



Title	Equivariant K-theoretic Euler classes and maps of representation spheres
Author(s)	Komiya, Katsuhiro
Citation	Osaka Journal of Mathematics. 2001, 38(1), p. 239-249
Version Type	VoR
URL	https://doi.org/10.18910/9897
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

Komiya, K.
 Osaka J. Math.
38 (2001), 239–249

EQUIVARIANT K -THEORETIC EULER CLASSES AND MAPS OF REPRESENTATION SPHERES

Dedicated to Professor Fuichi Uchida on his 60th birthday

KATSUHIRO KOMIYA

(Received April 1, 1999)

0. Introduction

Let G be a compact Lie group, and U a unitary representation of G . The unit sphere SU of U is called a *representation sphere* of G . In this paper we study G -maps between finite dimensional representation spheres of G .

In Atiyah and Tall [2], Bartsch [3], tom Dieck and Petrie [4], Komiya [5], [6], Liulevicius [7], and Marzantowicz [8], the equivariant K -theory is successfully employed for the study of G -maps. In [2] and [4], that is employed for the study of degrees of G -maps between representation spheres. In [3], [5], [6] and [7], that is employed to obtain necessary conditions for the existence of G -maps. In [8], the equivariant K -theoretic Lefschetz number is defined.

The main tool in this paper is also the equivariant K -theory. We give the definitions of the Thom class $tU \in K_G(U)$ of U and the Euler class $eU \in K_G(\text{pt}) = R(G)$, and then show that if there exists a G -map $f : SU \rightarrow SW$ between representation spheres SU and SW then $eW = z(f) \cdot eU$ for some element $z(f) \in R(G)$ (Theorem 1.2). Using this equality, we show that if G is connected then the degree of f is uniquely determined only by U and W (Theorem 4.1).

If G is compact abelian, then the degree of f is more explicitly discussed. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle group of complex numbers with absolute value 1, and \mathbb{Z}_n be the cyclic group of order n considered as a subgroup of S^1 . For any integer i , let $V_i = \mathbb{C}$ be a complex representation of S^1 and \mathbb{Z}_n given by $(z, v) \mapsto z^i v$ for $z \in S^1$ (or \mathbb{Z}_n) and $v \in V_i$. A compact abelian group G decomposes into a cartesian product

$$G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l},$$

where $T^k = S^1 \times \cdots \times S^1$, the cartesian product of k copies of S^1 . Letting γ be a sequence $(a_1, \dots, a_k, b_1, \dots, b_l)$ of integers, denote by V_γ the tensor product

$$V_{a_1} \otimes \cdots \otimes V_{a_k} \otimes V_{b_1} \otimes \cdots \otimes V_{b_l},$$

which is considered as a representation of G . Let Γ be the set of sequences

$$\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$$

of integers a_i ($1 \leq i \leq k$) and b_j with $0 \leq b_j \leq n_j - 1$ ($1 \leq j \leq l$). The set $\{V_\gamma \mid \gamma \in \Gamma\}$ gives a complete set of irreducible unitary representations of G , and any unitary representation U of G decomposes into a direct sum

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)},$$

where $u(\gamma)$ is a nonnegative integer and $V_\gamma^{u(\gamma)}$ denotes the direct sum of $u(\gamma)$ copies of V_γ .

Let

$$W = \bigoplus_{\gamma \in \Gamma} V_\gamma^{w(\gamma)}$$

be a second unitary representation of G with $\dim U = \dim W$. Let $|\gamma| = a_1 + \dots + a_k + b_1 + \dots + b_l$ for $\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$. If there exists a G -map $f : SU \rightarrow SW$, then we obtain

$$\deg f \cdot \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} = \prod_{\gamma \in \Gamma} |\gamma|^{w(\gamma)} + nr$$

for $n = \text{g.c.d.}\{n_1, \dots, n_l\}$ and some integer r (Theorem 4.2).

