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ON MODULES WITH EXTENDING PROPERTIES

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We have defined the extending property of uniform submodules and of direct sums of independent submodules in [5]. We also have studied modules with lifting property in [4].

In this note, we shall give results dual to those in [4] for the extending properties. Finally, we shall give the completely forms of modules with extending property of uniform submodules over a Dededind domain.

1 Definitions

Throughout this paper we assume that a ring R has the identity element and every module M is a unitary right R-module. We recall here defintions in [5].

If $\mathrm{End}_R(M)$ is a local ring, we call M a *completely indecomposable*. We denote the *socle* and an *injective envelope* of M by $S(M)$ and $E(M)$, respectively. Let $T=\sum_K \oplus T_\alpha$. If a submodule L of T is contained in $\sum_J \oplus T_\alpha$ for some finite subset J of K, we say L is *finitely contained* (briefly f.c.) (with respect to $\sum_K \oplus T_\alpha$). It is clear that this defintion depends on the direct decomposition of T. We have studied a cyclic hollow module in [3]. We note that the concept dual to a cyclic hollow module is a uniform module with non-zero socle.

If a submodule N of M is essential in M, we indicate it by $M_e\supseteq N$. Let $\{C_\gamma\}_I$ be set of independent submodules with certain property (*). If there exists a set of independent submodules $\{N_\gamma\}_I$ such that $N_{\gamma_e}\supseteq C_\gamma$ for all $\gamma\in I$ and $\sum_I \oplus N_\gamma$ is a direct summand of M, we say *the direct sum of* $\{C_\gamma\}_I$ *with* (*) *is essentially extended to a direct summand of* M. If every direct sum of independent submodules with (*) is essentially extended to a direct summand of M, then we say M has *the extending property of direct sums of independent submodules with* (*). Especially, if $S(M)=\sum_I \oplus C_\gamma$ and $M=\sum_I \oplus N_\gamma$ in the above, we say M has *the extending property of direct decompositions of* $S(M)$. Next, we consider a case of $|I|$ (=the cardinal of I)=1. In this case we say M has *the extending property of submodules with* (*).

In order to get good results, we always assume T_1 *is completely indecomposable* in the above when $|I|=1$ and C_1 is uniform.

If for any finite subset J of I, $\sum_J \oplus C_\delta$ is a direct summand of M, $\sum_I \oplus C_\alpha$ is called a *locally direct summand of M* [6]. Finally we quote here the definition in [5].

(M-I) *Every monomorphism of M into itself is an isomorphism.*
We refer the reader for other definitions to [5].

2 Extending property on direct sums

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M=\sum_I \oplus M_\alpha$. We shall study the extending property of M when M_α is uniform. We note that almost results in this section and the next one are dual to those in [4].

First we shall give the proposition dual to [3], Proposition 2.

Proposition 1. *Let N be an R-module with extending property of uniform module. Then every direct summand has the same property.*

Proof. Let $N=N_1\oplus N_2$ and A a uniform submodule in N_1. Then $N=K_1\oplus K_2$ with $K_1 {}_e\supseteq A$. Since K_1 has the exchange property by [8], Proposition 1, $N=K_1\oplus N_1'\oplus N_2$ and $N_1=N_2'\oplus(N_1\cap(K_1\oplus N_2))$. Let $x\neq 0$ be in $N_1\cap(K_1\oplus N_2)$ and $x=k_1+n_2$; $k_1\in K_1$, $n_1\in N_2$. Since $k_1\neq 0$, there exists r in R such that $k_1r\neq o\in A$. Hence, $xr-k_1r=n_2r\in N_1\cap N_2=0$. Therefore, $A\subseteq_e N_1\cap(K_1\oplus N_2)$.

REMARK. In the above proof, we know that A is esentially extended to a direct summand of N_1 without assumption "uniform on A", if K_1 has the exchange property.

Corollary 1. *Let N be as above. If the Goldie dimension of N is finite,* $N=\sum_{i=1}^n \oplus N_i$ *and the* N_i *are uniform and completely indecomposable modules.*

Corollary 2. *Let N be an R-module with* $S(N)\neq 0$. *We assume* $S(N)=\sum_{i=1}^n \oplus A_i$; *the* A_i *are simple and that every simple submodule in* $S(N)$ *is essentially extended to a completely indecomposable and direct summand of N. Then* $N=\sum_{i=1}^n \oplus N_i\oplus K$; *the* N_i *are uniform and* $\sum_{i=1}^n \oplus S(N_i)=S(N)$, $S(K)=0$.

Next we study the dual to [4], Theorems 1 and 2. Let $\{M_\alpha\}_I$ be a set of completely indecomposable and uniform modules with non-zero socles and $M=\sum_I \oplus M_\alpha$. Since we have obtained Proposition 1 which is dual to [3], Proposition 2, we have

Theorem 1. *Let* $\{M_\alpha\}_I$ *and M be as above. We assume* $\{M_\alpha\}_I$ *is locally* *semi-T-nilpotent. Then the following conditions are equivalent*:

1) *M has the extending property of simple modules.*

2) *M has the extending property of submodules in $S(M)$.*

3) *Every direct summand of M has the above property.*

Theorem 2. *Let $\{M_\alpha\}_I$ be a set of uniform modules* (not necessarily completely indecomposable) *and $M=\sum_I \oplus M_\alpha$. Then the following conditions are equivalent*:

1) *M has the extending property of direct sums of two independent submodules.*

2) *M has the extending property of direct sums of finite independent submodules.*

3) *Let N_1 and N_2 be any two independent submodules of M. Then the projection of $N_1 \oplus N_2$ to N_1 is extended to an element in* $\mathrm{End}_R(M)$.

