A characterization of the closable parts of pre-Dirichlet forms by hitting distributions

Kuwae, Kazuhiro

Osaka Journal of Mathematics. 29(4) P.771-P.788

10.18910/9896

https://doi.org/10.18910/9896

1992

Osaka University Knowledge Archive: OUKA
http://ir.library.osaka-u.ac.jp/dspace/
A CHARACTERIZATION OF THE CLOSABLE PARTS OF PRE-DIRICHLET FORMS BY HITTING DISTRIBUTIONS

KAZUHIRO KUWAE

(Received October 15, 1991)

1. Introduction

Let $X$ be a locally compact separable metric space with an extra point $\Delta$ such that $X_\Delta \equiv X \cup \{\Delta\}$ is a one point compactification and let $m$ be a positive Radon measure with $\text{supp}[m] = X$. When $X$ is compact, $\Delta$ is adjoined as an isolated point. For a subset $B$ of $X$, we denote $B_\Delta = B \cup \{\Delta\}$. We consider a $C_0$-regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ having a nice core $\mathcal{L}$ (see Section 2) and $M = (\Omega, \mathcal{F}_t, X_t, P_x, x \in X)$ the associated $m$-symmetric Hunt process. We say that a subset $B$ of $X$ is $\mathcal{E}_1$-polar if it is of zero capacity. Let $\{T_t, t \geq 0\}$ be the $L^2$-semigroup associated with $(\mathcal{E}, \mathcal{F})$. We say that a Borel set $B$ of $X$ is $T_t$-invariant if $T_t(I_B u) = I_B T_t u$ for any $u \in L^2(X, m)$, and $t > 0$. $(\mathcal{E}, \mathcal{F})$ is called irreducible if for any $T_t$-invariant set $B$, $B$ or $X \setminus B$ is $m$-negligible. A Borel set $B$ of $X$ is $M$-invariant if $P_x(X_t \in B_\Delta, X_{t-} \in B_\Delta$, for any $t > 0) = 1$, for any $x \in B$. M. Fukushima-K. Sato-S. Taniguchi [10] investigated the closable part of general symmetric bilinear form on a real Hilbert space. They characterized the closable part of a pre-Dirichlet form under the changes of underlying measures and gave a necessary and sufficient condition for the closability. They used the analytic characterization of the time changed Dirichlet space formulated in K. Kuwae-S. Nakao [12]. In these mentioned articles assumed is that $(\mathcal{E}, \mathcal{F})$ is either transient or irreducible in order to make a reduction to the transient case, but the irreducibility is not easily checked.

In this paper, we will not assume the irreducibility of $(\mathcal{E}, \mathcal{F})$ nor its transience. In Section 2 and Section 3 we prepare some quasi-notions and decomposition theorems of the state space $X$. In particular, we give a decomposition

$$X = X^{(c)} + X^{(d)} + N,$$

where $X^{(c)}$ (resp. $X^{(d)}$) is an $M$-invariant conservative (resp. dissipative) part of $X$, and $N$ is a properly exceptional set. In Section 4 we give a characterization of the regular Dirichlet space associated with the time changed process using the above decomposition. In Section 5 we fix a closed set $Y$ and consider the space $\mathcal{C}|_Y = \{u \in C_0(Y); u|_Y = u_1 \}$, for some $u \in \mathcal{C}$. We then introduce, for each
choice of a finely closed Borel set $F$ with $F \subset Y$, a pre-Dirichlet form $\mathcal{A}_F$ with domain $\mathcal{C}_Y$ defined by

$$\mathcal{A}_F(u, u) = \mathcal{E}(H_F u, H_F u), \ u \in \mathcal{C}_Y,$$

where $u$ is a function appearing in the definition of $\mathcal{C}_Y$ and $H_F u(x) = E_x[u(X_{\sigma_F})]$, $\sigma_F$ being the hitting time of $F$. Suppose $\mu$ is a positive Radon measure on $X$ and $Y = \text{supp} [\mu]$. Using the characterization of time changed Dirichlet space in Section 4, we prove that the closable part of $(\mathcal{A}_Y, \mathcal{C}_Y)$ on $L^2(Y; \mu)$ is $(\mathcal{A}_{\mathcal{F}_0}, \mathcal{C}_Y)$ where $\mathcal{F}_0$ is the quasi-support of the smooth part of $\mu$, generalizing a result of [10]. As a consequence, we can generalize the closability criterion of [10] (Theorem 5.4).

The author would like to express his thank to Professor M. Fukushima for helpful advice.

2. Quasi-notions

As in Section 1, let $X$ be a locally compact separable metric space with an extra point $\Delta$ such that $X_\Delta$ is a one point compactification and $m$ be a positive Radon measure with $\text{supp} [m] = X$. For a Borel measure $\gamma$ on $X$ and Borel functions $f$ and $g$ on $X$, we denote $(f, g)_\gamma = \int_X f(x) g(x) \gamma(dx)$ if this integral makes sense. Let $C_0(X)$ be the family of continuous functions with compact support. Consider a dense subalgebra $\mathcal{C}$ of $C_0(X)$ satisfying the following two properties:

(C. 1) For any compact set $K$ and relatively compact open set $G$ with $K \subset G \subset X$, there exists $f \in \mathcal{C}$ such that $0 \leq f \leq 1$ and $f = 1$ on $K$ and $f = 0$ on $X - G$.

(C. 2) For any $\varepsilon > 0$ there exists a real function $\varphi_\varepsilon(t)$ satisfying that $\varphi_\varepsilon(t) = t$ for any $t \in [0, 1]$, $-\varepsilon \leq \varphi_\varepsilon(t) \leq 1 + \varepsilon$ for any $t$, and $0 \leq \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \leq t - s$ for $s \leq t$, and $\varphi_\varepsilon(f) \in \mathcal{C}$ whenever $f \in \mathcal{C}$.

Let $(\mathcal{E}, \mathcal{F})$ be a $C_0$-regular Dirichlet space on $L^2(X, m)$ possessing $\mathcal{C}$ as its core, namely $\mathcal{C}$ is dense in $(\mathcal{E}_1, \mathcal{F})$, where $\mathcal{E}_1$ is defined by

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{\mathcal{M}}, \ u, v \in \mathcal{F}.$$

Let $M = (\Omega, \mathcal{F}, \{X_t, P_x: x \in X\}$ be the $m$-symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$. The capacity associated with $(\mathcal{E}, \mathcal{F})$ will be called the $\mathcal{E}_1$-capacity; for any open set $G$,

$$\mathcal{E}_1\text{-Cap}(G) = \inf \{\mathcal{E}_1(u, u); u \in \mathcal{F}, u \geq 1 \text{ m-a.e. on } G\}$$

and, for any subset $A$ of $X$,

$$\mathcal{E}_1\text{-Cap}(A) = \inf \{\mathcal{E}_1\text{-Cap}(G); A \subset G, \text{ open}\}.$$

It is well-known that for any compact set $K$,

$$\mathcal{E}_1\text{-Cap}(K) = \inf \{\mathcal{E}_1(u, u); u \in \mathcal{C}, u \geq 1 \text{ on } K\}.$$
A set $B \subseteq X$ is called $\mathcal{E}_1$-polar if $\mathcal{E}_1$-$\text{Cap}(B)=0$. A statement $\Gamma$ depending on $x \in A$ is said to hold $\mathcal{E}_1$-q.e. on $A$ (abbreviated to q.e. on $A$) if there exists an $\mathcal{E}_1$-polar set $N$ such that $\Gamma$ is true for $x \in A - N$. A function $f: X \to [-\infty, \infty]$ is called $\mathcal{E}_1$-quasi-continuous (abbreviated to quasi-continuous) if for any $\varepsilon > 0$ there exists an open set $G$ such that $\mathcal{E}_1$-$\text{Cap}(G) < \varepsilon$ and $f |_{X-G}$ is continuous. An increasing sequence of closed sets $\{F_n\}$ is called $\mathcal{E}_1$-nest (abbreviated to nest) if $\lim \mathcal{E}_1$-$\text{Cap}(X - F_n) = 0$. Let $\mathcal{M}$ be the space of positive Radon measures on $X$ and let $\mathcal{M}_0 = \{\nu \in \mathcal{M}; \nu \text{ charges no } \mathcal{E}_1\text{-polar set}\}$. As in [9], we use following notations: For set $A, B \subseteq X$, we denote $A \subset B$ q.e. (resp. $A = B$ q.e.) if the set $A - B$ (resp. $A \Delta B$) is $\mathcal{E}_1$-polar. Here $A \Delta B$ is the symmetric difference. Similarly we can define $A \subset B$ $\nu$-a.e. if $\nu(A - B) = 0$ for $\nu \in \mathcal{M}$. We say that a set $A$ is a q.e. (resp. $\nu$-a.e.) version of a set $B$ or $A$ is q.e. (resp. $\nu$-a.e.) equivalent to $B$ if $A = B$ q.e. (resp. $\nu$-a.e.). We call a set $E \subseteq X$ quasi-open if

