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HAUSDORFF CONDITION FOR BROWN-PETERSON COHOMOLOGIES

Dedicated to Professor Hirosi Toda on his sixtieth birthday

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Let X be a CW-spectrum and $\{X_\lambda\}$ be the set of all finite subspectra of X. For any CW-spectrum E we have a natural morphism $\pi\colon E^*X \to \varprojlim_\lambda E^*X_\lambda$ induced by the incusions $X_\lambda \subset X$. In [1, Theorem 1.8] Adams showed that the canonical morphism π is always an epimorphism. By making use of Adams' result we first prove that the spectral sequence $E_2^{s,t} = \varprojlim_\lambda{}^s E^*X_\lambda \Rightarrow E^*X$ collapses (Theorem 1), which has implicitly been given by Hikida [5]. Given a CW-spectrum E and an abelian group G we are interested in the Anderson dual spectrum $\nabla E(G)$ related by a universal coefficient sequence $0 \to \mathrm{Ext}(E_{*-1}X, G) \to \nabla E(G)^*(X) \to \mathrm{Hom}(E_*X, G) \to 0$. For the canonical morphism $\pi\colon \nabla E(G)^*X \to \varprojlim_\lambda \nabla E(G)^*X_\lambda$, a diagram chasing shows easily that $\mathrm{Ker}\,\pi \cong \varprojlim_\lambda{}^1 \mathrm{Hom}(E_{*-1}X_\lambda, G)$ and $\mathrm{Coker}\,\pi \cong \varprojlim_\lambda{}^2 \mathrm{Hom}(E_{*-1}X_\lambda, G)$. Since the latter part is trivial by Adams' result, we can observe that there exists a short exact sequence $0 \to \varprojlim_\lambda{}^1 \nabla E(G)^{*-1}X_\lambda \to \nabla E(G)^*X \to \varprojlim_\lambda \nabla E(G)^*X_\lambda \to 0$ (Theorem 4). This result was proved in [6, Theorem 1.1] under the restriction that E is of finite type.

A cohomology group E^*X is said to be Hausdorff if $\pi\colon E^*X \to \varprojlim_\lambda E^*X_\lambda$ is an isomorphism. We next give a sufficient condition under which E^*X is Hausdorff, when E is a wedge sum $\vee_\alpha E_\alpha$ or a direct limit $\varinjlim_k E_k$ of p-local CW-spectra of finite type (Proposition 8 and Theorem 9).

Let BP denote the Brown-Peterson spectrum, and $BP\langle n\rangle$, $E(n)$, $n \geq 0$, be the BP-related spectra. Their coefficient groups are $BP_* = Z_{(p)}[v_1, \cdots, v_n, \cdots]$, $BP\langle n\rangle_* = Z_{(p)}[v_1, \cdots, v_n]$ and $E(n)_* = Z_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. Using Wilson's splitting theorem we can restate Quillen's theorem [7, Theorem 5.7] that the torsion subgroup of BP^*Y is generated as a BP^*-module by torsion elements of non-negative degree if Y is a based finite CW-complex (Proposition 10). Combining

Theorem 9 with this result we finally give a sufficient condition on a based countable CW-complex X under which $(v_n^{-1}BP)^m X$ and $E(n)^m X$ are both Hausdorff (Theorem 13).

1. Hausdorffness of cohomologies E^*X

1.1. Let X be a CW-spectrum and $\{X_\alpha\}$ be a directed system of subspectra ordered by the inclusions whose union $\cup X_\alpha$ is the whole X. Let W denote the wedge sum $\vee_\gamma X_\gamma$, and W_α the wedge sum $\vee_{\gamma\leq\alpha} X_\gamma$ for each α. Note that W_α is finite if X_α is finite. Given a CW-spectrum E the set $\{E^*W_\alpha\}$ of cohomology groups forms an inverse system such that

$$E^*W \cong \varprojlim_\alpha E^*W_\alpha \quad \text{and} \quad \varprojlim_\alpha{}^q E^*W_\alpha^- = 0 \qquad \text{for any} \quad q\geq 1 \tag{1.1}$$

(see [8, Proposition 1]).

