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Osaka University
THE MCKAY NUMBERS OF A SUBGROUP
OF $GL(N; Q)$ CONTAINING $SL(N; Q)$

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(Received July 2, 1997)

1. Introduction

For a finite group $G$, a prime number $p$, a non-negative integer $k$, and a $p$-block $B$ of $G$, we put

$$m_p(k, G, B) = |\{ \zeta \in \text{Irr}(G) | \nu(\zeta(1)) = k, \zeta \in B \}|,$$

where $\text{Irr}(G)$ is the set of irreducible complex characters of $G$ and $\nu$ is the exponential valuation of some splitting field of $G$ with $\nu(p) = 1$. The sum $m_p(k, G) = \sum_B m_p(k, G, B)$ over all $p$-blocks of $G$ is called the $k$-th McKay number of $G$.

Let $GL = GL(n, q) = GL(n, GF(q))$ be the general linear group of degree $n$ over the finite field $GF(q)$ with $q$ elements, where $q = p^e$ is a power of the prime $p$. Let

$$L_h = L_h(n, q) = \{ x \in GL(n, q) | \det(x) \in U_h \},$$

where $U_h$ is the subgroup of the multiplicative group $F_1$ of $GF(q)$ of order $h$. (Thus $h$ is a divisor of $q - 1$.) In particular, $L_{q-1}(n, q)$ is $GL(n, q)$ and $L_1(n, q)$ is the special linear group $SL(n, q)$. In general, $L_h(n, q)$ satisfies

$$GL(n, q) \supset L_h(n, q) \supset SL(n, q).$$

Moreover, we denote by $PL_h = PL_h(n, q)$ the factor group of $L_h(n, q)$ modulo its center $Z(L_h(n, q))$.

In Section 4 of this paper, we write $m_p(k, G, B)$ concretely in terms of several invariants of partitions, where $G = L_h(n, q)$ or $G = PL_h(n, q)$.

In Section 5, we show the Alperin-McKay conjecture [1] holds for $L_h$ and $PL_h$. Note that for $L_h$ or $PL_h$, every $p$-block is of defect 0 or maximal defect. Thus it suffices to prove the following. If a $p$-block $B$ of $G$ is not of defect zero, then for a Sylow $p$-subgroup $P$ of $G$ and the $p$-block $b$ of the normalizer $N_G(P)$ of $P$ corresponding to $B$ by Brauer's first main theorem, we have

$$m_p(0, G, B) = m_p(0, N_G(P), b).$$
Section 2 is devoted to stating several preliminary results, and Section 3 is devoted to a parametrization of irreducible characters of $L_h$. Notations are standard. See, for example, [7].

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2. Preliminaries

In this section, we mention several definitions and results which are important when studying irreducible characters of $GL$ and related groups.

2.1. Polynomials and Simplices. Let $n$ be a fixed integer. For each positive integer $k$, denote by $F_k$ the multiplicative group of $GF(q^k)$. We take $K$ to be a fixed copy of $F_n$ and regard $F_k$ as a subgroup of $K$ for each $k$ with $k \leq n$. For each positive integer $k$ and $h$, let $\hat{F}_k$ and $\hat{U}_h$ denote the complex character group of $F_k$ and $U_h$, respectively.

Suppose $k$ and $l$ are positive integers and $k$ divides $l$. Then $F_k \leq F_l$ and we have a surjective homomorphism $N_{lk}: F_l \to F_k$ given by

$$N_{lk}(\rho) = \rho^{n_{lk}} \quad \text{for all } \rho \in F_l,$$

where $n_{lk} = |F_l|/|F_k| = (q^l - 1)/(q^k - 1)$. Defining $I_{kl}: \hat{F}_k \to \hat{F}_l$ by

$$I_{kl}(\psi)(\rho) = \psi(\rho^{n_{lk}}) \quad (\psi \in \hat{F}_k, \rho \in F_l),$$

we can embed $\hat{F}_k$ in $\hat{F}_l$. In this way, we embed $\hat{F}_k$ in $\hat{K}$ for each integer $k$ with $1 \leq k \leq n$. This embedding is well-defined (See Lemma 3.1 in [5]).

Lemma 2.1 (Lemma 3.2 in [5]). For integers $k, l$ with $k \mid l$, under the above identifications, the surjection: $\psi \mapsto \psi^{n_{lk}}$ from $\hat{F}_l$ to $\hat{F}_k$ is the same map as the restriction of characters.

In the same way, $\hat{U}_h$ is embedded in $\hat{F}_k$ and in $\hat{K}$.

Definition and Notations.

1. Let $\sigma$ denote the Frobenius map $\rho \mapsto \rho^q$ on $K$, and $\hat{\sigma}$ the corresponding action on $\hat{K}$.

2. An irreducible polynomial $f$ over $GF(q)$ with the degree less than $n$ will be identified with its set of roots in $GF(q^n)$, which forms a $\sigma$-orbit. If $f(x) \neq x$, then $f$ is a $\sigma$-orbit in $K$. If $\rho$ is an element of this orbit, we write $f = \langle \rho \rangle$.

3. A simplex $g$ over $GF(q)$ is a $\hat{\sigma}$-orbit in $\hat{K}$. If $\psi \in g$, we write $g = \langle \psi \rangle$.

We denote by $F$ the set of irreducible polynomials regarded as $\sigma$-orbits in $K$, and
McKay Numbers

by \( G \) the set of simplices over \( GF(q) \). By the degree \( \text{deg}(f) \) of an irreducible polynomial \( f \), or \( \text{deg}(g) \) of simplex \( g \), we mean the cardinality of the orbit concerned.

If we fix an isomorphism between \( K \) and \( \hat{K} \), then \( \hat{F}_k \) and \( \hat{U}_h \) correspond to \( F_k \) and \( U_h \), respectively. Moreover, \( \mathcal{F} \) and \( \mathcal{G} \) correspond bijectively, and then a polynomial and the corresponding simplex have the same degree.

### 2.2. Partitions.

Let \( \mu = (a_1^1, a_1^2, \ldots, a_\delta^l) \) be a partition of \( n \). Here we put \( a_1 > a_2 > \ldots > a_\delta > 0 \) and \( l_i \neq 0 \) is the multiplicity of \( a_i \) as a part of \( \mu \). (Thus \( n = l_1 a_1 + l_2 a_2 + \ldots + l_\delta a_\delta \).) For convenience sake, we also write \( \mu = (j^{m_j}) \), where \( m_{a_i} = l_i \), and \( m_j = 0 \) if \( j \neq a_i \) for any \( i \).

