Title	On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres. I
Author(s)	Nagura, Toshinobu
Citation	Osaka Journal of Mathematics. 1981, 18(1), p. 115-145
Version Type	VoR
URL	https://doi.org/10.18910/9875
rights	
Note	

Nagura, T.
Osaka J. Math.
18 (1981), 115-145

ON THE JACOBI DIFFERENTIAL OPERATORS ASSOCIATED TO MINIMAL ISOMETRIC IMMERSIONS OF SYMMETRIC SPACES INTO SPHERES I

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(Received September 17, 1979)

Contents

Introduction

Let $F: M \to \bar{M}$ be a minimal isometric immersion of a compact Riemannian manifold M. For a variation $\{F_t\}$ of F the second variation of the volume $V(t)$ of $F_t(M)$ is described by a differential operator $\tilde{S}$, called the Jacobi differential operator, on the normal bundle as

$$\left.\frac{d^2V(t)}{dt^2}\right|_{t=0} = \int_M \langle \tilde{S}(E^N), E^N \rangle dx ,$$

where E^N denotes the infinitesimal normal variation of $\{F_t\}$ (see section 1). The Jacobi differential operator $\tilde{S}$ is self-adjoint and strongly elliptic. Therefore the index and the nullity of F are obtained from the spectra of $\tilde{S}$. Here the index and the nullity are defined as those of the Hessian at F of the volume integral on the space of immersions of M into $\bar{M}$ modulo diffeomorphisms of M. For the study of minimal isometric immersions it seems to be important to study $\tilde{S}$ and its spectra. However there have been few studies on these problems except for the recent works of Hasegawa and others. Hasegawa [4] studies the spectral geometry of minimal submanifolds.

Let M be a compact symmetric space, $\bar{M}$ a unit sphere, and F an equivariant

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*) This study is partially supported by Yukawa Foundation

minimal isometric immersion. Under this situation we study the Jacobi differential operator $\tilde{S}$, applying the representation theory of compact Lie groups. In section 1 we recall some results on minimal isometric immersions. In section 2 we study equivariant isometric immersions of compact homogeneous spaces and their Killing nullities (see Hsiang and Lawson [6] p. 14 for Killing nullities). In section 3 we study equivariant minimal isometric immersions of compact symmetric spaces into unit spheres. And we compute the Jacobi differential operator $\tilde{S}$ in this case (Theorem 1). In section 4, recalling some results on invariant differential operators, we give some propositions, which give criterions in order that our operator $\tilde{S}$ reduces to the Casimir operator. In section 5 the problem of computing the spectra of $\tilde{S}$ is reduced to the eigenvalue problems for certain linear mappings S_σ of finite dimensional vector spaces (Theorem 3).

In the forthcoming papers we shall study the linear mappings S_σ in detail under certain conditions, and study the index and the nullity of minimally immersed spheres into spheres.

The author would like to express his sincere gratitude to Professor M. Takeuchi and Professor S. Murakami for their valuable suggestions and encouragements.

1. Preliminaries

1.1. Let (M, g) be an n-dimensional compact connected Riemannian manifold without boundary, and $(\bar{M}, \bar{g})$ an m-dimensional Riemannian manifold. Let $F: M \rightarrow \bar{M}$ be an isometric immersion of M into $\bar{M}$. We consider the tangent space $T_x(M)$ of M at $x \in M$ as a vector subspace of the tangent space $T_{F(x)}(\bar{M})$ of $\bar{M}$ at $F(x) \in \bar{M}$. We denote by $N_x(M)$ the orthogonal complement of $T_x(M)$ in $T_{F(x)}(\bar{M})$, which is called the *normal space* of the immersed submanifold M of $\bar{M}$ at x. Let $T(M)$ (resp. $T(\bar{M})$) be the tangent bundle of M(resp. of $\bar{M}$). We denote by $T(\bar{M})|_M$ the bundle induced by F from $T(\bar{M})$. The bundle $N(M) = \bigcup_{x \in M} N_x(M)$ is called the *normal bundle* of M. We denote by $\mathfrak{X}(M)$ (resp. $\Gamma(N(M))$) the space of all C^∞ cross-sections of $T(M)$ (resp. of $N(M)$).

Let $B: T_x(M) \times T_x(M) \rightarrow N_x(M)$ be the second fundamental form of M, and $A: N_x(M) \times T_x(M) \rightarrow T_x(M)$ the Weingarten form of M. The second fundamental form B is a symmetric bilinear mapping and A_v, $v \in N_x(M)$, is a self-adjoint linear mapping of $T_x(M)$. Let ∇(resp. $\bar{\nabla}$) be the Riemannian connection of M(resp. $\bar{M}$). Let D be the normal connection of M. For any vector fields X, $Y \in \mathfrak{X}(M)$ and for any normal vector field $\xi \in \Gamma(N(M))$, we have the following equations (cf. Kobayashi and Nomizu [7] Vol. II Chap. 7 section 3):

$$(1.1.1) \qquad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

(1.1.2) $\bar{\nabla}_X\xi = -A_\xi X + D_X\xi$,

(1.1.3) $\bar{g}(\xi, B(X, Y)) = g(A_\xi X, Y)$.

We denote by H the mean curvature of M. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_x(M)$. Then we have

$$H_x = \sum_{i=1}^n B(e_i, e_i) .$$

The isometric immersion $F: M \to \bar{M}$ is said to be *minimal*, if the mean curvature H of M vanishes identically.

1.2. Let $\bar{R}$ be the curvature tensor of $\bar{M}$. For $x \in M$ we define linear mappings $\tilde{A}$ and $\tilde{R}$ of $N_x(M)$ as follows:

(1.2.1) $\tilde{A}(v) = \sum_{i,j=1}^n \bar{g}(v, B(e_i, e_j))B(e_i, e_j)$,

(1.2.2) $\tilde{R}(v) = \sum_{i=1}^n (\bar{R}(e_i, v)e_i)^N$ for $v \in N_x(M)$,

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_x(M)$ and $(\bar{R}(*, *)*)^N$ denotes the normal component of $\bar{R}(*, *)*$. The linear mappings $\tilde{A}$ and $\tilde{R}$ are independent of the choice of an orthonormal basis.

If $\bar{M}$ is a space of constant sectional curvature k, we have for any vector fields X, Y and Z on $\bar{M}$ (cf. Kobayashi and Nomizu [7] Vol. I p. 203):

$$\bar{R}(X, Y)Z = k(\bar{g}(Z, Y)X - \bar{g}(Z, X)Y) .$$

Therefore we have

(1.2.3) $\tilde{R}(v) = -nkv$ for $v \in N_x(M)$.

We denote by Δ the Laplace operator on $N(M)$ (cf. Simons [10] p. 64). Let $\{E_1, \cdots, E_n\}$ be an orthonormal local basis of $T(M)$ on a neighborhood of $x \in M$. Then we have

(1.2.4) $\Delta f(x) = \sum_{i=1}^n (D_{E_i}D_{E_i}f)(x) - \sum_{i=1}^n (D_{\nabla_{E_i}E_i}f)(x)$ for $f \in \Gamma(N(M))$.

We define a differential operator $\tilde{S}$, called the *Jacobi differential operator*, on $N(M)$ as follows:

(1.2.5) $\tilde{S} = -\Delta - \tilde{A} + \tilde{R}$.

Let I be an open interval containing $0 \in \boldsymbol{R}$. A 1-parameter family $\{F_t\}_{t\in I}$ of immersions of M into $\bar{M}$ is called a *variation* of F, if $F = F_0$ and if the mapping $f: I \times M \to \bar{M}$, defined by $f(t, x) = F_t(x)$, is differentiable. The *variation vector field* E of the variation $\{F_t\}_{t\in I}$ is defined by

$$E_x = df\left(\left(\frac{\partial}{\partial t}\right)_{(0,x)}\right).$$

Proposition 1.2.1 (cf. Simons [10] p. 73). *Let $F: M \to \bar{M}$ be a minimal isometric immersion, $\{F_t\}_{t\in I}$ a variation of F, and E the variation vector field of $\{F_t\}$. We denote by $V(t)$ the volume of M with respect to the Riemannian metric induced by the immersion F_t. Let E^N be the normal component of E, which is a cross-section of $N(M)$. Then we have*

$$\text{(1.2.6)} \qquad \left.\frac{d^2V(t)}{dt^2}\right|_{t=0} = \int_M g(\tilde{S}(E^N), E^N)dx\,,$$

where dx is the Riemannian measure of (M, g).

The vector space $\Gamma(N(M))$ is a pre-Hilbert space with the inner product $(\ ,\ )$:

$$(f, f') = \int_M g(f, f')dx \qquad \text{for } f, f' \in \Gamma(N(M))\,.$$

We denote by $L^2(N(M))$ the completion of $\Gamma(N(M))$. We consider $\Gamma(N(M))$ as a linear subspace of $L^2(N(M))$. The Jacobi differential operator $\tilde{S}$ is a self-adjoint strongly elliptic operator on $\Gamma(N(M))$. Therefore we have

Proposition 1.2.2 (cf. Simons [10] p. 74). (1) *The Jacobi differential operator $\tilde{S}$ is diagonalizable in the sense that there exists a complete orthonormal system $\{e_\alpha\}_{\alpha\in A}$ of $L^2(N(M))$ such that each e_α is contained in $\Gamma(N(M))$ and that each e_α is an eigenvector of $\tilde{S}$.*

(2) *Each eigenspace of $\tilde{S}$ is finite dimensional. Let*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$$

be the eigenvalues of $\tilde{S}$. Then the sequence $\{\lambda_i\}_{i=1,2\cdots}$ is divergent to ∞.

REMARK 1.2.1. By Proposition 1.2.2 the spectra of $\tilde{S}$ acting on $\Gamma(N(M))$ coincide with ones of $\tilde{S}$ acting on $\Gamma(N(M))^C$, the complexification of $\Gamma(N(M))$.

We define a bilinear form $I(\ ,\ )$ on $\Gamma(N(M))$ as follows:

$$I(V, W) = \int_M g(\tilde{S}(V), W)dx \qquad \text{for } V, W \in \Gamma(N(M))\,.$$

The *index* and the *nullity* of F are those of the bilinear form $I(\ ,\ )$. By Proposition 1.2.1 and 1.2.2 the index of F is the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues of $\tilde{S}$, and the nullity of F is the dimension of the 0-eigenspace of $\tilde{S}$.

2. Equivariant isometric immersions

2.1. In section 2 we assume the followings. Let G be a compact connected Lie group, and K a closed subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup K. Let $\langle\ ,\ \rangle$ be an $Ad(G)$-invariant inner product on $\mathfrak{g}$. Then we have an orthogonal decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$. We denote by M the quotient space G/K. We canonically identify $\mathfrak{p}$ with the tangent space $T_o(M)$ of M at $o=\pi(e)$, where π is the natural projection of G onto $M=G/K$. We also denote by $\langle\ ,\ \rangle$ the G-invariant Riemannian metric on M which coincides with the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}=T_o(M)$. Let $F\colon (M, c\langle\ ,\ \rangle)\to\bar{M}$ be an isometric immersion for some $c>0$ which is equivariant in the following sense: There exists a Lie group homomorphism ρ of G into $I(\bar{M})$, the group of all isometries of $\bar{M}$, such that $F(x(yK))=\rho(x)F(yK)$ for $x, y\in G$. We also denote by $\langle\ ,\ \rangle$ the Riemannian metric on $\bar{M}$. Moreover we assume that the image $F(M)$ of M does not coincide with $\bar{M}$.

