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ASYMPTOTIC BEHAVIOR OF HITTING RATES FOR ABSORBING STABLE MOTIONS IN A HALF SPACE

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1. Introduction

For a transient Brownian motion (B_t, P_x) in \mathbf{R}^d ($d \geq 3$) and the Lebesgue measure $m(dx) = dx$ on \mathbf{R}^d , under the σ -finite measure $P_m = \int dx P_x$, F. Spitzer [12] gave the asymptotic expansion of order 2 for $P_m(0 \leq T_A < t)$ as $t \rightarrow \infty$ related with the first hitting time T_A for a compact subset A of \mathbf{R}^d involving the capacity $C(A)$. Since $\{B_t\}$ is stationary under P_m , this result introduces limit theorems for an *equilibrium process*, which is a stationary independent Markov particle system, related with the number of particles hitting the compact set A . Moreover similar results were obtained by R.K. Gettoor [2] for rotation invariant α -stable processes in \mathbf{R}^d and by S.C. Port and C.J. Stone [11] for general Lévy processes. The higher order expansions were obtained by Le Gall [8] for Brownian motion in \mathbf{R}^d ($d \geq 3$) and by S.C. Port [9], [10] for general α -stable processes in \mathbf{R}^d .

In general for a Markov process which does not have an invariant measure, it is possible to realize a stationary Markov process with the same transition probability by extending the probability space and considering new paths which are born at random time ([1], [3] and [4]). The distribution (which may not be a probability measure in general) is called a *Kuznetsov measure* [7]. By using this measure we can construct a stationary Markov particle system, which is called an *equilibrium process with immigration*. This process is also constructed by letting new particles immigrate according to a Poisson random measure (see [5]).

In our previous paper [5] for the absorbing Brownian motion $\{B_t^0(t)\}$ in a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$, we considered the same type problem under the Kuznetsov measure which has the Lebesgue measure dx on H as the invariant measure. We gave the asymptotic behavior of the hitting rate for a compact subset of H . Moreover by applying the obtained result to the equilibrium process with immigration, we also gave limit theorems for the particle system.

We want to extend the above results for absorbing stable processes in a half space H . However there are a few kind of absorbing stable processes in H . For

instance, one could imagine the killed process just before the original stable process in \mathbf{R}^d starting from a point $x \in H$ jumps into the other half space H^c , or consider the case that components are independent and d -th component is the killed process, and so on.

In the present paper we consider the another one, which seems to be more natural in the analytic sense. Let $0 < \alpha < 2$. The process is the time changed process $w^{-,\alpha}(t) = B^0(y^{\alpha/2}(t))$ of an absorbing Brownian motion $B^0(t)$ in H starting from x by an increasing stable process $y^{\alpha/2}(t)$ in $(0, \infty)$ starting from 0 with exponent $\alpha/2$, which are independent. We denote the distribution of the process $\{w^{-,\alpha}(t)\}$ by $P_x^{-,\alpha}$. This process is called “*absorbing stable motion $(w^{-,\alpha}(t), P_x^{-,\alpha})$ with exponent α in a half space H* ”. We usually omit the super-script “ α ”; (w^-, P_x^-) and simply call *absorbing α -stable motion in H* . One can easily see that the transition density $p_t^-(x, y) = p_t^{-,\alpha}(x, y)$ is given by the following :

$$p_t^-(x, y) = p_t^\alpha(y - x) - p_t^\alpha((\tilde{y}, -y_d) - x)$$

where $y = (\tilde{y}, y_d) \in H$ and $p_t^\alpha(x)$ is a density of a distribution of a rotation invariant α -stable process in \mathbf{R}^d at time $t > 0$, whose characteristic function $\varphi_t(z)$ is given as

$$\varphi_t(z) = \int_{\mathbf{R}^d} e^{izx} p_t^\alpha(x) dx = \exp[-c|z|^\alpha]$$

with some constant $c > 0$. Since $p_t^\alpha(x)$ is rotation invariant and C^∞ in $x \in \mathbf{R}^d$, one can write

$$\begin{aligned} p_t^-(x, y) &= p_t^\alpha(y - x) - p_t^\alpha(y - (\tilde{x}, -x_d)) \\ &= - \int_{-x_d}^{x_d} \partial_d p_t^\alpha((\tilde{y} - \tilde{x}, y_d + v)) dv \end{aligned}$$

and also write

$$p_t^-(x, y) = - \int_{-y_d}^{y_d} \partial_d p_t^\alpha((\tilde{x} - \tilde{y}, x_d + v)) dv,$$

where ∂_d denotes the partial differential operator in d -th component.

In order to define a Kuznetsov measure associated with $(w^-(t), P_x^-)$, we need an *entrance law* $(\nu_t)_{t>0}$ which is a family of σ -finite measures satisfying that $\nu_s P_{t-s}^- = \nu_t$ for $s < t$, where $(P_t^-)_{t \geq 0}$ is the transition semi-group of $(w^-(t), P_x^-)$. The density of an entrance law at a boundary point $\tilde{x} \in \partial H = \mathbf{R}^{d-1}$ is defined as

$$\tilde{\nu}_t^x(y) = \partial_d p_t^-((\tilde{x}, x_d), y)|_{x_d=0+} = -2\partial_d p_t^\alpha((\tilde{y} - \tilde{x}, y_d)).$$

For a σ -finite measure μ on ∂H , the density of an entrance law at the boundary is

defined as

$$\nu_t^\mu(y) = \int_{\mathbf{R}^{d-1}} \nu_t^{\tilde{x}}(y) \mu(d\tilde{x}),$$

and moreover set

$$m(y) = m^{\mu, \alpha}(y) = \int_0^\infty \nu_t^\mu(y) dt,$$

which will become the density of the invariant measure for the Kuznetsov measure, of course, if they are well-defined.