$$\inf \{\mathcal{E}_1$-$\text{Cap}(E \Delta G); \ G \text{ open}\} = 0$$

and a set $F$ is called quasi-closed if $X - F$ is quasi-open. It is easy to see that the notion of quasi-open (resp.-closed) is stable under q.e. equivalence and a set $E$ is quasi-open (resp.-closed) if and only if there exists a nest $\{F_n\}$ such that $E \cap F_n$ is an open (resp. a closed) subset of $F_n$. Any countable union and finite intersection of quasi-open sets are quasi-open and any countable intersection and finite union of quasi-closed sets are quasi-closed. A function $f: X \to [-\infty, \infty]$ is quasi-continuous if and only if for any open set $I \subseteq [-\infty, \infty], f^{-1}(I)$ is quasi-open. In particular, for a quasi-open and quasi-closed set $B$, the indicator function $I_B$ is quasi-continuous (B. Fuglede [4]). For two outer capacities $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ on $X$, we write $\mathcal{C}^{(1)} < \mathcal{C}^{(2)}$ if for any decreasing sequence of relatively compact open sets $\{A_n\}$

$$\lim_{n \to \infty} \mathcal{C}^{(2)}(A_n) = 0 \implies \lim_{n \to \infty} \mathcal{C}^{(1)}(A_n) = 0.$$ 

Then $\mathcal{C}^{(2)}$-polarity, $\mathcal{C}^{(2)}$-quasi-open set, $\mathcal{C}^{(2)}$-quasi-continuity are inherited to the corresponding notions relative to $\mathcal{C}^{(1)}$. We say that $\mathcal{C}^{(2)}$ is equivalent to $\mathcal{C}^{(1)}$ if $\mathcal{C}^{(2)} < \mathcal{C}^{(1)}$ and $\mathcal{C}^{(1)} < \mathcal{C}^{(2)}$.

For $\nu \in \mathcal{M}_0$, a set $\hat{Y} \subseteq X$ is called a quasi-support of $\nu$ if $\hat{Y}$ is a quasi-closed $\nu$-a.e. version of $X$ and $\overline{\nu} \subseteq \hat{Y}$ q.e. for any quasi-closed $\hat{Y}$ which is a $\nu$-a.e. version of $X$. Let $Y = \text{supp } [\nu]$ be the topological support of $\nu$. Then $\overline{\nu} \subseteq \hat{Y}$ q.e.. The existence of quasi-support of $\nu \in \mathcal{M}_0$ up to $\mathcal{E}_1$-polar set is guaranteed ([4], [10]). For $\nu \in \mathcal{M}_0$, denote by $q$-$\text{supp } [\nu]$ the quasi-support of $\nu$. We let $\mathcal{M}_0 = \{\nu \in \mathcal{M}_0; \mathcal{E}_1$-$\text{Cap}(X - q$-$\text{supp } [\nu])=0\}$. For $\nu \in \mathcal{M}_0$, there exists a unique (up to an $\mathcal{E}_1$-polar set) positive continuous additive functional (abbreviated to
PCAF) $A_t$ of $\mathcal{M}$ characterized by

$$
\langle \nu, f \rangle = \lim_{t \to 0^+} \frac{1}{t} E_t \left[ \int_0^t f(X_s) \, dA_s \right], \quad f \in \mathcal{B}^+(X),
$$

where $\mathcal{B}^+(X)$ denotes the family of all non-negative Borel functions on $X$ and $\langle \nu, f \rangle$ stands for $\int_X f(x) \nu(dx)$. $E_t$ denotes integration by $P_s(d\omega) = \int_X P_s(d\omega) \gamma(dx)$ for a Borel measure $\gamma$ on $X$. $\nu$ is called Revuz measure of $A_t$. We put $Y_A = \{ x \in X - N_A; P_x(A_t > 0 \text{ for any } t > 0) = 1 \}$, where $N_A$ is the defining exceptional set for $A_t$. $Y_A$ is called the support of $A_t$. In [9], Fukushima and LeJan proved that the support of PCAF associated with $\nu^c_A t$ is a quasi-support of $\nu$.

A set $B \subseteq X_A$ is called nearly Borel measurable if for any probability measure $\nu$ on $X_A$ there exist Borel sets $B_1, B_2 \subseteq X$ with $B_1 \subseteq B \subseteq B_2$ such that $P_s(X_t \iota^B_2 = B_1$ for some $t \geq 0) = 0$. A set $E \subseteq X$ is called finely open if for each $x \in E$ there exists nearly Borel set $B = B(x)$ with $X - E \subseteq B \subseteq X$ such that $P_x(\sigma_B > 0) = 1$. Here $\sigma_B = \inf \{ t > 0; X_t \subseteq B \}$. A set $F$ is finely closed if $X - F$ is finely open. For a set $A$ we denote $A' = \{ x \in X; P_x(\sigma_A = 0) = 1 \}$ the regular set for $A$. A nearly Borel set $F$ is finely closed if and only if $F^c \subseteq F$. We say that a set $E$ is q.e. finely open (resp. q.e. finely closed) if there exists a finely open (resp. finely closed) nearly Borel set $E$ with $E = E^c \text{ q.e.}$ A function $u: X \to [\infty, \infty]$ is called finely continuous q.e. if there exists an $E_1$-polar finely closed set $N$ such that $u$ is finely continuous and nearly Borel measurable on $X - N$. A set $N$ is called properly exceptional if $N$ is $m$-negligible Borel set and $X - N$ is $\mathcal{M}$-invariant. A function $u: X \to [\infty, \infty]$ is finely continuous q.e. if and only if there exists a properly exceptional set $N$ such that $u$ is finely continuous an Borel measurable on $X - N$ (Lemma 4.2.6 in [6]). We collect generalizations of some assertions in [6].

Lemma 2.1. (i) For a quasi-open set $E$ and a quasi-continuous function $u: X \to [\infty, \infty]$,

$$u \geq 0 \text{ m-a.e. on } E \text{ if and only if } u \geq 0 \text{ q.e. on } E.$$ (ii) For a quasi-open set $E$,

$$E_1 - \text{Cap}(E) = \inf_{u \in \mathcal{L}_E} E_1(u, u), \quad \text{where } \mathcal{L}_E = \{ u \in \mathcal{F}; u \geq 1 \text{ m-a.e. on } E \}.$$

(iii) A quasi-open m-negligible set $E$ is $E_1$-polar.

Proof. (i) The "if" part is trivial. We show the "only if" part. Let $\{ F' \}$ and $\{ F_k \}$ be nests such that $E \cap F_k$ is open in $F_k$ and $u |_{F_k}$ is continuous. We put $F_k = \sup \{ m |_{F_k} \}$. Then $\{ F_k \}$ is an $m$-regular nest, namely $m(U(x) \cap F_k) > 0$, for any $x \in F_k$ and any open neighbourhood $U(x)$ of $x$. The rest of the proof is the same as in Lemma 3.1.3 in [6].

(ii) By (i) and Theorem 3.3.1 in [6], (ii) is clear in case $E_1 - \text{Cap}(E) < \infty$. 


We show that $\mathcal{E}_1\text{-}\text{Cap}(E) = \infty$ implies $\mathcal{L}_E = \emptyset$. Suppose $\mathcal{L}_E \neq \emptyset$ and $\mathcal{E}_1\text{-}\text{Cap}(E) = \infty$. Then there exists unique element $e_E \in \mathcal{L}_E$ which attains the infimum. Let $\{G_n\}$ be an increasing sequence of relatively compact open sets such that $X = \bigcup_{n=1}^\infty G_n$. Then there exists unique element $e_{E\cap G_n} \in \mathcal{L}_{E\cap G_n}$ satisfying $\mathcal{E}_1\text{-}\text{Cap}(E \cap G_n) = \mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$, because $\mathcal{E}_1\text{-}\text{Cap}(E \cap G_n) < \mathcal{E}_1\text{-}\text{Cap}(G_n) < \infty$. Since $\mathcal{E}_1\text{-}\text{Cap}$ is a Choquet capacity, $\mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n}) \approx \mathcal{E}_1\text{-}\text{Cap}(E) = \infty$ as $n \to \infty$.

On the other hand, $\mathcal{E}_1(e_E, e_E) = \mathcal{E}_1(e_E - e_{E\cap G_n}, e_E - e_{E\cap G_n}) + \mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$, because $\mathcal{E}_1(e_{E\cap G_n}, v) = \mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$ for any $v \in \mathcal{F}$, $v = 1$ q.e. on $E \cap G_n$, where $\mathcal{F}$ is an $m$-a.e. quasi-continuous version of $v$. This is a contradiction. (iii) is a trivial consequence of (ii). The proof is complete.

**Theorem 2.2.** (i) A set $E$ is quasi-open if and only if $E$ is q.e. finely open.

(ii) A function $u: X \to [-\infty, \infty]$ is quasi-continuous if and only if $u$ is finely continuous q.e.