We define an action σ of G on $\Gamma(N(M))$ by

$$(\sigma(x)\tilde{f})(yK)=d(\rho(x))\tilde{f}(x^{-1}y)\qquad \text{for } \tilde{f}\in\Gamma(N(M)) \text{ and } x, y\in G,$$

where $d(\rho(x))$ denotes the differential of the isometry $\rho(x)$. We define an action of G on $\Gamma(T(\bar{M})|_M)$ in the same way as for $\Gamma(N(M))$, where $\Gamma(T(\bar{M})|_M)$ is the space of all C^∞ cross-sections of $T(\bar{M})|_M$. We also denote by σ the action of G on $\Gamma(T(\bar{M})|_M)$. Then we have by the equivariance of F

$$\begin{cases} \Delta\circ\sigma(x)=\sigma(x)\circ\Delta\,, \\ \tilde{A}\circ\sigma(x)=\sigma(x)\circ\tilde{A}\,, \\ \tilde{R}\circ\sigma(x)=\sigma(x)\circ\tilde{R}\,. \end{cases}$$

Therefore we have

$$\tilde{S}\circ\sigma(x)=\sigma(x)\circ\tilde{S}. \tag{2.1.1}$$

Moreover if F is minimal, each eigenspace of $\tilde{S}$ is G-invariant.

Put $U=N_o(M)$. Then K acts on U by the differential of $\rho(k)$, $k\in K$, at $F(o)$. We denote by ϕ this action of K on U. We denote by E the vector bundle $G\times_K U$ associated with G by ϕ. Put

$$C^\infty(G;\,U)_K=\left\{f\colon G\to U\quad C^\infty \text{ mapping};\, f(xk)=\phi(k)^{-1}f(x) \text{ for } x\in G \text{ and } k\in K\right\}.$$

The space $\Gamma(E)$ of C^∞ cross-sections of E is identified with $C^\infty(G;\,U)_K$ by the following correspondence:

$$C^\infty(G;\,U)_K\ni f\mapsto\tilde{f}\in\Gamma(E),\ \tilde{f}(xK)=x\circ f(x)\qquad \text{for } x\in G\,, \tag{2.1.2}$$

where $x\circ f(x)$ is the image of $(x, f(x))\in G\times U$ by the natural projection $G\times U\to$

$G\times_K U$. We define an action L of G on $C^\infty(G; U)_K$ as follows:

(2.1.3) $\quad (L_x f)(y) = f(x^{-1}y) \qquad$ for $f\in C^\infty(G; U)_K$ and $x, y\in G$.

Put $V=T_{F(o)}(\bar{M})$ and $W=T_o(M)$. Then K also acts on V (resp. W) by the differential of $\rho(k)$(resp. of k), $k\in K$, at $F(o)$ (resp. at o). We denote by J (resp. H) the associated vector bundle $G\times_K V$(resp. $G\times_K W$). We define a space $C^\infty(G; V)_K$(resp. $C^\infty(G; W)_K$) and an action L of G on $C^\infty(G; V)_K$(resp. on $C^\infty(G; W)_K$) in the same way. We can identify $T(\bar{M})|_M$(resp. $N(M)$ and $T(M)$) with J(resp. E and H) and $\Gamma(T(\bar{M})|_M)$(resp. $\Gamma(N(M))$ and $\mathfrak{X}(M)$) with $C^\infty(G; V)_K$(resp. $C^\infty(G; U)_K$ and $C^\infty(G; W)_K$) in the following way.

Proposition 2.1.1. (1) *The vector bundle homomorphism*

$$\iota\colon J\to T(\bar{M})|_M,\ \iota(x\circ v) = d(\rho(x))v \qquad \textit{for } x\in G \textit{ and } v\in V,$$

is an isomorphism, and ι *induces an isomorphism of* E*(resp.* H*) onto* $N(M)$ *(resp.* $T(M)$*).*

(2) *Also denoting by* ι *the isomorphism of* $C^\infty(G; V)_K$ *onto* $\Gamma(T(\bar{M})|_M)$ *induced from* $\iota\colon J\to T(\bar{M})|_M$*, the following diagram is commutative:*

$$\begin{array}{ccc} C^\infty(G; V)_K & \xrightarrow{\ \iota\ } & \Gamma(T(\bar{M})|_M) \\ \big\downarrow L_x & & \big\downarrow \sigma(x) \\ C^\infty(G; V)_K & \xrightarrow{\ \iota\ } & \Gamma(T(\bar{M})|_M) \end{array} \qquad \textit{for } x\in G.$$

The isomorphism $\iota\colon C^\infty(G; V)_K\to\Gamma(T(\bar{M})|_M)$ *induces an isomorphism of* $C^\infty(G; U)_K$ *(resp.* $C^\infty(G; W)_K$*) onto* $\Gamma(N(M))$ *(resp.* $\mathfrak{X}(M)$*).*

For $f\in C^\infty(G; V)_K$ we denote by $\tilde{f}$ the image of f by the isomorphism ι.

2.2. For $x\in G$ we define a diffeomorphism τ_x of M by $\tau_x(yK)=xyK$. Then τ_x is an isometry of $(M, \langle\ ,\ \rangle)$. For $X\in\mathfrak{g}$ we denote by X^* the infinitesimal transformation on M which generates the 1-parameter group of transformations $\tau_{\exp tX}$ on M. We define differential operators $\tilde{A}_0$ and Δ_0 on $N(M)$ as follows:

(2.2.1) $$\tilde{A}_0(\tilde{f}) = \sum_{i=1}^{n+p} B(E_i^*, A_{\tilde{f}}E_i^*),$$

(2.2.2) $$\Delta_0(\tilde{f}) = \sum_{i=1}^{n+p} D_{E_i^*}D_{E_i^*}(\tilde{f}) \qquad \text{for } \tilde{f}\in\Gamma(N(M)),$$

where $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$. The operators $\tilde{A}_0$ and Δ_0 are independent of the choice of an orthonormal basis of $\mathfrak{g}$.

Proposition 2.2.1. *For the operators* $\tilde{A}_0$ *and* $\tilde{A}$ *we have the following equation:*

(2.2.3) $\quad c\tilde{A} = \tilde{A}_0\,.$

Proof. Choose an orthonormal basis $\{E_1, \cdots, E_{n+p}\}$ of $\mathfrak{g}$ with the property that $\{E_1, \cdots, E_n\}$ (resp. $\{E_{n+1}, \cdots, E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. $\mathfrak{k}$). Then $\left\{\frac{1}{\sqrt{c}}(E_1^*)_o, \cdots, \frac{1}{\sqrt{c}}(E_n^*)_o\right\}$ is an orthonormal basis of $T_o(M)$ and $(E_{n+1}^*)_o = \cdots = (E_{n+p}^*)_o = 0$. For $x \in G$ put $F_i = Ad(x)E_i$, $i=1, 2, \cdots, n+p$. Then $\{F_1, \cdots, F_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$, and we have

$$(F_i^*)_{xK} = \left.\frac{d(\exp t(\mathrm{Ad}(x)E_i)\cdot xK)}{dt}\right|_{t=0}$$
$$= \left.\frac{d(x(\exp tE_i)\cdot o)}{dt}\right|_{t=0} = d\tau_x(E_i^*)_o\,.$$

Therefore $\left\{\frac{1}{\sqrt{c}}(F_1^*)_{xK}, \cdots, \frac{1}{\sqrt{c}}(F_n^*)_{xK}\right\}$ is an orthonormal basis of $T_{xK}(M)$ and $(F_{n+1}^*)_{xK} = \cdots = (F_{n+p}^*)_{xK} = 0$. For $v \in N_{xK}(M)$ we have

$$\tilde{A}_0(v) = \sum_{i=1}^{n+p} B((F_i^*)_{xK}, A_v((F_i^*)_{xK}))$$
$$= c\sum_{i=1}^{n} B\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}, A_v\left(\frac{1}{\sqrt{c}}\left((F_i^*)_{xK}\right)\right)\right).$$

By (1.1.3) we have

$$A_v\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}\right) = \sum_{j=1}^{n}\left\langle A_v\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}\right), \frac{1}{\sqrt{c}}(F_j^*)_{xK}\right\rangle \frac{1}{\sqrt{c}}(F_j^*)_{xK}$$
$$= \sum_{j=1}^{n}\left\langle v, B\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}, \frac{1}{\sqrt{c}}(F_j^*)_{xK}\right)\right\rangle \frac{1}{\sqrt{c}}(F_j^*)_{xK}\,.$$

Hence we have by (1.2.1)

$$\tilde{A}_0(v) = c\sum_{i,j=1}^{n}\left\langle v, B\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}, \frac{1}{\sqrt{c}}(F_j^*)_{xK}\right)\right\rangle \times$$
$$B\left(\frac{1}{\sqrt{c}}(F_i^*)_{xK}, \frac{1}{\sqrt{c}}(F_j^*)_{xK}\right)$$
$$= c\tilde{A}(v)\,.$$

Q.E.D.

Proposition 2.2.2. *If the curve $c(t) = \exp tX\cdot o$ is a geodesic of M for any $X \in \mathfrak{p}$, we have*

(2.2.4) $\quad c\Delta = \Delta_0\,.$

Proof. Fix $x \in G$ and let $\{E_1, \cdots, E_{n+p}\}$ and $\{F_1, \cdots, F_{n+p}\}$ be orthonormal bases in the proof of Proposition 2.2.1. Then we have for $\tilde{f} \in \Gamma(N(M))$

(2.2.5) $$(\Delta_0 \tilde{f})(xK) = \sum_{i=1}^{n} (D_{F_i^*} D_{F_i^*} \tilde{f})(xK) .$$

We have

$$(F_i^*)_{x(\exp sE_i)\cdot o} = \frac{d\{\exp t(\mathrm{Ad}(x)E_i)\cdot(x(\exp sE_i)\cdot o)\}}{dt}\bigg|_{t=0} = \frac{d\{x(\exp(t+s)E_i)\cdot o\}}{dt}\bigg|_{t=0} .$$

Hence the curve $x(\exp tE_i)\cdot o$ is an integral curve of F_i^*. Since the curves $x(\exp tE_i)\cdot o$, $i=1, \cdots, n$, are geodesics, then

(2.2.6) $$\nabla_{(F_i^*)_{xK}} F_i^* = 0 .$$

Let U be a normal neighborhood of xK. Let X_i, $i=1, \cdots, n$, be the vector fields on U adapted to $(F_i^*)_{xK}$, i.e. $(X_i)_q = \tau^q_{xK}(F_i^*)_{xK}$, where τ^q_{xK} is the parallel translation along the unique geodesic segment in U which joins xK and q. Then there exists $\varepsilon>0$ such that $(X_i)_{x(\exp tE_i)\cdot o} = (F_i^*)_{x(\exp tE_i)\cdot o}$ for $-\varepsilon<t<\varepsilon$. Hence $(D_{X_i}\tilde{f})(x(\exp tE_i)\cdot o) = (D_{F_i^*}\tilde{f})(x(\exp tE_i)\cdot o)$ for $\tilde{f} \in \Gamma(N(M))$ and $-\varepsilon<t<\varepsilon$. Hence we have

(2.2.7) $$(D_{X_i} D_{X_i} \tilde{f})(xK) = (D_{F_i^*} D_{F_i^*} \tilde{f})(xK) .$$

We have by (1.2.4), (2.5.5), (2.2.6) and (2.2.7)

$$\begin{aligned}(\Delta\tilde{f})(xK) &= \sum_{i=1}^{n} (D_{\frac{1}{\sqrt{c}}X_i} D_{\frac{1}{\sqrt{c}}X_i} \tilde{f})(xK) \\ &= \frac{1}{c}\sum_{i=1}^{n} (D_{X_i} D_{X_i} \tilde{f})(xK) \\ &= \frac{1}{c}\sum_{i=1}^{n} (D_{F_i^*} D_{F_i^*} \tilde{f})(xK) \\ &= \frac{1}{c}(\Delta_0 \tilde{f})(xK) ,\end{aligned}$$

which proves (2.2.4). Q.E.D.

REMARK 2.2.1. Suppose that the pair (G, K) is a Riemannian symmetric pair and that the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is invariant under the involutive automorphism of $\mathfrak{g}$ associated to the pair (G, K). Then the condition of Proposition 2.2.2 is satisfied (cf. Helgason [5] pp. 174–177).

In what follows, for a Riemannian symmetric pair (G, K) the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ will be always assumed to have the above property.

2.3. In this subsection we moreover assume that the equivariant isometric immersion $F: (M, c\langle\ ,\ \rangle) \to \bar{M}$ is minimal and that $\bar{M}$ is compact.