We shall investigate the case that $\mu(d\tilde{x}) = d\tilde{x}$ is the uniform measure. In this case $\nu_t(y) = \nu_t^{\tilde{x}}(y)$ is independent of \tilde{y} , so we can write

$$\nu_t(y) = \nu_t(y_d) = -2 \int_{\mathbf{R}^{d-1}} \partial_d p_t^\alpha((\tilde{x}, y_d)) d\tilde{x}.$$

Moreover this $\nu_t(y_d)$ satisfies the scaling property ;

$$\nu_t(y_d) = t^{-2/\alpha} \nu_1(t^{-1/\alpha} y_d),$$

because $\partial_d p_t^\alpha(x) = t^{-(d+1)/\alpha} \partial_d p_1^\alpha(t^{-1/\alpha} x)$ holds by the scaling property of $p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha} x)$. Hence the density of the invariant measure is also independent of \tilde{y} , i.e., $m(y) = m(y_d)$, which is given as

$$m(y_d) = m^\alpha(y_d) = c_\alpha y_d^{\alpha-2}$$

with a positive constant

$$c_\alpha = \alpha \int_0^\infty u^{1-\alpha} \nu_1(u) du = -2\alpha \int_0^\infty du u^{1-\alpha} \int_{\mathbf{R}^{d-1}} \partial_d p_1^\alpha(\tilde{v}, u) d\tilde{v}.$$

Note that this integral is finite because there exists a constant $C > 0$ such that

$$|\partial_d p_1^\alpha(x)| \leq C(1 \wedge x_d \wedge x_d |x|^{-2-d-\alpha}) \quad \text{for all } x \in H.$$

In fact if we set $q^{\alpha/2}(u)$ be a density of the distribution of $y^{\alpha/2}(1)$, then

$$p_1^\alpha(x) = \int_0^\infty q^{\alpha/2}(u) (2\pi u)^{-d/2} \exp\left[-\frac{|x|^2}{2u}\right] du.$$

One can see that

$$\partial_d p_1^\alpha(x) = - \int_0^\infty q^{\alpha/2}(u) (2\pi)^{-d/2} u^{-d/2-1} x_d \exp\left[-\frac{|x|^2}{2u}\right] du.$$

Thus by using a well-known result ; $q^{\alpha/2}(u) \leq C_1(u^{-1-\alpha/2} \wedge \exp[-C_2/u^{C_3}])$ for $u > 0$ with some constants $C_1, C_2, C_3 > 0$, the above estimate is obtained.

We define the entrance law $\nu_t(dy) = \nu_t(y_d)dy$ and set $m(dy) = m(y_d)dy$. For a fixed extra point $\Delta \notin H$, set $H_\Delta = H \cup \Delta$. We introduce a path space W , the set of all maps $w : \mathbf{R} \rightarrow H_\Delta$ such that there is a nonempty open interval $(\alpha(w), \beta(w))$ on which w is H -valued right continuous and has left-hand limit, with $w(t) = \Delta$ if $t \leq \alpha(w)$ or $t \geq \beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t) = \Delta$ for all t . On this path space the *excursion law* Q^0 and the *Kuznetsov measure* Q_m are defined by the following :

$$Q^0 = \lim_{t \downarrow 0} \int_0^\infty \nu_t(dx) P_x^-, \quad Q_m = \int_{-\infty}^\infty \theta_{-s}(Q^0) ds.$$

Then $(w(t), Q_m)$ is a stationary Markov process with the invariant measure m and the same transition probability as the absorbing stable motion $(w^-(t), P_x^-)$.

The equilibrium process with immigration (X_t, \mathbf{P}) associated with $(w(t), Q_m)$ is defined by the following :

$$\begin{aligned} \Omega & \text{ is the space of counting measures } \omega = \sum_n \delta_{w_n}, w_n \in W, , \\ X_t(\omega) & = \omega(t)|_H \text{ for } \omega \in \Omega, , \\ \mathcal{F} & = \sigma(X_s : s < \infty), \mathcal{F}_t = \sigma(X_s : s < t) , \\ \mathbf{P} & \text{ is the } Q_m\text{-Poisson measure on } \Omega, \end{aligned}$$

i.e., the distribution of the Poisson random measure with intensity Q_m . Then (X_t, \mathbf{P}) is a stationary Markov process such that

$$\mathbf{E} [\exp\{-\langle X_t, f \rangle\}] = \exp [-\langle m, 1 - e^{-f} \rangle]$$

for some positive measurable functions f (for the general definition about equilibrium processes with immigration, see [5]).

In §2 we state our main results, that is, asymptotic behavior of $Q_m(0 \leq \sigma_B < t)$ as $t \rightarrow \infty$ associated with a hitting time $\sigma_B = \inf\{t > 0 : w(t) \in B\}$ ($= \infty$ (if $\{\cdot\} = \emptyset$) for a compact subset B of H , and limit theorems for (X_t, \mathbf{P}) .

In §3 we give the outline of the proofs. The basic manner is the same as in our previous paper [5]. However estimates are slightly difficult, so we need several techniques for the calculus.

In §4 we give some additional information about the absorbing stable motion in a half space, that is, the potential kernel and the generator. This section is independent of §2 and §3. It's just our own curiosity.

2. Main Results

We always fix $0 < \alpha < 2$ and usually omit the super-script “ α ”.