Let Y_α denote the cofiber of the map $W_\alpha \to X_\alpha$ induced by the inclusions $X_\gamma \subset X_\alpha$. Then we have a short exact sequence $0 \to \{E^*X_\alpha\} \to \{E^*W_\alpha\} \to \{E^{*+1}Y_\alpha\} \to 0$ of inverse systems, since the cofiber sequence $\Sigma^{-1}Y_\alpha \to W_\alpha \to X_\alpha$ splits. This yields a four-term exact sequence

$$0 \to \varprojlim_\alpha E^*X_\alpha \to E^*W \to \varprojlim_\alpha E^{*+1}Y_\alpha \to \varprojlim_\alpha{}^1 E^*X_\alpha \to 0 \tag{1.2}$$

and isomorphisms

$$\varprojlim_\alpha{}^q E^{*+1}Y_\alpha \xrightarrow{\cong} \varprojlim_\alpha{}^{q+1} E^*X_\alpha \qquad \text{for any} \quad q\geq 1\,, \tag{1.3}$$

by making use of (1.1).

Let Y be the cofiber of the map $W \to X$ induced by the inclusions $X_\gamma \subset X$. Obviously Y is the union of the directed system of subspectra Y_α. We inductively construct a CW-spectrum W_{s+1} and a map $W_{s+1} \to X_s$ and form a cofiber sequence $W_{s+1} \to X_s \to X_{s+1}$, by setting $X_{-1}=X$ and $W_0=W=\vee_\gamma X_\gamma$. Let $\tilde{X}_s$ be the cofiber of the inclusion $X=X_{-1}\subset X_s$, and $\tilde{X}_\infty$ the union $\cup \tilde{X}_s$. Then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X_s \arrow[r] \arrow[d] & \tilde{X}_s \arrow[d] \\
X \arrow[r] & X_{s+1} \arrow[r] \arrow[d] & \tilde{X}_{s+1} \arrow[d] \\
 & \Sigma W_{s+1} \arrow[r, equal] & \Sigma W_{s+1}
\end{tikzcd}$$

involving four cofiber sequences. We have an equivalence $\tilde{X}_\infty \to \Sigma X$ because $\varinjlim_s \pi_* X_s = 0$.

We now observe a spectral sequence associated with the increasing filtration $*=\tilde{X}_{-1}\subset\tilde{X}_0\subset\cdots\subset\tilde{X}_s\subset\cdots$, whose union is $\tilde{X}_\infty$ (see [2] or [3]). Given a CW-spectrum E we set

$$\begin{aligned} Z_r^{s,t} &= \mathrm{Ker}\,\{E^{s+t+1}(\tilde{X}_s/\tilde{X}_{s-1})\to E^{s+t+2}(\tilde{X}_{s+r-1}/\tilde{X}_s)\} \\ B_r^{s,t} &= \mathrm{Im}\,\{E^{s+t}(\tilde{X}_{s-1}/\tilde{X}_{s-r})\to E^{s+t+1}(\tilde{X}_s/\tilde{X}_{s-1})\} \\ E_r^{s,t} &= Z_r^{s,t}/B_r^{s,t} \qquad \text{for each } r,\quad 1\leq r\leq\infty\,. \end{aligned}$$

This spectral sequence converges to $E^*X\cong E^{*+1}\tilde{X}_\infty$, and its E_1-term is $E_1^{s,t}=E^{s+t+1}(\tilde{X}_s/\tilde{X}_{s-1})\cong E^{s+t}(W_s)$. The differential $d_r^{s,t}\colon E_r^{s,t}\to E_r^{s+r,t-r+1}$ is defined to be the composition $Z_r^{s,t}/B_r^{s,t}\to Z_r^{s,t}/Z_{r+1}^{s,t}\cong B_{r+1}^{s+r,t-r+1}/B_r^{s+r,t-r+1}\to Z_r^{s+r,t-r+1}/B_r^{s+r,t-r+1}$. Therefore the differential $d_1^{s,t}$ is just the composition $E^{s+t}(W_s)\to E^{s+t+1}(X_s)\to E^{s+t+1}(W_{s+1})$. By making use of (1, 2) and (1.3) it is easily verified that

$$E_2^{s,t}\cong\varprojlim_\alpha{}^s E^tX_\alpha\,. \tag{1.4}$$

1.2. Let X be a CW-spectrum and $\{X_\lambda\}$ be the set of all finite subspectra of X. For any CW-spectrum E the inclusions $X_\lambda\subset X$ induce a natural morphism

$$\pi\colon E^*X\to\varinjlim_\lambda E^*X_\lambda\,, \tag{1.5}$$

which is always an epimorphism (see Adams [1, Theorem 1.8]).