Proof. By Theorem 4.3.2 in [6], (ii) follows from (i). We show (i). Suppose that $E$ is quasi-open and $\{F_n\}$ is a nest such that $E \cap F_n$ is open in $F_n$ for each $n$. There exists a properly exceptional set $N \supset \bigcap_{n=1}^\infty (X - F_n)$ satisfying $P_s(\lim_{n \to \infty} \sigma_{Y - F_n} = \infty) = 1$ for any $x \in X - N$,

by (4.3.5) in [6], $E - N$ is then finely open and Borel measurable. Conversely suppose $E$ is q.e. finely open. Then there exists a finely open and nearly Borel set $\tilde{E}$ with $E = \tilde{E}$ q.e.. For a strictly positive bounded $f \in L^1(X; m)$, we put

$$v(x) = E_s[\int_0^\infty e^{-t} f(X_t) \, dt].$$

Then $v \in \mathcal{F}$ and quasi-continuous by Theorem 4.3.2 in [6]. Further $v > 0$ on $\tilde{E}$ and $v = 0$ q.e. on $X - \tilde{E}$. Hence we get $\tilde{E} = v^{-1}(0, \infty)$ q.e. which implies that $E$ is quasi-open. The proof is complete.

A universally measurable function $h: X \to [0, \infty]$ is said to be $\alpha$-excessive if $e^{-\alpha t} \rho_t h(x) / \mathcal{M}(h)(x)$, $\alpha > 0$, $x \in X (\alpha \geq 0)$. It is known that $\alpha$-excessive function ($\alpha \geq 0$) is nearly Borel measurable and finely continuous.

**Corollary 2.3.** For each $\alpha \geq 0$, $\alpha$-excessive function is quasi-continuous.

**3. Ergodic decomposition into $M$-invariant sets**

As in Section 2, $(\mathcal{E}, \mathcal{F})$ is a $C_0$-regular Dirichlet space possessing $\mathcal{C}$ as its core. Let $\{T_t, t \geq 0\}$ be the $L^2$-semigroup associated with $(\mathcal{E}, \mathcal{F})$. In this section we give a relation of $T_t$-invariant set and $M$-invariant set.

**Lemma 3.1.** If a nearly Borel set $B$ is $T_t$-invariant and simultaneously
quasi-open and quasi-closed, then there exists a properly exceptional set \( N \) such that both \( B-N \) and \( X-B-N \) are \( M \)-invariant and quasi-open.

Proof. Denote by \( p_t \) the transition kernel of \( M \). Since \( I_B \) is a quasi-continuous function, we get

\[
p_t I_B u = I_B p_t u \quad \text{q.e. for } u \in \mathcal{B}(X) \cap L^2(X; m) \text{ for each } t > 0,
\]

where \( \mathcal{B}(X) \) is the family of positive Borel functions on \( X \). Approximating 1 by \( h_n \in \mathcal{B}(X) \cap L^2(X; m) \) with \( h_n \uparrow 1 \), we have

\[
p_t I_B = I_B p_t 1 \quad \text{q.e. for each } t > 0,
\]
or equivalently

\[
p_t I_B = 0 \quad \text{q.e. on } X-B \text{ and } p_t I_{X-B} = 0 \quad \text{q.e. on } B \text{ for each } t > 0.
\]

Since \( I_B \) is quasi-continuous, the map \( t \mapsto I_{B}(X_t) \) is right continuous and has left limit \( I_{\Delta}(X_t^-) \) \( \mathcal{P}_t \)-a.s. for q.e. \( x \in X \). Thus we have

\[
P_s(I_{X_t} \in B_\Delta) \text{ for any } t \geq 0, \quad X_{t^-} \in B_\Delta \text{ for any } t > 0 = 1 \quad \text{q.e. } x \in B.
\]

Similarly

\[
P_s(I_{X_t} \in (X-B)_\Delta) \text{ for any } t \geq 0, \quad X_{t^-} \in (X-B)_\Delta \text{ for any } t > 0 = 1 \quad \text{q.e. } x \in X-B.
\]

By Theorem 4.2.1 in [6] there exists an appropriate properly exceptional set \( N \) such that \( B_1 = B-N \) and \( B_2 = X-B-N \) are \( M \)-invariant. Since quasi-notions are invariant under q.e. equivalence, \( B_1 \) and \( B_2 \) are also quasi-open and quasi-closed sets. The proof is complete.

The next Corollary was proven in [7] under the local property.

**Corollary 3.2.** A Borel set \( B \) is \( T_t \)-invariant if and only if there exists a quasi-open and quasi-closed set \( B_1 \) (resp. \( B_2 \)) which is an \( M \)-invariant m-a.e. version of \( B \) (resp. \( X-B \)) and a properly exceptional set \( N \) such that \( X=B_1+B_2+N \).

Proof. The "if" part is trivial. We only show the "only if" part. Suppose that \( B \) is \( T_t \)-invariant. Then there exists an m-a.e. version \( \tilde{B} \) of \( B \) such that \( I_{\tilde{B}} \) is quasi-continuous (implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (vi) \( \Rightarrow \) (v) of Theorem 2 in [7]). Since \( \tilde{B} \) is also \( T_t \)-invariant, we have the assertion by Lemma 3.1. The proof is complete.

For a strictly positive \( f \in L^1(X; m) \), the sets \( C_f = \{ x \in X; G_f(x) = \infty \} \) and \( D_f = \{ x \in X; G_f(x) < \infty \} \) are \( T_t \)-invariant (Theorem 1.5.8 in [14]). Here \( G_f = \int_0^\infty T_t f \, dt \). Further \( C_f \) and \( D_f \) are independent of the choice of \( f \) up to \( m \)-negligible sets. Hence by Corollary 3.2 the whole space \( X \) admits a decomposi-
tion $X=X^{(c)}+X^{(d)}+N$, where $X^{(c)}$ (resp. $X^{(d)}$) is an $M$-invariant $m$-a.e. version of $C_r$ (resp. $D_r$) and $N$ is a properly exceptional set.

**Lemma 3.3.** Let $h$ be an excessive function. Then $p_t h=h$ q.e. on $X^{(c)}$ for each $t>0$.

Proof. This lemma follows from Corollary 2.3 and Theorem 1 in [1]. For the convenience of readers we give a direct proof. Suppose that $f\in L^r(X; m)$ is $m$-a.e. strictly positive on $X$. Then $Rf(x)=E_x[\int_0^\infty f(X_t) \, dt]=\infty$ q.e. on $X^{(c)}$ by Lemma 2.1 (i) and Corollary 2.3. Put $h_n=h\wedge n$. Then $h_n$ is an excessive function. By resolvent equation $R_p h_n=R_q h_n+(p-q) R_p R_q h_n=0$, we get

$$(h_n-qR_q h_n, Rf)_{m, X^{(c)}}\leq \lim_{n\to 0} (h_n-qR_q h_n, R_p f)_m$$

$$= \lim_{n\to 0} (h_n-pR_p h_n, R_q f)_m$$

$$\leq \frac{n}{q} \langle m, f \rangle < \infty .$$

Hence we have $qR_q h_n=h_n$ q.e. on $X^{(c)}$. Letting $n\to \infty$, we have $qR_q h=h$ q.e. on $X^{(c)}$. The proof is complete.

We say that the Dirichlet space $(\mathcal{E}, \mathcal{F})$ is transient if there exists a bounded $g\in L^r(X; m)$ with $g>0$ $m$-a.e. such that $Gg<\infty$ $m$-a.e. and $(\mathcal{E}, \mathcal{F})$ is recurrent if it is non-transient and irreducible ([8], [14]). The restricted process $M\upharpoonright_{X^{(c)}}$ (resp. $M\upharpoonright_{X^{(d)}}$) is transient (resp. conservative). $(\mathcal{E}, \mathcal{F})$ is transient if and only if $m(X^{(c)})=0$. If $(\mathcal{E}, \mathcal{F})$ is irreducible then $m(X^{(c)})=0$ or $m(X^{(d)})=0$, namely $(\mathcal{E}, \mathcal{F})$ is transient or recurrent. $X^{(c)}$ (resp. $X^{(d)}$) is called the conservative (resp. dissipative) part of $M$ ([1], [3], [5], [11]).