We write \( |\mu| = n \) to indicate that \( \mu \) is a partition of \( n \). Moreover \( l(\mu) = \sum l_i \) is the length of \( \mu \), \( \Lambda(\mu) = \gcd(l_1, l_2, \ldots, l_\delta) \), \( \Lambda(\mu) = \gcd(a_1, a_2, \ldots, a_\delta) \), \( \delta(\mu) = \delta \) is the number of distinct parts in \( \mu \), and \( n'(\mu) = \sum \binom{\delta}{2} l_i \). The partition conjugate to \( \mu \) is denoted by \( \mu' \). Let \( \mathcal{P} \) be the set of partitions of all nonnegative integers \( n \). Here we regard \( (0) \) as the only partition of \( 0 \).

### 2.3. Applications of the Clifford theory.

Let \( G \) be a finite group and \( H \) be a normal subgroup of \( G \). For \( \zeta \in \text{Irr}(H) \) we denote by \( T_G(\zeta) \) the stabilizer of \( \zeta \) in \( G \) and set

\[
\text{Irr}(G \mid \zeta) = \left\{ \chi \in \text{Irr}(G) \mid (\chi|_H, \zeta)_H = \frac{1}{|H|} \sum_{x \in H} \chi(x)\zeta(x^{-1}) \neq 0 \right\}.
\]

For \( \chi \in \text{Irr}(G) \), let

\[
\text{Irr}(H|\chi) = \{ \zeta \in \text{Irr}(H) \mid (\chi|_H, \zeta)_H \neq 0 \}.
\]

**Theorem 2.2** (Chapter 3, Theorem 3.8 in [7]). Let \( \zeta \in \text{Irr}(H) \) and \( T = T_G(\zeta) \). For \( \chi \in \text{Irr}(G \mid \zeta) \), we have

\[
\chi|_H = c \left( \sum_{x \in T \setminus G} \zeta^x \right),
\]

where \( c \) is some positive integer.

**Theorem 2.3** (Chapter 3, Theorem 5.12 in [7]). Let \( \zeta \in \text{Irr}(H) \) and \( T = T_G(\zeta) \). If \( \zeta \) extends to an irreducible character \( \eta \) of \( T \), then we have

\[
\text{Irr}(T \mid \zeta) = \{ \theta \eta \mid \theta \in \text{Irr}(T/H) \} \quad \text{and}
\]

\[
\text{Irr}(G \mid \zeta) = \{ (\theta \eta)^G \mid \theta \in \text{Irr}(T/H) \}.
\]
Theorem 2.4. With the above notation, each one of the following conditions implies that \( \zeta \) is extendible to an irreducible character of \( T \):

1. (Chapter 3, Theorem 5.11 in [7]) \( T/H \) is cyclic.
2. \( \zeta \) is of degree 1, and \( T = S \rtimes H \), where \( S \) is a certain group.

Lemma 2.5. If \( G/H \) is cyclic then the following hold.

1. In the restriction of irreducible characters of \( G \) to \( H \), the multiplicity of each irreducible constituent is 1.
2. Two irreducible characters of \( G \) either have the same restrictions to \( H \), or have restrictions without common irreducible constituents.
3. For \( \zeta \in \text{Irr}(H) \), \( \chi \in \text{Irr}(G \mid \zeta) \), we have \( |\text{Irr}(H \mid \chi)| = |G|/|H||\text{Irr}(G \mid \zeta)| \).

Proof. Let \( T = T_G(\zeta) \). Note that \( T/H \) is also cyclic.

1. \( \text{Irr}(T/H) \) has only characters of degree 1. By Theorems 2.4(1) and 2.3, \( c \) in Theorem 2.2 is 1.
2. It is clear from (1) and Theorem 2.2.
3. By Theorem 2.3, we have \( |\text{Irr}(G \mid \zeta)| = |\text{Irr}(T/H)| = |T/H| \), and by Theorem 2.2, we obtain \( |\text{Irr}(H \mid \chi)| = |G/T| \). Thus the equality holds.

2.4. \( p \)-blocks. Let \( G \) be a finite group, \( G_{p'} \) the set of elements of \( G \) whose orders are prime to \( p \), and \( \text{Cl}(G_{p'}) \) the set of conjugate classes of \( G \) contained in \( G_{p'} \). For \( C \in \text{Cl}(G_{p'}) \), let \( \bar{C} \) be the sum of all elements of \( C \) in the group algebra of \( G \) over \( \mathbb{C} \), and \( d(C) \) the defect of \( C \), i.e., \( d(C) = \nu(|C_G(x)|) \) for \( x \in C \). For a \( p \)-block \( B \) of \( G \), let \( d(B) \) be the defect of \( B \). For \( \chi \in \text{Irr}(G) \) and \( C \in \text{Cl}(G_{p'}) \), we define \( \omega_{\chi}(\bar{C}) \) by

\[
\omega_{\chi}(\bar{C}) = |C|\chi(x)/\chi(1)
\]

with \( x \in C \). Let \( p \) be the valuation ideal of \( \nu \), i.e., \( p \) is the set of elements in the field such that the values of \( \nu \) on them are positive.

Theorem 2.6 (Chapter 3, Theorem 6.28 in [7]). Assume that \( \chi, \chi' \in \text{Irr}(G) \) belong to \( p \)-blocks of the same defect \( d \). Then \( \chi \) and \( \chi' \) belong to the same \( p \)-block if and only if

\[
\omega_{\chi}(\bar{C}) \equiv \omega_{\chi'}(\bar{C}) \pmod{p}
\]

for any \( C \in \text{Cl}(G_{p'}) \) with \( d(C) = d \).

Theorem 2.7 (Chapter 3, Theorem 6.29 in [7]). Let \( B \) be a \( p \)-block of \( G \) and let \( \chi \in \text{Irr}(G) \) belong to \( B \). Then the following three conditions are equivalent to each other.
Therefore the number of $p$-blocks of defect 0 is the number of characters satisfying the condition (2) above.

3. A parametrization of irreducible characters of $L_h(n, q)$

In this section, we treat $\text{Irr}(L_h)$. We remark that the center $Z(L_h)$ of $L_h$ is isomorphic to $U_{\gcd(q-1, nh)}$. In particular, $Z(GL) \simeq F_1$.

3.1. A parametrization of $\text{Irr}(GL)$ and $\text{Irr}(L_h)$. Green [3] showed in 1955 how an irreducible complex character of $GL$ is given by a partition-valued function $\lambda : G \rightarrow \mathcal{P}$ which satisfies

\begin{equation}
\sum_{g \in G} |\lambda(g)| \deg(g) = n.
\end{equation}

In this subsection, we identify the set of all such functions with $\text{Irr}(GL)$. An account of how such a function determines an irreducible character may be found in Section 3 in [4] and Chapter IV in [6], too.