A cohomology group E^*X is said to be *Hausdorff* if π is an isomorphism. The following collapsing theorem has implicitly been obtained by Hikida [5].

Theorem 1. *Let X be a CW-spectrum and $\{X_\alpha\}$ be a set of subspectra whose union is X. Assume that E^*X_α is Hausdorff for each α. Then the spectral sequence $E_2^{s,*}=\varprojlim_\alpha{}^s E^*X_\alpha\Rightarrow E^*X$ collapses.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & E^*(\tilde{X}_{s+1}/\tilde{X}_{s-1}) \arrow[d, "\tilde{\varphi}"] \arrow[r, phantom, "\cong"] \arrow[dr, "\tilde{\eta}"] & E^*(X_{s+1}/X_{s-1}) \arrow[dr, "\eta"] & \\ E^*(\tilde{X}_\infty/\tilde{X}_{s-1}) \arrow[r, "\tilde{\varphi}'"] & E^*(\tilde{X}_s/\tilde{X}_{s-1}) \arrow[r, "\tilde{\varphi}''"] & E^{*+1}(\tilde{X}_\infty/\tilde{X}_s) \arrow[r, phantom, "\cong"] & E^*(X_s)\,. \end{tikzcd}$$

Then we have an isomorphism

$$Z_2^{s,*}/Z_\infty^{s,*}=\mathrm{Im}\,\tilde{\varphi}/\mathrm{Im}\,\tilde{\varphi}'\cong\mathrm{Im}\,\tilde{\eta}\cong\mathrm{Im}\,\eta\,.$$

Therefore it is sufficient to show that the morphism η is trivial, in order to observe that all the differentials d_r, $r\geq 2$, are trivial. Notice that the composition $\pi\cdot\eta\colon E^*(X_{s+1}/X_{s-1})\to E^*(X_s)\to\varprojlim_\alpha E^*(X_{s,\alpha})$ is trivial because $E^*(X_{s+1,\alpha})\to E^*(X_{s,\alpha})$ is trivial for each α. We here consider the commutative diagram

$$\begin{array}{ccccc} E^{*-1}(X_{s-1}) & \xrightarrow{\varphi'} & E^{*-1}(W_s) & \xrightarrow{\varphi''} & E^{*}(X_s) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\ 0 \to \varprojlim_{\alpha} E^{*-1}(X_{s-1,\alpha}) & \to & \varprojlim_{\alpha} E^{*-1}(W_{s,\alpha}) & \to & \varprojlim_{\alpha} E^{*}(X_{s,\alpha}) \end{array}$$

with exact rows. The central arrow π_2 is an isomorphism by (1.1), and the left one π_1 is an epimorphism by virtue of (1.5) because $E^{*+s}(X_{s-1,\alpha})$ is Hausdorff whenever $E^*(X_\alpha)$ is Hausdorff. Hence it is easily checked that $\operatorname{Ker}\pi\cdot\varphi''=\operatorname{Ker}\varphi''$. This implies that $\operatorname{Ker}\pi\cdot\eta=\operatorname{Ker}\eta$. Therefore we can show that the morphism η is trivial as desired.

As an immediate result we have

Corollary 2. *Let X be a CW-spectrum and $\{X_\lambda\}$ be the set of all finite subspectra of X. For a fixed integer m, $E^m X$ is Hausdorff if and only if $\varprojlim_\lambda^q E^{m-q}X_\lambda=0$ for all $q\geq 1$.*

1.3. Let E be a CW-spectrum and G an arbitrary abelian group G. Then there exists a CW-spectrum $\nabla E(G)$ related by a universal coefficient sequence

$$(1.6)\qquad 0\to \operatorname{Ext}(E_{*-1}X, G)\to \nabla E(G)^*X\to \operatorname{Hom}(E_*X, G)\to 0$$

for any CW-spectrum X (see [9]). The spectrum $\nabla E(G)$ is the function spectrum $F(E, \nabla S(G))$.