Without loss of generality, we shall assume that the space $\mathcal{F}$ consists of $\mathcal{E}$-quasi-continuous functions, two functions which equal $\otimes$-q.e. being identified. For each non-trivial $\nu\in \mathcal{M}_0$, the symmetric form $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ on $L^r(X; m)$ defined by

$$\begin{aligned}
(3.3) & \left\{ \begin{array}{l}
\mathcal{F}^\nu = \mathcal{F} \cap L^r(X; \nu), \\
\mathcal{E}^\nu(u, v) = \mathcal{E}(u, v) + \langle u, v \rangle
\end{array} \right.
\end{aligned}$$

is a $C_0$-regular Dirichlet form having $C$ as its core (see the proof of Lemma 3.1 in [10]). $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ is called $\nu$-killed Dirichlet space. Denote by $\mathcal{M}^\nu$ the $m$-symmetric Hunt process associated with $(\mathcal{E}^\nu, \mathcal{F}^\nu)$. Let $A_\nu^*$ be the PCAF associated with $\nu$. The set $C_\nu^* = \{x\in X; E_x[\int_0^\infty e^{-\lambda t} f(X_t) \, dt]=\infty \}$ and $D_\nu^* = \{x\in X; E_x[\int_0^\infty e^{-\lambda t} f(X_t) \, dt]<\infty \}$ are $T_\nu^*$-invariant set for $f\in L^r(X; m), f>0$ $m$-
a.e. on $X$. Since $\mathcal{E}^\cap$-Cap is equivalent to $\mathcal{E}^\cap$-Cap (Lemma 2.3 in [12]), we can denote by $X^{\nu}(c), X^{\nu}(d)$ the $\mathcal{M}^\nu$-invariant $\mathcal{E}^\cap$-quasi-open and $\mathcal{E}^\cap$-quasi-closed $m$-a.e. version of $C^\gamma, D^\gamma$ respectively. Put $B^\nu = \{x \in X; P_x(A^n^\nu > 0) > 0\}$.

**Proposition 3.4.** (i) For $\nu \in \mathcal{M}_0$, $X^{\nu}(c) \subset B^\nu$ q.e. if and only if the $\nu$-killed Dirichlet space $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ on $L^2(X; m)$ is transient.

(ii) In the above case the $\nu$-killed extended Dirichlet space $\mathcal{F}^\nu$ is complete by $\mathcal{E}^\nu$-norm. $\mathcal{E}^\nu$-capacity is equivalent to $\mathcal{E}^\nu$-capacity.

Proof. The proof of (ii) is the same as in Lemma 2.3 in [12]. We show (i). The "if" part is trivial. We show the "only if" part. Applying Lemma 3.3 to $M^\nu$ with $\lambda = 1$, we have

$$E_x[e^{-\lambda t}] = 1 \text{ q.e. } x \in X^{\nu}(c) \text{ for each } t > 0,$$

namely

$$P_x(A^n^\nu = 0) = 1 \text{ q.e. } x \in X^{\nu}(c).$$

which, combined with the assumption $X^{\nu}(c) \subset B^\nu$ q.e., implies $X^{\nu}(c) = \emptyset$ q.e.. The proof is complete.

**Corollary 3.5.** If $(\mathcal{E}, \mathcal{F})$ is irreducible, $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ is transient.

Proof. Suppose $(\mathcal{E}, \mathcal{F})$ is irreducible. Then $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ is irreducible. Hence $X^{\nu}(c) = \emptyset$ q.e. or $X^{\nu}(c) = X$ q.e.. Suppose $X^{\nu}(c) = X$ q.e.. Then by (3.4)

$$P_x(A^n^\nu = 0) = 1 \text{ q.e. } x,$$

which contradicts the non-triviality of $\nu$. The proof is complete.

**Corollary 3.6.** If $\nu \in \mathcal{M}_0$, $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ is transient.

Proof. By Corollary 3.5 in [9], $q$-supp $[\nu] \subset B^\nu$ q.e.. Hence $B^\nu = X$ q.e.. The proof is complete.

4. **Time changed Dirichlet space**

In this section we give a characterization of the time changed Dirichlet space without irreducibility as in Fitzsimmons [2]. Fix $\mu \in \mathcal{M}_0$. Let $A^\mu$ be the associated PCAF with $\mu$. Put $B^\mu = \{x \in X; P_x(A^n^\mu > 0) > 0\}$ and $\bar{Y} = q$-supp $[\mu]$.

**Lemma 4.1.** $B^\mu$ and $X - B^\mu$ have $M$-invariant q.e. versions.

Proof. It is easy to see that the function $u(x) = P_x(A^n^\mu > 0)$ is excessive and hence $B^\mu$ is finely open and nearly Borel. Put $B^\mu_n = \{x \in X; P_x(A^n^\mu > 0) \geq \frac{1}{n}\}$. Then $B^\mu_n$ is a finely closed and nearly Borel set. For each $n$ and $x \in X - B^\mu$, we
have

\[ 1 = P_x(A_x^\mu = 0) \]
\[ = P_x(A_x^\mu = 0; \sigma_x^\mu < \infty) + P_x(A_x^\mu = 0; \sigma_x^\mu = \infty) \]
\[ = P_x(A_x^\mu(\theta_x^\mu) = 0; \sigma_x^\mu < \infty) + P_x(A_x^\mu = 0; \sigma_x^\mu = \infty) \]
\[ = \mathbb{E}[P_{x \sigma_x^\mu}(A_x^\mu = 0; \sigma_x^\mu < \infty) + P_x(A_x^\mu = 0; \sigma_x^\mu = \infty)] \]
\[ \leq (1 - \frac{1}{n}) P_x(\sigma_x^\mu < \infty) + P_x(\sigma_x^\mu = \infty) \]
\[ = 1 - \frac{1}{n} P_x(\sigma_x^\mu < \infty). \]

Letting \( n \to \infty \), we get \( P_x(\sigma_x^\mu < \infty) = 0 \) for any \( x \in X - B^a \). In particular, \( X - B^a \) is \( T_7 \)-invariant and finely open. Since \( B^a \) is also finely open, we can find by Theoreme 2.2 and Lemma 3.1 a properly exceptional set \( N \) such that \( X - B^a - N \) and \( B^a - N \) are \( M \)-invariant. The proof is complete.

By the above lemma we may assume that \( X^{(\alpha)} - B^a \) and \( X^{(\alpha)} \cap B^a \) are \( M \)-invariant. For each \( \alpha > 0 \), we let \( \nu_\alpha = \alpha \mu + I_{X^{(\alpha)} - B^a} m \). Then \( \nu_\alpha \in \mathcal{M}_0 \), \( A_{\nu_\alpha} = \alpha A_\mu + \int_0^1 I_{X^{(\alpha)} - B^a}(X_s) \, ds \) and \( X^{(\alpha)} \in X^{(\alpha)} \subseteq B^a \) q.e.. By Proposition 3.4, we see that the extended Dirichlet space \( (E^{(\alpha)}, \mathcal{F}_{\nu_\alpha}) \) can be defined as the \( E^{(\alpha)} \)-completion of \( E^{(\alpha)} \) and that \( E^{(\alpha)} \)-capacity is equivalent to \( E_\nu \)-capacity. Note that the spaces \( \mathcal{F}_{\nu_\alpha} \) and \( \mathcal{F}_{\nu_\beta} \) is independent of \( \alpha > 0 \). We denote \( \mathcal{F}_{\nu} \) (resp. \( \mathcal{F}_{\nu}^\gamma \)) instead of \( \mathcal{F}_{\nu_\alpha} \) (resp. \( \mathcal{F}_{\nu_\beta} \)). Without loss of generality, we shall assume that every element of \( \mathcal{F}_{\nu} \) is \( E_\nu \)-quasi-continuous. We let \( \mathcal{F}_x^{(\nu)} = \{ u \in \mathcal{F}_{\nu} ; u = 0 \text{ q.e. on } Y \} \). This is a closed subspace of \( \mathcal{F}_{\nu} \) and the Hilbert space \( (E^{(\alpha)}, \mathcal{F}_{\nu}) \) admits the orthogonal decomposition

\[ \mathcal{F}_{\nu} = \mathcal{F}_{x \nu} \oplus \mathcal{H}_{\nu}^{\nu}, \]

where \( \mathcal{H}_{\nu}^{\nu} \) is the orthogonal complement of \( \mathcal{F}_{x \nu} \) with respect to \( E^{(\alpha)} \). Denote by \( \mathcal{P}_{\nu} \) the orthogonal projection on \( \mathcal{H}_{\nu}^{\nu} \). Note that the space \( \mathcal{H}_{\nu}^{\nu} \) and the projection \( \mathcal{P}_{\nu} \) are independent of \( \alpha > 0 \). Indeed for any \( u \in \mathcal{H}_{\nu}^{\nu} \) and \( \beta > 0 \),

\[ E^{(\beta)}(u, v) = E^{(\nu)}(u, v) + (\beta - \alpha)(u, v), \quad v \in \mathcal{F}_{x \nu}, \]

because \( \mu(X - Y) = 0 \) ([6]). Hence \( u \in \mathcal{H}_{\nu}^{\nu} \). Consequently \( \mathcal{P}_{\nu} \) is also independent of \( \alpha > 0 \). We may omit the index \( \alpha \) from \( \nu_\alpha \). We notice that, for \( f, g \in \mathcal{F}_{\nu} \), \( \mathcal{P}_{\nu} f = \mathcal{P}_{\nu} g \) if and only if \( f = g \) q.e. on \( \overline{Y} \).