We explain properties of characters of $GL$ which we need in this paper. We denote by $\deg(\lambda)$ the degree of a character $\lambda$. Let $\lambda' : G \rightarrow \mathcal{P}$ be the function such that $\lambda'(g)$ is the partition conjugate to $\lambda(g)$ for all $g \in G$. We denote by $\xi(\lambda)$ the product of all elements of $\langle \psi \rangle \in G$, i.e. $\xi(\langle \psi \rangle) = \psi^{|\lambda|}$ where $d = \deg(\langle \psi \rangle)$. It is clear that $\xi(\langle \psi \rangle) \in F_1$.

**Theorem 3.1.** Let $\lambda : G \rightarrow \mathcal{P}$ be an irreducible character of $GL$.

1. (p.444 in [3], (6.7) in IV of [6]) $\nu(\deg(\lambda)) = e \sum_{g \in G} \deg(g)n'(\lambda'(g))$.
2. (Example 2 in IV of [6], Theorem 5.4 in [5]) The restriction of $\lambda$ to $Z(GL)$ is a multiple of $\prod_{g \in G} \xi(g)^{|\lambda(g)|}$.

Let $\alpha \in \tilde{F}_1$ and $\langle \psi \rangle \in G$. We define the parallel translation $\tau_\alpha : G \rightarrow G$ as $\tau_\alpha \langle \psi \rangle = \langle \alpha \psi \rangle$. Moreover, we define an action of $\alpha \in \tilde{F}_1$ on $\text{Irr}(GL)$ as follows. For any irreducible character $\lambda : G \rightarrow \mathcal{P}$ of $GL$, the character $\lambda^\alpha$ is defined by $\lambda^\alpha(\langle \psi \rangle) = \lambda(\tau_\alpha \langle \psi \rangle) = \lambda(\langle \alpha \psi \rangle)$ for any $\langle \psi \rangle \in G$.

**Theorem 3.2** (Proposition 5.2 in [5]). Let $\lambda, \chi \in \text{Irr}(GL)$. Then $\lambda$ and $\chi$ have the same restrictions to $L_h$ if and only if $\lambda^\alpha = \chi$ for some $\alpha \in \tilde{U}_{(q-1)/h}$.

For any irreducible character $\lambda : G \rightarrow \mathcal{P}$ of $GL$, we denote by $\lambda_{\infty}$ the $\tilde{U}_{(q-1)/h}$.
orbit of $\text{Irr}(GL)$ containing $\lambda$. Let $\text{Irr}(L_h \mid \lambda_{\infty})$ denote $\text{Irr}(L_h \mid \lambda)$. (This notation is well defined by virtue of Theorem 3.2.)

**Theorem 3.3.**

1. $\text{Irr}(L_h) = \bigcup_{\lambda_{\infty}} \text{Irr}(L_h \mid \lambda_{\infty})$ (disjoint), where this union is over all $\hat{U}_{(q-1)/h}$-orbits in $\text{Irr}(GL)$. Moreover, for each $\lambda \in \text{Irr}(GL)$, we have

$$|\text{Irr}(L_h \mid \lambda_{\infty})| = \frac{q - 1}{h|\lambda_{\infty}|}.$$

2. For $\lambda \in \text{Irr}(GL)$, the followings hold.

   (i) For any $\varphi \in \text{Irr}(L_h \mid \lambda_{\infty})$,

   $$\nu(\text{deg}(\varphi)) = e \sum_{g \in G} \text{deg}(g)n'(\lambda'(g)).$$

   (ii) The restriction of each character in $\text{Irr}(L_h \mid \lambda_{\infty})$ to $Z(L_h)$ is a multiple of

   $$\prod_{g \in G} \hat{\xi}(g)^{\lambda(g)(q-1)/\text{gcd}(q-1,hn)}.$$

**Proof.** (1) Since $GL/L_h$ is a cyclic group, the first half is clear from Lemma 2.5(2).

Therefore, for $\lambda \in \text{Irr}(GL)$ and $\zeta \in \text{Irr}(L_h \mid \lambda)$, we have $\text{Irr}(GL \mid \zeta) = \lambda_{\infty}$. So we have the latter half by Lemma 2.5(3).

(2)(i) By Theorem 2.2 and Lemma 2.5(1), for $\zeta \in \text{Irr}(L_h \mid \lambda_{\infty})$,

$$\text{deg}(\lambda) = |T_{GL}(\zeta)\backslash GL|\text{deg}(\zeta).$$

Here, $|T_{GL}(\zeta)\backslash GL|$ divides $|L_h \backslash GL| = (q - 1)/h$ which is prime to $p$. So the $p$-part of $\text{deg}(\zeta)$ equals that of $\text{deg}(\lambda)$. From Theorem 3.1(1), we have (i).

(ii) The irreducible constituent of a restriction of each character in $\text{Irr}(L_h \mid \lambda_{\infty})$ to $Z(L_h)$ equals the irreducible constituent of $\lambda|Z(L_h)$. By $Z(L_h) \cong U_{\text{gcd}(q-1,nh)}$, Lemma 2.1 and Theorem 3.1(2), (ii) holds.

By using the above, we can count the number of characters of $L_h$.

**3.2. A parametrization of $\text{Irr}(L_h)$ by polynomials.** In order to count irreducible characters effectively, we parametrize $\text{Irr}(GL)$ and $\text{Irr}(L_h)$ by polynomials over $GF(q)$.

Fix an isomorphism from $K$ to $\hat{K}$. Then, as is seen in 2.1, we have the bijection from $\mathcal{F}$ to $\hat{G}$. Therefore elements in $\text{Irr}(GL)$ are parametrized by partition-valued
functions $\lambda : \mathcal{F} \to \mathcal{P}$ which satisfy

\[(3.2) \sum_{f \in \mathcal{F}} |\lambda(f)| \deg(f) = n.\]

We denote polynomials in $\mathcal{F}$ by $f_1, f_2, \cdots$. Let $\lambda'(f_i) = (j^m(i,j))$ where $m(i,j)$ is a non negative integer. For $\lambda : \mathcal{F} \to \mathcal{P}$ with (3.2), we define the sequence of polynomials

\[(3.3) \left( \prod_i f_i^{m(i,1)}, \prod_i f_i^{m(i,2)}, \cdots \right).\]

Then it is easy to see that this gives a bijection from the set of partition-valued functions with (3.2) to the set of sequences $(h_1, h_2, \cdots)$ of monic polynomials over $GF(q)$ which satisfy the following.