Let X be a CW-spectrum and $\{X_\lambda\}$ be the set of all finite subspectra of X. We here consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Ext}(E_{*-1}X, G) & \longrightarrow & \nabla E(G)^*X & \longrightarrow & \operatorname{Hom}(E_*X, G) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} & \\ 0 \longrightarrow & \varprojlim_\lambda \operatorname{Ext}(E_{*-1}X_\lambda, G) & \longrightarrow & \varprojlim_\lambda \nabla E(G)^*X_\lambda & \longrightarrow & \varprojlim_\lambda \operatorname{Hom}(E_*X_\lambda, G) & \end{array}$$

with exact rows. The right arrow π_2 is an isomorphism, and the left one π_1 satisfies that $\operatorname{Ker}\pi_1\cong\varprojlim_\lambda^1 \operatorname{Hom}(E_{*-1}X_\lambda, G)$ and $\operatorname{Coker}\pi_1\cong\varprojlim_\lambda^2 \operatorname{Hom}(E_{*-1}X_\lambda, G)$. As is easily seen, $\operatorname{Ker}\pi\cong\operatorname{Ker}\pi_1$ and $\operatorname{Coker}\pi\cong\operatorname{Coker}\pi_1$. Therefore we have a short exact sequence

$$(1.7)\qquad 0\to\varprojlim_\lambda^1 \operatorname{Hom}(E_{*-1}X_\lambda, G)\to \nabla E(G)^*X\xrightarrow{\pi}\varprojlim_\lambda \nabla E(G)^*X_\lambda\to 0$$

and

$$(1.8)\qquad \varprojlim_\lambda^2 \operatorname{Hom}(E_*X_\lambda, G)=0\,,$$

since the morphism π is always an epimorphism by (1.5).

Lemma 3. *Let X be a CW-spectrum and G an abelian group. Then* $\varprojlim_{\lambda}^{q} \operatorname{Hom}(E_*X_\lambda, G)=0$ *for all* $q\geq 2$.

Proof. We use the cofiber sequence $W_{s+1,\lambda}\to X_{s,\lambda}\to X_{s+1,\lambda}$ of finite CW-spectra for each λ, which is constructed by setting $X_{-1,\lambda}=X_\lambda$ and $W_{0,\lambda}=\vee_{\mu\leq\lambda}X_\mu$. This induces a short exact sequence

$$0\to \operatorname{Hom}(E_*X_{s,\lambda}, G)\to \operatorname{Hom}(E_*W_{s+1,\lambda}, G)\to \operatorname{Hom}(E_{*+1}X_{s+1,\lambda}, G)\to 0\,.$$

Notice that $\varprojlim_{\lambda}^{q} \operatorname{Hom}(E_*W_{s+1,\lambda}, G)=0$ for all $q\geq 1$ (see [8, Proposition 1]). Then it is shown that $\varprojlim_{\lambda}^{q} \operatorname{Hom}(E_{*+1}X_{s+1,\lambda}, G)\cong \varprojlim_{\lambda}^{q+1} \operatorname{Hom}(E_*X_{s,\lambda}, G)$ for any $q\geq 1$, and hence $\varprojlim_{\lambda}^{q} \operatorname{Hom}(E_*X_\lambda, G)\cong \varprojlim_{\lambda}{}_{z} \operatorname{Hom}(E_{*+q-2}X_{q-3,\lambda}, G)$ for each $q\geq 1$. By applying (1.8) to X_{q-3} we obtain the result as desired.

We now prove the following result, which was given in [6, Theorem 1.1] under the restriction that E is of finite type.

Theorem 4. *Let E be an arbitrary CW-spectrum and G any abelian group. Let X be a CW-spectrum and $\{X_\lambda\}$ be the set of all finite subspectra of X. Then,*
i) *there exists a short exact sequence*

$$0\to \varprojlim_{\lambda}^{1} \nabla E(G)^{*-1}X_\lambda \to \nabla E(G)^*X\to \varprojlim_{\lambda} \nabla E(G)^*X_\lambda\to 0\,,$$

ii) $\varprojlim_{\lambda}^{1} \nabla E(G)^*X_\lambda\cong \varprojlim_{\lambda}^{1} \operatorname{Hom}(E_*X_\lambda, G)$, *and*
iii) $\varprojlim_{\lambda}^{q} \nabla E(G)^*X_\lambda=0$ *for any* $q\geq 2$.

Proof. Use the long exact sequence $\cdots\to\varprojlim_{\lambda}^{q} \operatorname{Ext}(E_{*-1}X_\lambda, G)\to \varprojlim_{\lambda}^{q} \nabla E(G)^*X_\lambda\to\varprojlim_{\lambda}^{q} \operatorname{Hom}(E_*X_\lambda, G)\to\varprojlim_{\lambda}^{q+1} \operatorname{Ext}(E_{*-1}X_\lambda, G)\to\cdots$ and the isomorphisms $\varprojlim_{\lambda}^{q} \operatorname{Ext}(E_*X_\lambda, G)\cong\varprojlim_{\lambda}^{q+2} \operatorname{Hom}(E_*X_\lambda, G)$ for any $q\geq 1$. The results ii) and iii) are now immediate from Lemma 3. Moreover the result i) is obtained from (1.7) combined with ii).