4) *Let N_i be as in* 3). *Then any element in* $\mathrm{Hom}_R(N_1, N_2)$ *is extended to an element in* $\mathrm{End}_R(M)$.

In this case, for every direct summand K of M, there exists a subset J of I such that $M=K\oplus\sum_J \oplus M_\gamma$. If the N_i in 3) *are direct summands of M, so is $N_1 \oplus N_2$. Further, if f is a monomorphism of N_1 to N_2, then* $\mathrm{im}\, f$ *is a direct summand of N_2* (see Remark 2 in §4).

Proof. Let N be a submodule of M. Then we can find, by Zorn's lemma, a subset J of I such that $\{M_\gamma, N\}_J$ is independent and $M\ {}_e{\supseteq}\ N\oplus\sum_J\oplus M_\gamma$.

1)→3). Let N_i be as in 3). Then by 1) we have a decomposition $M=T_1\oplus T_2\oplus T_3$ with $T_i\ {}_e{\supseteq}\ N_i$ $(i=1, 2)$. The projection of M onto T_1 is the desired extension.

3)→2). First we assume that S_1 and S_2 are independent and $M\ {}_e{\supseteq}\ S_1\oplus S_2$. Then there exists a subset J of I such that $M\ {}_e{\supseteq}\ S_1\oplus\sum_J\oplus M_\gamma$ $(=L)$. Let $f: L\to\sum_J\oplus M_\gamma$ be the projection. Then there exists an element g in $\mathrm{End}_R(M)$ with $g|L=-f$. Let π_J be the projection of M onto $\sum_J\oplus M_\gamma$ with respect to $M=\sum_J\oplus M_\gamma\oplus\sum_{I-J}\oplus M_\varepsilon$. Put $F=\pi_J g$ and $M_{I-J}(F)=\{x+F(x)\,|\,x\in\sum_{I-J}\oplus M_\varepsilon\}$ (cf. the proof of [4], Theotcm 2). Then $M=M_{I-J}(F)\oplus\sum_J\oplus M_\gamma$. Let $s\in S_1$. Then $s=\pi_J(s)+\pi_{I-J}(s)$ and $o=F(s)=F\pi_J(s)+F\pi_{I-J}(s)=-\pi_J(s)+F\pi_{I-J}(s)$. Hence, S_1 is essential in $M_{I-J}(F)$ for $S_1\oplus\sum_J\oplus M_\gamma\subseteq_e M_{I-J}(F)=\oplus\sum_J\oplus M_\gamma$. Since $S_1\subseteq_e M_{I-J}(F)$ and $S_1\cap S_2=0$, $S_2\cap M_{I-J}(F)=0$ and $S_2\oplus M_{I-J}(F)\subseteq_e M$. Further $M=M_{I-J}(F)\oplus\sum_J\oplus M_\gamma$ and $M_{I-J}(F)\approx\sum_{I-J}\oplus M_\varepsilon$. Therefore, we can obtain similarly to the above that $M=M_{I-J}(F)\oplus M_J(F')$ and $M_J(F')\subseteq_e S$. We note that $M_{I-J}(F)\approx\sum_{I-J}\oplus M_\varepsilon$ and the condition 3) is valid for a direct summand. Thus, we can prove 1)and 2) by the first part and induction on the number

of independent submodules.
2)→1). It is clear.
1)→4). Let f be in $\mathrm{Hom}_R(N_1, N_2)$. Then $N_1\oplus N_2=N_1(f)\oplus N_2$. There exists a decomposition $M=T_1\oplus T_2\oplus T_3$ such that $N_1(f)\subseteq_e T_1$ and $N_2\subseteq_e T_2$. Then $-\pi_2|N_1$ is the desired extension of f, where π_2: $M\to T_2$ is the projection.
4)→1). We shall quote the same argument as 3)→2). We use the same notations. Let $M_e\supseteq S_1\oplus\sum_J\oplus M_\gamma$. Since $S_1\cap\sum_J\oplus M_\gamma=0$, $\pi_{I-J}|S_1$ is an isomorphism. Put $S_1'=\mathrm{im}(\pi_{I-J}|S_1)$. Then $S_1=\{a+f(a)|a\in S_1', f=\pi_J(\pi_{I-J}|S_1)^{-1}: S_1'\to\sum_J\oplus M_\gamma\}$. Let $g\in\mathrm{End}_R(M)$ be an extension of f. Put $G=\pi_J g\pi_{I-J}$ and $M_{I-J}(G)=\{b+G(b)\,|\,b\in\sum_{I-J}\oplus M_\delta\}$. Then $M=M_{I-J}(G)\oplus\sum_J\oplus M_\gamma$ and $M_{I-J}(G)_e\supseteq S_1$. Similarly, we obtain $M=M_{I-J}(G)\oplus M_J(G')$ and $M_J(G')_e\supseteq S_2$. For the remaining parts, we assume S_1 is a direct summand of M. Then $S_1=M_{I-J}(G)$ and so $M=S_1\oplus\sum_I\oplus M_\gamma$. Let $f\in\mathrm{Hom}_R(N_1, N_2)$ be a monomorphism. Then $N_1(f)\cap N_1=0$ and so $N_1\oplus N(f)$ is a direct summand of $N_1\oplus N_2$. Let π be the projection of $N_1\oplus N_2$ onto N_2. Then $\mathrm{im}\, f=\pi(N_1(f))$ is a direct summand of N_2.

