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THE CHERN CHARACTER HOMOMORPHISM OF THE COMPACT SIMPLY CONNECTED EXCEPTIONAL GROUP E_6

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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0. Introduction

Let F_4 and E_6 be the compact, 1-connected representatives of the respective local classes. As in [22] there is an involutive automorphism θ of E_6 such that the subgroup consisting of fixed points of θ is F_4. Thus the quotient E_6/F_4 forms a compact symmetric space, which is denoted by EIV in É. Cartan's notation. For brevity we shall write EIV instead of E_6/F_4.

The ordinary cohomology and complex K-theory of three spaces F_4, E_6 and EIV are well understood (see §1). Moreover, the Chern character homomorphism of F_4 was described explicitly in [20]. The purpose of this paper is to study those of E_6 and EIV. Our results are stated as follows (for notations used below, see §1):

Theorem 1. *The Chern character homomorphism*

$$ch\colon K^*(E_6)=\Lambda_Z(\beta(\rho_1),\beta(\rho_2),\beta(\Lambda^2\rho_1),\beta(\Lambda^3\rho_1),\beta(\Lambda^2\rho_6),\beta(\rho_6))$$
$$\to H^*(E_6;\boldsymbol{Q})=\Lambda_{\boldsymbol{Q}}(x_3,x_9,x_{11},x_{15},x_{17},x_{23})$$

is given by

$$ch(\beta(\rho_1))=6x_3+\frac{1}{2}x_9+\frac{1}{20}x_{11}+\frac{1}{168}x_{15}+\frac{1}{480}x_{17}+\frac{1}{443520}x_{23}$$
$$ch(\beta(\rho_2))=24x_3-\frac{3}{10}x_{11}+\frac{3}{28}x_{15}-\frac{31}{221760}x_{23}$$
$$ch(\beta(\Lambda^2\rho_1))=150\,x_3+\frac{11}{2}x_9-\frac{1}{4}x_{11}-\frac{101}{168}x_{15}-\frac{229}{480}x_{17}-\frac{2021}{443520}x_{23}$$
$$ch(\beta(\Lambda^3\rho_1))=1800\,x_3-\frac{27}{2}x_{11}-\frac{153}{28}x_{15}+\frac{6789}{24640}x_{23}$$
$$ch(\beta(\Lambda^2\rho_6))=150\,x_3-\frac{11}{2}x_9-\frac{1}{4}x_{11}-\frac{101}{168}x_{15}+\frac{229}{480}x_{17}-\frac{2021}{443520}x_{23}$$
$$ch(\beta(\rho_6))=6\,x_3-\frac{1}{2}x_9+\frac{1}{20}x_{11}+\frac{1}{168}x_{15}-\frac{1}{480}x_{17}+\frac{1}{443520}x_{23}\,.$$

Theorem 2. *The Chern character homomorphism*

$$\begin{aligned} ch\colon K^*(EIV) &= \Lambda_{\boldsymbol{Z}}(\beta(\rho_1-\rho_6),\ \beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) \\ &\to H^*(EIV;\boldsymbol{Q}) = \Lambda_{\boldsymbol{Q}}(x_9,\ x_{17}) \end{aligned}$$

is given by

$$ch(\beta(\rho_1-\rho_6)) = x_9+\frac{1}{240}x_{17}$$

$$ch(\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) = 11\,x_9-\frac{229}{240}x_{17}\,.$$

The paper has the following organization. In §1 we collect some facts which we need. §2 contains various computations and consequently we obtain certain data. §3 is devoted to prove Theorems 1 and 2.

The author would like to thank Professor H. Minami for his kind advice.

1. Preliminaries

In this section we recollect some results on the cohomology and K-theory of our spaces F_4, E_6 and EIV.

Let us begin with the cohomology of compact Lie groups. Throughout the paper G stands for a compact, 1-connected, simple Lie group of rank l. Then the rational cohomology ring of G is an exterior algebra generated by primitive elements of degrees $2m_i-1$, $1\leqq i\leqq l$, where the m_i are certain integers such that $2=m_1\leqq m_2\leqq\cdots\leqq m_l$ (see [5]). Since G is parallelizable, one can utilize the Poincaré duality theorem for choosing elements

$$x_k\in H^k(G;\boldsymbol{Z}),\ k=2m_i-1,\ 1\leqq i\leqq l\,,$$

which satisfy the following conditions:

(1.1) (i) *x_k is not divisible in $H^k(G;\boldsymbol{Z})$;*

(ii) *The image of x_k under the coefficient group homomorphism $H^k(G;\boldsymbol{Z})\to H^k(G;\boldsymbol{Q})$ induced by the natural inclusion $\boldsymbol{Z}\to\boldsymbol{Q}$ belongs to $PH^k(G;\boldsymbol{Q})$, where P denotes the primitive module functor;*

(iii) *The cup product*

$$x_{2m_1-1}\,x_{2m_2-1}\cdots x_{2m_l-1}$$

generates the infinite cyclic group $H^n(G;\boldsymbol{Z})$, where

$$n=\sum_{i=1}^{l}(2m_i-1)=\dim G\,.$$

We will use the same symbol x_k to denote the image of x_k under the homomorphism $H^k(G;\boldsymbol{Z})\to H^k(G;\boldsymbol{Q})$.