We assume that \( \mu \) is non-trivial. Put \( Y = \text{supp}[\mu] \). Define a symmetric bilinear form on \( L^2(Y; \mu) \) by

\[ (4.1) \]

\[ \begin{align*}
\{ \mathcal{F}_{\nu} = \{ u \in L^2(Y; \mu) ; u = v |_Y \text{ (}\mu\text{-a.e. on } Y \text{ for some } v \in \mathcal{F}_{\nu}\} \\
E^{(\nu)}(u, u) = E^{(\nu)}(\mathcal{P}_{\nu} u, \mathcal{P}_{\nu} v), \text{ for } u \in \mathcal{F}_{\nu}, v \in \mathcal{F}_{\nu} \text{ s.t. } u = v |_Y \text{ (}\mu\text{-a.e.)} \}
\end{align*} \]
where $v|_Y$ is the restriction of function $v$ to $Y$ and $E^\gamma u(v, v) = E^\gamma u(v, v) - (v, v)$ for $v \in \mathcal{D}_\gamma^\gamma$. $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$ is a well defined closed symmetric form on $L^2(Y; \mu)$.

**Theorem 4.2.** $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$ is the Dirichlet space on $L^2(Y; \mu)$ associated with the time changed process $M^\tau=(X_\tau, P_\mu)_x\in \mathcal{F}$. Here $\tau_1 = \inf \{ t > 0; A^\mu_t > t \}$. $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$ is $C_0$-regular and has the core $C|_Y = \{ u \in C_0(Y); \text{for some } v \in C, u = v|_Y \}$.

Proof. First we show that $C|_Y$ is a core of $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$. For $u \in \mathcal{D}_\gamma^\gamma$, there exists $v \in \mathcal{D}_\gamma^\gamma$ such that $u = v|_Y$ $\mu$-a.e.. Since $C$ is a core of $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$, there exists $\{ v_n \} \subset C$ such that $\lim_{n \to \infty} E^\gamma(u_n - v, v_n - v) = 0$. By (4.1) we get

$$\lim_{n \to \infty} E^\gamma_\gamma(u_n - v|_Y, u - v|_Y) = \lim_{n \to \infty} E^\gamma_\gamma(\mathcal{P}^\gamma(v_n - v), \mathcal{P}^\gamma(v - v_n))$$

$$\leq \lim_{n \to \infty} E^\gamma(v_n - v, v_n - v) = 0.$$

For $u \in C_0(Y)$, there exists $w \in C_0(X)$ such that $u = w|_Y$ $\mu$-a.e.. Since $w$ is uniformly approximated by an element of $C$, $u$ is uniformly approximated by an element of $C|_Y$.

Next we show that, for $u \in \mathcal{B}_0(Y) \cap L^2(Y; \mu)$ and $v \in \mathcal{D}_\gamma^\gamma$,

$$\begin{aligned}
\{ \mathcal{R}_u u \in \mathcal{D}_\gamma^\gamma \\
\mathcal{E}^\gamma_\gamma(\mathcal{R}_u u, v) = (u, v)_\mu
\end{aligned}
$$

(4.2)

where $\mathcal{R}_u u(x) = E_\tau \int_0^\infty e^{-sA^\mu_t} u(X_t) dA^\mu_t$, $x \in \mathcal{Y}$, is the resolvent kernel for $M^\tau$. We introduce the kernel $V_\mu$ on $X$ by

$$V_\mu f(x) = E_\tau \int_0^\infty e^{-sA^\mu_t} f(X_t) dA^\mu_t, x \in X, f \in \mathcal{B}_0(X) .$$

(4.3)

Take now $u \in \mathcal{B}_0(Y) \cap L^2(Y; \mu)$ and let $u$ be any bounded Borel extension of $u$ to $X$. Then $\mathcal{R}_u u = V_\mu u|_Y$. Applying Theorem 2.4 and Corollary 2.7 in [12] to $A^\mu$ and $A^\mu_t$, $E_\tau \int_0^\infty e^{-sA^\mu_t} u(X_t) dA^\mu_t$, $x \in X$ is seen to be a quasi-continuous version of $0$-order potential $U^\gamma(u|_Y)$ with respect to $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$. Note that only the transience of $(E_\gamma^\gamma, \mathcal{D}_\gamma^\gamma)$ is used and the irreducibility condition is irrelevant in the proof of Theorem 2.4 and Corollary 2.7 in [12]. By Lemma 4.1 and the identity $P_\mu(A^\mu_t = 0,$ for any $t > 0) = 1$ q.e. $x \in X - B^\mu$, we conclude that $V_\mu u$ is a quasi-continuous version of $U^\gamma(u|_Y)$, and accordingly $\mathcal{R}_u u \in \mathcal{D}_\gamma^\gamma$ and moreover $V_\mu u = \mathcal{D}^\gamma u \in \mathcal{D}_\gamma^\gamma$. Let $v$ be an element of $\mathcal{D}_\gamma^\gamma$ such that $v = v|_Y$ $\mu$-a.e.. Noting that $\mathcal{D}^\gamma f = f$ $\mu$-a.e. on $Y$ for each $f \in \mathcal{D}_\gamma^\gamma$, we have

$$\begin{aligned}
\mathcal{E}^\gamma_\gamma(\mathcal{R}_u u, v) &= \mathcal{E}^\gamma_\gamma(\mathcal{R}_u u, v) + \alpha(\mathcal{R}_u u, v)_\mu \\
&= E^\gamma (V_\mu u, \mathcal{P}^\gamma v) + \alpha \mathcal{P}^\gamma V_\mu u, \mathcal{P}^\gamma v)_\mu \\
&= E^\gamma (V_\mu u, \mathcal{P}^\gamma v) = E^\gamma (U^\gamma(u|_Y), \mathcal{P}^\gamma v)
\end{aligned}$$
The proof is complete.

For each $u \in \mathcal{B}_+ (X)$, we denote $H \gamma u (x) = E_x [u (X \sigma \gamma)]$.

**Corollary 4.3.** $H \gamma v$ is a quasi-continuous version of $\mathcal{L}^\gamma v$ for each $v \in \mathcal{F}^\gamma_*$ and the time changed Dirichlet space $(\mathcal{F}^\gamma_* , \mathcal{E}^\gamma_*)$ is given by

$$
\mathcal{F}^\gamma_* = \{ u \in L^2 (Y; \mu); u = v \big|_Y \mu \text{-a.e. on } Y \text{ for some } v \in \mathcal{F}^\gamma_n \}
$$

$$
\mathcal{E}^\gamma_*(u, u) = \mathcal{E}(H \gamma v, H \gamma v), \text{ for } u \in \mathcal{F}^\gamma_* , v \in \mathcal{F}^\gamma_* \text{ s.t. } u = v \big|_Y \mu \text{-a.e.}
$$

Proof. Since $\gamma dB \mu$ q.e., we get $H \gamma v (x) = E_x [v (X_m)] = 0 \text{ q.e. } x \in X - B^\mu$. Therefore the latter assertion holds. Next we show the first assertion. We may assume that $v \in \mathcal{F}^\gamma_n$ is non-negative. Put $v_n = v \land n$. Noting that $\sigma \gamma (\omega) = \inf \{ t > 0; A^\mu_t (\omega) > 0 \}$, we get from (4.3)

$$
H \gamma v_n (x) = \lim_{m \to \infty} mV^\gamma_m v_n (x).
$$

On the other hand $mV^\gamma_m v_n = \mathcal{L}^\gamma v_n \mu$ is $\mathcal{E}^\gamma$-convergent to $\mathcal{L}^\gamma v_n \in \mathcal{F}^\gamma_n$ as $m \to \infty$ because $mR^\gamma_m (v_n \big|_Y)$ is $\mathcal{E}^\gamma_\alpha$-convergent to $v_n \big|_Y \in \mathcal{F}^\gamma_Y$ as $m \to \infty$. We get $H \gamma v_n = \mathcal{L}^\gamma v_n$ q.e.. Since $\mathcal{L}^\gamma v_n$ is $\mathcal{E}^\gamma$-convergent to $\mathcal{L}^\gamma v \in \mathcal{F}^\gamma_Y$ as $n \to \infty$, we have

$$
H \gamma v = \lim_{n \to \infty} H \gamma v_n
$$

$$
= \lim_{n \to \infty} \mathcal{L}^\gamma v_n = \mathcal{L}^\gamma v \text{ q.e.}
$$

The proof is complete.

By Theorem 4.2 we can get next result in the similar manner as in Section 4 in [12].

**Theorem 4.4.** (i) For a Borel set $B \subset Y$,

$$
\mathcal{E}^\gamma_\alpha \text{-Cap} (B \cap \bar{Y}) = 0 \text{ if and only if } \mathcal{E}_1 \text{-Cap} (B \cap \bar{Y}) = 0.
$$

(ii) For any decreasing sequence of open sets $A_n$, $\mathcal{E}_1 \text{-Cap} (A_n) \downarrow 0$ implies $\mathcal{E}^\gamma_\alpha \text{-Cap} (A_n \cap \bar{Y}) \downarrow 0$. In case $\mu \in \mathcal{M}_0$, $\mathcal{E}_1 \text{-Cap}$ is equivalent to $\mathcal{E}^\gamma_\alpha \text{-Cap}$.