In Marzantowicz [9, Theorems 2.2, 2.5], he obtained the same kind of results as this for $G = T^k$ or $G = \mathbb{Z}_p^k$ ($= \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$) with p prime. To obtain the result, he employed the Borel cohomology theory. In the Borel cohomology theory the Euler class resides in $H_G^*(\text{pt}; \mathcal{K}) = H^*(BG; \mathcal{K})$. The choice of the coefficients \mathcal{K} practically depends on G considered. For example, if $G = T^k$ then \mathcal{K} is \mathbb{Z} , and if $G = \mathbb{Z}_p^k$ then \mathcal{K} is \mathbb{Z}_p . This means that the Borel cohomological Euler classes are not so systematically treated as the equivariant K -theoretic Euler classes. In fact, [9] treated the cases of $G = T^k$ and $G = \mathbb{Z}_p^k$ separately, but in this paper we can simultaneously treat all of compact abelian groups as in Theorem 4.2.

As long as the author knows, the results about G -maps obtained by using the Borel cohomology theory are almost only for the cases of $G = T^k$ and $G = \mathbb{Z}_p^k$.

There seems one more advantage of the equivariant K -theoretic Euler classes. If $G = T^k$ for example, then a unitary representation U of G decomposes into a direct sum

$$U = \bigoplus_{\Gamma} (V_{a_1} \otimes \dots \otimes V_{a_k})^{u(a_1, \dots, a_k)}$$

where the direct sum \oplus_{Γ} is taken over all sequences (a_1, \dots, a_k) of integers. Denote

by $e'U$ the Borel cohomological Euler class of U . We see

$$eU = \prod_{\Gamma} (1 - x_1^{a_1} \cdots x_k^{a_k})^{u(a_1, \dots, a_k)} \quad \text{in} \quad R(T^k) = \mathbb{Z}[x_1, \dots, x_k]_L,$$

$$e'U = \prod_{\Gamma} (a_1 x_1 + \cdots + a_k x_k)^{u(a_1, \dots, a_k)} \quad \text{in} \quad H^*(BT^k; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k],$$

where $\mathbb{Z}[\dots]_L$ denotes the Laurent polynomial ring and $\mathbb{Z}[\dots]$ the ordinary polynomial ring. The degree of eU (as a polynomial) is higher than that of $e'U$. This means that eU contains much more information than $e'U$.

In the final section of this paper we show that the K -theoretic Euler class distinguishes the isomorphism class of real representations of S^1 (Theorem 5.1). But for the Borel cohomology theory we easily have examples of two nonisomorphic representations U and W of S^1 with $e'U = e'W$.

1. Thom classes and Euler classes

Let G be a compact Lie group. The equivariant K -ring $K_G(X)$ of a *compact* G -space X is defined to be the Grothendieck ring of the isomorphism classes of complex G -vector bundles over X (see Atiyah [1], Segal [10]). If X has a distinguished base point x_0 , then $\tilde{K}_G(X)$ is the kernel of the homomorphism $i^* : K_G(X) \rightarrow K_G(x_0)$ induced from the inclusion $i : x_0 \rightarrow X$. If X is noncompact but locally compact, then $K_G(X)$ is defined to be $\tilde{K}_G(X^+)$, where X^+ is the one-point compactification of X .

Given a complex G -vector bundle $E \rightarrow X$ over a compact G -space X , there is the Thom isomorphism $\Psi : K_G(X) \rightarrow K_G(E)$. In what follows we only consider the case in which X is the one-point space $\{\text{pt}\}$. In this case the total space of a G -vector bundle is a complex representation U of G , and the Thom isomorphism is an isomorphism $\Psi : R(G) \rightarrow K_G(U)$, since $K_G(\text{pt})$ is canonically identified with the complex representation ring $R(G)$ of G . The *Thom class* $tU \in K_G(U)$ of U is defined as $tU = \Psi(1)$, where $1 \in R(G)$ is the 1-dimensional trivial representation. The *Euler class* $eU \in R(G)$ is defined as $eU = s^*(tU)$, where $s^* : K_G(U) \rightarrow K_G(\text{pt}) = R(G)$ is the homomorphism induced from the zero section $s : \{\text{pt}\} \rightarrow U$. eU can also be given as $eU = \sum_i (-1)^i \Lambda^i U$, where $\Lambda^i U$ is the i -th exterior power of U . The Euler class is multiplicative, i.e., $e(U \oplus W) = eU \cdot eW$ for two complex representations U and W .