Theorem 3 (cf. [5], Theorem 22). *Let $\{M_\alpha\}_I$ and M be as above. Then the following conditions are equivalent:*

1) *M has the extending property of finite direct sums of f.c. uniform modules.*

2) *$\mathrm{Hom}_R(A_\alpha, M_\beta)$ is extended to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for any $\alpha\neq\beta$ in I and $A_\alpha\subseteq M_\alpha$.*

Proof. 1)→2). (cf. the proof of [5], Lemma 34). Let f be in $\mathrm{Hom}_R(A_\alpha, M_\beta)$ and put $A(f)=\{a+f(a)\,|\in A_\alpha\}$. We consider the direct sum $A(f)\oplus M_\beta$. Then there exists a decomposition $M=S_\alpha\oplus S_\beta\oplus S$ such that $S_{\alpha\, e}\supseteq A(f)$ and $S_{\beta\, e}\supseteq M_\beta$. Since M_β is a direct summand of M, $S_\beta=M_\beta$. Let π: $T\to S_\beta=M_\beta$ be the projection for the decomposition $T=S_\alpha\oplus M_\beta\oplus S$. Then $-\pi\,|\,M_\alpha$ is an extension of f.
2)→1). Let $N=\sum_{i=1}^n\oplus N_i$ in M with N_i f.c. uniform. Then we may assume $N\subseteq\sum_{i=1}^m\oplus M_i\langle\oplus M$. Hence, we assume $M=\sum_{i=1}^m\oplus M_i$. We assume there exists a set of uniform direct summands T_i of M for $i<k$ such that $T_{i\,e}\supseteq N_i$, $T_i\approx M_{\rho(i)}$ and $M=\sum_{i=1}^k\oplus T_i\oplus\sum_{j=k+1}^m\oplus M_{\rho(j)}$, where ρ is a permutation of $\{1, 2, \cdots, m\}$. Let π_p be the projection of M onto T_p or $M_{\rho(p)}$ for the above decomposition. Since $\bigcap_p\ker(\pi_p|N_{k+1})=0$, $\pi_q|N_{k+1}$ is an isomorphism for some q. If $q\notin\{k+1, \cdots, n\}$, put $L=\bigcap_{s\geqslant k+1}\ker(\pi_s|N_{k+1})\neq 0$. Then $L\subseteq\sum_{i=1}^k\oplus T_i$ and $\sum_{i=1}^k\oplus N_i\subseteq_e\sum_{i=1}^k\oplus T_i$, which is a contradiction. Hence, we may assume $q=k+1$. Then if we put $N_{k+1}'=\mathrm{im}(\pi_{k+1}|N_{k+1})\subseteq M_{\rho(k+1)}$, $N_{k+1}=\{f_1(a)+f_2(a)+\cdots+f_k(a)+a+f_{k+2}(a)\cdots+f_m(a)\,|\,a\in N_{k+1}'\}$, where $f_s=\pi_s(\pi_{k+1}|N_{k+1})^{-1}$ (cf. the proof of [5], Theorem 10).

Since, $T_i \approx M_{\rho(i)}$, there exists a set of homorphisms $\{g_j \in \mathrm{Hom}_R(M_{\rho(k+1)}, K_j)$ $(K_j = T_j$ or $K_j = M_{\rho(j)})\}$ such that $g_j | N_{k+1}' = f_j$. Put $M_{\rho(k+1)}(g) = \{g_1(b) + \cdots + g_k(b) + b + g_{k+2}(b) + \cdots + g_n(b) \mid b \in M_{\rho(k+1)}\}$. Then $M = \sum_{i=1}^{k} \oplus T_i \oplus M_{\rho(k+1)}(g) \oplus \sum_{p \geqslant k+2} \oplus M_{\rho(p)}$. It is clear that $N_{k+1} \subseteq_e M_{\rho(k+1)}(g) \approx M_{\rho(k+1)}$. Therefore, we can prove the theorem by induction.

Corollary 1 (cf. [5], Theorem 18). *We assume each M_α is uniform and completely indecomposable and further $\{M_\alpha\}_I$ is a locally semi-T-nilpotent. We put $M = \sum_I \oplus M_\alpha$. Then the following conditions are equivalent*:

1) *M has the extending property of direct sums of f.g. uniform modules.*

2) $\mathrm{Hom}_R(A_\alpha, M_\beta)$ *is extended to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for $\alpha \neq \beta$ in I and any f.g. submodule A_α of M_α.*

Proof. 1)→2). We can use the same argument as the proof of 1)→2) in the theorem.
2)→1). Let $\{A_\alpha\}_I$ be a set of independent and *f.g.* uniform submodules of M with $M\, _e\!\supseteq \sum_I \oplus A_\alpha$. We may assume I is a well ordered set and we shall use the same argument in the proof of [4], Theorem 1. We assume, for each $\kappa \leqslant \beta < \alpha$, that there exist direct summands T_κ such that $T_\kappa\, _e\!\supseteq A_\kappa$ and $\sum_{\kappa<\beta} \oplus T_\kappa$ is a locally direct summand of M. Then $M = \sum_{\beta<\alpha} \oplus T_\beta \oplus T$ and $T_\beta\, _e\!\supseteq A_\beta$, since $\{M_\alpha\}_I$ is semi-T-nilpotent [6]. We may assume $T = \sum_P \oplus N_\delta$, each N_δ is isomorphic to a module in $\{M_\alpha\}_I$ by [2] and [7]. Let $\pi: M \to T$ and $\pi_\delta: T \to N_\delta$ be the projections of M and T, respectively. Since $(\sum_{\beta<\alpha} \oplus A_\beta) \cap A_\alpha = 0$ and $\sum_{\beta<\alpha} \oplus T\, _e\!\supseteq \sum_{\beta<\alpha} A_\beta$, $\pi | A_\alpha$ is an isomorphism. A_α being *f.g*, uniform, $\pi_\delta \pi | A_\alpha$ is an isomorphism for some δ. Making use of the method in the proof of the theorem, we obtain $M = \sum_{\beta<\alpha} \oplus T_\beta \oplus \sum_{\delta' \neq \delta} \oplus T_{\delta'} \oplus T_\delta(f)$ and $T_\delta(f)\, _e\!\subseteq A_\alpha$. Hence, we have proved 2)→1) by transfinite induction.