Let E be a Killing vector field on $\bar{M}$ and E^N the normal component of the restriction of E to M. The dimension of the space $\{E^N; E$ is a Killing vector field on $\bar{M}\}$ is called the *Killing nullity* of F. We have $\tilde{S}(E^N)=0$ (Simons [10] p. 74). Hence the nullity is not less than the Killing nullity. Let $I(\bar{M}, M)$ be the group of isometries of $\bar{M}$ which leave $F(M)$ invariant. Then $I(\bar{M}, M)$ is a closed subgroup of $I(\bar{M})$. Since $\bar{M}$ is compact, the Killing nullity of F is equal to dim $I(\bar{M})/I(\bar{M}, M)$.

Proposition 2.3.1. *Assume that $\bar{M}$ is a compact connected Riemannian homogeneous space and that the equivariant isometric immersion F: $M\to\bar{M}$ is minimal. Then the Killing nullity of F is strictly positive.*

Proof. If the Killing nullity is equal to 0, then dim $I(\bar{M})=\dim I(\bar{M}, M)$. Since $\bar{M}$ is connected, the group $I(\bar{M}, M)$ is transitive on $\bar{M}$ (cf. Helgason [5] p. 114). Therefore we have $F(M)=I(\bar{M}, M)(F(M))=\bar{M}$, which is a contradiction. Q.E.D.

3. Equivariant minimal isometric immersions into spheres

3.1. In section 3 the assumptions and the notation are the same as in subsection 2.1. Moreover we assume that V is a Euclidean vector space with an inner product $\langle\ ,\ \rangle$ and that $\bar{M}$ is the unit sphere S of V with the center 0, the origin of V. Since the isometric immersion F: $M\to S$ is equivariant, there exists an orthogonal representation ρ: $G\to GL(V)$ such that $\rho(k)v_0=v_0$ for any $k\in K$, where $v_0=F(o)$.

We identify the tangent space of V with V itself in a canonical way. Then we have $d(\rho(x))=\rho(x)$ for $x\in G$. Since the induced bundle $T(V)|_M$ is trivial, we consider $\Gamma(T(V)|_M)$, the space of all C^∞ cross-sections of $T(V)|_M$, as the space of all V-valued C^∞ functions on M.

Under the above identification we have an orthogonal decomposition of the tangent space $T_{v_0}(V)$ as follows:

$$T_{v_0}(V) = V^0+V^T+V^N\,, \tag{3.1.1}$$

where $V^0=\boldsymbol{R}v_0$, $V^T=T_o(M)$ and $V^N=N_o(M)$. By Proposition 2.1.1 we have the following proposition.

Proposition 3.1.1. (1) *The vector bundle homomorphism*

$$\iota\colon G\times_K V\to T(V)|_M\,,\ \iota(x\circ v) = \rho(x)v \qquad \textit{for } x\in G \textit{ and } v\in V\,,$$

is an isomorphism, and ι induces an isomorphism of $G\times_K V^N$(resp. $G\times_K V^T$) onto $N(M)$ (resp. $T(M)$).

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} C^\infty(G;V)_K & \xrightarrow{\iota} & \Gamma(T(V)|_M) \\ \downarrow L_x & & \downarrow \sigma(x) \\ C^\infty(G;V)_K & \xrightarrow{\iota} & \Gamma(T(V)|_M) \end{array} \qquad \textit{for } x \in G.$$

The isomorphism ι: $C^\infty(G;V)_K \to \Gamma(T(V)|_M)$ *induces an isomorphism of* $C^\infty(G;V^N)_K$ (*resp.* $C^\infty(G;V^T)_K$) *onto* $\Gamma(N(M))$ (*resp.* $\mathfrak{X}(M)$).

For $f \in C^\infty(G;V)_K$ we denote $\iota(f)$ by $\tilde{f}$. We denote by S the operator of $C^\infty(G;V^N)_K$ corresponding to $\tilde{S}$ by the isomorphism ι.

Let $\bar{\nabla}$ be the connection in $T(V)|_M$ induced from the flat connection in $T(V)$. Then we have for $f \in C^\infty(G;V)_K$ and a vector field $Y \in \mathfrak{X}(M)$

$$\bar{\nabla}_Y \tilde{f} = Y\tilde{f}, \tag{3.1.2}$$

where we consider $\tilde{f}$ as a V-valued function on M. For $X \in \mathfrak{g}$ we denote by $\hat{X}$ the right invariant vector field on G such that $\hat{X}_e = X_e$, where we consider $\mathfrak{g}$ as the Lie algebra of left invariant vector fields on G and e is the unit element of G.

Lemma 3.1.2. *We have*

$$\bar{\nabla}_{X^*}\tilde{f} = \iota(\hat{X}f + d\rho(Ad(*^{-1})X)f) \qquad \textit{for } f \in C^\infty(G;V)_K \textit{ and } X \in \mathfrak{g}. \tag{3.1.3}$$

Here $d\rho(Ad(*^{-1})X)f$ *is the* V*-valued* C^∞ *function defined by*

$$(d\rho(Ad(*^{-1})X)f)(x) = d\rho(Ad(x^{-1})X)f(x),$$

$d\rho$ *is the differential of the homomorphism* ρ, *and* X^* *denotes the infinitesimal transformation which generates the* 1*-parameter group of transformations* $\tau_{\exp tX}$.

Proof. Let g be an element of $C^\infty(G;V)_K$ such that $\tilde{g} = \bar{\nabla}_{X^*}\tilde{f}$. By (2.1.2) and Proposition 3.1.1 we have for $f \in C^\infty(G;V)_K$ and $x \in G$

$$\tilde{f}(xK) = \iota(x \circ f(x)) = \rho(x)f(x).$$

Hence we have by (3.1.2)

$$\begin{aligned} g(x) &= \rho(x)^{-1}(\bar{\nabla}_{X^*}\tilde{f})(xK) = \rho(x)^{-1}X^*_{xK}\tilde{f} \\ &= \rho(x)^{-1}\lim_{t\to 0}\frac{1}{t}(\tilde{f}((\exp tX)xX) - \tilde{f}(xK)) \\ &= \rho(x)^{-1}\lim_{t\to 0}\frac{1}{t}\{\rho((\exp tX)x)f((\exp tX)x) - \rho(x)f(x)\} \\ &= \lim_{t\to 0}\frac{1}{t}\{\rho(\exp t(\mathrm{Ad}(x^{-1})X))f((\exp tX)x) - f((\exp tX)x) \\ &\qquad + f((\exp tX)x) - f(x)\} \\ &= d\rho(\mathrm{Ad}(x^{-1})X)f(x) + (\hat{X}f)(x). \end{aligned}$$

This proves the lemma. Q.E.D.

REMARK 3.1.1. Since left translations of G are commutative with right translations of G, we have $\hat{X}f \in C^\infty(G; V)_K$. Therefore we have $d\rho(\mathrm{Ad}(*^{-1})X)f \in C^\infty(G; V)_K$.

Lemma 3.1.3. (1) *We have for $X \in \mathfrak{g}$ and $f \in C^\infty(G; V^N)_K$*

$$D_{X^*}\tilde{f} = \iota(\hat{X}f + \{d\rho(\mathrm{Ad}(*^{-1})X)f\}^N), \tag{3.1.4}$$

$$-A_{\tilde{f}}X^* = \iota(\{d\rho(\mathrm{Ad}(*^{-1})X)f\}^T), \tag{3.1.5}$$

where we denote by g^N (resp. g^T) the V^N-component (resp. V^T-component) of $g \in C^\infty(G; V)_K$ with respect to the decomposition (3.1.1).

(2) *We have for $X \in \mathfrak{g}$ and $f \in C^\infty(G; V^T)_K$*

$$B(X^*, \tilde{f}) = \iota(\{d\rho(\mathrm{Ad}(*^{-1})X)f\}^N). \tag{3.1.6}$$

Proof. The lemma is an easy consequence of (1.1.1), (1.1.2), Proposition 3.1.1 and Lemma 3.1.2. Q.E.D.

For the differential operators $\tilde{A}_0$ and Δ_0 defined in subsection 2.2, we obtain the following two propositions.

Proposition 3.1.4. *We have for $f \in C^\infty(G; V^N)_K$*

$$\tilde{A}_0(\tilde{f}) = \iota\Big(-\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^T\}^N\Big), \tag{3.1.7}$$

where $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$.

Proof. Applying Lemma 3.1.3, we have

$$\begin{aligned}\tilde{A}_0(\tilde{f}) &= \sum_{i=1}^{n+p} B(E_i^*, A_{\tilde{f}}E_i^*)\\ &= \sum_{i=1}^{n+p} \iota(-(d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^T)^N).\end{aligned}$$

Put $\mathrm{Ad}(x)E_i = \sum_{i=1}^{n+p} a^j{}_i(x)E_j$ for $x \in G$. Then $(a^i{}_j(x))_{i,j=1,\cdots,n+p}$ is an orthogonal matrix. We have for $x \in G$

$$\begin{aligned}&\sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^T)^N(x)\\ &= \sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(x^{-1})E_i)\{d\rho(\mathrm{Ad}(x^{-1})E_i)f(x)\}^T)^N\\ &= \sum_{j,k=1}^{n+p}\Big(\sum_{i=1}^{n+p} a^j{}_i(x^{-1})a^k{}_i(x^{-1})\{d\rho(E_j)(d\rho(E_k)f(x))^T\}^N\Big)\\ &= \sum_{j=1}^{n+p}\{d\rho(E_j)(d\rho(E_j)f)^T\}^N(x).\end{aligned}$$

Therefore

$$\tilde{A}_0(\tilde{f}) = \iota(-\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^T\}^N) .$$

Q.E.D.

Proposition 3.1.5. *We have for* $f \in C^\infty(G; V^N)_K$

$$\Delta_0 \tilde{f} = \iota(\sum_{i=1}^{n+p} E_i E_i f + 2\sum_{i=1}^{n+p} (d\rho(E_i)(E_i f))^N + \sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N) , \tag{3.1.8}$$

where $\{E_1, \cdots, E_{n+p}\}$ *is an orthonormal basis of* $\mathfrak{g}$.

Proof. Applying Lemma 3.1.3, we have

$$\begin{aligned}\Delta_0 \tilde{f} &= \sum_{i=1}^{n+p} D_{E_i^*} D_{E_i^*} \hat{f} \\ &= \iota(\sum_{i=1}^{n+p} (\hat{E}_i(\hat{E}_i f + \{d\rho(\mathrm{Ad}(*^{-1})E_i f)\}^N) \\ &\qquad + \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f + \{d\rho(\mathrm{Ad}(*^{-1})(E_i)f\}^N)\}^N)) \\ &= \iota(\sum_{i=1}^{n+p} \hat{E}_i \hat{E}_i f + \sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N \\ &\qquad + \sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f)\}^N \\ &\qquad + \sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)\{d\rho(\mathrm{Ad})*^{-1})E_i)f\}^N\}^N) .\end{aligned}$$

We have (cf. Takeuchi [12] p. 51)

$$\sum_{i=1}^{n+p} \hat{E}_i \hat{E}_i = \sum_{i=1}^{n+p} E_i E_i . \tag{3.1.9}$$

Put $\mathrm{Ad}(x)E_i = \sum_{j=1}^{n+p} a^j{}_i(x) E_j$. Then we have for $x \in G$

$$\begin{aligned}(\hat{E}_i)_x &= dr_x(E_i)_e = dl_x(dl_{x^{-1}} dr_x(E_i)_e) \\ &= dl_x(\mathrm{Ad}(x^{-1})E_i)_e \\ &= \sum_{i=1}^{n+p} a^j{}_i(x^{-1}) dl_x(E_j)_e \\ &= \sum_{i=1}^{n+p} a^i{}_j(x)(E_j)_x ,\end{aligned} \tag{3.1.10}$$

where r_x(resp. l_x) denotes the right translation (resp. left translation) by $x \in G$. We obtain

$$\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}(x) = d\rho(\mathrm{Ad}(x^{-1})E_i)f(x) \tag{3.1.11}$$

$$= \sum_{j=1}^{n+p} a^j{}_i(x^{-1}) d\rho(E_j) f(x)$$

$$= \sum_{j=1}^{n+p} a^i{}_j(x) d\rho(E_j) f(x)\,.$$

By (3.1.11) and (3.1.10) we have

$$\sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N$$

$$= \sum_{i,j=1}^{n+p} ((\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + a^i{}_j \{d\rho(E_j)(\hat{E}_i f)\}^N)$$

$$= \sum_{i,j=1}^{n+p} (\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(E_k f)\}^N$$

$$= \sum_{i,j=1}^{n+p} (\hat{E}_i a^i{}_j)(d\rho(E_j)f)^N + \sum_{j=1}^{n+p} \{d\rho(E_j)(E_j f)\}^N\,.$$