1. The constant term of each $h_i$ does not equal to 0, and
2. $\sum j \deg(h_j) = n$, i.e., $\mu = (j^{\deg(h_i)})$ is a partition of $n$.

From now on, we identify the set of such sequences with $\text{Irr}(GL)$.

Let $g \in G$ correspond to an irreducible monic polynomial $f(x) = x^d + b_1x^{d-1} + \cdots + b_d$ over $GF(q)$ such that $b_d \neq 0$, and let $\alpha \in \bar{F}_1$ correspond to $\rho \in F_1$. As in 2.1, we regard an irreducible polynomial as the $\sigma$-orbit consisting of its roots in $K$. Because $\tau_\alpha(g)$ is the $\sigma$-orbit obtained by multiplying all elements of the $\sigma$-orbit $g$ by $\alpha$, it corresponds to the $\sigma$-orbit obtained by multiplying all roots of $f$ by $\rho$, i.e., we have

\[(3.4) \tau_\rho(f(x)) = x^d + \rho b_1x^{d-1} + \rho^2 b_2x^{d-2} + \cdots + \rho^d b_d.\]

We apply this notation when $f(x)$ is reducible, too. If $\rho$ is a primitive $m$-th root of unity, then

\[(3.5) f(x) = \tau_\rho(f(x)) \Leftrightarrow b_k = 0 \quad \text{if} \quad m \nmid k.\]

In particular, if this condition holds, then we have $m \mid d$ because $b_d \neq 0$.

Let $\lambda = (h_1, h_2, \cdots) \in \text{Irr}(GL)$ and put $h_i(x) = x^{d_i} + b_{i,1}x^{d_i-1} + \cdots + b_{i,d_i}$. Note that $b_{i,d_i} \neq 0$ for any $i$. The action of $\alpha$ on $\lambda$ corresponds to the action of $\rho$ in such a way that

\[(3.6) \lambda^\rho = (\tau_{\rho^{-1}}(h_1), \tau_{\rho^{-1}}(h_2), \cdots).\]

If $\rho$ is a primitive $m$-th root of 1, then

\[(3.7) \lambda = \lambda^\rho \Leftrightarrow b_{i,j} = 0 \quad \text{if} \quad m \nmid j.\]

In particular, if $\rho$ stabilizes $\lambda$, then $m \mid \gcd(\deg(h_1), \deg(h_2), \cdots)$. 
By our identification, \( \lambda_\infty \) equals the \( U_{(q-1)/h} \)-orbit of \( \text{Irr}(GL) \) containing \( \lambda \).

We restate Theorems 3.2 and 3.3 by using the above notation.

**Corollary 3.4.**

(1) \( \lambda, \chi \in \text{Irr}(GL) \) have the same restrictions to \( L_h \) if and only if \( \lambda^p = \chi \) for some \( p \in U_{(q-1)/h} \).

(2) \( \text{Irr}(L_h) = \bigcup_{\lambda_\infty} \text{Irr}(L_h \mid \lambda_\infty) \) (disjoint),
where this union is over all \( U_{(q-1)/h} \)-orbits in \( \text{Irr}(GL) \). Moreover, we have

\[
|\text{Irr}(L_h \mid \lambda_\infty)| = \frac{q-1}{h|\lambda_\infty|}
\]

(3) For \( \lambda = (h_1, h_2, \cdots) \in \text{Irr}(GL) \) with \( h_i(x) = x^d_i + b_{i,1}x^{d_i-1} + \cdots + b_{i,d_i} \) and \( b_{i,d_i} \neq 0 \), the following hold.
(i) For any \( \varphi \in \text{Irr}(L_h \mid \lambda_\infty) \),

\[
\nu(\deg(\varphi)) = e \sum_j \binom{j}{2} \deg(h_j) = en'(\mu),
\]

where \( \mu \) is the partition \((j^{\deg(h_j)})\) of \( n \).

(ii) The restriction of each character in \( \text{Irr}(L_h \mid \lambda_\infty) \) to \( Z(L_h) \) is a multiple of the irreducible character of \( U_{\gcd(q-1,nh)} \) corresponding to

\[\left\{(-1)^n \prod_j \left(\frac{b_{j,d_j}}{j}\right)^{(q-1)/\gcd(q-1,nh)}\right\}\]

**Proof.** (1) and (2) are clear from Theorems 3.2 and 3.3.

(3) Let us regard \( \lambda \) as a function from \( \mathcal{F} \) to \( \mathcal{P} \). We denote polynomials in \( \mathcal{F} \) by \( f_1, f_2, \cdots \). We write \( \lambda'(f_i) = (j^{m(i,j)}) \) where \( m(i,j) \) is a non negative integer. Then \( h_j = \prod_i f_i^{m(i,j)} \) and \( \deg(h_j) = \sum_i \deg(f_i)m(i,j) \).

(i) By Theorem 3.3(2)(i), the \( p \)-part of the degree of each character in \( \text{Irr}(L_h \mid \lambda_\infty) \)
equals

\[
e \sum_{f \in \mathcal{F}} \deg(f)n'(\lambda'(f)) = e \sum_i \deg(f_i) \sum_j \binom{j}{2} m(i,j)
= e \sum_j \binom{j}{2} \sum_i \deg(f_i)m(i,j) = e \sum_j \binom{j}{2} \deg(h_j).
\]

(ii) If \( g \in \mathcal{G} \) corresponds to \( f \in \mathcal{F} \), then \( \xi(g) \) corresponds to the product of all roots of \( f \), i.e., \( (-1)^{\deg(f)} f(0) \). We remark that \((j^{\deg(h_j)})\) is a partition of \( n \). Then the irreducible constituent of the restriction of \( \lambda \) to \( Z(L_h) \) corresponds to

\[
\prod_{f \in \mathcal{F}} \left\{(-1)^{\deg(f)} f(0)\right\}^{\lambda(f)(q-1)/\gcd(q-1,nh)}
\]
3.3. \( p \)-blocks of \( L_h(n, q) \). By Theorem 4 of [2], a defect group of any \( p \)-block of \( L_1 = SL(n, q) \) is a Sylow \( p \)-subgroup or trivial subgroup. The same argument as for \( GL \) in the last paragraph of Section 4 in [2] yields that the same is true for \( L_h \). Therefore the defect \( d(B) \) of the block \( B \) of \( L_h \) equals to 0 or \( nL_h = \frac{en(n-1)}{2} \). In the later case, we say that \( B \) is of maximal defect. By Theorem 2.7, any \( p \)-block of defect 0 has a character the \( p \)-part of whose degree equals that of \( L_h \), i.e. \( p^{\frac{en(n-1)}{2}} \). On the other hand, characters in any \( p \)-block of the maximal defect have \( p \)-parts of degree less than \( p^{\frac{en(n-1)}{2}} \).