Corollary 2 (cf. [5], Corollary 8). *Let $\{M_\alpha\}_I$ be a set of uniform modules with non-zero socles and $M = \sum_I \oplus M_\alpha$. Then the following conditions are equivalent*:

1) *M has the extending property of finite direct sum of simple modules.*

2) $\mathrm{Hom}_R(S(M_\alpha), S(M_\beta))$ *is extended to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for any $\alpha \neq \beta$ in I.*

Proof. It is clear from the proof of Theorem 3.

REMARK. Let R be a local self-injective ring with maximal ideal $J(R)$ not T-nilpotent. Put $\{R_n = E\}_n$. Then E_n satisfies 2) in Theorem 3. Hence,

$\sum_n \oplus E_n$ has the extending property of finite direct sum of *f.c.* uniform modules, however $\sum_n \oplus E_n$ does not have the extending property of infinite direct sums (cf. Theorem 4 below).

Let $\{f_n \in \mathrm{Hom}_R(T, N_n)\}_n$ be a set of homorphisms. If $f_n(t)=o$ for $t \in T$ and almost n, $\{f_n\}$ is called *summable*.

Theorem 4 (cf. [5], Theorem 22). *Let $\{M_\alpha\}_I$ be a set of completely indecomposable and uniform modules and $M=\sum_I \oplus M_\alpha$. Then the following conditions are equivalent*:

1) *M has the extending property of direct sums of independent uniform submodules.*

2) *$\{M_\alpha\}_I$ is locally semi-T-nilpotent and for any set of summable homomorphisms $\{f_\beta \in \mathrm{Hom}_R(A_\alpha, M_\beta)\}_{\beta \neq \alpha}$ $(\alpha, \beta \in I)$ there exists a set of summable homomorphisms $\{F_\beta \in \mathrm{Hom}_R(M_\alpha, M_\beta)\}$, which are extensionss of $\{f_\beta\}$, where A_α is a submodule of M_α.*

Proof. 1)→2). We know from the proof of [5], Theorem 22 that $\{M_\alpha\}_I$ is locally semi-T-nilpotent. Let $\boldsymbol{F}=\{f_\beta\}$ be any set of summable homomorphisms in $\{\mathrm{Hom}_R(A_\alpha, M_\beta)\}_{\beta \neq \alpha}$ and $A_\alpha \subseteq M_\alpha$. Since $\boldsymbol{F}$ is summable, $A_\alpha(\boldsymbol{F})=\{a+\sum f_\beta(a) \mid a \in A_\alpha\}$ is an R-submodule of M and $M \supseteq_e A(\boldsymbol{F}) \oplus \sum_{\beta \neq \alpha} \oplus M_\beta$. Then we have a direct decomposition $M=M'_\alpha \oplus \sum_{\beta \neq \alpha} \oplus M_\beta$ by 1). Let $\pi_\beta\colon M \to M_\beta$ be the projection. Then $\{F_\beta = -\pi_\beta \mid M_\alpha\}$ is the desired set.
2)→1). Let $M=\sum_I \oplus N_\alpha$ be any decomposition as in the theorem and B a uniform submodule of M. Let $\pi_\alpha\colon M \to N_\alpha$ be projections for each $\alpha \in I$. Then $\bigcap_\alpha \ker \pi_\alpha = 0$. Let $b \neq o$ be in B and $b=\sum_{i=1}^n \pi_{\alpha_i}(b)$. Then $b \in \bigcap_{\beta \neq \alpha_i} \ker \pi_\beta$. Hence, there exists π_{β_i} such that $\pi_{\alpha_i} \mid B$ is an isomorphism. Now, from 2), the proofs of Theorem 3 and Corollary 1 and the above remark, we can obtain 1) by making use of transfinite induction.

3 Modules with extending properties

In the preceeding section, we have studied modules with direct decomposition. In this section we shall study some relationships between modules with extending property and direct decomposition of the modules.

Theorem 5. *Let M be an R-module. We assume*

a) *$S(M)$ is essential in M, and*

b) $\mathrm{End}_R(S(M))$ *is extended to* $\mathrm{End}_R(M)$.