Since the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is $\mathrm{Ad}(G)$-invariant, we have

$$(\hat{E}_i a^i{}_j)(x) = \lim_{t\to 0} \frac{1}{t} (\langle \mathrm{Ad}((\exp tE_i)x)E_j,\ E_i\rangle - \langle \mathrm{Ad}(x)E_j,\ E_i\rangle)$$

$$= \lim_{t\to 0} \frac{1}{t} \langle \mathrm{Ad}(\exp tE_i)\mathrm{Ad}(x)E_j - \mathrm{Ad}(x)E_j,\ E_i\rangle$$

$$= \langle \mathrm{ad}(E_i)\mathrm{Ad}(x)E_j,\ E_i\rangle$$

$$= -\langle \mathrm{Ad}(x)E_j,\ \mathrm{ad}(E_i)E_i\rangle = 0$$

Therefore we obtain

(3.1.12) $$\sum_{i=1}^{n+p} \hat{E}_i \{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N = \sum_{i=1}^{n+p} \{d\rho(E_i)(E_i f)\}^N\,.$$

We have by (3.1.10) and (3.1.11)

(3.1.13) $$\sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)(\hat{E}_i f)\}^N$$

$$= \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(E_k f)\}^N$$

$$= \sum_{j=1}^{n+p} \{d\rho(E_j)(E_j f)\}^N\,.$$

We have by (3.1.11)

(3.1.14) $$\sum_{i=1}^{n+p} \{d\rho(\mathrm{Ad}(*^{-1})E_i)\,\{d\rho(\mathrm{Ad}(*^{-1})E_i)f\}^N\}^N$$

$$= \sum_{i=1}^{n+p} \{\sum_{j=1}^{n+p} a^i{}_j d\rho(E_j)\{\sum_{k=1}^{n+p} a^i{}_k d\rho(E_k)f\}^N\}^N$$

$$= \sum_{j,k=1}^{n+p} \sum_{i=1}^{n+p} a^i{}_j a^i{}_k \{d\rho(E_j)(d\rho(E_k)f)^N\}^N$$

$$= \sum_{j=1}^{n+p} \{d\rho(E_j)(d\rho(E_j)f)^N\}^N .$$

We obtain (3.1.8) by (3.1.9), (3.1.12), (3.1.13) and (3.1.14). Q.E.D.

3.2. In the rest of this section we moreover assume that the equivariant isometric immersion $F: (M, c\langle\ ,\ \rangle) \to S$ is minimal. Let Δ_M be the Laplace operator of the Riemannian manifold $(M, \langle\ ,\ \rangle)$ acting on functions. Then we have (cf. Wallach [13] p. 20)

$$\Delta_M = \sum_{i=1}^{n+p} (E_i{}^*)^2 ,$$

where $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$. Hence the Laplace operator $\Delta_M(c)$ of $(M, c\langle\ ,\ \rangle)$ is given by the following equation:

$$\text{(3.2.1)} \qquad \Delta_M(c) = \frac{1}{c} \sum_{i=1}^{n+p} (E_i{}^*)^2 .$$

Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis of V and $(x_1, \cdots, x_N)$ the coordinate system on V with respect to $\{e_1, \cdots, e_N\}$. Put $F=(f_1, \cdots, f_N)$, i.e. $f_i(xK) =\langle e_i, F(xK)\rangle$. Then it is known (Takahashi [11] p. 383) that

$$\text{(3.2.2)} \qquad \Delta_M(c) f_i = -n f_i , \qquad i=1, \cdots, N .$$

We define an action L of G on $C^\infty(M)$, the space of C^∞ functions on M, as follows:

$$(L_x f)(yK) = f(x^{-1}yK) \qquad \text{for } x, y \in G \text{ and } f \in C^\infty(M) .$$

Proposition 3.2.1. *Let $\rho: G \to GL(V)$ be an orthogonal representation of G. Let $F: (M, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho(x)F(o)$, be an equivariant miniaml isometric immersion. If F is full, i.e. if the image $F(M)$ of M is not contained in any great spheres, then the following equation holds*:

$$\text{(3.2.3)} \qquad \sum_{i=1}^{n+p} d\rho(E_i) d\rho(E_i) = -nc 1_V ,$$

where 1_V denotes the identity transformation of V.

Proof. Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis of V and put $F=(f_1, \cdots, f_N)$ with respect to this basis. We define a linear mapping $\phi: V \to C^\infty(M)$ by $\phi(v)(xK) = \langle v, F(xK)\rangle$ for $v \in V$ and $x \in G$. Then the subspace $\phi(V)$ of $C^\infty(M)$ is spanned by $f_1, \cdots, f_N$. We have for $x, y \in G$ and $v \in V$

$$\begin{aligned} \phi(\rho(x)v)(yK) &= \langle \rho(x)v, F(yK)\rangle = \langle v, \rho(x^{-1})F(yK)\rangle \\ &= \langle v, F(x^{-1}yK)\rangle = \phi(v)(x^{-1}yK) \\ &= (L_x \phi(v))(yK) . \end{aligned}$$

Hence ϕ is a G-module homomorphism. Let ψ: $G\to GL(\phi(V))$ be a representation defined by $\psi(x)=L_x|_{\phi(V)}$. Then we have for $X\in\mathfrak{g}$

(3.2.4) $\quad d\psi(X)=-X^*$.

We assert that $\dim\phi(V)=N$. If the assertion is not true, there exist real numbers $c_1, \cdots, c_N$, which are not all equal to zero, such that $\sum_{i=1}^{N} c_i f_i=0$. Then the image $F(M)$ is contained in the hyperplane $\sum_{i=1}^{N} c_i x_i=0$, which is a contradiction. Therefore ϕ: $V\to\phi(V)$ is a G-module isomorphism. It follows from (3.2.4), (3.2.1) and (3.2.2) that

$$\begin{aligned}\sum_{i=1}^{n+p} d\psi(E_i)d\psi(E_i)f_k &= \sum_{i=1}^{n+p} E_i{}^*E_i{}^*f_k \\ &= c\Delta_M(c)f_k=-ncf_k\,.\end{aligned}$$

Hence we have $\sum_{i=1}^{n+p} d\psi(E_i)d\psi(E_i)=nc1_{\phi(V)}$, where $1_{\phi(V)}$ denotes the identity transformation of $\phi(V)$. Since ϕ: $V\to\phi(V)$ is a G-module isomorphism, we have

$$\sum_{i=1}^{n+p} d\rho(E_i)d\rho(E_i)=-nc1_V\,.$$

Q.E.D.

REMARK 3.2.1. Suppose that the linear isotropy representation of G/K is irreducible. Let ρ: $G\to GL(V)$ be a real spherical representation of (G, K), i.e. ρ is an irreducible orthogonal representation of G such that there is a unit vector $v\in V$ with the property that $\rho(k)v=v$ for any $k\in K$. Then we can construct a full equivariant minimal isometric immersion of $M=G/K$ in the following way. Let S be the unit sphere of V with the center 0. Define a mapping F: $M\to S$ by $F(xK)=\rho(x)v$ for $x\in G$. Then there exists a positive number c such that F: $(M, c\langle\ ,\ \rangle)\to S$ is a minimal isometric immersion (cf. Wallach [13] p. 21).

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^C$ the complexification of $\mathfrak{g}$. For a linear subspace $\mathfrak{u}$ of $\mathfrak{g}$ we denote by $\mathfrak{u}^C$ the complex linear subspace of $\mathfrak{g}^C$ generated by $\mathfrak{u}$. Let $\mathfrak{r}$ be the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{t}$. A non-zero element $\lambda\in\mathfrak{t}$ is a root, if and only if there exists a non-zero element $X\in\mathfrak{g}^C$ such that $[H, X]=\sqrt{-1}\langle\lambda, H\rangle X$ for any $H\in\mathfrak{t}$. Choosing a linear order in $\mathfrak{t}$, we denote by $\mathfrak{r}^+$ the set of all positive roots. Put $\delta=\frac{1}{2}\sum_{\lambda\in\mathfrak{r}^+}\lambda$.

Let (G, K) be a Riemannian symmetric pair and $D(G, K)$ the set of all equivalence classes of complex spherical representations of (G, K). Recall that an irreducible complex representation ϕ: $G\to GL(W)$ is called a complex spherical representation of (G, K), if there exists a non-zero vector $w\in W$ such that

$\phi(k)w=w$ for any $k\in K$. For a complex irreducible representation $\phi: G\to GL(W)$, we denote by $[\phi]$ the equivalence class to which ϕ belongs. For $[\phi]\in D(G, K)$ we denote by $\mathfrak{o}_{[\phi]}(M)$ the subspace of $C^\infty(M)^C$ generated by G-submodules of $C^\infty(M)^C$ which are isomorphic to ϕ, where $C^\infty(M)^C$ is the complexification of $C^\infty(M)$ (We will not distinguish G-modules and representations of G). Then $\mathfrak{o}_{[\phi]}(M)$ is isomorphic to ϕ as G-module and the Laplace operator Δ_M acts on $\mathfrak{o}_{[\phi]}(M)$ as a scalar operator $c_{[\phi]}$. The scalar $c_{[\phi]}$ is given by $-\langle\Lambda+2\delta, \Lambda\rangle$, where Λ is the highest weight of ϕ (cf. Takeuchi [12] p. 20, p. 207).

If the Riemannian symmetric pair (G, K) is of rank 1, there exists a dominant integral form Λ_0 such that the highest weight Λ of each complex spherical representation ϕ is given by $\Lambda=k\Lambda_0$ for some non-negative integer k (cf. Takeuchi [12] p. 166). Hence the scalar $c_{[\phi]}$ is given by $-\langle k\Lambda_0+2\delta, k\Lambda_0\rangle = -(k^2\langle\Lambda_0, \Lambda_0\rangle+2k\langle\delta, \Lambda_0\rangle)$. Since both $\langle\Lambda_0, \Lambda_0\rangle$ and $\langle\delta, \Lambda_0\rangle$ are positive, it follows that $c_{[\phi]}\neq c_{[\phi']}$ for $[\phi]$, $[\phi']\in D(G, K)$ with $[\phi]\neq[\phi']$. Therefore we have the following lemma.

Lemma 3.2.2. *If (G, K) is a Riemannian symmetric pair of rank* 1, *then each eigenspace of the Laplace operator Δ_M acting on $C^\infty(M)^C$ is irreducible.*

Proposition 3.2.3. *Assume that (G, K) is a Riemannian symmetric pair of rank* 1. *Let $\rho: G\to GL(V)$ be an orthogonal representation and the mapping $F: (M, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho(x)F(o)$, an equivariant minimal isometric immersion. If F is full, the complexification $\rho: G\to GL(V^C)$ of ρ is irreducible. Therefore $\rho: G\to GL(V)$ is irreducible.*

Proof. Put $F=(f_1\cdots, f_N)$ as in the proof of Proposition 3.2.1. We also denote by $\langle\ ,\ \rangle$ the Hermitian inner product on V^C which is the extension of the inner product $\langle\ ,\ \rangle$ on V. Let $\phi: V^C\to C^\infty(M)^C$ be the $\boldsymbol{C}$-linear mapping defined by $\phi(v)(xK)=\langle v, F(xK)\rangle$ for $v\in V^C$ and $x\in G$. We assert that $\{f_1, \cdots, f_N\}$ is linear independent over $\boldsymbol{C}$. If the assertion is not true, there exist complex numbers $c_1, \cdots, c_N$, which are not all equal to zero, such that $\sum_{i=1}^N c_i f_i=0$. Put $c_i=a_i+\sqrt{-1}\,b_i$, where a_i and b_i are real numbers. Then at least one of the equations $\sum_{i=1}^N a_i x_i=0$ and $\sum_{i=1}^N b_i x_i=0$ defines a hyperplane. Since every f_i is real valued, the image $F(M)$ is contained in this hyperplane. This is a contradiction. Hence by the proof of Proposition 3.2.1 we have that $\phi: V^C\to\phi(V^C)$ is a G-module isomorphism and that $\Delta_M f=-ncf$ for $f\in\phi(V^C)$. Therefore it follows from Lemma 3.2.2 that $\phi(V^C)$ is an irreducible G-module. Hence $\rho: G\to GL(V^C)$ is irreducible. Q.E.D.