Let B be a compact subset of H and $\pi_B(dx)$ be the capacitary measure of B with respect to the Lebesgue measure dx on H , i.e., π_B is supported by B and satisfies that

$$P_x^-(T_B < \infty) = \int g^-(x, y)\pi_B(dy),$$

with the potential kernel

$$g^-(x, y) = \int_0^\infty p_t^-(x, y)dt.$$

With respect to the measure $m(dx) = m(x_d)dx$ ($m(u) = m^\alpha(u) = c_\alpha u^{\alpha-2}$), we have the following : The transition density $p_t^m(x, y) = p_t^-(x, y)/m(x)$, the *capacitary measure* $\pi_B^m(dx) = m(x)\pi_B(dx)$, the *capacity* $C^m(B) = \pi^m(1)$, the *co-capacitary measure* $\hat{\pi}_B^m(dx) = Q_m(w(\tau_B) \in dx : 0 < \tau_B < 1) = Q^0(w(\sigma_B) \in dx : \sigma_B < \infty)$ and the *co-capacity* $\hat{C}^m(B) = Q_m(0 < \tau_B < 1) = Q^0(\sigma_B < \infty)$, where $\tau_B(w) = \inf\{t \in \mathbf{R} : w(t) \in B\} (= \infty \text{ if } \{\cdot\} = \emptyset)$. In this case w^- is not symmetric relative to m , thus in general $C^m(B) \neq \hat{C}^m(B)$ (see §3.3 and §3.4 in [5]).

We shall use a symbol $Q^0[\cdot]$ as the integral by the measure Q^0 .

Our main result is the following :

Theorem 1. *Let B be a compact subset of H with a positive co-capacity. Then it holds that*

$$Q_m(0 \leq \sigma_B < t) = t \hat{C}^m(B) + f(t)$$

with

$$f(t) = \begin{cases} C_\alpha \Phi(B)t^{2-3/\alpha} + o(t^{2-3/\alpha}) & \text{if } d = 1, 3/2 < \alpha < 2, \\ \Phi(B) \log t + o(\log t) & \text{if } d = 1, \alpha = 3/2, \\ O(1) & \text{if } d = 1, 0 < \alpha < 3/2 \text{ or } d \geq 2 \end{cases}$$

as $t \rightarrow \infty$, where

$$C_\alpha = 1 + \frac{3 - \alpha}{2\alpha - 3} \quad (3/2 < \alpha < 2)$$

and

$$\Phi(B) = -\frac{2\alpha p_1^{\alpha''}(0)}{3 - \alpha} \int_B x \pi_B(dx) Q^0[w(\sigma_B) : \sigma_B < \infty] \quad (3/2 \leq \alpha < 2)$$

if $d = 1$.

Furthermore we also have the following : Set $P_m^- = \int_H m(dx)P_x^-$.

Theorem 2. *Let B be a compact subset of H with a positive co-capacity. Then*

$$P_m^-(T_B < t) = \begin{cases} \frac{\alpha^2}{(3-\alpha)(2\alpha-3)}\Psi(B)t^{2-3/\alpha} + O(1) & (3/2 < \alpha < 2) \\ \Psi(B) \log t + O(1) & (\alpha = 3/2) \\ O(1) & (0 < \alpha < 3/2) \end{cases}$$

and

$$Q^0(t \leq \sigma_B < \infty) = \frac{C_{1,\alpha}}{t^{3/\alpha-1}} + o\left(\frac{1}{t^{3/\alpha-1}}\right)$$

as $t \rightarrow \infty$, where

$$\Psi(B) = -2 \int_{\mathbf{R}^{d-1}} \partial_d^2 p_1^\alpha((\tilde{v}, 0)) d\tilde{v} \int_B y_d \pi_B(dy)$$

and

$$C_{1,\alpha} = \begin{cases} \frac{\alpha}{3-\alpha}\Psi(B) + \Phi(B) & \text{if } d = 1, 3/2 < \alpha < 2, \\ \frac{\alpha}{3-\alpha}\Psi(B) & \text{otherwise.} \end{cases}$$

Let $(\{X_t\}_{t \in \mathbf{R}}, \mathbf{P})$ be the equilibrium process associated with $(w(t), Q_m)$. Set $N_t^B = \{w \in W : 0 \leq \sigma_B(w) < t\}$. Then $X(N_t^B)$ is the number of particles hitting B during time interval $[0, t)$. Now we have the following result :

Theorem 3. *Let B be a compact subset of H with a positive co-capacity. Then*

$$\frac{X(N_t^B)}{t} \rightarrow \widehat{C}^m(B) \quad P\text{-a.s. and in } L^1(\mathbf{P})$$

as $t \rightarrow \infty$. Moreover

$$\frac{X(N_t^B) - t\widehat{C}^m(B)}{\sqrt{t}} \rightarrow N(0, \widehat{C}^m(B)) \quad \text{in law}$$

as $t \rightarrow \infty$.

3. Proofs

Proof of Theorem 1. By the same computations as in [5] we see that the following :

$$\begin{aligned} & Q_m(0 \leq \sigma_B < t) - t \widehat{C}^m(B) \\ &= P_m^-(T_B < t) - t Q^0(t \leq \sigma_B < \infty) - Q^0[\sigma_B : \sigma_B < t]. \end{aligned}$$

Moreover

$$P_m^-(T_B < t) = \int_H m(dx) \int_B \pi_B(dy) \int_0^t p_s^-(x, y) ds + P_m^-(T_B < t, T_B \circ \theta_t < \infty),$$

$$Q^0(t \leq \sigma_B < \infty) = Q^0(\sigma_B \circ \theta_t < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty)$$

and

$$Q^0[\sigma_B : \sigma_B < t] = \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_B \pi_B(dy) \int_0^t s p_s^-(x, y) ds + g(t)$$

with

$$g(t) = Q^0[\sigma_B : \sigma_B < t] - \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_B \pi_B(dy) \int_0^t s p_s^-(x, y) ds.$$