Then the following conditions are equivalent:

1) *M has the extending property of simple modules.*

2) *M contains a submodule M' as follows*:

i) $M'=\sum_I \oplus M_\alpha$: *the M_α are uniform and completely indecomposable and* $S(M')=S(M)$. ($\mathrm{End}_R(S(M'))$ *is extended to* $End_R(M')$).

ii) *M' is a locally direct summand of M and has the extending property of finite direct sums of simple submodules. In this case, M has the extending property of finite direct sums of simple modules.*

Proof. 2)→1). Let S be a finite direct sum of simple submodule in M. Then $S\subseteq \sum_{i=1}^n \oplus S(M_{\alpha_i})$ by i). Since $\sum_{i=1}^n \oplus M_{\alpha_i}$ is a direct summand of M by ii) and has the extending property of finite direct sums of simple submodules by Corollary 2 to Theorem 3, M has the same property
1)→2). We can use the argument dual to the proof 2)→1) of [4], Theorem 3. Let $\boldsymbol{N}$ be the set of submodules in N' of M such that $N'=\sum_{J'} \oplus N_\gamma$; the N_γ are uniform and completely indecomposable and N' is a locally direct summand of M. Let N be maximal in $\boldsymbol{N}$. We shall show $N\subseteq_e M$. There exists a submodule A of $S(M)$ such that $M\,_e\!\supseteq N\oplus A$ by a). Let K be any finite subset of J and put $N_0=\sum_K \oplus N_\gamma$. Then $M=N_0\oplus P$ and $S(M)=S(N_0)\oplus \sum_{J-K} \oplus S(N_\gamma)\oplus A$. Let π' be the projection of $S(M)$ onto $S(N_0)$ and $\pi=\pi'\,|\,S(P)$. Then we obtain $f\in \mathrm{Hom}_R(P, N_0)$ such that $f\,|\,S(P)=-\pi$ by b). Hence, $M=P(f)\oplus N_0$ and $S(P(f))=\sum_{J-K} \oplus S(N_\delta)\oplus A$. If $A\neq 0$, there exists a direct summand T of $P(f)$ with $S(T)=A$ by Proposition 1. Hence, $\sum N_\gamma+T$ is a locally direct summand of M, which contradicts the maximality of N. Therefore, $A=0$. Let N_1 and N_2 be in $\{N_\gamma\}_I$. Then $M=N_1\oplus N_2\oplus M_0$ and we know from b) $\mathrm{Hom}_R(S(N_1), S(N_2))$ is extended to $\mathrm{Hom}_R(N_1, N_2)$. Hence, N has the extending property of finite direct sums of simple submodules by Corollary 2 to Theorem 3.

Theorem 6. *Let M be an R-module. We assume that $M\,_e\!\supseteq S(M)$ and every uniform direct summand of M is artinian. Then M has the extending property of simple modules if and only if M contains a submodule M' satisfying the following.*

1) *$M\,_e\!\supseteq M'$ and so $S(M)=S(M')$.*

2) *$M'=\sum_I \oplus M_\alpha$ with M_α uniform.*

3) *$\sum_I \oplus M_\alpha$ is a locally direct summand of M.*

4) *M' has the extending property of simple module.*

Proof. We note that every artinian module satisfies (M-I) and the theorem is dual to [4], Theorem 4. Therefore, we can prove the theorem by making use of argument dual to the proof of [4], Theorem 4.

Corollary 1. *Let R be a right artinian ring such that every indecomposable R-injective module is artinian. Then M has the extending property of simple* *modules* (resp. *of direct sum of simple modules*) *if and only if M contains a submodule M' satisfying* 1)~4) (resp. 1)~3) and 4') *M' has the extending property of direct decompositions of* $S(M')$).

Proof. The first part is clear from the theorem. We know from the assumption that every uniform module is completely indecomposable and satisfies (M-I). Furthermore, every set of uniform modules is T-nilpotent by [1], Lemma 11. We assume M has the extending property of direct sums of two simple modules. Then M has the extending property of simple modules. Hence, M has a submodule M' with 1)~3). From the proof of Case 1) of [4], Theorem 3, its duality and [5], Theorem 23, M' satisfies 4'). The converse is clear (see the proof [4], Theorem 3).

Corollary 2.[1] *Let R be a Dedekind domain and let M be a torsion R-module. Then M has the extending property of simple modules if and only if M contains a submodule M' satisfying* 1)~4).

REMARKS 1. Let Z be the ring of integers and p a prime. Then $\sum_{i=1}^{\infty}\oplus Z/p^i$ is a locally direct summand of $\prod_i Z/p^i$. Any submodule M of $\prod_i Z/p^i$ containing essentially $\sum_{i=1}^{\infty}\oplus Z/p^i$ has the extending property of simple module. M is a direct sum of indecomposable and uniform modules if and only if $M=\sum_{i=1}^{\infty}\oplus Z/p^i$ by [1] and [6].

2. Let R be any ring and $\{S_\alpha\}_I$ a set of simple R-modules. Then $\sum_I\oplus S_\alpha$ is a locally direct summand of $\prod_\alpha S_\alpha$. Hence, every R-submodule $T({}_e\supseteq\sum_I\oplus S_\alpha)$ of $\prod_I S_\alpha$ satisfies the conditions in Theorem 6. This example shows that 4') in Corollary 1 does not imply the extending property of decomposition of $S(M)$.

4 Modules over Dededind domains

Let R a Dedekind domain. We have determined the types of R-modules which have the extending property of direct sums of uniform modules in [5], Theorem 31. We shall determine the types of R-modules which have the extending property of uniform modules.

We put $Q=E(R)$ and $E(p)=E(R/p)$, where p is prime. Let M be a torsion free and uniform module. Then we may assume $M\supseteq R$. $Q/R\supseteq M/R=\sum_p\oplus p^{-n(p)}R$, where $n(p)$ is finite or infinite. Put $\boldsymbol{P}=\{p_i\,|\,n_i(p_i)<\infty\}$ for M and we denote M by $F(\boldsymbol{P})$. Then M is completely indecomposable if and only if $\boldsymbol{P}$ is a singleton or empty (i.e. $M=Q$). We note $F(\boldsymbol{P})_p\neq Q$ for $p\in\boldsymbol{P}$ and $F(\boldsymbol{P})_q=Q$ for $q\notin\boldsymbol{P}$. An R-module N is called p^∞-divisible if $p^nN=N$ for all

1) Added in proof. We shall show $M=M'$ in the forth comming paper

n and we denote the unique maximal p^{∞}-divisible submodule of N by $N[p]$.