REMARK 3.2.2. Assume that (G, K) is a Riemannian symmetric pair of rank 1. Then full equivariant minimal isometric immersions of $M=G/K$ into spheres are in one-to-one correspondence with complex spherical representations of (G, K). In fact a complex spherical representation of (G, K) corresponds to a full equivariant minimal isometric immersion $F: (M, c\langle\ ,\ \rangle)\to S$ by Proposition 3.2.3. Conversely since (G, K) is of rank 1, every zonal spherical function is real-valued (Do Carmo and Wallach [3] p. 98). Therefore every complex spherical representation of (G, K) is the complexification of a real spherical representation of (G, K). Hence a full equivariant minimal isometric immersion corresponds to a complex spherical representation of (G, K) (Remark 3.2.1).

3.3. In this subsection we assume that (G, K) is a Riemannian symmetric pair.

Theorem 1. *Let ρ: $G\to GL(V)$ be an orthogonal representation and F: $(M, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho(x)F(o)$, a full equivariant minimal isometric immersion. Then we have for $f\in C^\infty(G; V^N)_K$*

$$Sf=-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_if-2c_\rho f+2\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N+2\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N\Big), \tag{3.3.1}$$

where $c_\rho=-nc$ and $\{E_1, \cdots, E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$.

Proof. Since the condition of Proposition 2.2.2 is satisfied (Remark 2.2.1), it follows from (1.2.5), (1.2.3), (2.2.3) and (2.2.4) that $\tilde{S}=-\frac{1}{c}(\Delta_0+\tilde{A}_0+nc1_{\Gamma(N(M))})$, where $1_{\Gamma(N(M))}$ is the identity transformation of $\Gamma(N(M))$. Hence we have by (3.1.7) and (3.1.8)

$$\tilde{S}\tilde{f}=\iota\Big(-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_if+2\sum_{i=1}^{n+p}(d\rho(E_i)(E_if))^N+\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N-\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^T\}^N-c_\rho f\Big)\Big).$$

Applying (3.2.3), we have

$$\begin{aligned}\sum_{i=1}^{n+p}&\{d\rho(E_i)(d\rho(E_i)f)^T\}^N\\&=\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)\}^N-\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N-\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N\\&=-ncf-\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N-\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N.\end{aligned}$$

In the above equation $(d\rho(E_i)f)^0$ denotes the V^0-component of $d\rho(E_i)f$ with respect to the orthogonal decomposition (3.1.1). Since $d\rho(\mathfrak{g})v_0=V^T$, we have $\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^0\}^N=0$. Hence we have

$$Sf=-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_if+2\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N$$
$$+2\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N-2c_\rho f\Big).$$

Q.E.D.

REMARK 3.3.1. It follows from Remark 3.1.1, (3.1.9), (3.1.12) and (3.1.14) that $\sum_{i=1}^{n+p}E_iE_if$, $\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N$, $\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N\in C^\infty(G;V^N)_K$ for $f\in C^\infty(G;V^N)_K$. Moreover each of the above three operators is commutative with L_x for all $x\in G$.

We define an operator S_1: $C^\infty(G;V^N)_K\to C^\infty(G;V^N)_K$ by

$$S_1f=\sum_{i=1}^{n+p}\{d\rho(E_i)(E_if)\}^N+\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N$$
$$\text{for } f\in C^\infty(G;V^N)_K.$$

By Proposition 3.1.1 the operator S_1 corresponds to a first order differential operator on $N(M)$. We denote by $\tilde{S}_1$ the corresponding differential operator on $N(M)$. If $S_1=0$, the operator S reduces to the simple operator

$$-\frac{1}{c}\Big(\sum_{i=1}^{n+p}E_iE_i-2c_\rho 1_{C^\infty(G;V^N)_K}\Big),$$

where $1_{C^\infty(G;V^N)_K}$ is the identity transformation of $C^\infty(G;V^N)_K$. The following lemma provides a sufficient condition for $S_1=0$. In fact this condition is also necessary (see Proposition 4.2.2).

Lemma 3.3.1. *If $(d\rho(X)v)^N=0$ for $X\in\mathfrak{p}$ and $v\in V^N$, then we have $S_1=0$.*

Proof. Choose an orthonormal basis $\{E_1,\cdots,E_{n+p}\}$ of $\mathfrak{g}$ such that $\{E_1,\cdots,E_n\}$ (resp. $\{E_{n+1},\cdots,E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. of $\mathfrak{k}$). We have for $x\in G$, $f\in C^\infty(G;V^N)_K$ and E_i, $i=n+1,\cdots,n+p$,

$$\begin{aligned}(E_if)(x)&=\lim_{t\to0}\frac{1}{t}(f(x(\exp tE_i))-f(x))\\&=\lim_{t\to0}\frac{1}{t}(\rho(\exp-tE_i)f(x)-f(x))\\&=-d\rho(E_i)f(x).\end{aligned}$$

Hence

$$S_1 f=\sum_{i=1}^{n+p}\{d\rho(E_i)(E_i f)\}^N+\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N$$
$$=\sum_{i=1}^{n}\{d\rho(E_i)(E_i f)\}^N-\sum_{i=n+1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)\}^N$$
$$+\sum_{i=1}^{n}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N+\sum_{i=n+1}^{n+p}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N .$$

Since V^N is invariant under $\rho(k)$ for $k\in K$, we have $(d\rho(E_i)f)^N=d\rho(E_i)f$, $i=n+1,\cdots,n+p$. Therefore we have

$$S_1 f=\sum_{i=1}^{n}\{d\rho(E_i)(E_i f)\}^N+\sum_{i=1}^{n}\{d\rho(E_i)(d\rho(E_i)f)^N\}^N .$$

Thus we obtain the proposition. Q.E.D.

REMARK 3.3.2. In the following cases the operator S_1 vanishes.

(1) The case of the minimal isometric immersion of S^n induced from the representation ρ_2, which is defined as follows: When $(G, K)=(SO(n+1), SO(n))$, the highest weight ϕ_1 of the canonical representation of $SO(n+1)$ has the property of Λ_0 in the proof of Lemma 3.2.2. Our representation ρ_2 is the real spherical representation whose complexification has the highest weight $2\phi_1$ (Remark 3.2.2).

(2) The cases of minimal symmetric R-spaces (see Nagura [8]), which include (1) as a special case.

3.4. Let N be a connected Riemannian manifold and $\tilde{N}$ the universal Riemannian covering manifold of N. Then we have by the universal property

Lemma 3.4.1. *For each isometry $x\in I(N)$ there exists an isometry $\tilde{x}\in I(\tilde{N})$ such that $\pi\circ\tilde{x}=x\circ\pi$, where $\pi\colon \tilde{N}\to N$ is the covering map.*

In this subsection we assume that G acts on M almost effectively. This means that $\mathfrak{k}$ does not contain any trivial ideals of $\mathfrak{g}$.

Proposition 3.4.2. *Let $\tilde{M}$ be the universal Riemannian covering manifold of M. If the equivariant minimal isometric immersion $F\colon (M, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho(x)F(o)$, is full and if dim G=dim $I(\tilde{M})$, then the Killing nullity of F is equal to $\frac{m(m-1)}{2}-\dim G$. Here $m=\dim V$.*

Proof. Let $I^0(S, M)$ be the identity component of $I(S, M)$. By the argument in subsection 2.3 it is sufficient to show that $\dim I^0(S, M)=\dim G$. It is trivial that $I^0(S, M)$ contains $\rho(G)$. Put $K'=\{x\in G;\ \rho(x)F(o)=F(o)\}$. Since F is an immersion, $\dim K'=\dim K$ and hence the Lie algebra of K' coincides with $\mathfrak{k}$. Therefore G acts on V almost effectively and we have

$$(3.4.1) \qquad \dim \rho(G) = \dim G .$$

Since the image $F(M)$ of M is the orbit of G through $F(o)$, $F(M)$ is a regular submanifold of S. Let $I^0(F(M))$ be the identity component of $I(F(M))$, the group of all isometries of the Riemannian manifold $F(M)$. Since F is full, we may consider $\rho(G)$ as a closed subgroup of $I^0(F(M))$. It follows from Lemma 3.4.1, the assumption of the proposition and (3.4.1) that

$$\dim I^0(F(M)) \leqq \dim I(\tilde{M}) = \dim \rho(G) .$$

Therefore we have

$$I^0(F(M)) = \rho(G) .$$

Let A be an element of $I^0(S, M)$. Since $F(M)$ is a regular submanifold of S, A induces an isometry of $F(M)$, which is contained in $I^0(F(M))$. Then there exists an element $x \in G$ such that the actions $\rho(x)$ and A coincide on $F(M)$. Since F is full, we have $A = \rho(x)$. Therefore $I^0(S, M)$ coincides with $\rho(G)$. Thus we obtain the proposition. Q.E.D.

REMARK 3.4.1. The condition $\dim G = \dim I(\tilde{M})$ is satisfied, when the pair (G, K) is an almost effective Riemannian symmetric pair and when G is semi-simple.

4. Invariant differential operators

4.1. Let G be a connected Lie group and K a closed subgroup of G. We assume that the quotient space $M = G/K$ is reductive, i.e. the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{k}$ of K and an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$. We identify $\mathfrak{p}$ with the tangent space $T_o(M)$ at the origin $o \in M$.

Let ϕ: $K \to GL(U)$ be a real (or complex) representation and put $\xi = G \times_K U$. For each $x \in G$ we define an automorphism α_x: $\xi \to \xi$ by

$$\alpha_x(y \circ u) = xy \circ u \qquad \text{for } y \in G \text{ and } u \in U .$$

We also denote by α_x the automorphism α_x of $\Gamma(\xi)$, the space of all C^∞ cross-sections of ξ, defined by $(\alpha_x \tilde{f})(yK) = \alpha_x(\tilde{f}(x^{-1}yK))$ for $\tilde{f} \in \Gamma(\xi)$ and $y \in G$. We have for $\tilde{f} \in \Gamma(\xi)$, $\tilde{a} \in C^\infty(M)$ and $x, y \in G$

$$\begin{aligned}(\alpha_x(\tilde{a}\tilde{f}))(yK) &= \alpha_x(\tilde{a}(x^{-1}yK)\tilde{f}(x^{-1}yK)) \\ &= \tilde{a}(x^{-1}yK)\alpha_x(\tilde{f}(x^{-1}yK)) \\ &= (\tau_{x^{-1}}{}^*\tilde{a})(yK)(\alpha_x\tilde{f})(yK) .\end{aligned}$$

Hence we obtain

$$(4.1.1) \qquad \alpha_x(\tilde{a}\tilde{f}) = (\tau_{x^{-1}}{}^*\tilde{a})(\alpha_x\tilde{f}) .$$

Put

$$C^\infty(G;\,U)_K = \left\{ \begin{array}{l} f\colon G\to U,\quad C^\infty \text{ mapping}; f(xK)=\phi(k^{-1})f(x) \\ \qquad\qquad\qquad\qquad \text{for } x\in G \text{ and } k\in K \end{array} \right\}.$$