Let $f : SU \rightarrow SW$ be a G -map between representation spheres. The radial extension of f induces a homomorphism $f^* : K_G(W) \rightarrow K_G(U)$. Since $K_G(U)$ is a free $R(G)$ -module over just one generator tU , there exists a unique element $z(f) \in R(G)$ such that $f^*(tW) = z(f) \cdot tU$.

We consider an element of $R(G)$ as a class function $G \rightarrow \mathbb{C}$, and observe $z(f)(1)$, where 1 is the identity element of G .

Theorem 1.1 (cf. Atiyah-Tall [2, IV. §1]). *Let $f : SU \rightarrow SW$ be a G -map between representation spheres of a compact Lie group G . Then*

$$z(f)(1) = \begin{cases} \deg f & \text{if } \dim U = \dim W \\ 0 & \text{if } \dim U \neq \dim W \end{cases}$$

where $\deg f$ is the Brouwer degree of f with respect to the canonical orientation of SU and SW induced from the complex structure.

Proof. Let DU denote the unit disk of U . Consider the following commutative diagram:

$$\begin{array}{ccc} K_G(W) = K_G(DW, SW) & \xrightarrow{f_1^*} & K_G(DU, SU) = K_G(U) \\ F \downarrow & & \downarrow F \\ K(DW, SW) & \xrightarrow{f_2^*} & K(DU, SU) \\ ch \downarrow & & \downarrow ch \\ \mathbb{Z} \cong H^{ev}(DW, SW; \mathbb{Z}) & \xrightarrow{f_3^*} & H^{ev}(DU, SU; \mathbb{Z}) \cong \mathbb{Z}, \end{array}$$

where f_i^* is the homomorphism induced from f , F is the homomorphism which forgets the G -action, ch is the Chern character which has its image in the cohomology with integer coefficients in this case, and $H^{ev}(\)$ denotes the direct sum of even dimensional parts.

We see

$$\begin{aligned} ch \circ F \circ f_1^*(tW) &= ch \circ F(z(f) \cdot tU) \\ &= ch(z(f)(1) \cdot F(tU)) \\ &= z(f)(1), \end{aligned}$$

since $F(tU)$ is the canonical generator of $K(DU, SU) \cong \mathbb{Z}$, and on the other hand we also see

$$\begin{aligned} f_3^* \circ ch \circ F(tW) &= f_3^*(1) \\ &= \begin{cases} \deg f & \text{if } \dim U = \dim W \\ 0 & \text{if } \dim U \neq \dim W. \end{cases} \end{aligned}$$

By the commutativity of the diagram we can obtain the theorem. \square

Given a G -map $f : SU \rightarrow SW$, there is a commutative diagram:

$$\begin{array}{ccc}
 K_G(W) & \xrightarrow{f^*} & K_G(U) \\
 & \searrow s^* & \swarrow s^* \\
 & K_G(\text{pt}) = R(G) &
 \end{array}$$

From this we see in $R(G)$,

$$eW = s^*(tW) = s^*f^*(tW) = z(f) \cdot eU.$$

Thus we obtain

Theorem 1.2. *If there exists a G -map $f : SU \rightarrow SW$ between representation spheres of a compact Lie group G , we obtain in $R(G)$,*

$$eW = z(f) \cdot eU.$$

Corollary 1.3. *If $eU = 0$ and $eW \neq 0$, then there exists no G -map $SU \rightarrow SW$.*

The converse of Theorem 1.2 does not hold. We give an example in the following section. That shows that the answer to the question by Marzantowicz [9, Problem 2.6] is negative in K -theoretic version.

2. Representation rings of compact abelian groups

The complex representation ring $R(G)$ of a compact abelian group $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ is isomorphic to $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L/Y$, where $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$ is the Laurent polynomial ring with indeterminates $x_1, \dots, x_k, y_1, \dots, y_l$, and Y is the ideal generated by $1 - y_j^{n_j}$ ($1 \leq j \leq l$). For a Laurent polynomial

$$\varphi = \varphi(x_1, \dots, x_k, y_1, \dots, y_l) \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L,$$

we denote by $[\varphi]$ the element in $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L/Y$ represented by φ . Through the isomorphism $R(G) \cong \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L/Y$,

$$\begin{aligned}
 V_\gamma &= [x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}], \\
 eV_\gamma &= [1 - x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}]
 \end{aligned}$$

for $\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$.