Let $\{A_\alpha\}$ be a directed system of abelian groups and G be any abelian group. Then there exists a four-term exact sequence

$$\begin{aligned}0\to\varprojlim_{\alpha}^{1} \operatorname{Hom}(A_\alpha, G)\to \operatorname{Pext}(\varinjlim_{\alpha} A_\alpha, G)\to&\varprojlim_{\alpha} \operatorname{Pext}(A_\alpha, G)\\ &\to\varprojlim_{\alpha}^{2} \operatorname{Hom}(A_\alpha, G)\to 0\end{aligned}$$

(see [6, Proposition 1.4]). Hence we have

Corollary 5. *For a fixed integer m, $\nabla E(G)^m X$ is Hausdorff if G is algebraically compact or if $E_{m-1}X$ is pure projective.*

Proof. Our hypothesis implies that $\mathrm{Pext}(E_{m-1}X, G)=0$ and hence $\varprojlim_{\lambda}^1 \mathrm{Hom}(E_{m-1}X_\lambda, G)=0$.

Corollary 6. *For a fixed integer m, $\nabla E(G)^m X$ is Hausdorff if G is torsion free and* $\mathrm{Ext}(E_{m-1}X/\mathrm{Tor}, G)=0$ *where* Tor *stands for the torsion subgroup.*

Proof. If $\mathrm{Ext}(E_{m-1}X/\mathrm{Tor}, G)=0$, then $\varprojlim_{\lambda}^1 \mathrm{Hom}(E_{m-1}X_\lambda/\mathrm{Tor}, G)=0$, which means that $\varprojlim_{\lambda}^1 \mathrm{Hom}(E_{m-1}X_\lambda, G)=0$ when G is torsion free.

2. Brown-Peterson and *BP*-related cohomologies

2.1. Let E be a p-local CW-spectrum. We simply write ∇E instead of $\nabla E(G)$ when $G=Z_{(p)}$, the integers localized at p. If E is of finite type, then the canonical map $E\to\nabla^2 E$ is an equivalence (see [4] or [9]) where $\nabla^2 E$ stands for $\nabla(\nabla E)$.

Let $\{E_\alpha\}$ be a family of p-local CW-spectra of finite type where α runs over an arbitrary indexing set, and E be the wedge sum $\vee E_\alpha$. For any finite CW-spectrum Y we have a commutative diagram

$$\begin{CD}
0 @>>> \bigoplus_\alpha \mathrm{Ext}(\nabla E_{\alpha *-1} Y, Z_{(p)}) @>>> \bigoplus_\alpha E_\alpha^* Y @>>> \bigoplus_\alpha \mathrm{Hom}(\nabla E_{\alpha *} Y, Z_{(p)}) @>>> 0\\
@. @VVV @VVV @VVV \\
0 @>>> \mathrm{Ext}(\nabla E_{*-1} Y, Z_{(p)}) @>>> \nabla^2 E^* Y @>>> \mathrm{Hom}(\nabla E_* Y, Z_{(p)}) @>>> 0
\end{CD}$$

involving two universal coefficient sequences (1.6), in which the vertical arrows are induced by the inclusions $E_\alpha\subset E$. Applying again the universal coefficient sequences (1.6) for any finite CW-spectrum Y, we obtain natural isomorphisms

$$\begin{aligned}(2.1)\qquad &\mathrm{Hom}(\nabla E_{\alpha *} Y, Z_{(p)})\cong \mathrm{Hom}(\mathrm{Hom}(E_\alpha^* Y, Z_{(p)}), Z_{(p)}),\\ &\mathrm{Ext}(\nabla E_{\alpha *-1} Y, Z_{(p)})\cong \mathrm{Ext}(\mathrm{Ext}(E_\alpha^* Y, Z_{(p)}), Z_{(p)})\end{aligned}$$

and moreover a natural exact sequence

$$\begin{aligned}(2.2)\quad 0 \to \mathrm{Hom}(\mathrm{Hom}(E^* Y, Z_{(p)}), Z_{(p)}) &\to \mathrm{Hom}(\nabla E_* Y, Z_{(p)})\\ &\to \mathrm{Hom}(\mathrm{Ext}(E^{*+1} Y, Z_{(p)}), Z_{(p)}).\end{aligned}$$