Then as in subsection 2.1 we have the isomorphism $\iota\colon C^\infty(G;\,U)_K\to\Gamma(\xi)$, $(\iota(f))(xK)=x\circ f(x)$, and the following commutative diagram:

$$\begin{array}{ccc} C^\infty(G;\,U)_K & \xrightarrow{\ \iota\ } & \Gamma(\xi) \\ \downarrow L_x & & \downarrow \alpha_x \\ C^\infty(G;\,U)_K & \xrightarrow{\ \iota\ } & \Gamma(\xi)\,. \end{array}$$

We denote by $\tilde{f}$ the image $\iota(f)$ of f. Put

$$C^\infty(G)_K = \{a\in C^\infty(G)\colon a(xk)=a(x) \qquad \text{for } x\in G \text{ and } k\in K\}\,.$$

Then the pull back $\pi^*\colon C^\infty(M)\to C^\infty(G)_K$ is an isomorphism, where $\pi\colon G\to M= G/K$ is the natural projection. We denote by $\tilde{a}$ the inverse image $\pi^{*-1}(a)$ of $a\in C^\infty(G)_K$. For $f\in C^\infty(G;\,U)_K$ and $a\in C^\infty(G)_K$ we have $af\in C^\infty(G;\,U)_K$ and

$$(4.1.2)\qquad \iota(af)=\tilde{a}\tilde{f}\,.$$

Let $\psi\colon K\to GL(V)$ be a real (or complex) representation and put $\eta=G\times_K V$. We define automorphisms $\beta_x\colon \eta\to\eta$ and $\beta_x\colon \Gamma(\eta)\to\Gamma(\eta)$ in the same manner as for ξ. Let $\mathrm{Diff}_h(\xi,\,\eta)$ be the set of all h-th order differential operators from ξ to η. A differential operator $D\in\mathrm{Diff}_h(\xi,\,\eta)$ is said to be *invariant,* if $D\circ\alpha_x=\beta_x\circ D$ for every $x\in G$. Let D be an h-th order differential operator from ξ to η. Then for each $p\in M$ the symbol $\sigma_h(D)$ of D defines an h-th order homogeneous polynomial mapping from the cotangent space $T_p{}^*(M)$ to $\mathrm{Hom}(\xi_p,\,\eta_p)$ (cf. Palais [9] p. 62), where $\mathrm{Hom}(\xi_p,\,\eta_p)$ denotes the vector space of all linear mappings from ξ_p to η_p.

Let ${}^t(d\tau_x)$ be the transposed mapping of the differential $d\tau_x$ of τ_x, $x\in G$. Then we have for $\tilde{a}\in C^\infty(M)$ and $x,\,y\in G$

$$(4.1.3)\qquad d(\tau_{x^{-1}}{}^*\tilde{a})_{xyK} = \tau_{x^{-1}}{}^*(d\tilde{a})_{yK} = {}^t(d\tau_{x^{-1}})(d\tilde{a})_{yK}\,.$$

Proposition 4.1.1. *Assume that a differential operator $D\in\mathit{Diff}_h(\xi,\,\eta)$ is invariant. Then we have for $x,\,y\in G$, $v\in T_{yK}{}^*(M)$ and $\omega\in\xi_{yK}$*

$$(4.1.4)\qquad \sigma_h(D)({}^t(d\tau_{x^{-1}})v)(\alpha_x(\omega)) = \beta_x(\sigma_h(D)(v)(\omega))\,.$$

Proof. Take $\tilde{a}\in C^\infty(M)$ (resp. $\tilde{f}\in\Gamma(\xi)$) which satisfies $\tilde{a}(yK)=0$ and $d\tilde{a}_{yK}=v$(resp. $\tilde{f}(yK)=\omega$). Then we have

$$(\tau_{x^{-1}}{}^*\tilde{a})(xyK)=\tilde{a}(yK)=0$$

and

$$(\alpha_x\tilde{f})(xyK) = \alpha_x(\tilde{f}(yK)) = \alpha_x(\omega)\,.$$

By (4.1.3) we have

$$d(\tau_{x^{-1}}{}^*\tilde{a})_{xyK} = {}^t(d\tau_{x^{-1}})(d\tilde{a})_{yK} = {}^t(d\tau_{x^{-1}})v\,.$$

Applying (4.1.1), we have

$$\alpha_x\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big) = \frac{1}{h!}(\tau_{x^{-1}}{}^*\tilde{a})^h(\alpha_x\tilde{f})\,.$$

Hence it follows from the definition of the symbol $\sigma_h(D)$ and the invariance of D that

$$\begin{aligned}\sigma_h(D)({}^t(d\tau_{x^{-1}})v)(\alpha_x(\omega)) &= D\Big(\frac{1}{h!}(\tau_{x^{-1}}{}^*\tilde{a})^h(\alpha_x\tilde{f})\Big)(xyK)\\ &= D\Big(\alpha_x\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big)\Big)(xyK)\\ &= \beta_x\Big(D\Big(\frac{1}{h!}\tilde{a}^h\tilde{f}\Big)(yK)\Big)\\ &= \beta_x(\sigma_h(D)(v)(\omega))\,.\end{aligned}$$

Q.E.D.

Corollary 1. *Assume that $D\in Diff_h(\xi, \eta)$ is invariant. If $\sigma_h(D)_o=0$, then $\sigma_h(D)=0$.*

Proof. The corollary is an immediate consequence of the proposition. Q.E.D.

If D is a first order differential operator, the symbol $\sigma_1(D)_p$, $p\in M$, defines a bilinear mapping from $T_p{}^*(M)\times\xi_p$ to η_p. We also denote by $\sigma_1(D)_p$ the linear mapping from $T_p{}^*(M)\otimes\xi_p$ to η_p induced from the bilinear mapping $\sigma_1(D)_p$. We have easily the following corollary.

Corollary 2. *If a differential operator $D\in Diff_1(\xi, \eta)$ is invariant, then the linear mapping $\sigma_1(D)_o$: $\mathfrak{p}^*\otimes U=T_o{}^*(M)\otimes\xi_o\to\eta_o=V$ is a K-module homomorphism, i.e. for each $k\in K$*

$$\sigma_1(D)_o\circ{}^t\mathrm{Ad}_{\mathfrak{p}}(k^{-1})\otimes\phi(k) = \psi(k)\circ\sigma_1(D)_o\,,$$

where the action $Ad_{\mathfrak{p}}(k)$ is the restriction of $Ad(k)$ to $\mathfrak{p}$ and $\mathfrak{p}^*$ denotes the dual space of $\mathfrak{p}$.

4.2. In this subsection the assumptions and the notation are the same as in subsection 3.3.

The differential operator $\tilde{S}_1$ on $N(M)$ defined in subsection 3.3 is invariant by Remark 3.3.1. Choose an orthonormal basis $\{E_1, \cdots, E_{n+p}\}$ of $\mathfrak{g}$ such that $\{E_1, \cdots, E_n\}$ (resp. $\{E_{n+1}, \cdots, E_{n+p}\}$) is an orthonormal basis of $\mathfrak{p}$ (resp. $\mathfrak{k}$). Let $\{\phi_1, \cdots, \phi_{n+p}\}$ be the basis of the dual space of $\mathfrak{g}$ dual to $\{E_1, \cdots, E_{n+p}\}$. We consider $\{\phi_1, \cdots, \phi_n\}$ as a basis of $T_o^*(M)$. Then we obtain

Lemma 4.2.1. *We have for $\phi_i \in T_o^*(M)$, $i=1, \cdots, n$, and $v \in V^N$*

$$\sigma_1(\tilde{S}_1)(\phi_i)(v) = (d\rho(E_i)v)^N . \tag{4.2.1}$$

Proof. Let N be an open neighborhood of $o \in M$ such that $\pi^{-1}(N)$ is diffeomorphic to $N \times K$, where $\pi: G \to G/K$ is the natural projection. Let $(x_1, \cdots, x_n)$ be the local coordinate system on N defined by $x_i(\exp(\sum_{j=1}^n s_jE_j)K) = s_i$ for $-\varepsilon < s_i < \varepsilon$, where ε is some positive number. For $v \in V^N$ we define a V^N-valued C^∞ function α_v on $\pi^{-1}(N)$ by

$$\alpha_v(\exp(\sum_{j=1}^n s_jE_j)k) = \rho(k^{-1})v \qquad \text{for } k \in K .$$

Taking $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$, put

$$N' = \{\exp(\sum_{j=1}^n s_jE_j)K;\ -\varepsilon' < s_j < \varepsilon'\} .$$

Then there exists a V^N-valued C^∞ function α'_v on G such that $\alpha_v = \alpha'_v$ on $\pi^{-1}(N')$. We define a V^N-valued C^∞ function β_v on G by

$$\beta_v(x) = \int_K \rho(k)\alpha'_v(xk)dk \qquad \text{for } x \in G ,$$

where dk denotes the normalized Haar measure of K. Then $\beta_v \in C^\infty(G; V^N)_K$. In fact we have for $x \in G$ and $h \in K$

$$\begin{aligned}
\beta_v(xh) &= \int_K \rho(k)\alpha'_v(xhk)dk \\
&= \int_K \rho(h^{-1}(hk))\alpha'_v(xhk)dk \\
&= \rho(h^{-1})\int_K \rho(hk)\alpha'_v(xhk)dk \\
&= \rho(h^{-1})\beta_v(x) .
\end{aligned}$$

We have for $x = \exp(\sum_{j=1}^n s_jE_j)h\ (-\varepsilon' < s_j < \varepsilon')$

$$\begin{aligned}
\beta_v(x) &= \rho(h^{-1})\int_K \rho(k)\alpha'_v(\exp(\sum_{j=1}^n s_jE_j)k)dk \\
&= \rho(h^{-1})\int_K v\,dk = \rho(h^{-1})v .
\end{aligned}$$

Therefore $\tilde{\beta}_v(o)=\iota(e\circ\beta_v(e))=v$. Take $\tilde{f}_i\in C^\infty(M)$ such that $\tilde{f}_i=x_i$ on N' and then take $f_i\in C^\infty(G)_K$ such that $\pi^*\tilde{f}_i=f_i$. Then $\tilde{f}_i(o)=0$ and $(d\tilde{f}_i)_o=\phi_i$. We have by (4.1.2)

$$\begin{aligned}\sigma_1(\tilde{S}_1)(\phi_i)(v) &= \tilde{S}_1(\tilde{f}_i\tilde{\beta}_v)(o) = \tilde{S}_1(\iota(f_i\beta_v))(o)\\ &= \iota(S_1(f_i\beta_v))(o) = S_1(f_i\beta_v)(e)\\ &= \sum_{j=1}^{n+p}\{d\rho(E_j)(E_j(f_i\beta_v))(e)\}^N .\end{aligned}$$

We have by (3.1.13)

$$\begin{aligned}&\sum_{j=1}^{n+p}\{d\rho(E_j)(E_j(f_i\beta_v))(e))\}^N\\ &= \sum_{j=1}^{n+p}\{d\rho(\mathrm{Ad}(*^{-1})E_j)(\hat{E}_j(f_i\beta_v))(e)\}^N\\ &= \sum_{j=1}^{n+p}\{d\rho(E_j)\{(\hat{E}_jf_i)(e)\beta_v(e)+f_i(e)(\hat{E}_j\beta_v)(e)\}\}^N\\ &= (d\rho(E_i)v)^N .\end{aligned}$$

This proves (4.2.1). Q.E.D.

Proposition 4.2.2. *The following three conditions are equivalent*:

(1) $(d\rho(X)v)^N=0$ *for* $X\in\mathfrak{p}$ *and* $v\in V^N$.

(2) $\tilde{S}_1=0$.

(3) $\sigma_1(\tilde{S}_1)=0$.

Proof. Lemma 3.3.1 shows that (1) implies (2). It is evident that (2) implies (3). Lemma 4.2.1 shows that (3) implies (1). Q.E.D.

The vector spaces V^N and $\mathfrak{p}\otimes V^N$ are K-modules in a natural manner. Since K is compact, we may decompose V^N(resp. $\mathfrak{p}\otimes V^N$) into a direct sum of irreducible K-modules.