Furthermore

$$\begin{aligned} & \int_H m(dx) \int_B \pi_B(dy) \int_0^t p_s^-(x, y) ds \\ &= \int_B \pi_B(dy) \left\{ t \int_t^\infty \nu_u(y_d) du + \int_0^t u \nu_u(y_d) du \right\}, \\ Q^0(\sigma_B \circ \theta_t < \infty) &= \int_B \pi_B(dy) \int_t^\infty \nu_u(y_d) du \end{aligned}$$

and

$$\lim_{u \downarrow 0} \int_H \nu_u(dx) \int_B \pi_B(dy) \int_0^t s p_s^-(x, y) ds = \int_B \pi_B(dy) \int_0^t s \nu_s(y_d) ds$$

Therefore we have

$$\begin{aligned} & Q_m(0 \leq \sigma_B < t) - t \widehat{C}^m(B) \\ &= P_m^-(T_B < t, T_B \circ \theta_t < \infty) + t Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - g(t). \end{aligned}$$

Moreover we can show that

$$(3.1) \quad P_m^-(T_B < t, T_B \circ \theta_t < \infty) = \begin{cases} O(t^{2-4/\alpha} \log t) & \text{if } d = 1, 3/2 \leq \alpha < 2, \\ O(t^{-1/\alpha}) & \text{otherwise} \end{cases}$$

$$= o(1)$$

and

$$(3.2) \quad \begin{aligned} & Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \\ &= \begin{cases} \Phi(B)/t^{3/\alpha-1} + o(1/t^{3/\alpha-1}) & \text{if } d = 1, 3/2 \leq \alpha < 2, \\ O(t^{-3/\alpha}) & \text{otherwise} \end{cases} \end{aligned}$$

as $t \rightarrow \infty$. By further computations it can be shown that

$$(3.3) \quad g(t) = \begin{cases} -\frac{3-\alpha}{2\alpha-3}\Phi(B)t^{2-3/\alpha} + o(t^{2-3/\alpha}) & (d = 1, 3/2 < \alpha < 2), \\ -\Phi(B)\log t + o(\log t) & (d = 1, \alpha = 3/2), \\ O(1) & (\text{otherwise}) \end{cases}$$

as $t \rightarrow \infty$. Hence our claim follows. □

To prove the above equations (3.1), (3.2) and (3.3) we need several lemmas. Recall that

$$\begin{aligned} |\partial_d p_1^\alpha(x)| &\leq C(1 \wedge |x_d| \wedge |x_d||x|^{-2-d-\alpha}) \\ &\leq C(1 \wedge |x|^{-1-d-\alpha}) \\ &\leq C(1 \wedge |\tilde{x}|^{-1-d-\alpha}) \quad \text{for all } x = (\tilde{x}, x_d) \in \mathbf{R}^d. \end{aligned}$$

It can be seen that

Lemma 1. *For each fixed $h > 0$, $\int_H p_h^-(x, y)m(dx) = \int_H p_h^-(x, y)x_d^{\alpha-2}dx$ is bounded in $y \in H$ and $\int_H P_x^-(T_B < h)x_d dx$ is finite.*

Proof. By $\partial_d p_h^\alpha(x) = h^{-(d+1)/\alpha} \partial_d p_1^\alpha(h^{-1/\alpha}x)$ we have

$$\begin{aligned} |\partial_d p_h^\alpha(x)| &\leq Ch^{-(d+1)/\alpha}(1 \wedge |h^{-1/\alpha}\tilde{x}|^{-1-d-\alpha}) \\ &\leq C_h(1 \wedge |\tilde{x}|^{-1-d-\alpha}) \end{aligned}$$

for some suitable constant $C_h > 0$. Hence

$$\begin{aligned} \int_{\{0 < x_d < 1\}} p_h^-(x, y)x_d^{\alpha-2}dx &= \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^1 dx_d x_d^{\alpha-2} \int_{-x_d}^{x_d} (-2)\partial_d p_h^\alpha((\tilde{x} - \tilde{y}, y_d + v))dv \\ &\leq 2 \int_0^1 dx_d x_d^{\alpha-2} \int_{-x_d}^{x_d} dv \int_{\mathbf{R}^{d-1}} |\partial_d p_h^\alpha((\tilde{x}, y_d + v))|d\tilde{x} \\ &\leq 2C_h \int_0^1 dx_d x_d^{\alpha-1} \int_{\mathbf{R}^{d-1}} (1 \wedge |\tilde{x}|^{-1-d-\alpha})d\tilde{x} < \infty \end{aligned}$$

and

$$\begin{aligned} \int_{\{x_d \geq 1\}} p_h^-(x, y)x_d^{\alpha-2}dx &\leq \int_H p_h^-(x, y)dx \\ &\leq \int_H p_h^\alpha(y-x)dx \\ &\leq 1. \end{aligned}$$

Thus the first claim follows. Moreover since $w^-(t)$ is bounded on a finite time interval, for each $h > 0$, there is a bounded set K_h in H such that for all $x \in B$,

$$P_x^-(T_B < h) \leq 2P_x^-(w^-(h) \in K_h) = 2 \int_{K_h} p_h^-(x, y) dy.$$

Therefore

$$\begin{aligned} \int_H P_x^-(T_B < h) x_d dx &\leq 2 \int_{K_h} dy \int_H x_d p_h^-(x, y) dx \\ &= -2 \int_{K_h} dy \int_{-y_d}^{y_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^\infty x_d \partial_d p_h^\alpha((\tilde{x} - \tilde{y}, x_d + v)) dx_d \\ &= 2 \int_{K_h} dy \int_{-y_d}^{y_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^\infty p_h^\alpha((\tilde{x} - \tilde{y}, x_d + v)) dx_d \\ &\hspace{15em} \text{(by integration-by-parts)} \\ &= 2 \int_{K_h} dy \int_{-y_d}^{y_d} dv \int_H p_h^\alpha((\tilde{x}, x_d + v)) dx \\ &\leq 4 \int_{K_h} y_d dy < \infty \end{aligned} \quad \square$$

From this lemma one can get the following estimates :

Lemma 2. *For each fixed $h > 0$, there is a constant $C_h > 0$ such that*

$$P_m^-(t \leq T_B < t + h) \leq C_h t^{-1/\alpha} \quad \text{and} \quad Q^0(t \leq \sigma_B < t + h) \leq C_h t^{-3/\alpha}$$

for all $t > 0$.