We here recall Zeeman's Theorem [10, Theorem 1 ii)], which says that the natural embedding $\oplus P_\alpha\to(\oplus P_\alpha)^{**}$ is an isomorphism if P_α is a free $Z_{(p)}$-module for each α, where $\mathrm{Hom}(\mathrm{Hom}(A, Z_{(p)}), Z_{(p)})$ is shortly written A^{**} for any $Z_{(p)}$-module A. Let $\{A_\alpha\}$ be a family of finitely generated $Z_{(p)}$-modules. Zeeman's Theorem implies easily that the canonical morphism

$$(2.3)\qquad \bigoplus_\alpha A_\alpha^{**} \to (\bigoplus_\alpha A_\alpha)^{**}$$

is an isomorphism. Let T_α denote the torsion subgroup of A_α. Choose a free resolution $0\to R_\alpha\to P_\alpha\to T_\alpha\to 0$, then we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[r] & \bigoplus_\alpha R_\alpha \arrow[r] \arrow[d] & \bigoplus_\alpha P_\alpha \arrow[d] \\ 0 \arrow[r] & \mathrm{Hom}(\mathrm{Ext}(\bigoplus_\alpha T_\alpha, Z_{(p)}), Z_{(p)}) \arrow[r] & (\bigoplus_\alpha R_\alpha)^{**} \arrow[r] & (\bigoplus_\alpha P_\alpha)^{**} \end{tikzcd}$$

with exact rows. Applying Zeeman's Theorem again it follows immediately that $\mathrm{Hom}(\mathrm{Ext}(\bigoplus_\alpha T_\alpha, Z_{(p)}), Z_{(p)})=0$, and hence

$$\mathrm{Hom}(\mathrm{Ext}(\bigoplus_\alpha A_\alpha, Z_{(p)}), Z_{(p)}) = 0\,. \tag{2.4}$$

Lemma 7. *Let Y be a finite CW-spectrum and E be a wedge sum of p-local CW-spectra of finite type. Then there exists a natural exact sequence*

$$0 \to \mathrm{Tor}\, E^*Y \to E^*Y \to \mathrm{Hom}(\nabla E_*Y, Z_{(p)}) \to 0$$

where Tor E^*Y *denotes the torsion subgroup of* E^*Y.

Proof. Consider the commutative square

$$\begin{tikzcd} \bigoplus_\alpha (E_\alpha^*Y)^{**} \arrow[r] \arrow[d] & \bigoplus_\alpha \mathrm{Hom}(\nabla E_{\alpha*}Y, Z_{(p)}) \arrow[d] \\ (E^*Y)^{**} \arrow[r] & \mathrm{Hom}(\nabla E_*Y, Z_{(p)})\,. \end{tikzcd}$$

The top arrow is an isomorphism by (2.1) and the left one is an isomorphism by (2.3). Moreover the bottom one is also an isomorphism, by putting (2.2) and (2.4) together. Therefore the right arrow becomes an isomorphism. Making use of this isomorphism and (2.1) we obtain an exact sequence

$$0 \to \bigoplus_\alpha \mathrm{Tor}\, E_\alpha^*Y \to \bigoplus_\alpha E_\alpha^*Y \to \mathrm{Hom}(\nabla E_*Y, Z_{(p)}) \to 0$$

as desired.

2.2. Let X be a CW-spectrum and E be a wedge sum of p-local CW-spectra of finite type. Consider the exact sequences $0\to \mathrm{Tor}\, E^*X_\lambda \to E^*X_\lambda \to \mathrm{Hom}(\nabla E_*X_\lambda, Z_{(p)}) \to 0$ for all the finite subspectra X_λ of X. By use of Lemma 3 and Theorem 4 ii) we have an exact sequences

$$\varprojlim_\lambda{}^1 \mathrm{Tor}\, E^*X_\lambda \to \varprojlim_\lambda{}^1 E^*X_\lambda \to \varprojlim_\lambda{}^1 \nabla^2 E^*X_\lambda \to \varprojlim_\lambda{}^2 \mathrm{Tor}\, E^*X_\lambda \to \varprojlim_\lambda{}^2 E^*X_\lambda \to 0\,, \tag{2.5}$$

and isomorphisms

$$\varprojlim_\lambda{}^q \mathrm{Tor}\, E^*X_\lambda \cong \varprojlim_\lambda{}^q E^*X_\lambda \qquad \text{for any} \quad q\geq 3\,. \tag{2.6}$$