Lemma 3.5. The number of blocks of \( L_h \) of defect 0 is \( h \). Moreover, for a non-negative integer \( k \) and any block \( B \) of \( L_h \) of defect 0, we have

\[
m_p(k, L_h, B) = \begin{cases} 1, & \text{if } k = \frac{en(n-1)}{2}; \\ 0, & \text{otherwise}. \end{cases}
\]

Proof. The latter half is clear by Theorem 2.7.

Let \( \lambda_a = (1,1,\ldots,1, x-a) \in \text{Irr}(GL) \) for \( a \in F_1 \). By Theorem 2.7 and Corollary 3.4(3)(i), the set of \( p \)-blocks of defect 0 corresponds bijectively to

\[
\{ \zeta \in \text{Irr}(L_h) \mid \nu(\zeta(1)) = \nu(|L_h|) = \frac{en(n-1)}{2} \} = \bigcup_{a \in F_1} \text{Irr}(L_h \mid \lambda_a) = \bigcup_{(\lambda_a)\infty} \text{Irr}(L_h \mid (\lambda_a)\infty) \quad \text{(disjoint)}
\]

where the last union is over all \( U_{(q-1)/h} \)-orbit consisting of characters \( \lambda_a \). Because \( \lambda_a \) is stabilized only by 1, \( |(\lambda_a)\infty| = (q-1)/h \). Therefore, the number of \( U_{(q-1)/h} \)-orbit consisting of characters \( \lambda_a \) is \( h \), and \( |\text{Irr}(L_h \mid (\lambda_a)\infty)| = 1 \) by Corollary 3.4(2). Therefore the number of blocks of defect 0 is \( h \).

For characters belonging to \( p \)-blocks of maximal defect, we can determine their distribution to \( p \)-blocks by looking at the values at \( C \)'s for all \( C \in Cl((L_h)p') \).
with $C_{L_h}(x)(x \in C)$ containing a Sylow $p$-subgroup of $L_h$. This is possible because of Theorem 2.6. Since an element of $L_h$ satisfying this condition is in the center $Z(L_h)$ of $L_h$, it is enough to see the character values on $Z(L_h)$. Moreover $Z(L_h) \simeq U_{\gcd(q-1,nh)}$ is a cyclic group whose order is prime to $p$. So, it is enough to look at their actual values, not those modulo $p$. Therefore we have the following.

**Lemma 3.6.** Let $\zeta, \zeta' \in \text{Irr}(L_h)$ belong to $p$-blocks of non-zero defect. Then $\zeta$ and $\zeta'$ belong to the same block if and only if $\omega_\zeta(x) = \omega_{\zeta'}(x)$ for all $x \in Z(L_h)$.

By this lemma, we can determine distribution of characters to $p$-blocks of $L_h$ of maximal defect by looking at the irreducible constituent of their restriction to $Z(L_h)$. Therefore $p$-blocks of maximal defect are parametrized by the element of $Z(L_h)$. Because $Z(L_h) \simeq U_{\gcd(q-1,nh)}$, $p$-blocks of $L_h$ of maximal defect are parametrized by the element of $U_{\gcd(q-1,nh)}$. The number of blocks of $L_h$ of maximal defect is $\gcd((q-1), nh)$.

We fix an isomorphism $Z(L_h) \simeq U_{\gcd(q-1,nh)}$, and identify them via the isomorphism. We denote by $B_a$ the $p$-block of $L_h$ of maximal defect corresponding to $a \in U_{\gcd(q-1,nh)}$.

In particular, the principal block is $B_1$. Moreover, $B_1$ is the set of characters in blocks of non-zero defect of $L_h$ such that restrictions of those to $Z(L_h)$ equal to multiples of the trivial character. So these characters are regarded as characters of $L_h/Z(L_h) = PL_h$. Therefore, we can identify $B_1$ with the only $p$-block $B_0$ of maximal defect of $PL_h$. On the other hand, by Corollary 3.4 and the proof of Lemma 3.5, the number of $p$-blocks of defect zero of $PL_h$ is $\gcd(q-1, n)(q-1)/\gcd(q-1, nh)$.

Let $\lambda = (h_1, h_2, \cdots) \in \text{Irr}(GL)$ and let $a_i$ be the constant term of $h_i$. All characters in $\text{Irr}(L_h|\lambda_{\infty})$ have the same restrictions to $Z(L_h)$. So, all constituents belong to the same $p$-block. By Corollary 3.4(3), characters in $\text{Irr}(L_h|\lambda_{\infty})$ belong to $B_a$ if and only if

$$a = \left((-1)^n \prod_j (a_j)^{j}\right)^{(q-1)/\gcd(q-1, nh)}$$

**Lemma 3.7** (Lemma 2.5 in [8]). Let $a_i$ $(1 \leq i \leq \delta)$ be positive integers, $A = \gcd(a_1, a_2, \cdots, a_\delta)$, and $a \in F_1$. Then

$$|\{x_1, x_2, \cdots, x_\delta \in F_1^\delta \mid x_1^{a_1}x_2^{a_2} \cdots x_\delta^{a_\delta} = a\}| = (q-1)^{\delta-1} \beta(A, a)$$

where $\beta(A, a)$ is the number of solutions in $F_1$ to the equation $x^A = a$, i.e.,

$$\beta(A, a) = \begin{cases} \gcd(q-1, A), & \text{if } a \in U_{(q-1)/\gcd(q-1, A)}; \\ 0, & \text{otherwise}. \end{cases}$$
Let $a$ be in $U_{\gcd(q-1,nh)}$ and $\mu = (a_1^l, a_2^l, \cdots, a_5^l)$ be a partition. By the above lemma, we have

\[(3.9) \{ (x_1, x_2, \cdots, x_5) \in F_1^5 \mid ((-1)^{x_1} x_1 a_1^l x_2 a_2^l \cdots x_5 a_5^l)^{(q-1)/\gcd(q-1,nh)} = a \} \]

\[= \{(x_1, x_2, \cdots, x_5) \in F_1^5 \mid \{((-1)^{x_1} x_1 a_1^l x_2 a_2^l \cdots x_5 a_5^l)^{(q-1)/\gcd(q-1,nh)} = a \} \}

\[= (q-1)^{\delta-1} \beta \left( \frac{(q-1)A(\mu)}{\gcd(q-1,nh)}, a \right) \]

4. The McKay numbers of $L_h$

For a partition $\mu = (a_1^l, a_2^l, \cdots, a_5^l)$ of $n$, $a$ be in $U_{\gcd(q-1,nh)}$, and a positive integer $s$, we denote by $\text{Irr}(GL, \mu, a, s)$ the set of irreducible characters $\lambda = (h_1, h_2, \cdots)$ of $GL$ satisfying the following.