In the preceeding sections we have assumed that a direct summand of M which is an extension of a uniform submodule is completely indecomposable. In the following, *we shall drop this assumption.* We consider only the extending property of uniform modules and so we call it simply *the extending property.*

Theorem 7 (cf. [5], Theorem 31). *Let R be a Dedekind domain and M an R-module. Then M has the extending property of uniform module if and only if M is one of the following.*

1) *M is torsion and $M(p)=M_1^{(J_1)}\oplus M_2^{(J_2)}$* and $||M_1|-|M_2||\leqslant 1$, *where $M_i(\subset E(p))$ is completely indecomposable ($|M_1|=\infty$ means that M_1 is injective).*

2) *$E\oplus M_\beta$, where E is injective and M_β is torsion free and uniform or zero. Here $|M_1|$ means the composition length of M_1, $M_1^{(J)}$ means the direct sum of $|J|$-copies of M_1, and $M(p)$ is ihe p-primary component of M.*

We shall prove the theorem by making use of several lemmas below.

First, we recall here useful lemma in [5], which we have used above.

Lemma 1 *Let M be an R-module (R is any ring). We assume $M=M_1\oplus M_2\oplus M_3$, N is a submodule of M_1 and $f\in \mathrm{Hom}_R(N, M_2)$. If there exists a direct summand T of M such that $M=T\oplus M_2\oplus M_3$ and $T\supseteq N(f)=\{n+f(n)\,|\,n\in N\}$, then f is extended to an element in $\mathrm{Hom}_R(M_1, M_2)$. Conversely, we assume $M=T\oplus T'$. Let A be a submodule of T and $g\in \mathrm{Hom}_R(A, T')$. If g is extended to an element in $\mathrm{Hom}_R(T, T')$, we have a decomposition $M=T''\oplus T'$ such that $T''\supseteq A(f)=\{a+f(a)\,||a\in A\}$.*

Proof. Let π_T and π_{M_2} be the projections of M with respect to the decompositions $M=T\oplus M_2\oplus M_3$ and $M=M_1\oplus M_2\oplus M_3$, respectively. Then $\pi_{M_2}(\pi_T|M_1)\in \mathrm{Hom}_R(M_1, M_2)$ is an extension of f. Put $T''=T(g)=\{t+g(t)\,|\,t\in T\}$ for the second assertion.

Lemma 2. *Let M be an R-module with the extending property* (resp. *of cyclic and uniform module*) *and $M=T_1\oplus T_2$. Then* 1) *if M is torsion free, T_1 has the extending property* (resp. *of cyclic and uniform module*).

2) *If T_2 is the torsion submodule $M(t)$ of M, T_1 and T_2 have the above property.*

Proof. 1) Let N be a uniform submodule (resp. cyclic and uniform submodule) of T_1. Then there exists a decomposition $M=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq N$. Let $\pi_i\colon M\to T_i$ be the projection. Then since $\ker\pi_2\supseteq N$, $\pi_2(L_1)=0$. Hence, $L_1\subseteq T_1$ and $T_1=L_1\oplus T_1\cap L_2$. 2) Let N be a uniform submodule (resp. cyclic and uniform submodule) of M. Then $M=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq N$. If $N\subseteq T_2$, $L_1\subseteq T_2$. Hence, T_2 has the extending property. We assume $N\subseteq T_1$. Then L_1 is torsion free. Hence, $T_2\subseteq L_2$ and $L_2=T_2\oplus(T_1\cap L_2)$. Now, $M=T_1\oplus T_2=$

$L_1 \oplus (T_1 \cap L_2) \oplus T_2$. Therefore, $T_1 = \pi_1(L_1) \oplus \pi_1(T_1 \cap T_2)$ and $\pi_1(L_1)_e \supseteq N$.

Since R is a hereditary ring, every R-module M contains the unique maximal injective submodule E, say $M = E \oplus K$ and K is reduced.

Lemma 3. *Let $M = E \oplus K$ be as above. If M has the extending property, then K does.*

Proof. Let N be a uniform submodule of K. Then $M = L_1 \oplus L_2$ and $L_{1\,e} \supseteq N$. Since L_1 is indecomposable, L_1 is either injective or reduced. If L_1 is injective, L_1 has the exchange property by [9], and so there exists a direct summand K_1 of K with $K_{1\,e} \supseteq N$ by Remark after Proposition 1. We assume L_1 is reduced. Let E' be the unique maximal injective submodule of L_2. Then $E \supseteq E'$ and $M = L_1 \oplus L_2' \oplus E'$, $L_2' \subseteq L_2$. Accordingly, $E = E' \oplus (E \cap (L_1 \oplus L_2'))$ and the injective module $E \cap (L_1 \oplus L_2')$ has the exchange property. However, L_1 and L_2' are reduced. Hence, $E \cap (L_1 \oplus L_2') = 0$ and so $E = E'$. Therefore, $M = L_1 \oplus L_2' \oplus E = K \oplus E$. Since $N \subseteq K \cap L_1$, $K = \pi(L_1) \oplus \pi(L_2')$ and $\pi(L_1)_e \supseteq N$, where $\pi: M \to K$.