EXAMPLE 2.1 (in which the converse of Theorem 1.2 does not hold). Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$ is the cyclic group of order 2. We see

$$R(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}[x, y]/(1 - x^2, 1 - y^2)$$

where $(1 - x^2, 1 - y^2)$ is the ideal generated by $1 - x^2$ and $1 - y^2$. For $i, j \in \{0, 1\}$, $V_{(i,j)}$ is the one-dimensional complex representation on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts in such a way that $((s, t), v) \mapsto s^i t^j v$ for $(s, t) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $v \in V_{(i,j)} = \mathbb{C}$. Let $U = V_{(1,0)} \oplus V_{(1,0)}$ and $W = V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. We see in $R(\mathbb{Z}_2 \times \mathbb{Z}_2)$

$$\begin{aligned} eU &= eV_{(1,0)} \cdot eV_{(1,0)} = [(1 - x)^2] = [2(1 - x)], \\ eW &= eV_{(1,0)} \cdot eV_{(0,1)} \cdot eV_{(1,1)} = [(1 - x)(1 - y)(1 - xy)] = 0. \end{aligned}$$

Thus eU divides eW , but we can see that there can not exist a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -map $f : SU \rightarrow SW$. Assume that such $f : SU \rightarrow SW$ exists, and let $H_1 = \mathbb{Z}_2 \times \{1\}$, $H_2 = \{1\} \times \mathbb{Z}_2$. Restricting f to the fixed point set of the H_2 -action, we obtain an H_1 -map $f^{H_2} : S(U^{H_2}) \rightarrow S(W^{H_2})$. Since $U^{H_2} = V_{(1,0)} \oplus V_{(1,0)}$ and $W^{H_2} = V_{(1,0)}$, the existence of f^{H_2} contradicts the Borsuk-Ulam theorem.

Lemma 2.2. *Let $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$, and let*

$$\varphi = \varphi(x_1, \dots, x_k, y_1, \dots, y_l) \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$$

be a representative of

$$z \in R(G) \cong \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L / Y,$$

i.e., $z = [\varphi]$. Then

$$z(1) = \varphi(1, \dots, 1, 1, \dots, 1),$$

where 1 in the left hand side is the identity element of G , and 1's in the right hand side are the numerical 1.

Proof. There are complex representations U and W of G such that $z = U - W$. Decompose U and W into the direct sums of irreducible representations,

$$U = \bigoplus_{\gamma \in \Gamma} V_{\gamma}^{u(\gamma)} \quad \text{and} \quad W = \bigoplus_{\gamma \in \Gamma} V_{\gamma}^{w(\gamma)},$$

and let

$$\begin{aligned} \varphi_1 &= \varphi_1(x_1, \dots, x_k, y_1, \dots, y_l) = \sum_{\gamma \in \Gamma} u(\gamma)(\mathbf{xy})^{\gamma}, \\ \varphi_2 &= \varphi_2(x_1, \dots, x_k, y_1, \dots, y_l) = \sum_{\gamma \in \Gamma} w(\gamma)(\mathbf{xy})^{\gamma}, \end{aligned}$$

where $(\mathbf{xy})^{\gamma} = x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_l^{b_l}$ for $\gamma = (a_1, \dots, a_k, b_1, \dots, b_l)$. Then $z = [\varphi_1] - [\varphi_2]$ in $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L / Y$, and

$$z(1) = \dim U - \dim W$$

$$= \varphi_1(1, \dots, 1, 1, \dots, 1) - \varphi_2(1, \dots, 1, 1, \dots, 1).$$

Since

$$\varphi_1 - \varphi_2 = \varphi + \sum_{j=1}^l \psi_j \cdot (1 - y_j^{n_j})$$

for some $\psi_j \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$, we obtain $z(1) = \varphi(1, \dots, 1, 1, \dots, 1)$. \square

3. Nonvanishing of Euler classes

In this section we give a necessary and sufficient condition for the nonvanishing of Euler classes of representations of G . We first consider the case $G = \mathbb{Z}_n$. The set $\{V_i \mid 0 \leq i \leq n-1\}$ gives a complete set of irreducible complex representations of \mathbb{Z}_n , and $R(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}[x]/(1-x^n)$. If U is a complex representation of \mathbb{Z}_n decomposed into a direct sum $\bigoplus_{i=0}^{n-1} V_i^{u_i}$ for nonnegative integers u_i , then we see

$$(3.1) \quad eU = \prod_{i=0}^{n-1} (eV_i)^{u_i} = \left[\prod_{i=0}^{n-1} (1-x^i)^{u_i} \right].$$