Proof. Since

$$\begin{aligned} \int_H p_t^-(y, z) dy &= - \int_{-z_d}^{z_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty \partial_d p_t^\alpha((\tilde{y} - \tilde{z}, y_d + v)) dy_d \\ &= \int_{-z_d}^{z_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{y} p_t^\alpha((\tilde{y} - \tilde{z}, v)) \\ &= t^{-1/\alpha} \int_{-z_d}^{z_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{v} p_1^\alpha((\tilde{v}, t^{-1/\alpha} v)) \\ &\leq t^{-1/\alpha} \int_{-z_d}^{z_d} dv \int_{\mathbf{R}^{d-1}} d\tilde{v} p_1^\alpha((\tilde{v}, 0)) \\ &\leq C' t^{-1/\alpha} z_d \left(C' = 2 \int_{\mathbf{R}^{d-1}} p_1^\alpha((\tilde{v}, 0)) d\tilde{v} \right), \end{aligned}$$

we have

$$\begin{aligned}
 P_m^-(t \leq T_B < t+h) &\leq \int_H P_x^-(T_B \circ \theta_t < h)m(dx) \\
 &= \int_H m(dx) \int_H dy p_t^-(x,y) P_y^-(T_B < h) \\
 &\leq 2 \int_H dy \int_{K_h} dz p_h^-(y,z) \int_H m(dx) p_t^-(x,y) \\
 &= 2 \int_{K_h} dz \int_H m(dx) p_{t+h}^-(x,z) \\
 &= 2 \int_{K_h} dz \int_H dy p_t^-(y,z) \int_H m(dx) p_h^-(x,y) \\
 &\leq C'_h \int_{K_h} dz \int_H p_t^-(y,z) dy \quad \text{by Lemma 1} \\
 &\leq C_h t^{-1/\alpha} \quad \text{by the above inequality,}
 \end{aligned}$$

where K_h is the same compact set as in the proof of the previous lemma. Furthermore it is easy to see that $\nu_1(y_d) \leq C y_d$ for some constant $C > 0$ (cf. see the next lemma). Hence

$$\begin{aligned}
 Q^0(t \leq \sigma_B < t+h) &\leq Q^0(\sigma_B \circ \theta_t < h) \\
 &= \int_H \nu_t(dx) P_x^-(T_B < h) \\
 &= t^{-2/\alpha} \int_H dx \nu_1(t^{-1/\alpha} x_d) P_x^-(T_B < h) \\
 &\leq t^{-3/\alpha} \int_H x_d P_x^-(T_B < h) dx.
 \end{aligned}$$

By the previous lemma our claim follows. □

REMARK 1. From this lemma we see that $P_m^-(T_B < t) = O(t^{1-1/\alpha} \wedge 1)$ if $\alpha \neq 1$, $= O(\log t)$ if $\alpha = 1$ as $t \rightarrow \infty$.

Note that

$$\partial_d p_t^\alpha(x) = t^{-(d+1)/\alpha} \partial_d p_1^\alpha(t^{-1/\alpha} x)$$

and

$$\partial_d p_1^\alpha((\tilde{x}, x_d)) = \partial_d^2 p_1^\alpha((\tilde{x}, 0)) x_d + \frac{1}{6} \partial_d^4 p_1^\alpha((\tilde{x}, h x_d)) x_d^3$$

for some $h \in (0, 1)$. Thus we can see the following lemma :

Lemma 3. *There exist suitable constants $C_1, C_2 > 0$ such that*

$$p_t^-(x, y) \begin{cases} \leq C_1 x_d(x_d + 2y_d)t^{-(d+2)/\alpha} & \text{for all } t > 0, \\ = -2\partial_d^2 p_1^\alpha((\tilde{y} - \tilde{x}, 0))x_d y_d t^{-(d+2)/\alpha} + O(t^{-(d+4)/\alpha}) & \text{as } t \rightarrow \infty, \end{cases}$$

where the $O(t^{-(d+4)/\alpha})$ -constant is finite whenever (x, y) is bounded, and

$$v_t(y_d) \begin{cases} \leq C_2 y_d t^{-3/\alpha} & \text{for all } t > 0, \\ = -2 \int_{\mathbf{R}^{d-1}} \partial_d^2 p_1^\alpha((\tilde{v}, 0)) d\tilde{v} y_d t^{-3/\alpha} + O(t^{-5/\alpha}) & \text{as } t \rightarrow \infty, \end{cases}$$

where the $O(t^{-5/\alpha})$ -constant is finite whenever (x, y) is bounded.