Proposition 4.2.3. *If any irreducible component of* $\mathfrak{p}\otimes V^N$ *is not isomorphic to any irreducible component of* V^N, *then* $S_1=0$.

Proof. Since the representation $\mathrm{Ad}_\mathfrak{p}$: $K\rightarrow GL(\mathfrak{p})$ is orthogonal, the contragradient representation of $\mathrm{Ad}_\mathfrak{p}$ coincides with itself. Hence it follows from Corollary 2 for Proposition 4.1.1 and Schur's lemma (cf. Chevalley [2] p. 182) that $\sigma_1(\tilde{S}_1)_o=0$. Therefore we have our proposition by the above proposition. Q.E.D.

5. Reduction to the finite dimensional eigenvalue problems

5.1. Let G be a compact connected Lie group and K a closed subgroup of G. We denote by M the quotient space G/K. The G-invariant Riemannian metric $\langle\ ,\ \rangle$ on M is the same as in subsection 2.1. Let $D(G)$ be the set of equivalence classes of complex irreducible representations of G. For a complex irreducible representation $\sigma\colon G\to GL(W)$ we denote by $\sigma^*\colon G\to GL(W^*)$ the contragradient representation of σ on the dual space W^* of W. Let $C^\infty(G)^C$ be the space of $\boldsymbol{C}$-valued C^∞ functions on G. We define actions L_x and R_x of G on $C^\infty(G)^C$ by the followings:

$$(L_xf)(y)=f(x^{-1}y),\ (R_xf)(y)=f(yx)\qquad \text{for } f\in C^\infty(G)^C.$$

For $[\sigma]\in D(G)$ let $\mathfrak{D}^L{}_{[\sigma]}(G)$ (resp. $\mathfrak{D}^R{}_{[\sigma]}(G)$) be the subspace of $C^\infty(G)^C$ generated by G-submodules of $C^\infty(G)^C$ which are isomorphic to σ by the G-action L(resp. by the G-action R). Then we have $\mathfrak{D}^L{}_{[\sigma]}(G)=\mathfrak{D}^R{}_{[\sigma^*]}(G)$.

Let U be a complex vector space with a Hermitian inner product $\langle\ ,\ \rangle$ and $C^\infty(G;\ U)$ the space of U-valued C^∞ functions on G. We also denote by L_x (resp. R_x) the action of G on $C^\infty(G;\ U)$: $(L_xf)(y)=f(x^{-1}y)$(resp. $(R_xf)(y)=f(yx)$) for $f\in C^\infty(G;\ U)$. Note that our L_x(resp. R_x) is nothing but the tensor product $L_x\otimes 1_U$ (resp. $R_x\otimes 1_U$) on $C^\infty(G)^C\otimes U=C^\infty(G;\ U)$. Let $\sigma\colon G\to GL(W)$ be a complex irreducible representation. We define a multilinear mapping $\Phi^\sigma\colon W\times W^*\times U\to C^\infty(G;\ U)$ by

$$\Phi^\sigma(w,\ \omega,\ u)(x)=\omega(\sigma^{-1}(x)w)u\qquad \text{for } w\in W,\ \omega\in W^* \text{ and } u\in U.$$

We also denote by Φ^σ the induced linear mapping of $W\otimes W^*\otimes U$ to $C^\infty(G;\ U)$. We define an action $L_\sigma(x)$(resp. $R_{\sigma^*}(x)$) of G on $W\otimes W^*\otimes U$ by $L_\sigma(x)=\sigma(x)\otimes 1_{W^*}\otimes 1_U$(resp. $R_{\sigma^*}(x)=1_W\otimes\sigma^*(x)\otimes 1_U$). Then we have $\Phi^\sigma\circ L_\sigma(x)=L_x\circ\Phi^\sigma$ and $\Phi^\sigma\circ R_{\sigma^*}(x)=R_x\circ\Phi^\sigma$ for every $x\in G$.

Theorem 5.1.1 (cf. Takeuchi [12] p. 15). (1) *We consider* $W\otimes W^*\otimes U$ *(resp.* $C^\infty(G;\ U)$*) as a G-module with the G-action* L_σ*(resp. L). Then* Φ^σ *is a G-module isomorphism of* $W\otimes W^*\otimes U$ *onto* $\mathfrak{D}^L{}_{[\sigma]}(G)\otimes U$.

(2) *We consider* $W\otimes W^*\otimes U$*(resp.* $C^\infty(G;\ U)$*) as a G-module with the G-action* R_{σ^*}*(resp. R). Then* Φ^σ *is a G-module isomorphism of* $W\otimes W^*\otimes U$ *onto* $\mathfrak{D}^R{}_{[\sigma^*]}(G)\otimes U=\mathfrak{D}^L{}_{[\sigma]}(G)\otimes U$.

Let $\phi\colon K\to GL(U)$ be a unitary representation and $\langle\ ,\ \rangle$ the Hermitian inner product on U. Put $\xi=G\times_K U$. Then ξ has a natural Hermitian fibre metric, which will be also denoted by $\langle\ ,\ \rangle$. We define a subspace $C^\infty(G;\ U)_K$ of $C^\infty(G;\ U)$ by

$$C^\infty(G;\ U)_K=\left\{f\in C^\infty(G;\ U);\ f(xk)=\phi(k^{-1})f(x)\ \text{ for } x\in G \text{ and } k\in K\right\}.$$

We identify the space $\Gamma(\xi)$ of C^∞ cross-sections of ξ with $C^\infty(G;\ U)_K$. Then $C^\infty(G;\ U)_K$ is a G-module with the G-action L. We define a Hermitian inner product $\langle\ ,\ \rangle$ on $C^\infty(G;\ U)_K$ as follows:

$$\langle f, g\rangle = \int_G \langle f(x), g(x)\rangle dx\,,$$

where dx is the normalized Haar measure of G. Then we have

$$\langle L_x f, L_x g\rangle = \langle f, g\rangle \qquad \text{for every } x\in G\,.$$

The space $C^\infty(G; U)_K$ is a pre-Hilbert space. We denote by $L^2(\xi)$ the completion of $C^\infty(G; U)_K$. Identifying as $C^\infty(G; U)=C^\infty(G)^C\otimes U$, we define an action J of K on $C^\infty(G; U)$ by $J(k)=R_k\otimes\phi(k)$ for $k\in K$. Then we have

$$(5.1.1) \qquad C^\infty(G; U)_K = \{f\in C^\infty(G; U); J(k)f=f \qquad \text{for } k\in K\}\,.$$

For a complex irreducible representation σ: $G\rightarrow GL(W)$, we define an action J_σ of K on $W\otimes W^*\otimes U$ by $J_\sigma(k)=1_W\otimes\sigma^*(k)\otimes\phi(k)$. Then we have

$$(5.1.2) \qquad \Phi^\sigma\circ J_\sigma(k) = J(k)\circ\Phi^\sigma \qquad \text{for every } k\in K\,.$$

Let $\mathfrak{o}_{[\sigma]}(\xi)$ be the subspace of $C^\infty(G; U)_K$ generated by all G-submodules of $C^\infty(G; U)_K$ which are isomorphic to W. Then $\mathfrak{o}_{[\sigma]}(\xi)$ is a G-submodule of $\mathfrak{o}^L{}_{[\sigma]}(G)\otimes U$. Put

$$\mathfrak{o}(\xi) = \{f\in C^\infty(G; U)_K;\ \dim\{L_x f\colon x\in G\}_C<\infty\}\,,$$
$$D(G; K, \phi) = \left\{[\sigma]\in D(G);\ \sigma^*|_K\otimes\phi \text{ contains a trivial representation}\right\},$$

and

$$(W^*\otimes U)_0 = \{\alpha\in W^*\otimes U;\ (\sigma^*(k)\otimes\phi(k))(\alpha) = \alpha \qquad \text{for } k\in K\}\,.$$

Then $W\otimes(W^*\otimes U)_0$ is a G-module with the G-action L_σ. We have the following Peter-Weyl theorem for vector bundles.

Theorem 5.1.2. (Bott [1] p. 173). (1) *The G-module isomorphism* Φ^σ: $W\otimes W^*\otimes U\rightarrow\mathfrak{o}^L{}_{[\sigma]}(G)\otimes U$ *in* (1) *of Theorem* 5.1.1 *induces a G-module isomorphism of* $W\otimes(W^*\otimes U)_0$ *onto* $\mathfrak{o}_{[\sigma]}(\xi)$.

(2) *We have the following orthogonal decompositions*:

$$\mathfrak{o}(\xi) = \sum_{[\sigma]\in D(G;\,K,\phi)}\mathfrak{o}_{[\sigma]}(\xi) \ (\textit{algebraic direct sum}),$$
$$L^2(\xi) = \sum_{[\sigma]\in D(G;\,K,\phi)}\mathfrak{o}_{[\sigma]}(\xi) \ (\textit{direct sum as Hilbert space}).$$

We have the following theorem for an invariant differential operator.

Theorem 2. *Let D be an invariant differential operator on* ξ *and consider it as an operator on* $C^\infty(G; U)_K$*(see the commutative diagram in subsection* 4.1). *Let* σ: $G\rightarrow GL(W)$ *be an irreducible representation with* $[\sigma]\in D(G; K, \phi)$. *Then D leaves* $\mathfrak{o}_{[\sigma]}(\xi)$ *invariant and there exists a unique linear mapping* D_σ *of* $(W^*\otimes U)_0$ *such that*

$$D\circ\Phi^\sigma = \Phi^\sigma\circ(1_W\otimes D_\sigma)\,.$$

Proof. For $f\in\mathfrak{v}(\xi)$ the subspace $\{L_xDf\colon x\in G\}_C=\{DL_xf\colon x\in G\}_C$ of $C^\infty(G, U)$ is finite dimensional, and hence D leaves $\mathfrak{v}(\xi)$ invariant. It follows from Schur's lemma that every $\mathfrak{v}_{[\sigma]}(\xi)$ is invariant under D. Let D' be the linear mapping of $W\otimes(W^*\otimes U)_0$ corresponding to $D|\mathfrak{v}_{[\sigma]}(\xi)$ by the G-module isomorphism $\Phi^\sigma\colon W\otimes(W^*\otimes U)_0\to\mathfrak{v}_{[\sigma]}(\xi)$. Let $\{\alpha_1, \cdots, \alpha_{m_\sigma}\}$ be a basis of $(W^*\otimes U)_0$. We define linear mappings $f^i{}_j$, $i, j=1, 2, \cdots, m_\sigma$, of W as follows:

$$D'(w\otimes\alpha_j) = \sum_{i=1}^{m_\sigma} f^i{}_j(w)\otimes\alpha_i \qquad \text{for } w\in W\,.$$

Then we have for $x\in G$

$$\begin{aligned} D'(L_\sigma(x)(w\otimes\alpha_j)) &= D'(\sigma(x)w\otimes\alpha_j) \\ &= \sum_{i=1}^{m_\sigma} f^i{}_j(\sigma(x)w)\otimes\alpha_i\,. \end{aligned}$$

On the other hand we have

$$\begin{aligned} D'(L_\sigma(x)(w\otimes\alpha_j)) &= L_\sigma(x)(D'(w\otimes\alpha_j)) \\ &= \sum_{i=1}^{m_\sigma} \sigma(x)f^i{}_j(w)\otimes\alpha_i\,. \end{aligned}$$

Hence

$$f^i{}_j(\sigma(x)w) = \sigma(x)f^i{}_j(w), \qquad i, j = 1, \cdots, m_\sigma\,.$$

It follows from Schur's lemma that there exist complex numbers $c^i{}_j$, $i, j= 1, \cdots, m_\sigma$, such that $f^i{}_j=c^i{}_j1_W$. Hence we have

$$D'(w\otimes\alpha_j) = w\otimes(\sum_{i=1}^{m_\sigma} c^i{}_j\alpha_i)\,.$$

A linear mapping D_σ of $(W^*\otimes U)_0$ defined by

$$D_\sigma\alpha_j = \sum_{i=1}^{m_\sigma} c^i{}_j\alpha_i\,, \qquad j = 1, \cdots, m_\sigma\,,$$

is the required one. Q.E.D.