(iii) $\mathcal{E}^\gamma_\alpha \text{-Cap} (Y - \bar{Y}) = 0$.

(iii) There exists a Borel set $\bar{N}$ with $\mu (\bar{N}) = 0$ such that $Y - \bar{Y} \subset \bar{N}$ and $\bar{Y} - \bar{N}$ is $\mathcal{M}^\gamma$-invariant. And further the restricted process $M^\gamma_{t|\bar{Y}-\bar{N}}$ of the time changed process $M^\gamma$ is a Hunt process on $\bar{Y} - \bar{N}$ associated with the regular Dirichlet space $(\mathcal{E}^\gamma_* , \mathcal{F}^\gamma_*)$.

5. **Closable part of a pre-Dirichlet form on $\mathcal{C}|_Y$**

A non-negative definite symmetric bilinear form $\mathcal{A}$ on $\mathcal{C}$ is called a pre-
Dirichlet form if there exists a function $\varphi_z$ satisfying condition $(C.2)$ and $\mathcal{A}(\varphi_z(u), \varphi_z(u)) \leq \mathcal{A}(u, u)$ for any $u \in \mathcal{C}$. For a closed set $Y$, $\mathcal{C}|_Y = \{ u \in C_0(Y); u = \bar{u}|_Y \}$ for some $\bar{u} \in \mathcal{C}_Y$ satisfies $(C.2)$ and $(C.1)$ with respect to the relative topology on $Y$. A pre-Dirichlet form $(\mathcal{A}, \mathcal{C}|_Y)$ is said to be closable on $L^2(Y; \mu)$ for a positive Radon measure $\mu$ on $Y$ if $\mathcal{A}(u_n, w_n) \to 0$ whenever $\{ w_n \} \in \mathcal{C}|_Y$ is $\mathcal{C}$-Cauchy and $u_n \to 0$ in $L^2(Y; \mu)$. A pre-Dirichlet form $(\mathcal{A}, \mathcal{C}|_Y)$ is said to be the closable part of $(\mathcal{A}, \mathcal{C}|_Y)$ on $L^2(Y; \mu)$ if $(\mathcal{C}, \mathcal{C}|_Y)$ is closable on $L^2(Y; \mu)$ for a positive Radon measure $\mu$ on $Y$ with $Y = \text{supp}[\mu]$. Let $(\mathcal{E}, \mathcal{F})$ be a $C_0$-regular Dirichlet space as in Section 2. In general, a function $u$ defined m.a.e. is said to belong to the extended Dirichlet space $\mathcal{F}$ if there exists an $\mathcal{A}$-Cauchy sequence $\{ u_n \} \subset \mathcal{F}$ such that $u_n \to u$, m.a.e. as $n \to \infty$. In this case we define $\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(u_n, u_n)$. $\mathcal{E}(u, u)$ does not depend on the choice of $\{ u_n \}$ ([16]). It is easy to see that $u \in \mathcal{F}$ if and only if there exists an $\mathcal{E}$-Cauchy sequence $\{ v_n \} \subset \mathcal{C}$ such that $v_n \to u$, m.a.e. as $n \to \infty$, and that $\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(v_n, v_n)$ in this case.

Lemma 5.1. (i) $u \in \mathcal{F}$ has quasi-continuous version $\bar{u}$.
(ii) Every normal contraction operates on $(\mathcal{F}, \mathcal{E})$.
(iii) For a Borel set $B$, let $H_B \bar{u}(x) = E_x[\bar{u}(X_{\sigma_B})]$. Then $H_B \bar{u} \in \mathcal{F}$, for any $u \in \mathcal{F}$. Furthermore

\begin{align*}
\mathcal{E}(u, v) = \mathcal{E}(H_B \bar{u}, H_B \bar{v}) + \mathcal{E}((I-H_B) \bar{u}, (I-H_B) \bar{v}), \text{ for any } u, v \in \mathcal{F}.
\end{align*}

Proof. For each $g \in L^1(\mathbb{X}; m)$ with $g > 0$ m.a.e., the finite measure $g m$ belongs to $\mathcal{M}_m$. Hence the $g m$-killed Dirichlet space $(\mathcal{E}^{gm}, \mathcal{F}^{gm})$ is transient by Corollary 3.6. Denote by $\mathcal{F}^{gm}$ its extended Dirichlet space. By (4.1) the time changed Dirichlet space $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\mathbb{X}; gm)$ associated with the time changed process $M^t$ by the PCAF $A_t = \int_0^t g(X_i) \, dt$ is given by

\begin{align*}
\tilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v), \text{ for any } u, v \in \mathcal{F}.
\end{align*}

and $\mathcal{C}$ is a core of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Now the extended Dirichlet space $\tilde{\mathcal{F}}$, of this time changed Dirichlet space coincides with $\mathcal{F}$. We therefore get $\mathcal{F}(\mathbb{X}; gm) = \mathcal{F}(\mathbb{X}; gm) = \tilde{\mathcal{F}} = \mathcal{F}^{gm}$ by [16]. For each $u \in \mathcal{F}$, choose $g \in L^1(\mathbb{X}; m)$, $g > 0$ m.a.e. such that $u \in L^2(\mathbb{X}; gm)$. Then $u \in \mathcal{F}$ with this choice of $g$. Thus (i) follows from $C_0$-regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ and (ii) follows from that every normal contraction operates on $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. 

782 K. KUWAE

$K_uw_e$
Next we show (iii). For each Borel set $B$, we denote $\tilde{\mathcal{F}}_{X-B} = \{ \omega \in \tilde{\mathcal{F}}; \overline{\omega} = 0 \text{ q.e. on } B \}$. Then $\tilde{\mathcal{F}}$ admits the orthogonal decomposition as follows: For each $p > 0$,

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{X-B} \oplus \tilde{\mathcal{H}}_B,$$

where $\tilde{\mathcal{H}}_B$ is the orthogonal complement of $\tilde{\mathcal{F}}_{X-B}$ with respect to $\tilde{\mathcal{E}}_\rho = \tilde{\mathcal{E}} + p(\cdot, \cdot)_{\mathcal{F}_e}$. For each $u \in \tilde{\mathcal{F}}_e$ we denote $H_B^\rho \overline{u}(x) = E_x[e^{-pBx} \overline{u}(X_{\sigma_p})]$. Letting $M^\rho = (Y, P_\rho)$ and denoting by $\delta_B$ its hitting time, we see that $H_B^\rho \overline{u}(x) = E_x[e^{-pBx} \overline{u}(Y_\sigma x)]$ and hence $H_B^\rho \overline{u}$ is the quasi-continuous version of $P_\rho \tilde{\mathcal{H}}_B u$, where $P_\rho \tilde{\mathcal{H}}_B$ is the projection to $M^\rho$. Hence we have

$$\tilde{\mathcal{E}}_\rho(u, v) = \tilde{\mathcal{E}}_\rho(H_B^\rho \overline{u}, H_B^\rho \overline{v}) + \tilde{\mathcal{E}}_\rho((I-H_B^\rho) \overline{u}, (I-H_B^\rho) \overline{v}), \text{ for any } u, v \in \tilde{\mathcal{F}}_e.$$

Fix non-negative $u, v \in \mathcal{F}_e$. Choose $g \in L(X; m)$, $g > 0$, m.a.e. such that $u, v \in \mathcal{F}_e$. Consider the time changed Dirichlet space $(\mathcal{E}, \tilde{\mathcal{F}})$ with this choice of $g$.

Put $u_n = u \wedge n, v_n = v \wedge n$. Then $u_n, v_n \in \mathcal{F}$ and $u_n \to u, v_n \to v, n \to \infty$ in $\mathcal{F}_e$. Since $B - B'$ is $\mathcal{E}$-polar, $H_B^\rho u_n - H_B^\rho u_n \in \mathcal{F}_e$. Hence we have

$$\mathcal{E}(u_n, v_n) = \mathcal{E}(H_B^\rho \overline{u}_n, H_B^\rho \overline{v}_n) + \mathcal{E}((I-H_B^\rho) \overline{u}_n, (I-H_B^\rho) \overline{v}_n),$$

namely, $H_B^\rho \overline{u}_n$ is $\mathcal{E}_1$-Cauchy. We have $H_B^\rho \overline{u}_n \in \tilde{\mathcal{F}}$ and

$$\mathcal{E}(u_n, v_n) = \mathcal{E}(H_B^\rho \overline{u}_n, H_B^\rho \overline{v}_n) + \mathcal{E}((I-H_B^\rho) \overline{u}_n, (I-H_B^\rho) \overline{v}_n)+ \mathcal{E}(u_n, v_n) + \mathcal{E}((I-H_B^\rho) u_n, (I-H_B^\rho) v_n).$$

Since $u_n$ and $v_n$ are $\mathcal{E}_1$-convergent to $u, v$ as $n \to \infty$, we arrive at (5.1). The proof is complete.