From these lemmas one can establish the equations (3.1), (3.2) and (3.3) as follows:

$$\begin{aligned} & P_m^-(T_B < t, T_B \circ \theta_t < \infty) \\ & \leq P_m^-(T_B < [t] - 1, T_B \circ \theta_t < \infty) + P_m^-([t] - 1 \leq T_B < t) \\ & = \int_H m(dx) \int_H P_x^-(w^-(T_B) \in dy : T_B < [t] - 1) P_y^-(T_B < \infty) + O(1/t^{1/\alpha}) \\ & \hspace{25em} \text{by Lemma 2} \\ & = \int_B \pi_B(dz) \int_H m(dx) \int_{v \in B} \int_0^{[t]-1} d_s P_x^-(w^-(T_B) \in dv : T_B \leq s) \int_{t-s}^\infty p_u^-(v, z) du \\ & \quad + O(1/t^{1/\alpha}). \end{aligned}$$

By Lemma 3 the first term is equal or less than

$$\begin{aligned} & C \int_H m(dx) \int_{v \in B} \int_0^{[t]-1} d_s P_x^-(w^-(T_B) \in dv : T_B \leq s) (t-s)^{1-(d+2)/\alpha} \\ & = C \int_H m(dx) \int_0^{[t]-1} (t-s)^{1-(d+2)/\alpha} P_x^-(T_B \in ds) \\ & = C \left\{ \frac{P_m^-(T_B \leq t)}{t^{(d+2)/\alpha-1}} + \int_0^{[t]-1} \left(\frac{1}{(t-s)^{(d+2)/\alpha-1}} - \frac{1}{t^{(d+2)/\alpha-1}} \right) P_m^-(T_B \in ds) \right\}. \end{aligned}$$

By Lemma 2 the first term is equal to $O(t^{2-(d+3)/\alpha})$ if $1 < \alpha < 2$, $O(t^{-d-1} \log t)$ if $\alpha = 1$ and $O(t^{1-(d+2)/\alpha})$ if $0 < \alpha < 1$ (see Remark 1). The second term is equal to $O(t^{2-(d+3)/\alpha} \log t)$ ($(d+2)/2 \leq \alpha < 2$) and $O(t^{-1/\alpha})$ ($0 < \alpha < (d+2)/2$) as $t \rightarrow \infty$. In fact, if $(d+2)/2 \leq \alpha < 2$, i.e., $d = 1, 3/2 \leq \alpha < 2$, then $1/2 < 3/\alpha - 1 \leq 1$ and

$$\int_0^{[t]-1} \left(\frac{1}{(t-s)^{3/\alpha-1}} - \frac{1}{t^{3/\alpha-1}} \right) P_m^-(T_B \in ds)$$

$$\begin{aligned} &\leq \left(\frac{3}{\alpha} - 1\right) \sum_{k=1}^{[t]-1} \frac{k(t-k)^{3/\alpha-2}}{(t-k)^{3/\alpha-1}t^{3/\alpha-1}} P_m^-(k-1 \leq T_B < k) \\ &= O\left(t^{1-3/\alpha} \sum_{k=1}^{[t]-1} \frac{1}{(t-k)k^{1/\alpha-1}}\right) \\ &= O\left(t^{1-3/\alpha} \left(\frac{1}{t} \int_1^{t/2} u^{1-1/\alpha} du + t^{1-1/\alpha} \int_{t/2}^{t-1} \frac{du}{t-u}\right)\right) \\ &= O\left(t^{2-4/\alpha} \log t\right). \end{aligned}$$

If $0 < \alpha < (d+2)/2$, then $(d+2)/\alpha - 1 > 1$ and

$$\begin{aligned} &\int_0^{[t]-1} \left(\frac{1}{(t-s)^{(d+2)/\alpha-1} - \frac{1}{t^{(d+2)/\alpha-1}}\right) P_m^-(T_B \in ds) \\ &\leq \left(\frac{d+2}{\alpha} - 1\right) \sum_{k=1}^{[t]-1} \frac{kt^{(d+2)/\alpha-2}}{(t-k)^{(d+2)/\alpha-1}t^{(d+2)/\alpha-1}} P_m^-(k-1 \leq T_B < k) \\ &= O\left(\frac{1}{t} \sum_{k=1}^{[t]-1} \frac{1}{(t-k)^{(d+2)/\alpha-1}k^{1/\alpha-1}}\right) \\ &= O\left(\frac{1}{t} \left(t^{1-(d+2)/\alpha} \int_1^{t/2} \frac{du}{u^{1-1/\alpha}} + t^{1-1/\alpha} \int_{t/2}^{t-1} \frac{du}{(t-u)^{(d+2)/\alpha-1}\right)\right) \\ &= O(t^{-1/\alpha}). \end{aligned}$$

Therefore we have (3.1). Moreover by Lemma 3

$$\begin{aligned} &\int_{t-s}^\infty p_u^-(v, z) du - \frac{-2\alpha \partial_d^2 p_1^\alpha((\tilde{z} - \tilde{v}, 0)) z_d v_d}{(d+2-\alpha)t^{(d+2)/\alpha-1}} \\ &= -\frac{2\alpha \partial_d^2 p_1^\alpha((\tilde{z} - \tilde{v}, 0)) z_d v_d}{(d+2-\alpha)} \left(\frac{1}{(t-s)^{(d+2)/\alpha-1} - \frac{1}{t^{(d+2)/\alpha-1}}\right) \\ &\quad + O((t-s)^{1-(d+4)/\alpha}) \end{aligned}$$

as $t \rightarrow \infty$. Hence if we set

$$\Phi(B) = -\frac{2\alpha}{2+d-\alpha} \int_B \int_B \partial_d^2 p_1^\alpha((\tilde{x} - \tilde{y}, 0)) x_d y_d \pi_B(dx) Q^0(w(\sigma_B) \in dy : \sigma_B < \infty),$$