(1) The partition $j^{\deg(h_i)}$ equals $\mu$,
(2) $\text{Irr}(L_h \mid \lambda) \subseteq B_a$, and
(3) $\lambda$ is stabilized by $s$-th roots of 1 in $U_{(q-1)/h}$, but is not stabilized by $s'$-th roots of 1 for any $s' > s$ with $s \mid s'$, i.e., the restriction of $\lambda$ to $L_h$ has $s$ irreducible constituents.

Note that by (3.6) and (3.7) $\text{Irr}(GL, \mu, a, s)$ is closed under the action of $U_{(q-1)/h}$.

We denote by $\text{Irr}(GL, \mu, a)$ the set of irreducible characters $\lambda$ of $GL$ satisfying (1) and (2) of the above, i.e.,

$$\text{Irr}(GL, \mu, a) \cup \bigcup_{s \mid (q-1)/h} \text{Irr}(GL, \mu, a, s) \text{ (disjoint)}.$$ 

And we denote by $\text{Irr}(GL, \mu, a, s)$ the set of irreducible characters $\lambda$ of $GL$ satisfying (1),(2) above and the following.

(4) $\lambda$ is stabilized by $s$-th roots of 1 in $U_{(q-1)/h}$. (Thus $\lambda$ is stabilized by $s'$-th roots of 1 for any $s' > s$ with $s \mid s'$.)

This means that

$$\text{Irr}(GL, \mu, a, s) = \bigcup_{s \mid s'} \text{Irr}(L_h, \mu, a, s').$$ 

Moreover, we put

$$\text{Irr}(L_h, \mu, a, s) = \{ \zeta \in \text{Irr}(L_h) \mid \zeta \in \text{Irr}(L_h \mid \chi), \chi \in \text{Irr}(GL, \mu, a, s) \},$$

$$\text{Irr}(L_h, \mu, a) = \{ \zeta \in \text{Irr}(L_h) \mid \zeta \in \text{Irr}(L_h \mid \chi), \chi \in \text{Irr}(GL, \mu, a) \},$$

$$m(\mu, a, s) = |\text{Irr}(L_h, \mu, a, s)|,$$

$$m(\mu, a) = |\text{Irr}(L_h, \mu, a)|.$$
For an integer \( t > 1 \), we define \( \Pi(t) \) by

\[
\Pi(t) = \prod \left( 1 - \frac{1}{r^t} \right),
\]

where \( r \) runs over all prime numbers that divide \( t \). For example, for any positive integers \( i, j \), \( \Pi(2^i) = 3/4 \), \( \Pi(3^j) = 8/9 \), \( \Pi(2^i3^j) = 24/36 \), etc. For convenience, we put \( \Pi(1) = 1 \).

At first, we show the following lemma. For a divisor \( s \) of \( \Lambda(\mu) \), we put

\[
\gamma(\mu, s) = q^{\ell(\mu)/s - \delta(\mu)}. \]

Note that if \( \lambda = (h_1, h_2, \cdots) \in \text{Irr}(GL) \) is stabilized by \( s \)-th roots of 1 in \( U_{(q-1)/h} \), then \( s \) divides \( \gcd((q-1)/h, \deg(h_1), \deg(h_2), \cdots) \).

**Lemma 4.1.**

1. \( |\text{Irr}(GL, \mu, a, s)| = \gamma(\mu, s)(q-1)^{\delta(\mu)-1} \beta((q-1)A(\mu)/\gcd(q-1, nh), a). \)
2. Let \( \gcd(\Lambda(\mu), (q-1)/h) = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \) be the prime decomposition of \( \gcd(\Lambda(\mu), (q-1)/h) \), \( s \) be a divisor of \( \gcd(\Lambda(\mu), (q-1)/h) \) with the prime decomposition \( s = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k} \), and set \( c_i(1 \leq i \leq k) \) as follow. We put \( c_i = 0 \) if \( s_i = r_i \) and \( c_i = 1 \) if \( s_i < r_i \). Then

\[
m(\mu, a, s) = hs^2 \sum_{0 \leq d_i \leq c_i, 1 \leq i \leq k} (-1)^{d_1 + \cdots + d_k} \gamma(\mu, p_1^{s_1+d_1} \cdots p_k^{s_k+d_k}) \times (q-1)^{\delta(\mu)-2} \beta \left( \frac{(q-1)A(\mu)}{\gcd(q-1, nh)}, a \right) \]

Proof. (1) If \( (h_1, h_2, \cdots) \in \text{Irr}(GL, \mu, a) \) is stabilized by \( s \)-th roots of 1 in \( U_{(q-1)/h} \), then we may write

\[
h_{a, j}(x) = x^{l_j} + \sum_{i=0}^{t_j/s-1} b_{j,i} x^{is}
\]

for all \( j \) by (3.7). Moreover, because this character belongs to \( B_a \), by Corollary 3.4(3)(ii) we have

\[
((-1)^{n_1 a_1} b_{2,0} a_2 \cdots b_{6,0} a_6)^{(q-1)/\gcd(q-1, nh)} = a. \]

If \( i \neq 0 \), then the possible of \( b_{j,i} \) is any element in \( GF(q) \), and the number of all possible of the set of \( b_{j,0} \) is determined by (3.9). Thus
\[ |\text{Irr}(GL, \mu, a, s)| = \left( \prod_{j=1}^{s} q^{L_j/s-1} \right) (q - 1)^{\delta - 1} \beta \left( \frac{(q - 1)A(\mu)}{\gcd(q - 1, nh)}, a \right) \]

\[ = \gamma(\mu, s)(q - 1)^{\delta(\mu - 1)} \beta \left( \frac{(q - 1)A(\mu)}{\gcd(q - 1, nh)}, a \right). \]

(2) The above number includes characters stabilized by \( s' \)-th roots of 1 for some \( s < s' \) with \( s \mid s' \). Thus

\[ |\text{Irr}(GL, \mu, a, s)| \]

\[ = \sum_{0 \leq d_i \leq c_i \atop 1 \leq i \leq k} (-1)^{d_1 + \cdots + d_k} \gamma(\mu, p^{e_1+d_1} \cdots p^{e_k+d_k})(q - 1)^{\delta(\mu - 1)} \beta \left( \frac{(q - 1)A(\mu)}{\gcd(q - 1, nh)}, a \right). \]

Each \( U_{(q - 1)/h} \)-orbit in \( \text{Irr}(GL, \mu, a, s) \) has \( (q - 1)/hs \) elements. So each orbit gives \( s \) characters of \( L_h \) by Corollary 3.4(2). Consequently, all characters in \( \text{Irr}(GL, \mu, a, s) \) give irreducible characters of \( L_h \).