For a finely closed Borel set $F$ and a closed set $Y$ with $F \subset Y \subset X$, we introduce a symmetric bilinear form $(\mathcal{A}_F, C|_Y)$ by

$$\mathcal{A}_F(u, v) = \mathcal{E}(H_F u, H_F v) u, v \in C|_Y, u, v \in C, u = u|_Y, v = v|_Y.$$

Suppose $u_n, u_2 \in C$ and $u_1 = u_2$ on $Y$. Then $H_F u_1(x) = E_x[u_1(X_{\sigma_F})] = E_x[u_2(X_{\sigma_F})] = H_F u_2(x)$. Hence $(\mathcal{A}_F, C|_Y)$ is well-defined.

**Lemma 5.2.**

$$\mathcal{A}_F(u, u) = \inf \{ \mathcal{E}(v, v); v \in \mathcal{F}_e, u = \mathcal{B} \text{ q.e. on } F \}, u \in C|_Y.$$

Proof. For each $u \in C|_Y$, we take $v \in \mathcal{F}_e$ such that $u = v$ q.e. on $F$. Then there exists a properly exceptional set $N$ such that $u(x) = \mathcal{B}(x)$ for $x \in F - N$. Since $F - N$ is again finely closed Borel set of $M|_{X-N}$, we have $H_F u(x) =...$
Theorem 5.3. \((A, C|_Y)\) is a pre-Dirichlet form.

Proof. Let \(\varphi_\epsilon\) be the function described in \((C, 2)\). It suffices to show that

\[ A_\varphi(\varphi_\epsilon(u), \varphi_\epsilon(u)) \leq A_\varphi(u, u), \text{ for any } u \in C|_Y. \]

For each \(u \in C|_Y\),

\[ A_\varphi(\varphi_\epsilon(u), \varphi_\epsilon(u)) = \inf \{ \mathcal{E}(v, v); v \in \mathcal{F}_\epsilon, \varphi_\epsilon(u) = v \text{ q.e. on } F \} \]
\[ \leq \inf \{ \mathcal{E}(\varphi_\epsilon(w), \varphi_\epsilon(w)); v \in \mathcal{F}_\epsilon, \varphi_\epsilon(u) = \varphi_\epsilon(w) \text{ q.e. on } F \} \]
\[ \leq \inf \{ \mathcal{E}(\varphi_\epsilon(w), \varphi_\epsilon(w)); v \in \mathcal{F}_\epsilon, u = \varphi_\epsilon(w) \text{ q.e. on } F \} \]
\[ = A_\varphi(u, u). \]

The proof is complete.

Each \(\mu \in \mathcal{M}\) is uniquely decomposed as follows:

\[ \mu = \mu_0 + \mu_1 \]
\[ \mu_0 \in \mathcal{M}_0, \mu_1 = I_N \mu \text{ for some } E_1\text{-polar set } N. \]

\(\mu_0\) is called the smooth part of \(\mu\), (cf. Fukushima-Sato-Taniguchi [10]). We let \(Y = \text{supp}[\mu], Y_0 = \text{supp}[\mu_0]\) and \(\tilde{Y}_0 = \text{q-sup}[\mu_0]\). The \(E_1\)-polar set \(N\) is unique upto a \(\mu\)-negligible set. We may assume that \(N \subset Y\). Hence \(Y_0 \cup N \subset Y\). We state the main theorem in this section.

Theorem 5.4. (i) \((A_{\tilde{Y}_0}, C|_Y)\) is the closable part of \((A_Y, C|_Y)\) on \(L^2(Y; \mu)\).

(ii) Suppose that \(E_1\text{-Cap}(Y - \tilde{Y}_0) = 0\). Then \((A_Y, C|_Y)\) is closable on \(L^2(Y; \mu)\) and \(X^{(c)} - B^\mu = \emptyset \text{ q.e. }\)

(iii) Suppose \((A_Y, C|_Y)\) is closable on \(L^2(Y; \mu)\) and \(X^{(c)} - B^\mu = \emptyset \text{ q.e.}\). Then \(E_1\text{-Cap}(Y - \tilde{Y}_0) = 0\).

(iv) The closure \((A_{\tilde{Y}_0}, C|_Y)\) on \(L^2(Y; \mu)\) is associated with the Hunt process \(M^\mu = (X^i, P^\mu_x)\) such that

(a) "the law of \(X^\mu\) under \(P^\mu_x\) is the law of \(X^\mu_0\) under \(\hat{P}^\mu_{x_0}\)" for any \(x \in Y_0 - N, \)

(b) \(P^\mu_x(X^i_t = x, \text{ for any } t \geq 0) = 1, \text{ for any } x \in N, \)

(c) \(Y - Y_0 - N\) is an exceptional set for \(M^\mu,\)

where \(M^\mu_0 = (X^i, \hat{P}^\mu_{x_0})\) is the Hunt process associated with the time changed regular Dirichlet space \((\mathcal{F}^\mu_\mathcal{F}_2, \mathcal{F}^\mu_\mathcal{F}_2)\) on \(L^2(Y_0; \mu_0).\)

Remark. By Theorem 4.4 the condition (a) and (c) can be replaced by
"the law of $X_x$ under $P_x^a" = "the law of $X_{\tau_n^a}$ under $P^a_x$" for any $x \in \bar{Y}_0 - \bar{N}_0 - N$,
(c') $Y - \bar{Y}_0 - \bar{N}_0 - N$ is an exceptional set of $M^a$,
where $M^a = (X_{\tau^D_n}, P^a_x)_{x \in \bar{Y}_0}$ is the time changed process by the PCAF $A^a$ and $N_0$ is
a properly exceptional set of $M^a$.

To prove this theorem we prepare several lemmas as in [10].

Lemma 5.5. For a closed set $\hat{X} \subset X$, we let $\hat{m} \in \mathcal{M}$ with $\hat{X} = \text{supp} [\hat{m}]$ and
$(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ be another Dirichlet form on $L^2(\hat{X}; \hat{m})$ with $\mathcal{C}|\hat{X} \subset \hat{\mathcal{F}}$. Assume that $\hat{\mathcal{E}}(u, u) 
\leq \mathcal{E}(u, u)$, $u \in \mathcal{C}|\hat{x}$, $\mathcal{F} \in \mathcal{C}$, $u = \mathcal{F}|\hat{x}$. Then for any $\mathcal{E}$-polar set $N'$,
$$\hat{\mathcal{E}}_\alpha(I_{N'} \hat{x} u) = \frac{1}{\alpha} I_{N'} \hat{x} u, \quad \hat{m}-a.e. \text{ on } \hat{X} \text{ for any } u \in L^2(\hat{X}; \hat{m}),$$
where $\mathcal{E}_\alpha$ is the resolvent on $L^2(\hat{X}; \hat{m})$ associated with $\hat{\mathcal{E}}$.

Proof. The proof is the same as in Lemma 4.1 in [10].

Lemma 5.6. Let $(\mathcal{B}, \mathcal{C}|Y)$ be a closable pre-Dirichlet form on $L^2(Y; \mu)$ such that
$\mathcal{B}(u, u) \leq \mathcal{E}(u, u)$, $u \in \mathcal{C}|Y$, $u \in \mathcal{F}$, $u = \mathcal{F}|Y$. Then $(\mathcal{B}, \mathcal{C}|Y)$ is well-defined on
$L^2(Y_0; \mu_0)$ and closable on $L^2(Y_0; \mu_0)$.

Proof. The proof is same as in Lemma 4.2 in [10].

Lemma 5.7. $(\mathcal{A}_{\hat{Y}_0}, \mathcal{C}|Y)$ is the closable part of $(\mathcal{A}_Y, \mathcal{C}|Y)$ on $L^2(Y; \mu)$.