then by a similar way to [5] we can get

$$\begin{aligned} &\left|Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - \Phi(B)/t^{(d+2)/\alpha-1}\right| \\ &\leq C \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_0^{[t]-1} \left(\frac{1}{(t-s)^{(d+2)/\alpha-1} - \frac{1}{t^{(d+2)/\alpha-1}}\right) P_x^-(T_B \in ds) \end{aligned}$$

$$\begin{aligned}
 &+ C \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_0^{[t]^{-1}} (t-s)^{1-(d+4)/\alpha} P_x^-(T_B \in ds) + O(t^{-3/\alpha}) \\
 \leq & C \int_0^{[t]^{-1}} \left(\frac{1}{(t-s)^{(d+2)/\alpha-1}} - \frac{1}{t^{(d+2)/\alpha-1}} \right) Q^0(\sigma_B \in ds) \\
 &+ C \int_0^{[t]^{-1}} (t-s)^{1-(d+4)/\alpha} Q^0(\sigma_B \in ds) + O(t^{-3/\alpha}),
 \end{aligned}$$

where C is a finite constant. As in case of P_m^- we can see that the first term of the right hand side is equal to $O(t^{2-6/\alpha} \log t) = o(t^{1-3/\alpha})$ ($d = 1, 3/2 \leq \alpha < 2$) and $O(t^{-3/\alpha})$ (otherwise) as $t \rightarrow \infty$. Hence we have (3.2).

In order to prove (3.3) it is enough to show the following: For each $h > 0$,

$$g(t+h) - g(t) = \begin{cases} -\frac{h}{t^{3/\alpha-1}} \left(\frac{3}{\alpha} - 1 \right) \Phi(B) + h o \left(\frac{1}{t^{3/\alpha-1}} \right) & \text{if } d = 1, 3/2 \leq \alpha < 2, \\ h O \left(\frac{1}{t^{3/\alpha}} \right) & \text{otherwise} \end{cases}$$

as $t \rightarrow \infty$.

By the same manner as in [5] we have

$$\begin{aligned}
 &g(t+h) - g(t) \\
 &\leq t \{ Q^0(\sigma_B < t+h, \sigma_B \circ \theta_{t+h} < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \} \\
 &\quad + h Q^0(t \leq \sigma_B < t+h) \\
 &= \begin{cases} -\frac{h}{t^{3/\alpha-1}} \left(\frac{3}{\alpha} - 1 \right) \Phi(B) + h o \left(\frac{1}{t^{3/\alpha-1}} \right) & \text{if } d = 1, 3/2 \leq \alpha < 2, \\ h O \left(\frac{1}{t^{3/\alpha}} \right) & \text{otherwise} \end{cases}
 \end{aligned}$$

as $t \rightarrow \infty$ by Lemma 2 and equations (3.1), (3.2). A lower estimate is also given by the same way :

$$\begin{aligned}
 &g(t+h) - g(t) \\
 &\geq t \{ Q^0(\sigma_B < t+h, \sigma_B \circ \theta_{t+h} < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \} \\
 &\quad - h \{ Q^0(\sigma_B \circ \theta_t < \infty) - Q^0(\sigma_B \circ \theta_{t+h} < \infty) \} \\
 &= \begin{cases} -\frac{h}{t^{3/\alpha-1}} \left(\frac{3}{\alpha} - 1 \right) \Phi(B) + h o \left(\frac{1}{t^{3/\alpha-1}} \right) & \text{if } d = 1, 3/2 \leq \alpha < 2, \\ h O \left(\frac{1}{t^{3/\alpha}} \right) & \text{otherwise} \end{cases}
 \end{aligned}$$

as $t \rightarrow \infty$. Therefore the equation (3.3) follows.

Proof of Theorem 2. Form the above computations we have

$$P_m^-(T_B < t) = \int_B \pi_B(dy) \left\{ t \int_t^\infty \nu_u(y_d) du + \int_0^t u \nu_u(y_d) du \right\} + P_m^-(T_B < t, T_B \circ \theta_t < \infty),$$

and

$$Q^0(t \leq \sigma_B < \infty) = \int_B \pi_B(dy) \int_t^\infty \nu_u(y_d) du - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty)$$

Moreover by Lemma 3 we see that

$$\int_B \pi_B(dy) \int_t^\infty \nu_u(y_d) du = \frac{1}{t^{3/\alpha-1}} \frac{\alpha}{3-\alpha} \Psi(B) + O(t^{1-5/\alpha})$$

and that since $\int_0^1 s \nu_s(y_d) ds$ is bounded in $y \in B$,

$$\int_B \pi_B(dy) \int_0^t s \nu_s(y_d) ds = \begin{cases} \frac{\alpha}{2\alpha-3} \Psi(B) t^{2-3/\alpha} + O(1) & (3/2 < \alpha < 2) \\ \Psi(B) \log t + O(1) & (\alpha = 3/2) \\ O(1) & (0 < \alpha < 3/2) \end{cases}$$

as $t \rightarrow \infty$. Hence the desired result follows. □

Proof of Theorem 3. This result immediately follows from Theorem 1. The proof is the same as in [5]. It is a routine work. So we don't describe it in this paper.

4. About absorbing stable motion in a half space

In this final section we give some additional information about absorbing α -stable motion $(w^-(t), P_x^-)$ in a half space H .

Let $(w^0(t), P_x^0) = (w^{0,\alpha}(t), P_x^{0,\alpha})$ be the killed α -stable process in H just before the original α -stable process in \mathbf{R}^d , which is rotation invariant, jumps into H^c . Of course it has a transition density $p_t^0(x, y)$ such that $P_x^0(w^0(t) \in dy) = p_t^0(x, y) dy$. Then it holds that

$$p_t^-(x, y) \leq p_t^0(x, y) \quad \text{for all } t > 0, x, y \in H.$$

That is, paths w^- are killed more than w^0 , because before the stable process jumps into H^c , the absorbing Brownian motion could be killed.