Proposition 8. *Let X be a CW-spectrum and E be a wedge sum of p-local CW-spectra of finite type. For a fixed integer m we assume that $\nabla^2 E^m X$ is Hausdorff* *and that* $\varprojlim_\lambda^q \operatorname{Tor} E^{m-q}X_\lambda=0$ *for all* $q\geq 1$. *Then* $E^m X$ *is Hausdorff.*

Proof. By making use of (2.5), (2.6) and Corollary 2 we observe that $\varprojlim_\lambda^q E^{m-q}X_\lambda=0$ for each $q\geq 1$ under our hypothesis, and hence that $E^m X$ is Hausdorff.

Let $\{E_k\}_{k\geq 0}$ be a directed sequence of p-local CW-spectra of finite type and E be the wedge sum $\vee E_k$. When the mapping telescope is denoted by $\varinjlim_k E_k$, there exists a natural short exact sequence $0\to E^*Y\to E^*Y\to(\varinjlim_k E_k)^*Y\to 0$ for any finite CW-spectrum Y. We use (2.5), (2.6) and Corollary 2 again to obtain

Theorem 9. *Let* $\{E_k\}_{k\geq 0}$ *be a directed sequence of* p*-local* CW*-spectra of finite type and* E *denote the wedge sum* $\vee E_k$. *Let* X *be a* CW*-spectrum and* $\{X_\lambda\}$ *be the set of all finite subspectra of* X. *For a fixed integer* m *we assume that* $\nabla^2 E^m X$ *is Hausdorff and that* $\varprojlim_\lambda^q \operatorname{Tor} E^{m-q}X_\lambda=0=\varprojlim_\lambda^{q+1} \operatorname{Tor} E^{m-q}X_\lambda$ *for any* $q\geq 1$. *Then* $(\varinjlim_k E_k)^m X$ *is Hausdorff.*

Proof. Under our hypothesis we observe that $\varprojlim_\lambda^q E^{m-q}X_\lambda=0=\varprojlim_\lambda^{q+1} E^{m-q}X_\lambda$ for any $q\geq 1$. This implies that $\varprojlim_\lambda^q (\varinjlim_k E_k)^{m-q}X_\lambda=0$ for each $q\geq 1$. The desired result is now immediate.

2.3. Let BP be the Brown-Peterson spectrum for a fixed prime p. This is an associative and commutative ring spectrum whose coefficient ring is $BP_*=Z_{(p)}[v_1,\cdots,v_n,\cdots]$ with degree of $v_n=2(p^n-1)$. For any $n\geq 0$ there are BP-related spectra $BP\langle n\rangle$ constructed in [7]. These are associative BP-module spectra with $BP\langle n\rangle_*=Z_{(p)}[v_1,\cdots,v_n]$, and related by the cofiber sequences $\Sigma^{2(p^n-1)}BP\langle n\rangle\xrightarrow{\cdot v_n}BP\langle n\rangle\to BP\langle n-1\rangle$ where $\cdot v_n$ denotes the multiplication by v_n. Consider the map $g_n\colon BP\to BP\langle n\rangle$ inducing the canonical projection in homotopy. Based on Wilson's splitting theorem [7, Theorem 5.4] we get

(2.7) $g_n\colon BP^m X\to BP\langle n\rangle^m X$ *is split epic for* $m<2(p^n+\cdots+p+1)$ *if* X *is a based* CW*-complex,* and hence

(2.8) g_n; $BP^*Y\to BP\langle n\rangle^*Y$ *is split epic for all but finitely many degrees if* Y *is a based finite* CW*-complex.*

In [7, Theorem 5.7] Wilson proved Quillen's Theorem that BP^*Y is generated as a BP^*-module by elements of non-negative degree if Y is a based finite CW-complex. We here give faithfully an imitation of his proof to show

Proposition 10. *Let* Y *be a based finite* CW*-complex. Then the torsion* *subgroup* $\mathrm{Tor}BP^*Y$ *is generated as a* BP^**-module by torsion elements of non-negative degree.*

Proof. Let y be a torsion element of $BP^m Y$ for some negative degree m. Then we can find a positive integer n and an element z_n of $BP\langle n\rangle^{m+2(p^n-1)}Y$ such that $g_n(y)=v_n\cdot z_n$. Note that z_n is a torsion element. Since g_n is split epic in degree $m+2(p^n-1)$ by virtue of (2.7) we can pick a torsion element w_n of $BP^{m+2(p^n-1)}Y$ with $g_n(y)=g_n(v_n\cdot w_n)$. Continue this process to show that y is represented as a finite sum $\sum_{n\geq 1} v_n w_n$ where w_n are all torsion elements. Now the result follows by downward induction on the degree m of y.