Proof. This follows from the description of Corollary 4.3 of the time changed Dirichlet space as the proof of Lemma 4.3 in [10]. We give the proof for completeness. We let $\nu_0 = \mu_0 + I_{X(\nu_0) = 0}\mu$. Then the $\nu_0$-killed Dirichlet space $(\mathcal{F}_0, \mathcal{E}_0)$ is transient. Let $\mathcal{F}_0$ be the extended Dirichlet space of
$(\mathcal{F}_0, \mathcal{E}_0)$. We let $\mathcal{F}_{0X-\hat{Y}_0} = \{u \in \mathcal{F}_0 \mid u = 0 \text{ q.e. on } \hat{Y}_0\}$. Let $\mathcal{F}_0$ be the projec-
tion operator on the orthogonal complement of $\mathcal{F}_{0X-\hat{Y}_0}$ with respect to $\mathcal{E}_0$. Since $\mathcal{E}_0$-Cauchy sequence is an $\mathcal{E}$-Cauchy sequence, $\mathcal{F}_0 u \in \mathcal{F}_0$ for any $u \in \mathcal{F}_0$. Note that
$$\mathcal{A}_{\hat{Y}_0}(u, u) = \mathcal{E}(\mathcal{F}_0 u, \mathcal{F}_0 u), u \in \mathcal{C}|Y, \mathcal{F} \in \mathcal{C}, u = \mathcal{F}|Y.$$
Indeed if $\mu_0$ is non-trivial, (5.3) follows from Corollary 4.3. Suppose that $\mu_0$ is trivial. Then $\hat{Y}_0 = \phi$ q.e.. We have $\mathcal{F}_{0X-\hat{Y}_0} = \mathcal{F}_0$ and $\mathcal{E}_0(\mathcal{F}_0 u, \mathcal{F}_0 u) = 0$. On
the other hand, $P_x(\sigma_{\hat{Y}_0} = \infty) = 1$ q.e. $x \in X$. We get $H_{\hat{Y}_0} u = 0$ q.e.. Thus we have
(5.3).
If $\mu_0$ is trivial, the closability of $(\mathcal{A}_{\hat{Y}_0}, \mathcal{C}|Y)$ on $L^2(Y; \mu)$ is clear. If $\mu_0$ is
non-trivial, the closability follows from (5.3) and Theorem 4.2. The inequality
$\mathcal{A}_{\hat{Y}_0}(u, u) \leq \mathcal{A}_Y(u, u)$, $u \in \mathcal{C}|Y$ follows from (5.1) and $H_{\hat{Y}_0} H_Y u = H_{\hat{Y}_0} u$, $u \in \mathcal{C}|Y$. Let $(\mathcal{B}, \mathcal{C}|Y)$ is a closable pre-Dirichlet form with $\mathcal{B}(u, u) \leq \mathcal{A}_Y(u, u)$ for $u \in \mathcal{C}|Y$. 

Fix an \( f \in \mathcal{C}_{|Y} \). Then there exists \( f \in \mathcal{C} \) such that \( f = f_{|Y} \). Since \( \mathcal{C} \) is dense in \( \mathcal{E}^\gamma_0 \), there exists a sequence \( \{f_n\} \subset \mathcal{C} \) such that
\[
\lim_{n \to \infty} \mathcal{E}^\gamma_0 (f_n - \mathcal{D}_\gamma f, f_n - \mathcal{D}_\gamma f) = 0.
\]

We have
\[
(5.4) \quad \{f_n\} \text{ is an } \mathcal{E}\text{-Cauchy sequence and } f_n \to f \text{ in } L^2(Y_0; \mu_0).
\]

By (5.3), we see that
\[
\mathcal{A}_{\tilde{Y}_0}(f, f) = \mathcal{E}(\mathcal{D}_\gamma f, \mathcal{D}_\gamma f) = \lim_{n \to \infty} \mathcal{E}(f_n, f_n).
\]

It follows from (5.3) and (5.1) that \( \{f_n|_Y - f\} \subset \mathcal{C}_{|Y} \) is an \( \mathcal{B}\text{-Cauchy sequence and } f_n - f \to 0 \) in \( L^2(Y_0; \mu_0) \). By Lemma 5.6, we have that \( \mathcal{B}(f_n|_Y - f, f_n|_Y - f) \to 0 \). Therefore it holds that
\[
\mathcal{B}(f, f) = \lim_{n \to \infty} \mathcal{B}(f_n|_Y, f_n|_Y) \leq \lim_{n \to \infty} \mathcal{E}(f_n, f_n) = \mathcal{A}_{\tilde{Y}_0}(f, f).
\]

The proof is complete.

**Lemma 5.8.** Suppose \((\mathcal{A}_Y, \mathcal{C}_{|Y})\) is closable on \( L^2(Y; \mu) \). Then
\[
\mathcal{E}(H_Y u - H_{\tilde{Y}_0} u, H_Y u - H_{\tilde{Y}_0} u) = 0, \quad \text{for any } u \in \mathcal{C}.
\]

**Proof.** By Lemma 5.7 we have
\[
\mathcal{E}(H_Y u, H_Y u) \leq \mathcal{E}(H_{\tilde{Y}_0} u, H_{\tilde{Y}_0} u) \quad \text{for any } u \in \mathcal{C}.
\]

Hence by (5.1)
\[
\mathcal{E}(H_Y u - H_{\tilde{Y}_0} u, H_Y u - H_{\tilde{Y}_0} u)
= \mathcal{E}(H_Y u, H_Y u) - 2\mathcal{E}(H_Y u, H_{\tilde{Y}_0} u) + \mathcal{E}(H_{\tilde{Y}_0} u, H_{\tilde{Y}_0} u)
= \mathcal{E}(H_Y u, H_Y u) - \mathcal{E}(H_{\tilde{Y}_0} u, H_{\tilde{Y}_0} u) \leq 0.
\]

**Lemma 5.9.** Denote the closure of \((\mathcal{A}_{\tilde{Y}_0}, \mathcal{C}_{|Y})\) on \( L^2(Y; \mu) \) by \((\mathcal{A}_{\tilde{Y}_0}, \tilde{\mathcal{C}}_{|Y})\). Let \( \{G^\alpha_{\tilde{Y}_0}, \alpha > 0\} \) (resp. \( \{G^\alpha_{\tilde{Y}_0}, \alpha > 0\} \)) be the resolvent on \( L^2(Y; \mu) \) (resp. \( L^2(Y_0; \mu_0) \)) associated with \((\mathcal{A}_{\tilde{Y}_0}, \tilde{\mathcal{C}}_{|Y})\) (resp. \((\mathcal{E}^\gamma_0, \mathcal{D}_\gamma))\). Then
\[
(i) \quad G^\alpha_{\tilde{Y}_0}(I_N u) = \frac{1}{\alpha} I_N u, \quad \mu\text{-a.e. for any } u \in L^2(Y; \mu).
\]
\[
(ii) \quad G^\alpha_{\tilde{Y}_0} u = G^\alpha_{\tilde{Y}_0} u, \quad \mu_0\text{-a.e. on } Y_0 \text{ for any } u \in L^2(Y_0; \mu_0).
\]
\[
(iii) \quad \mathcal{A}_{\tilde{Y}_0}^{-1}\text{-Cap}(Y - Y_0 - N) = 0.
\]

**Proof.** (i) follows from Lemma 5.5. The proof of (ii) is same as in Lemma 4.5 in [10]. For compact set \( K \subset Y - Y_0 \) in \( Y \), there exists a relatively compact open set \( G \) in \( Y \) and \( f \in \mathcal{C}_{|Y} \) such that \( G \subset Y - Y_0 \) and \( 0 \leq f \leq 1, f = 1 \) on
$K, f=0$ on $Y-G$. Then $\mathcal{A}_{\bar{Y}_0}(f,f)=0$. By Lemma 5.7 we have

$$\mathcal{A}_{\bar{Y}_0,1} \text{-Cap}(K) = \inf \{ \mathcal{A}_{\bar{Y}_0,1}(u,u); u \in C, u \leq 1 \text{ on } K \} \leq (f,f)_\mu \leq \mu(G).$$

Hence we can get $\mathcal{A}_{\bar{Y}_0,1} \text{-Cap}(B) \leq \mu(B)$ for any Borel set $B \subset (Y-Y_0)$, which implies (iii). The proof is complete.

Proof of Theorem 5.4. (i) follows from Lemma 5.7 (iv) follows from Lemma 5.9. We show (ii). Suppose $\mathcal{E}_1 \text{-Cap}(Y-Y_0)=0$. Then $(\mathcal{A}_{Y}, C|_Y) = (\mathcal{A}_{\bar{Y}_0}, C|_{\bar{Y}_0})$. Hence $(\mathcal{A}_{Y}, C|_Y)$ is closable on $L^2(Y; \mu)$ and $Y \setminus (X^{(c)}-B^\#) = \bar{Y}_0 \setminus (X^{(c)}-B^\#) = \phi$ q.e., because $\bar{Y}_0 \subset B^\#$. Next we show (iii). Suppose $X^{(c)}-B^\# = \phi$ q.e. and $(\mathcal{A}_{Y}, C|_Y)$ is closable on $L^2(Y; \mu)$. Then $\nu_0 = \mu_0$. We get $H_Y u = H_{\bar{Y}_0} u$ q.e., because $\nu_0 = \mu_0$. By Lemma 5.8 we have $\mathcal{E}_1 \text{-Cap}(H_Y u = H_{\bar{Y}_0} u, H_Y u - H_{\bar{Y}_0} u) = 0$ for any $u \in C$, namely $H_Y u = H_{\bar{Y}_0} u$ q.e. for any $u \in C$. Hence we have that $Y-Y_0$ is $\mathcal{E}_1$-polar. The proof is complete.

References


Department of Mathematics
Osaka University, Toyonaka
Osaka 560, Japan

Present Address
Department of Mathematics
Faculty of Science
Kochi University
Kochi 780, Japan