Lemma 4. *Let $M = M_1 \oplus M_2$ be torsion free. We assume that the M_i are completely indecomposable uniform modules. Then M has the extending property of a cyclic uniform modules if and only if either M_1 or M is injective. In this case M has the extending property.*

Proof. "If" part is clear by [5], Theorem 31. We assume $M_i = F(\boldsymbol{P}_i)$ where $\boldsymbol{P}_i = \{p_i\}$. Then considering the multiplication by $x^{-m} (x \in p_1 - p_1^2)$ and using the proof 1)→2) of [5], Theorem 10, we have $x^{-m} M_1 \subseteq M_2$ or $x^{-m} M_2 \subseteq M_1$. Hence, either M_1 or M_2 is injective.

Lemma 5. *Let M be torsion free and reduced. If M has the extending property of cyclic uniform modules, then M is uniform.*

Proof. Since every direct summand of M has the extending property of cyclic uniform modules by Lemma 2, M has a direct summand $M_1 \oplus M_2$ with M_i uniform if M is not uniform. Let $M_i = F(\boldsymbol{P}_i)$, $i = 1, 2$. Since $M_1 \oplus M_2$ has the extending property of cyclic uniform modules, we may assume $M = M_1 \oplus M_2$. If $\boldsymbol{P}_1 \cap \boldsymbol{P}_2 \neq \emptyset$ (say $p \in \boldsymbol{P}_1 \cap \boldsymbol{P}_2$), $M_{1p} \neq Q$ and $M_{2p} \neq Q$. However M_p has the extending property of cyclic uniform modules, which is a contradiction by Lemma 4. Hence, there exists $p \in \boldsymbol{P}_1 - \boldsymbol{P}_2$. Put $N = \{x + x' \mid x \in R \subseteq M_1,\ x' = x \in R \subseteq M_2\}$. Then we obtain $M = L_1 \oplus L_2$ and $L_{1\,e} \supseteq N$. Now, $M[p] = M_2$ and so $M_1 \approx M/M[p] \approx L_1/L_1[p] \oplus L_2/L_2[p]$. Since M_1 is uniform, $L_1 = L_1[p]$ or $L_2 = L_2[p]$. We assume $L_1 = L_1[p]$. Then $L_2/L_2[p]$ is torson free. Since L_2 is uniform, $L_2[p] = 0$. Hence, $M_2 = M[p] = L_1[p] \oplus L_2[p]. = L_1$. However, $0 = N \cap M_2 = N \cap L_1 = N$, a contradiction. If $L_2 = L_2[p]$, $M_2 = L_2$ as above. Therefore, the identity map $R\to R$ is extended to $\mathrm{Hom}_R(M_1, M_2)$ by Lemma 1. We may assume from the first half that there exists $q\in \boldsymbol{P}_2-\boldsymbol{P}_1$. Then $g(M_{1q})=g(Q)\subseteq M_{2q}\subsetneqq Q$, a contradiction. Hence, M is uniform.

Next, we shall study torsion modules. If M_α is torsion and uniform, $M_\alpha\subseteq E(p)$ for some p. We indıcate it by $M_\alpha(p)$.

Lemma 6. *Let M be torsion. We assume $M=\sum_I\oplus M_\alpha$ with M_α uniform. Then M has the extending property if and only if $M=\sum_p\oplus(M_{\alpha_1}(p)^{(B_1)}\oplus M_{\alpha_2}(p)^{(B_2)})$ and $||M_{\alpha_1}(p)|-|M_{\alpha_2}(p)||\leqslant 1$ for each p.*

Proof. Since $E(p)$ is serial and M_α is completely indecomposable, we have the lemma by [5], Theorem 10.

Lemma 7. *Let M be torsion. If M has the extending property then M has the form in Lemma 6, provided M is reduced.*

Proof. Since M is torsion, every (indecomposable) uniform submodule is completely indecomposable. M is a direct sum of p-primary components $M(p)$ and it is clear that $M(p)$ has the extending property. Therefore, we may assume that M is p-primary and reduced. Let x be in M. Put $o(x)=\{r\in R\,|\,xr=o\}=p^n$ and put $n=n(x)$. We first show $\{n(x)\}_{x\in M}$ is bounded. Let $N=xR$ be a uniform submodule of M. Then $M=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq N$. Since M is reduced, $L_1=yR$. If $\{n(x)\}_{x\in M}$ is not bounded, there exists z in L_2 such that $n(z)\geqq n(y)+1$. Since L_2 has the extending porperty by Remark after Proposition 1, $L_2=L_2'\oplus L_3$ and $L_2'\,{}_e\!\supseteq zR$. $L_1\oplus L_2'$ has the extending property, too, which is a contradiction by Lemma 6. Hence, $M=\sum_I\oplus M_\alpha$ with M_α uniform as follows: We denote the bounded order of $\{n(x)\}_{x\in M}$ by m. Put $\boldsymbol{A}=\{\sum\oplus A_\alpha\subseteq M\,|\,A_\alpha=Ry$ and $n(y_\alpha)=m\}$. We can find a maximal submodule in $\boldsymbol{A}$ with respect to the member of direct components by Zorn's lemma, say $A=\sum\oplus A_\alpha$. Then we can find a submodule B of such that $M\,{}_e\!\supseteq A\oplus B$ and $B=\sum\oplus B_\beta$ with B_β uniform. Then $E(M)=\sum\oplus E(A_\alpha)\oplus\sum\oplus E(B_\beta)\,{}_e\!\supseteq M\,{}_e\!\supseteq A\oplus B$. Let $x\in M$ and $x=\sum x_\alpha+\sum x_\beta$; $x_\alpha\in E(A_\alpha)$ and $x_\beta\in E(B_\beta)$. Since $n(x)\leqslant m$, $n(x_\alpha)\leqslant m$ and $A_\alpha=\{z\,|\in E(A_\alpha), o(z)\leqslant m\}$. Hence, $\sum x_\alpha\in A$. Therefore, $M=A\oplus M\cap(\sum\oplus E(B_\beta))$ and $o(y)<m$ for any $y\in M\cap(\sum\oplus E(B_\beta))$ by the extending property and the maximality of A. Use induction. Hence, M is of the form in Lemma 6.