**Theorem 4.2.** For a partition \( \mu = (\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_s^{l_s}) \) of \( n \) and \( a \in U_{\gcd(q - 1, nh)} \),

\[ m(\mu, a) = h \left\{ \sum_{t \mid \gcd(\Lambda(\mu), (q - 1)/h)} t^2 \Pi(t) q^{\ell(\mu)/t - \delta(\mu)} \right\} \]

\[ \times (q - 1)^{\delta(\mu - 2)} \beta \left( \frac{(q - 1)A(\mu)}{\gcd(q - 1, nh)}, a \right). \]

**Proof.** We obtain \( m(\mu, a) \) by summing \( m(\mu, a, s) \) for all \( s \) dividing \( \gcd(\Lambda(\mu), (q - 1)/h) \), i.e., for all \( (s_1, \ldots, s_k) \) \( (0 \leq s_i \leq r_i) \). Hence we may write by the previous lemma,

\[ m(\mu, a) = h \left\{ \sum_{t \mid \gcd(\Lambda(\mu), (q - 1)/h)} e_t \gamma(\mu, t) \right\} (q - 1)^{\delta(\mu - 2)} \beta \left( \frac{(q - 1)A(\mu)}{\gcd(q - 1, nh)}, a \right), \]

for some \( e_t \). If \( t = p_1^{r_1} \cdots p_k^{r_k} \), then \( e_t \) is in fact, obtained as follows.

\[ e_t = \sum_{0 \leq d_i \leq c_i \atop 1 \leq i \leq k} (-1)^{d_1 + \cdots + d_k} p_1^{2(t_1 - d_1)} \cdots p_k^{2(t_k - d_k)}, \]
where \( c'_i = 0 \) if \( t_i = 0 \), and \( c'_i = 1 \) if \( t_i > 0 \). Therefore,

\[
e_t = \prod_{t_i \neq 0} (q^2 q^{2t_i} - q^{2t_i - 2}) = t^2 \Pi(t)
\]

Consequently, we have the statement of the theorem. \( \square \)

Note that each character \( \zeta \) in \( \text{Irr}(L_h, \mu, a) \) satisfies \( \nu(\zeta(1)) = en'(\mu) \). Therefore, we have the following theorem.

**Theorem 4.3.** For \( 0 \leq k < n(n - 1)/2 \),

\[
m_p(ek, L_h(n, q), B_a) = \sum' m(\mu, a),
\]

where the sum is taken over all partitions \( \mu \) of \( n \) such that \( n'(\mu) = k \). And if \( i \neq ek \) for any \( k \) with \( 0 \leq k < n(n - 1)/2 \), then \( m_p(ek, L_h(n, q), B_a) = 0 \).

Recall that \( \tilde{B}_0 \) is the unique \( p \)-block of maximal defect of \( PL_h \). Because we can identify \( B_1 \) with \( \tilde{B}_0 \), by Lemmas 3.7, 4.1, and Theorem 4.2, we have the following.

**Corollary 4.4.** For \( 0 \leq k < n(n - 1)/2 \),

\[
m_p(ek, PL_h(n, q), \tilde{B}_0) = \sum' m(\mu, 1)
\]

\[
= \sum' \left\{ \sum_{t|\gcd(A(\mu), (q-1)/h)} t^2 \Pi(t)q^{(\mu)t+t}\delta(\mu) \right\} \left( q-1 \right)^{\delta(\mu)-1} \frac{\gcd(q-1, A(\mu))}{\gcd(q-1, nh)}
\]

where the first sum is the same as in Theorem 4.3. And if \( i \neq ek \) for any \( k \) with \( 0 \leq k < n(n - 1)/2 \), then \( m_p(ek, PL_h(n, q), \tilde{B}_0) = 0 \).

5. The Alperin-McKay conjecture for \( L_h \)

In this section, we show the following theorem, i.e., we prove the Alperin-McKay conjecture for \( L_h \). The notations are the same as in the previous sections.

**Theorem 5.1.** For a Sylow \( p \)-subgroup \( P \) of \( L_h \), let \( b_a \) be the \( p \)-block of \( N = N_{L_h}(P) \) corresponding to the \( p \)-block \( B_a \) of maximal defect of \( L_h \). Then we have

\[
m_p(0, L_h, B_a) = m_p(0, N, b_a).
\]

Proof. We classify irreducible characters of \( L_h \) and \( N \) respectively by sequences \( \iota = (s_0, s_1, s_2, \ldots, s_k) \) of integers \( s_i \) such that \( 0 = s_0 < s_1 < s_2 < \cdots < s_k = n \) for some \( k \leq n \).
By Corollary 3.4, the degree of $\zeta \in \text{Irr}(L_h)$ is not divisible by $p$ if and only if $\zeta$ is in $\text{Irr}(L_h|_{\lambda_{\infty}})$ for some $\lambda = (h(x), 1, 1, \cdots) \in \text{Irr}(GL)$ where $h(x)$ is a polynomial of degree $n$. For given $\iota = (s_0, s_1, \cdots, s_k)$, we consider characters $(h(x), 1, 1, \cdots) \in \text{Irr}(GL)$ with

$$h(x) = x^n + \sum_{i=0}^{n-1} a_i x^i, \quad \text{where} \begin{cases} a_i \neq 0, & \text{if } i = s_j \text{ for some } 0 \leq j \leq k - 1; \\ a_i = 0, & \text{otherwise.} \end{cases}$$

Thus the number of characters of this type is $(q - 1)^k$. By (3.7), an element of $U_{(q-1)/h}$ stabilizes characters of this type if and only if it is a $\gcd(s_1 - s_0, \cdots, s_k - s_{k-1}, (q-1)/h)$-th roots of 1. By Corollary 3.4(2) the number of characters in $\text{Irr}(L_h)$ given by $\iota$ is

$$\gcd\left(s_1 - s_0, \cdots, s_k - s_{k-1}, \frac{q-1}{h}\right)^2 h(q-1)^{k-1}. $$

By (3.8) the above characters belong to a $p$-block $B_\alpha (a \in U_{\gcd(q-1,nh)})$ if and only if $a_0^{(q-1)/\gcd(q-1,nh)} = a$. Note that for any $a \in U_{\gcd(q-1,nh)}$ the number of solutions $a_0$ in $U_{q-1}$ to this equation is $(q-1)/\gcd(q-1,nh)$. Since this number does not depend on $a$, all $B_\alpha$'s have the same number of characters of this type given by $\iota$.

