l}\subset S_t$, and $S_l\subset S_t$. Thus we have $S_t=S_r=S_l$.

REMARK. A ring which satisfies the condition in Theorem 2.1 but is not a left PF-ring really exists. The example introduced by B. Osofsky, Example 2 [13] satisfies this condition. It is easy to check that all minimal right ideals are precisely x_iR, counting the right annihilator of every principal right ideal.

REMARK. Following Y. Utumi we say that a ring Λ is a left continuous ring if for every left ideal A of Λ there exists an idempotent e in Λ such that Λe is an essential extension of A and if for any left ideal B and any idempotent f, $B\cong\Lambda f$ implies $B=\Lambda e$ for some idempotent e. If Λ is a left self cogenerator and left continuous, it satisfies the condition. But the author does not know whether such a ring becomes a left PF-ring.

Proposition 2.1. *For a ring Λ the following three conditions are equivalent:*

1) *Λ is a quasi-Frobenius ring.*

2) *Λ is a left self cogenerator with the ascending chain condition for left annihilator ideals.*

3) *Λ is a left self cogenerator with the ascending chain condition for right annihilator ideals.*

Proof. The equivalence 1)$\Leftrightarrow$2) is clear, since every left ideal is an annihilator ideal by Lemma 2 [14]. Hence we need to show only 3)$\Rightarrow$1). Let $\mathfrak{l}_1\supset\mathfrak{l}_2\supset\cdots\supset\mathfrak{l}_n\supset\cdots$ be an arbitrary descending chain of left ideals. Then $r(\mathfrak{l}_1)\subset r(\mathfrak{l}_2)\subset\cdots\subset r(\mathfrak{l}_n)\subset\cdots$, and we have $r(\mathfrak{l}_k)=r(\mathfrak{l}_{k+1})=\cdots$ for some k by assumption. Since $\mathfrak{l}_i=l(r(\mathfrak{l}_i))$, we have $\mathfrak{l}_k=\mathfrak{l}_{k+1}=\cdots$. Thus Λ is left artinean and left self injective. Therefore Λ is a quasi-Frobenius ring.

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