Lemma 8. *Let M be an R-module. If M has the extending property of cyclic uniform module, then $M/M(t)$ does.*

Proof. Put $\bar{M}=M/M(t)$. Let $\bar{a}R$ be a uniform submodule of $\bar{M}$. Since $a\notin M(t)$, aR is a torsion free and uniform submodule of M. Hence, we have a decomposition $M=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq aR$. Then $L_2\supseteq M(t)$ and so $\bar{M}=\bar{L}_1\oplus L_2/M(t)$ and $\bar{L}_1\simeq\bar{L}_2\,{}_e\!\supseteq \bar{a}R$.

Lemma 9. *Let $M=M(t)\oplus M_\beta$ have the extending property and let M_β be torsion free and uniform. Then $M(t)$ is injective.*

Proof. Let N be a uniform submodule of M_β and $f\in \mathrm{Hom}_R(N, M(t))$. Then $M=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq N(f)$. Since L_1 is torsion free, $L_2=M(t)$. Then f is extended to an element in $\mathrm{Hom}_R(M_\beta, M(t))$ by Lemma 1. Hence, $M(t)$ is injective by Lemmas 2 and 7 and [5], Lemma 33.

Lemma 10. *Let $M=E\oplus T$ with E injective and torsion free and $T=E'\oplus T'$ with E' torsion and injective and T' torsion free. If T has the extending property, then M does.*

Proof. Let N be a uniform submodule of M. If N is torsion, $N\subseteq T$. Hence, N is essentially extended to a direct summand of M. Let N be torsion free. If $N\subseteq E\oplus E'$, N is essentially extended to a direct summand by [5], Proposition 1. We assume $N\not\subseteq E\oplus E'$. Let $\pi: M\to T'$. Then $\pi|N$ is an isomorphism and $N=\{x+f(x)\,|\,x\in\pi(N)\}$, $f\in\mathrm{Hom}_R(\pi(N), E\oplus E')$. Since T' has the extending property by Lemma 2, $T'=D_1\oplus D_2$ and $D_1\,{}_e\!\supseteq\pi(N)$. $E\oplus E'$ being injective, f is extended to $g\in\mathrm{Hom}_R(D_1, E\oplus E')$. Since $N=\pi(N)(g)$, $M=D_1(g)\oplus D_2\oplus E\oplus E'$ and $D_1(g)\supseteq N$ by Lemma 1.

Proof of Theorem 7. We assume M has the extending property. Let $M=E\oplus K$ with E injective and K reduced. Then K has the extending property by Lemma 3. Assume K is torsion. Then K is of the form 1) by Lemma 6. In this case every indecomposable module is completely indecomposable. Hence, every direct summand of M has the extending property by Remark after Proposition 1. We assume $E\neq 0$. If E is not torsion, K is injective by Lemma 9. If E is torsion, $K=0$ by Lemma 6. In either case, $K=0$ if $E\neq 0$ and M is injective and is of the form 2). Next, we assume K has a torsion free uniform submodule N. Then $K=L_1\oplus L_2$ and $L_1\,{}_e\!\supseteq N$. If L_2 is not torsion, L_2 contains a torsion free and uniform submodule N'. Let $K=L_2'\oplus L_3$ and $L_2'\,{}_e\!\supseteq N'$. Then since $L_2'\cap L_2\supseteq N'$, $L_2'\subseteq L_2$ (see the proof of Lemma 2). Hence, $K=L_1\oplus L_2'\oplus L_2''$ and L_1, L_2' are uniform and torsion free. Since $K(t)\subseteq L_2''$, $L_1\oplus L_2'$ is isomorphic to a direct summand of $K/K(t)$. $K/K(t)$ has the extending property of cyclic uniform modules by Lemma 8 and so does $L_1\oplus L_2'$ by Lemma 2, which is a contradiction by Lemma 5. Hence, $L_2=K(t)$ and $K(t)$ is injective by Lemma 9. Thus, M is of the form 2). Conversely, if M is of the form 1), M has the extending property by Lemma 6 and [9]. Let M be of the form 2) and $E=E(t)\oplus E'$. Then $E(t)\oplus K$ has the extending property by [5], Theorem 31. Therefore, $M=E'\oplus E(t)\oplus K$ has the extending property by Lemma 10.

REMARKS. 1. Let Z and p be as in Remark in §3. Let M be an essential extension of $(Z/p^2)^{(I)}$ in $\prod_I Z/p^2$. Then M has the extending property of simple modules, but not of uniform modules unless $M=(Z/p^2)^{(I)}$.

2. Let R be a commutative and noetherian ring and let $\{P_i\}_{i=1}^n$ be a set of distinct non maximal prime ideals in R. We put $M=\sum_{i=1}^n \oplus R/P_i$. Then every uniform submodule of M is contained in some R/P_i. Hence, M has the extending property of direct sums of independent submodules. We note that each R/P_i is neither completely indecomposable (if R/P_i is not local) nor quasi-injective and does not satifty (M-I) (see Theorem 2 and [5], Theorem 22). We further assume R is integral. If the conclusions of [5], Theorem 31 are true, then R is a Dedekind domain.

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