REMARK 5.1.1. If an invariant differential operator D on ξ is self-adjoint with respect to the inner product $\langle\ ,\ \rangle$, each $D|\mathfrak{v}_{[\sigma]}(\xi)$ is diagonalizable. If furthermore D is elliptic, every eigensection of D belongs to $\mathfrak{v}(\xi)$. Thus the problem of computing the spectra of D is reduced to the study of the eigenvalues of D_σ for each $[\sigma]\in D(G; K, \phi)$.

5.2. In this subsection the assumptions and the notation are the same as in subsection 3.3. Moreover we assume that the minimal isometric immersion $F\colon (M, c\langle\ ,\ \rangle)\to S$ is full. We also denote by $\langle\ ,\ \rangle$ the Hermitian inner product on V^C, the complexification of V, which is the extension of the inner product $\langle\ ,\ \rangle$ on V. Then the orthogonal representation $\rho\colon G\to GL(V)$ extends to the unitary representation $\rho\colon G\to GL(V^C)$. Let $(V^N)^C$ be the subspace of V^C generated by V^N and $\rho^N\colon K\to GL((V^N)^C)$ the unitary representation induced from $\rho\colon G\to GL(V^C)$. We may identify the complexification $\Gamma(N(M))^C$ of $\Gamma(N(M))$ with $C^\infty(G; (V^N)^C)_K$. Let $(V^T)^C$(resp. $(V^0)^C$) be the complex linear subspace of V^C generated by V^T(resp. V^0). We have the direct sum decomposition $V^C=(V^0)^C+(V^T)^C+(V^N)^C$. For $v\in V^C$ we denote by v^N the $(V^N)^C$-component of v with respect to this decomposition of V^C.

Let $\sigma\colon G\to GL(W)$ be a complex irreducible representation with $[\sigma]\in D(G; K, \rho^N)$. Put

$$(W^*\otimes(V^N)^C)_0 = \{\omega\in W^*\otimes(V^N)^C;\ (\sigma^*(k)\otimes\rho^N(k))(\omega)=\omega \text{ for } k\in K\}.$$

Let S' be the linear mapping of $W\otimes(W^*\otimes(V^N)^C)_0$ corresponding to $S|_{\mathfrak{D}_{[\sigma]}(N(M)^C)}$ by the G-isomorphism $\Phi^\sigma\colon W\otimes(W^*\otimes(V^N)^C)_0\to\mathfrak{D}_{[\sigma]}(N(M)^C)$, where $N(M)^C$ denotes the complexification of the normal bundle $N(M)$. Then we have by Theorem 1 and (2) of Theorem 5.1.1

$$\begin{aligned} S' = -\frac{1}{c}\Big(1_W\otimes\{(c_{\sigma^*}-2c_\rho)1_{W^*\otimes(V^N)^C}+2\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N \\ +2\sum_{i=1}^{n+p} 1_{W^*}\otimes\{d\rho(E_i)(d\rho(E_i)*)^N\}^N\}\Big)\,, \end{aligned}$$

where c_{σ^*} is the scalar determined by the Casimir operator $\sum_{i=1}^{n+p} d\sigma^*(E_i)d\sigma^*(E_i)$ of σ^*. Let c_σ be the scalar determined by the Casimir operator $\sum_{i=1}^{n+p} d\sigma(E_i)d\sigma(E_i)$ of σ. Then $c_{\sigma^*}=c_\sigma$. Put

$$\begin{aligned} S_\sigma = -\frac{1}{c}\{(c_\sigma-2c_\rho)1_{W^*\otimes(V^N)^C}+2\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N \\ +2\sum_{i=1}^{n+p} 1_{W_*}\otimes\{d\rho(E_i)(d\rho(E_i)*)^N\}^N\}\,. \end{aligned}$$

Then it follows from Remark 5.1.1, Theorem 2 and (2) of Theorem 5.1.2 that the problem of computing the spectra of $\tilde{S}$ is reduced to the eigenvalue problems of the linear mappings S_σ of $(W^*\otimes(V^N)^C)_0$ with $[\sigma]\in D(G; K, \rho^N)$.

Summarizing, we get the following theorem.

Theorem 3. *Let $F\colon (M, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho(x)F(o)$, be a full equivariant minimal isometric immersion of a compact symmetric space $M=G/K$ into a unit sphere S. For a complex irreducible representation $\sigma\colon G\to GL(W)$ with $[\sigma]\in D(G; K, \rho^N)$, let $\{\lambda_{\sigma;1},\cdots,\lambda_{\sigma;m_\sigma}\}$ be the eigenvalues of S_σ on $(W^*\otimes(V^N)^C)_0$. Then the spectra of the Jacobi differential operator $\tilde{S}$ are given by*

$$\bigcup_{[\sigma]\in D(G\,;\,K,\,\rho^N)} \{\underbrace{\lambda_{\sigma\,;\,1},\cdots,\lambda_{\sigma\,;\,1}}_{d_\sigma},\cdots,\underbrace{\lambda_{\sigma\,;\,m_\sigma},\cdots,\lambda_{\sigma\,;\,m_\sigma}}_{d_\sigma}\}\,,$$

where $d_\sigma = dim\ W$.

For a complex irreducible representation σ: $G\to GL(W)$ with $[\sigma]\in D(G; K, \rho^N)$, it follows from Remark 3.3.1 and Theorem 2 that each of the linear mappings $\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N$ and $\sum_{i=1}^{n+p} 1_{W^*}\otimes\{d\rho(E_i)(d\rho(E_i)*)^N\}^N$ leaves $(W^*\otimes(V^N)^C)_0$ invariant. For the study of the linear mapping S_σ it is important to study these linear mappings. We shall study these linear mappings.

Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$ and (,) the symmetric bilinear form on $\mathfrak{g}^C$ which is the $\boldsymbol{C}$-bilinear extension of the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Choose bases $\{F_1\cdots, F_{n+p}\}$ and $\{F'_1, \cdots, F'_{n+p}\}$ of $\mathfrak{g}^C$ with the property $(F_i, F'_j)=\delta_{ij}$. Let χ: $G\to GL(U)$ be an arbitrary unitary representation (not necessarily irreducible). We define a linear mapping $L(\chi, \rho)$ of $U\otimes V^C$ by

$$L(\chi, \rho) = \sum_{i=1}^{n+p} d\chi(F_i)\otimes d\rho(F'_i)\,.$$

The linear mapping $L(\chi, \rho)$ is independent of the choice of bases. In fact let $\{H_1, \cdots, H_{n+p}\}$ and $\{H'_1, \cdots, H'_{n+p}\}$ be bases of $\mathfrak{g}^C$ with $(H_i, H'_j)=\delta_{ij}$. Let $H_i=\sum_{k=1}^{n+p} a^k{}_i F_k$ and $H'_i=\sum_{h=1}^{n+p} b^i{}_h F'_h$, $i=1, \cdots, n+p$. Then we have

$$\delta_{ij} = (H_i, H'_j) = \sum_{k=1}^{n+p} a^k{}_i b^j{}_k\,.$$

Hence if we put $A=(a^i{}_j)_{i,j=1,\cdots,n+p}$ and $B=(b^i{}_j)_{i,j=1,\cdots,n+p}$, we have $B=A^{-1}$. Therefore we have

$$\begin{aligned}\sum_{i=1}^{n+p} d\chi(H_i)\otimes d\rho(H'_i) &= \sum_{k,h=1}^{n+p}\sum_{i=1}^{n+p} a^k{}_i b^i{}_h d\chi(F_k)\otimes d\rho(F'_h)\\ &= \sum_{k=1}^{n+p} d\chi(F_k)\otimes d\rho(F'_k)\,.\end{aligned}$$

We denote by $C_{\chi\otimes\rho}$(resp. C_χ and C_ρ) the Casimir operator of the representation $\chi\otimes\rho$ (resp. χ and ρ). Since $\sum_{i=1}^{n+p} d\chi(F_i)\otimes d\rho(F'_i)=\sum_{i=1}^{n+p} d\chi(F'_i)\otimes d\rho(F_i)$, we have

$$(5.2.1)\qquad 2L(\chi, \rho) = C_{\chi\otimes\rho} - C_\chi\otimes 1_{V^C} - 1_U\otimes C_\rho\,.$$

We obtain the following lemma by (5.2.1) and the fact that the Casimir operator commutes with the action of G.

Lemma 5.2.1. *We have*

$$(\chi\otimes\rho)(x)\circ L(\chi, \rho) = L(\chi, \rho)\circ(\chi\otimes\rho)(x) \qquad \textit{for } x\in G\,.$$

Put

$$(U\otimes V^C)_0=\{\omega\in U\otimes V^C;\ (\chi\otimes\rho)(k)\omega=\omega \quad \text{for } k\in K\}.$$

Then we have by the above lemma

$$L(\chi,\rho)((U\otimes V^C)_0)\subset(U\otimes V^C)_0. \tag{5.2.2}$$

Now we come back to our complex irreducible representation $\sigma: G\to GL(W)$. We denote by p_1 the projection to the first component of the following direct sum decomposition:

$$W^*\otimes V^C=(W^*\otimes(V^N)^C)+(W^*\otimes\{(V^T)^C+(V^0)^C\}).$$

Then we have

Lemma 5.2.2.

$$\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes(d\rho(E_i)*)^N=p_1\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes d\rho(E_i) \quad \textit{on } W^*\otimes V^C, \tag{5.2.3}$$

$$\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes d\rho(E_i)((W^*\otimes V^C)_0)\subset(W^*\otimes V^C)_0, \tag{5.2.4}$$

where $(W^*\otimes V^C)_0=\{\omega\in W^*\otimes V^C,\ (\sigma^*(k)\otimes\rho(k))\omega=\omega \quad \textit{for } k\in K\}$.

Proof. The first equality is trivial. Since $\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes d\rho(E_i)=L(\sigma^*,\rho)$, we have (5.2.4) by (5.2.2). Q.E.D.

Lemma 5.2.3. *We have*

$$\begin{aligned}&\rho(k)\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)v)^N\}^N \\ &=\sum_{i=1}^{n+p}(d\rho(E_i)\{d\rho(E_i)\rho(k)v\}^N)^N \quad \textit{for } k\in K \textit{ and } v\in V^C.\end{aligned} \tag{5.2.5}$$

Proof. For $k\in K$ the linear mapping $\rho(k)$ leaves $(V^N)^C$, $(V^T)^C$ and $(V^0)^C$ invariant respectively. Therefore we have

$$\begin{aligned}&\rho(k)\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)v)^N\}^N \\ &=\sum_{i=1}^{n+p}(\{\rho(k)d\rho(E_i)\rho(k^{-1})\}[\{\rho(k)d\rho(E_i)\rho(k^{-1})\}(\rho(k)v)]^N)^N \\ &=\sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(k)E_i)\{d\rho(\mathrm{Ad}(k)E_i)(\rho(k)v)\}^N)^N.\end{aligned}$$

Since $\{\mathrm{Ad}(k)E_1,\cdots,\mathrm{Ad}(k)E_{n+p}\}$ is an orthonormal basis of $\mathfrak{g}$, we have

$$\begin{aligned}&\sum_{i=1}^{n+p}(d\rho(\mathrm{Ad}(k)E_i)\{d\rho(\mathrm{Ad}(k)E_i)(\rho(k)v)\}^N)^N \\ &=\sum_{i=1}^{n+p}(d\rho(E_i)\{d\rho(E_i)(\rho(k)v)\}^N)^N.\end{aligned}$$

Q.E.D.

In the forthcoming papers we shall study the linear mappings

$$\sum_{i=1}^{n+p} d\sigma^*(E_i)\otimes d\rho(E_i): (W^*\otimes V^C)_0 \to (W^*\otimes V^C)_0$$

and

$$\sum_{i=1}^{n+p} \{d\rho(E_i)(d\rho(E_i)*)^N\}^N: (V^N)^C \to (V^N)^C.$$

These studies, together with Lemma 5.2.2 and Lemma 5.2.3, will give us informations on the linear mapping S_σ.

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