Let X be a based CW-complex which is countable. Then it has an increasing filtration $*=X_{-1}\subset X_0\subset\cdots\subset X_k\subset\cdots$ of finite subcomplexes such that $\bigcup X_k=X$.

Lemma 11. *Let X be a based countable CW-complex. Then*

$$\varprojlim_k{}^1 (\bigoplus_m \mathrm{Tor}\, BP^m X_k) = 0\,.$$

Proof. The inverse system $\{\bigoplus_{m\geq 0} \mathrm{Tor}\, BP^m X_k\}$ satisfies the Mittag-Leffler condition since each abelian group $\bigoplus_{m\geq 0} \mathrm{Tor}\, BP^m X_k$ is finite. Therefore the inverse system $\{(BP^*\otimes(\bigoplus_{m\geq 0} \mathrm{Tor}\, BP^m X_k)\}$ satisfies the Mittag-Leffler condition, too. This implies that $\varprojlim_k{}^1 BP^*\otimes(\bigoplus_{m\geq 0}\mathrm{Tor}\, BP^m X_k)=0$. On the other hand, Proposition 10 says that the natural morphism $BP^*\otimes(\bigoplus_{m\geq 0}\mathrm{Tor}\, BP^m X_k)\to\bigoplus_m \mathrm{Tor}\, BP^m X_k$ is epic. Hence it is easily shown that $\varprojlim_k{}^1(\bigoplus \mathrm{Tor}\, BP^m X_k)=0$.

Let $C_n Y$ denote the cokernel of the morphism $g_n\colon \bigoplus_m \mathrm{Tor}\, BP^m Y\to \bigoplus_m \mathrm{Tor}\, BP\langle n\rangle^m Y$. If Y is a based finite CW-complex, then the abelian group $C_n Y$ is finite by virtue of (2.8). As an immediate result we have

Corollary 12. *Let X be a based countable CW-complex. Then*

$$\varprojlim_k{}^1 (\bigoplus_m \mathrm{Tor}\, BP\langle n\rangle^m X_k) = 0 \qquad \textit{for each} \quad n\,.$$

Let E be an associative BP-module spectrum. Then we can form a CW-spectrum $v_n^{-1}E$ defined to be the mapping telescope $\varinjlim_k \Sigma^{-2k(p^n-1)} E$ of the map $\cdot v_n\colon E\to\Sigma^{-2(p^n-1)}E$. This is a weak associative BP-module spectrum such that $(v_n^{-1}E)_*X\cong v_n^{-1}BP_*\otimes_{BP_*}E_*X$. Particularly $v_n^{-1}BP\langle n\rangle$ is denoted by $E(n)$.

Theorem 13. *Let X be a based countable CW-complex and t be a fixed integer with $0\leq t<2(p-1)$. Assume that $H_m X\otimes Q=0$ for all m with $m\equiv t$* mod $2(p-1)$. *Then $(v_n^{-1}BP)^{m+1}X$ and $E(n)^{m+1}X$ are both Hausdorff when $m\equiv t$*

mod $2(p-1)$.

Proof. Set $E=\vee_k \Sigma^{-2k(p^n-1)}BP$ or $\vee_k \Sigma^{-2k(p^n-1)}BP\langle n\rangle$. Then $\pi_m \nabla E=0$ unless $m\equiv 0 \bmod 2(p-1)$. Thus our hypothesis on X implies that $\nabla E_m X\otimes Q=0$ if $m\equiv t \bmod 2(p-1)$. By means of Corollary 6 we see that $\nabla^2 E^{m+1}X$ is Hausdorff if $m\equiv t \bmod 2\,(p-1)$. By virtue of Lemma 11 and Corollary 12 we now apply Theorem 9 to obtain that $E^{m+1}X$ is Hausdorff when $m\equiv t \bmod 2(p-1)$.

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