Note that

$$p_t^-(x, y) = \int_0^\infty du q_t^{\alpha/2}(u) \int_{-x_d}^{x_d} \frac{y_d + v}{\sqrt{2\pi u^{d+2}}} \exp \left[-\frac{|\tilde{y} - \tilde{x}|^2 + (y_d + v)^2}{2u} \right] dv,$$

where $q_t^{\alpha/2}$ is the density of the distribution of $y^{\alpha/2}(t)$; the increasing $\alpha/2$ -stable process in $[0, \infty)$ starting from 0. The Laplace transform of $y^{\alpha/2}(t)$ is given as $\exp[-c'tr^{\alpha/2}]$ for $r \geq 0$ with some constant $c' > 0$ and the potential density is

$$\int_0^\infty q_t^{\alpha/2}(u)dt = \frac{1}{c'\Gamma(\alpha/2)}u^{\alpha/2-1} \quad \text{for } u > 0,$$

because the Laplace transforms coincide, where Γ is the gamma function. From these results one can easily see that the potential kernel $g^-(x, y)$ of $(w^-(t), P_x^-)$ is given as

$$g^-(x, y) = \int_0^\infty p_t^-(x, y)dt = \begin{cases} c_{1,1} \log \left| \frac{x+y}{x-y} \right| & (d = \alpha = 1) \\ c_{\alpha,d} \left\{ |y-x|^{\alpha-d} - (|\tilde{y}-\tilde{x}|^2 + (y_d+x_d)^2)^{(\alpha-d)/2} \right\} & (\text{otherwise}), \end{cases}$$

where

$$c_{1,1} = \frac{\sqrt{2}}{\pi c'}, \quad c_{\alpha,d} = \frac{2^{(1+d-\alpha)/2}\Gamma((d+2-\alpha)/2)}{\sqrt{\pi}(d-\alpha)\Gamma(\alpha/2)c'} \quad (d \neq 1 \text{ or } \alpha \neq 1).$$

Fix $d < p < d + \alpha$. We define a function space $D_p = D_p(H)$ on H by the following : $f \in D_p \Rightarrow f \in C^2(H)$, $|f(x)|, |\partial_d^2 f(x)| \leq C(1 \wedge x_d \wedge |x|^{-p})$, and other partial derivatives are bounded by $C(1 \wedge |x|^{-p})$ with some constant $C > 0$. Moreover for $f \in D_p$, we define a function \bar{f} on \mathbf{R}^d as

$$\bar{f}(x) = \begin{cases} f(x) & (x_d > 0), \\ f(\tilde{x}, 0+) = 0 & (x_d = 0), \\ -f(\tilde{x}, -x_d) & (x_d < 0). \end{cases}$$

Then the generator L^- on D_p of $(w^-(t), P_x^-)$ is given by the following:

$$\begin{aligned} L^- f(x) &= c \int_{\mathbf{R}^d} [\bar{f}(y+x) - \bar{f}(x)] \frac{dy}{|y|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(y+x) - f(x)] \frac{dy_d}{|y|^{d+\alpha}} \\ &\quad + c \int_{\mathbf{R}^{d-1}} \int_{x_d}^\infty [f(y+x) - f(\tilde{y} + \tilde{x}, y_d - x_d) - 2f(x)] \frac{dy_d}{|y|^{d+\alpha}} \end{aligned}$$

with a suitable constant $c > 0$. We can also write that if $0 < \alpha < 1$, then

$$L^- f(x) = c \int_{\mathbf{R}^d} [\bar{f}(y) - \bar{f}(x)] \frac{dy}{|y-x|^{d+\alpha}}$$

$$= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x)] K(x, y) dy_d - 2f(x) \int_0^\infty \frac{dy_d}{|y+x|^{d+\alpha}} \right\},$$

and that if $1 \leq \alpha < 2$, then

$$\begin{aligned} L^- f(x) &= c \int_{\mathbf{R}^d} [\bar{f}(y) - \bar{f}(x) - \nabla \bar{f}(x) \cdot (y-x) I_{U_1}(y-x)] \frac{dy}{|y-x|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x) - \nabla f(x) \cdot (y-x) I_{U_1}(y-x)] K(x, y) dy_d \right. \\ &\quad + \int_0^\infty [-2f(x) + \nabla f(x) \cdot (y+x) I_{U_1}(y+x) \\ &\quad \left. - \nabla f(x) \cdot (\tilde{y} + \tilde{x}, -y_d + x_d) I_{U_1}((\tilde{y} + \tilde{x}, -y_d + x_d))] \frac{dy_d}{|y+x|^{d+\alpha}} \right\}, \end{aligned}$$

where $\nabla f = (\partial_1 f, \dots, \partial_d f)$, $U_1 = \{x \in \mathbf{R}^d : |x| < 1\}$ and

$$K(x, y) = \frac{1}{|y-x|^{d+\alpha}} - \frac{1}{|(\tilde{y} - \tilde{x}, y_d + x_d)|^{d+\alpha}}.$$

In fact if set

$$P_t^- f(x) = P_t^{-,\alpha} f(x) = \int_H p_t^-(x, y) f(y) dy$$

and $(P_t) = (P_t^\alpha)$ be the transition semi-group of a rotation invariant α -stable process in \mathbf{R}^d , then $P_t^- f(x) = P_t \bar{f}(x)$ and of course (P_t^-) satisfies semi-group property. Hence we have $L^- f(x) = L \bar{f}(x)$ for $f \in D_p$, where $L = L^\alpha$ is the generator of the rotation invariant α -stable process in \mathbf{R}^d

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