Lemma 3.2. *For a complex representation $U = \bigoplus_{i=0}^{n-1} V_i^{u_i}$ of \mathbb{Z}_n , $eU \neq 0$ in $R(\mathbb{Z}_n)$ if and only if $U^{\mathbb{Z}_n} = \{0\}$, i.e., $u_0 = 0$.*

Proof. From (3.1) we easily see that $eU \neq 0$ implies $u_0 = 0$. Conversely assume $u_0 = 0$. Then

$$eU = \left[\prod_{i=1}^{n-1} (1-x^i)^{u_i} \right] \quad \text{in } \mathbb{Z}[x]/(1-x^n).$$

If $eU = 0$, then there is $\varphi(x) \in \mathbb{Z}[x]$ such that

$$(3.3) \quad \prod_{i=1}^{n-1} (1-x^i)^{u_i} = \varphi(x)(1-x^n) \quad \text{in } \mathbb{Z}[x].$$

We see

$$1 - x^i = - \prod_{j|i} \Phi_j(x),$$

where the product $\prod_{j|i}$ is taken over all divisors j of i , and $\Phi_j(x)$ is the j th cyclotomic polynomial which is irreducible in $\mathbb{Z}[x]$. From this the right hand side of (3.3) contains $\Phi_n(x)$ as a factor, but the left hand side does not. This is a contradiction. \square

For a subgroup H of G we obtain a homomorphism $\gamma_H : R(G) \rightarrow R(H)$ by restricting the action of G to H .

Lemma 3.4. *For any compact Lie group G ,*

$$\gamma = \bigoplus_H \gamma_H : R(G) \rightarrow \bigoplus_H R(H)$$

is injective, where the direct sum \bigoplus_H is taken over all finite cyclic subgroups H of G .

Proof. Assume that $\gamma(a) = 0$ for $a \in R(G)$. Then $\gamma_H(a) = 0$ in $R(H)$ for any finite cyclic subgroup H of G . To show $a = 0$, considering a as a class function, we will show $a(g) = 0$ for any $g \in G$. To show this we divide into two cases. First, if g is of finite order, i.e., $g^n = 1$ for some integer n , then g generates the cyclic group $\langle g \rangle$ of order n . By the assumption, we have $\gamma_{\langle g \rangle}(a) = 0$ in $R(\langle g \rangle)$. Hence

$$a(g) = \gamma_{\langle g \rangle}(a)(g) = 0.$$

Second, if g is of infinite order, then there exists a sequence of elements of finite order which converges to g . From the continuity of class function, we see $a(g) = 0$. \square

We obtain the following theorem by Lemmas 3.2 and 3.4.

Theorem 3.5. *For a complex representation U of a compact Lie group G , $eU \neq 0$ in $R(G)$ if and only if there exists a cyclic subgroup $\mathbb{Z}_n \leq G$ such that $U^{\mathbb{Z}_n} = \{0\}$.*

4. Degrees of G -maps

In this section we discuss the degree of a G -map between representation spheres.

Theorem 4.1. *Let U and W be unitary representations of a compact connected Lie group G with $\dim U = \dim W$ and $eU \neq 0$. If there exists a G -map $f : SU \rightarrow SW$, then*

- (i) $\deg f$ is uniquely determined only by U and W , and
- (ii) in particular, $\deg f = 0$ if $eW = 0$.

Proof. (i) Assume $g : SU \rightarrow SW$ is another G -map. From Theorem 1.2 we have $eW = z(f) \cdot eU$ and $eW = z(g) \cdot eU$, and hence $(z(f) - z(g)) \cdot eU = 0$ in $R(G)$. This implies $z(f) = z(g)$ since the restricting homomorphism $\gamma_{T^k} : R(G) \rightarrow R(T^k)$ is injective where T^k is a maximal torus of G , and since $R(T^k)$ has no zero divisor. Hence Theorem 1.1 shows $\deg f = z(f)(1) = z(g)(1) = \deg g$.
(ii) If $eW = 0$, we have $z(f) \cdot eU = 0$ and hence $z(f) = 0$. This implies $\deg f = z(f)(1) = 0$. \square

If G is compact abelian, the degree can be more explicitly discussed in the following theorem.