On the other hand, a Sylow $p$-subgroup $P$ of $L_h$ is conjugate to the subgroup of upper triangle matrices all of whose diagonal entries are 1. Thus we may assume that $N$ is the subgroup of upper triangle matrices in $L_h$.

But the degree of a character $\chi$ of $N$ is not divisible by $p$ if and only if the kernel of $\chi$ contains the commutator subgroup $P'$ of $P$. Therefore we may consider such characters as those of $M = N/P'$. Let $Q = P/P'$ and let $D$ be the set of elements in $N/P'$ corresponding to diagonal matrices in $N$. Then we have $M = D \times Q$. We denote an element $a$ in $D$ by $(a_1, a_2, \cdots, a_n)$ where $a_i \in F_1$ and $a_1 a_2 \cdots a_n \in U_h$, in such a way that the product of elements in $D$ is the component-wise product. We denote an element $b$ in $Q$ by $(b_1, b_2, \cdots, b_{n-1})$ where $b_i \in GF(q)$, and the product of elements in $Q$ is the component-wise sum. Thus the action $a$ on $b$ is given by

$$b^a = a^{-1}ba = (a_1^{-1}b_1 a_2, a_2^{-1}b_2 a_3, \cdots, a_{n-1}^{-1}b_{n-1} a_n).$$

Since $D$ and $Q$ are Abelian groups, every irreducible character of these groups is of degree 1, and we fix an isomorphism from $D$ (resp. $Q$) to the group of characters of $D$ (resp. $Q$).

We construct characters of $M$ by using Theorems 2.3 and 2.4.

For the above sequence $\iota = (s_0, \cdots, s_k)$, we consider $b = (b_1, b_2, \cdots, b_{n-1}) \in$
Irr(Q) such that

\[
\begin{cases}
  b_i = 0, & \text{if } i = s_j \text{ for some } 1 \leq j \leq k - 1; \\
  b_i \neq 0, & \text{otherwise}.
\end{cases}
\]

The number of such characters is \((q-1)^{n-k}\). Then, for \(a = (a_1, a_2, \ldots, a_n) \in \text{Irr}(D)\), \(b^a = b\) if and only if \(a_{s_j+1} = a_{s_j+2} = \cdots = a_{s_{j+1}} (0 \leq j \leq k - 1)\). And since \(a \in \text{Irr}(D)\), it is necessary that \(a_{s_1}^{-1} a_{s_2}^{-s_1} \cdots a_{s_k}^{-s_{k-1}} \in U_h\). By Lemma 3.7, the order of the stabilizer of \(b\) in \(D\) is

\[
(q - 1)^{k-1} \sum_{c \in U_h} \beta(m, c)
\]

\[
= (q - 1)^{k-1} \sum_{c \in U_{gcd(mh, q - 1)/gcd(m, q - 1)}} \text{gcd}(m, q - 1)
\]

\[
= (q - 1)^{k-1} \text{gcd}(mh, q - 1)
\]

where \(m = \text{gcd}(s_1 - s_0, s_2 - s_1, \ldots, s_k - s_{k-1})\). Since the order of \(D\) is \(h(q-1)^{n-1}\), the number of elements contained in each orbit is \((q-1)^{n-k}/\text{gcd}(m, (q-1)/h)\). Hence the number of orbits in the set of irreducible characters given by \(\iota\) is \(\text{gcd}(m, (q-1)/h)\).

From Theorems 2.3 and 2.4, the number of characters \(\chi\) of \(M\) such that the restriction of \(\chi\) to \(Q\) is a sum of certain irreducible characters all of which have the type given by \(\iota\) is

\[
\gcd \left( \frac{m}{h}, \frac{q - 1}{h} \right)^2 h(q - 1)^{k-1}
\]

\[
= \gcd \left( s_1 - s_0, s_2 - s_1, \ldots, s_k - s_{k-1}, \frac{q - 1}{h} \right)^2 h(q - 1)^{k-1}.
\]

The distribution of irreducible characters of \(N\) to \(p\)-blocks of maximal defect can be seen by comparing the irreducible constituent of the restriction to the center \(Z(M)\) of \(M\). Note that \(Z(N) = Z(M)\). Recall that the same is true for \(L_h\). See Lemma 3.6. We fix an irreducible character \(b\) of \(Q\) given by \(\iota\), and consider the distribution of the characters in \(\text{Irr}(M | b)\) to \(p\)-blocks. The center \(Z(M)\) of \(M\) is contained in the stabilizer \(T\) of \(b\) in \(D\) and on the other hand we have \(Q \cap Z(M) = \{1\}\). Thus, from Theorem 2.3, for an irreducible character \(\chi\) in \(\text{Irr}(M | b)\), there exists an extension \(\tilde{b}\) of \(b\) to \(T\) and an irreducible character \(\eta\) of \(T\) such that \(\chi = (\tilde{b}\eta)^M\). So, in order to look at the restriction of \(\chi\) to \(Z(M)\), we may consider that of \(\eta\) to \(Z(M)\). Since \(T\) is Abelian, by Theorems 2.3, 2.4, the characters in \(\text{Irr}(M | b)\) are distributed into \(p\)-blocks in such a way that all blocks of \(M\) of maximal defect have the same numbers of characters in \(\text{Irr}(M | b)\). Since the above argument can be applied for any character \(b\) of \(Q\) given by \(\iota\), all \(p\)-blocks of \(M\) of maximal defect have the same numbers of characters given by \(\iota\).
Let \( B_a \) and \( b_a \) be the same as in the statement of Theorem 5.1. For a fixed \( t \), the above argument shows that the numbers of characters of \( L_h \) given by \( t \) belonging to \( B_a \) and that of \( N \) given by \( t \) belonging to \( b_a \) are equal. Since \( t = (s_0, s_1, \cdots, s_k) \) is arbitrary, we have

\[
m_p(0, L_h, B_a) = m_p(0, N, b_a).
\]

We identify \( B_1 \) with \( B_0 \), and in the same way as we identify \( b_1 \) with the \( p \)-block \( b_0 \) of \( N_{PL_h}(P) \). Therefore, we have the following.

**Corollary 5.2.** For a Sylow \( p \)-subgroup \( P \) of \( L_h/Z(L_h) \), let \( b_0 \) be the \( p \)-block of \( \tilde{N} = N_{L_h/Z(L_h)}(P) \) corresponding to the \( p \)-block \( \tilde{B}_0 \) of maximal defect of \( L_h/Z(L_h) \). Then we have

\[
m_p(0, L_h/Z(L_h), \tilde{B}_0) = m_p(0, \tilde{N}, \tilde{b}_0).
\]

**References**