Theorem 4.2. *Let G be a compact abelian group, say, $G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$. Let*

$$U = \bigoplus_{\gamma \in \Gamma} V_{\gamma}^{u(\gamma)} \quad \text{and} \quad W = \bigoplus_{\gamma \in \Gamma} V_{\gamma}^{w(\gamma)}$$

be unitary representations of G with $\dim U = \dim W$. If there exists a G -map $f : SU \rightarrow SW$, then

$$\deg f \cdot \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} = \prod_{\gamma \in \Gamma} |\gamma|^{w(\gamma)} + nr$$

for $n = \text{g.c.d.}\{n_1, \dots, n_l\}$ and some integer r .

Proof. Theorem 1.2 gives $z(f) \cdot eU = eW$ in $R(G) \cong \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L / Y$. Let $\varphi \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$ be a representative of $z(f)$, i.e., $z(f) = [\varphi]$. The equality $z(f) \cdot eU = eW$ gives the following equality in $\mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$,

$$\varphi \cdot \prod_{\gamma \in \Gamma} (1 - (\mathbf{xy})^{\gamma})^{u(\gamma)} = \prod_{\gamma \in \Gamma} (1 - (\mathbf{xy})^{\gamma})^{w(\gamma)} + \sum_{j=1}^l \psi_j \cdot (1 - y_j^{n_j})$$

for some $\psi_j \in \mathbb{Z}[x_1, \dots, x_k, y_1, \dots, y_l]_L$. Substituting X for all of $x_1, \dots, x_k, y_1, \dots, y_l$, we obtain

$$(4.3) \quad \bar{\varphi}(X) \prod_{\gamma \in \Gamma} (1 - X^{|\gamma|})^{u(\gamma)} = \prod_{\gamma \in \Gamma} (1 - X^{|\gamma|})^{w(\gamma)} + \sum_{j=1}^l \bar{\psi}_j(X) (1 - X^{n_j})$$

in $\mathbb{Z}[X]_L$, where $\bar{\varphi}(X) = \varphi(X, \dots, X)$, $\bar{\psi}_j(X) = \psi_j(X, \dots, X)$. We note that

$$\prod_{\gamma \in \Gamma} (1 - X^{|\gamma|})^{u(\gamma)} = (1 - X)^{\sum_{\gamma \in \Gamma} u(\gamma)} \prod_{\gamma \in \Gamma} (1 + X + X^2 + \cdots + X^{|\gamma|-1})^{u(\gamma)},$$

and

$$\sum_{\gamma \in \Gamma} u(\gamma) = \dim U = \dim W = \sum_{\gamma \in \Gamma} w(\gamma).$$

Thus, dividing (4.3) by $(1 - X)^{\dim U}$, we obtain

$$\bar{\varphi}(X) \prod_{\gamma \in \Gamma} (1 + X + X^2 + \cdots + X^{|\gamma|-1})^{u(\gamma)} = \prod_{\gamma \in \Gamma} (1 + X + X^2 + \cdots + X^{|\gamma|-1})^{w(\gamma)} + A(X),$$

where

$$\begin{aligned} A(X) &= \frac{\sum_{j=1}^l \bar{\psi}_j(X)(1-X^{n_j})}{(1-X)^{\dim U}} \in \mathbb{Z}[X]_L \\ &= (1+X+X^2+\cdots+X^{n-1})B(X) \quad \text{for some } B(X) \in \mathbb{Z}[X]_L. \end{aligned}$$

Substituting 1 for X , we obtain the desired equality since $\bar{\varphi}(1) = \deg f$ from Theorem 1.1 and Lemma 2.2. \square

REMARK 4.4. In Theorem 4.2, if G is connected, i.e., $G = T^k$, then $n = 0$. If G is finite, i.e., $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$, and if $eU \neq 0$, then we see $\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \neq 0$ since $u(0, \dots, 0) = 0$ from Theorem 3.5. If G is not finite, then $\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)}$ can be zero even if $eU \neq 0$. For example, the representation $U = V_1 \otimes V_{-1}$ of $G = T^2$ gives an example for this. Essentially the same point as this is missed in [9]. If the last statement of [9, Theorem 2.2] would be correct, $\deg f$ would be always positive. This is incorrect.

5. Euler classes for representations of S^1

In this section we show that the Euler class distinguishes the isomorphism class of real representations of S^1 . The set $\{V_i \mid i \in \mathbb{Z}\}$ gives a complete set of irreducible complex representations of S^1 . We note that $R(S^1) \cong \mathbb{Z}[x]_L$, and that $V_i \cong V_{-i}$ as real representations.

Theorem 5.1. *Let U and W be two complex representations of S^1 . If $eU = eW \neq 0$ in $R(S^1)$, then $U \cong W$ as real representations.*

Proof. Decompose U and W into direct sums of irreducible representations as

$$U = \bigoplus_{i \in \mathbb{Z}} V_i^{u_i} \quad \text{and} \quad W = \bigoplus_{i \in \mathbb{Z}} V_i^{w_i}.$$

Here $u_0 = 0$ and $w_0 = 0$ since $eU \neq 0$ and $eW \neq 0$. Noting that $1 - x^i = -x^i(1 - x^{-i})$ in $\mathbb{Z}[x]_L$, we have

$$eU = (-1)^a x^b \prod_{i>0} (1 - x^i)^{u_i + u_{-i}},$$

where $a = \sum_{i<0} u_i$ and $b = \sum_{i<0} iu_i$. Factoring $1 - x^i$ into the cyclotomic polynomials $\Phi_j(x)$, we have

$$eU = (-1)^{a'} x^b \prod_{j>0} \Phi_j(x)^{c_j},$$

where $a' = a + \sum_{i>0}(u_i + u_{-i})$ and $c_j = \sum_{j|i}(u_i + u_{-i})$ is the sum of $u_i + u_{-i}$ with i a multiple of j . Similarly we have

$$eW = (-1)^r x^s \prod_{j>0} \Phi_j(x)^{t_j}$$

for some integers r, s , and $t_j = \sum_{j|i}(w_i + w_{-i})$. Since $\mathbb{Z}[x]_L$ is a unique factorization domain, $eU = eW$ implies $c_j = t_j$, i.e.,

$$\sum_{j|i}(u_i + u_{-i}) = \sum_{j|i}(w_i + w_{-i})$$

for all $j > 0$. From this we have $u_i + u_{-i} = w_i + w_{-i}$ for all $i > 0$, and hence $U \cong W$ as real representations. \square

References

- [1] M.F. Atiyah: Lectures on K -theory, Benjamin, New York-Amsterdam, 1967.
- [2] M.F. Atiyah and D.O. Tall: *Group representations, λ -rings and the J -homomorphism*, Topology **8** (1969), 253–297.
- [3] T. Bartsch: *On the genus of representation spheres*, Comment. Math. Helv. **65** (1990), 85–95.
- [4] T. tom Dieck and T. Petrie: *Geometric modules over the Burnside ring*, Invent. math. **47** (1978), 273–287.
- [5] K. Komiya: *Equivariant K -theory and maps between representation spheres*, Publ. RIMS, Kyoto Univ. **31** (1995), 725–730.
- [6] K. Komiya: *Equivariant maps between representation spheres of a torus*, Publ. RIMS, Kyoto Univ. **34** (1998), 271–276.
- [7] A. Liulevicius: *Borsuk-Ulam theorems and K -theory degrees of maps*, Springer Lecture Notes in Math. **1051** (1984), 610–619.
- [8] W. Marzantowicz: *The Lefschetz number in equivariant K -theory*, Bull. Acad. Pol. Sci. **25** (1977), 901–906.
- [9] W. Marzantowicz: *Borsuk-Ulam theorem for any compact Lie group*, J. London Math. Soc. (2) **49** (1994), 195–208.
- [10] G. Segal: *Equivariant K -theory*, Publ. Math. IHES **34** (1968), 129–151.

Department of Mathematics
 Yamaguchi University
 Yamaguchi 753-8512
 Japan
 e-mail: komiya@po.cc.yamaguchi-u.ac.jp