As in [5],

(1.2) *If* $G=F_4$, *then* $l=4$ *and* $(m_1, m_2, m_3, m_4)=(2, 6, 8, 12)$; *if* $G=E_6$, *then* $l=6$ *and* $(m_1, m_2, m_3, m_4, m_5, m_6)=(2, 5, 6, 8, 9, 12)$.

In this paper R stands for a commutative ring with a unit 1. If $\Lambda_R(\ \ )$ denotes an exterior algebra over R, then by (1.2)

$$H^*(F_4; \boldsymbol{Q}) = \Lambda_{\boldsymbol{Q}}(x_3, x_{11}, x_{15}, x_{23}) \tag{1.3}$$
$$H^*(E_6; \boldsymbol{Q}) = \Lambda_{\boldsymbol{Q}}(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}).$$

Here we quote a result of Araki [1, Proposition 2.5]:

Proposition 3. *$H^*(EIV; \boldsymbol{Z})$ has no torsion, and*

$$H^*(EIV; \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(x_9, x_{17})$$

where $x_k \in H^k(EIV; \boldsymbol{Z})$ ($k=9, 17$) is primitive.

As mentioned in §0, there is a fibration

$$F_4 \xrightarrow{j} E_6 \xrightarrow{q} EIV.$$

Consider the Leray-Serre spectral sequence for the cohomology with coefficients in R of this fibration. When $R=\boldsymbol{Q}$, it follows from (1.3) and Proposition 3 that the spectral sequence collapses. When $R=\boldsymbol{Z}/(p)$, by [1, Proposition 2.8] the spectral sequence collapses for every prime p. Therefore, when $R=\boldsymbol{Z}$, the spectral sequence collapses. This implies that

(1.4) (i) *The induced homomorphism $j^*: H^k(E_6; \boldsymbol{Z}) \to H^k(F_4; \boldsymbol{Z})$ satisfies*

$$j^*(x_k) = \begin{cases} x_k & \text{for } k=3, 11, 15, 23 \\ 0 & \text{for } k=9, 17; \end{cases}$$

(ii) *The induced homomorphism $q^*: H^k(EIV; \boldsymbol{Z}) \to H^k(E_6; \boldsymbol{Z})$ satisfies*

$$q^*(x_k) = x_k \qquad \text{for } k=9, 17.$$

In view of these circumstances, it seems natural to assert that

(1.5) *The induced homomorphism $\theta^*: H^k(E_6; \boldsymbol{Z}) \to H^k(E_6; \boldsymbol{Z})$ satisfies*

$$\theta^*(x_k) = \begin{cases} x_k & \text{for } k=3, 11, 15, 23 \\ -x_k & \text{for } k=9, 17. \end{cases}$$

This will be verified at the end of the next section.

Let T be a maximal torus of G. Consider a complex representation ρ of G, i.e., $\rho: G \to U(n)$, a continuous homomorphism of G into the unitary group, where n is the dimension of ρ. Since $\rho(T)$ is a torus subgroup of $U(n)$, there exists a maximal torus T' of $U(n)$ with $\rho(T)\subset T'$. Let T^n be the standard maximal torus of $U(n)$, i.e., the group of diagonal matrices in $U(n)$. Since any two maximal tori are conjugate, we have a commutative diagram

$$\begin{array}{ccccc} T & \rightarrow & T' & \rightarrow & T^n \\ i\downarrow & & \downarrow i' & & \downarrow i_n \\ G & \xrightarrow{\rho} & U(n) & \rightarrow & U(n) \end{array}$$

of continuous homomorphisms, where i, i', i_n are the inclusions and the lower right horizontal map is an inner automorphism of $U(n)$. For simplicity we denote by ρ the composite of the lower horizontal maps and also that of the upper horizontal maps. Since any inner automorphism of a connected compact Lie group induces a self-map of its classifying space which is homotopic to the identity, we have a homotopy commutative diagram

$$\begin{array}{ccc} BT & \xrightarrow{B\rho} & BT^n \\ Bi\downarrow & & \downarrow Bi_n \\ BG & \xrightarrow{B\rho} & BU(n)\,. \end{array}$$

Let $t_1, \cdots, t_n$ be the standard base for $H^2(BT^n; \boldsymbol{Z})$. Then the elements

$$\mu_j = B\rho^*(t_j)\in H^2(BT; \boldsymbol{Z}),\ 1\leqq j\leqq n\,,$$

are called the weights of ρ.

Let $L(T)$ be the Lie algebra of T and $L(T)^*=\mathrm{Hom}(L(T), \boldsymbol{R})$ the dual of $L(T)$. Denote by $(\ ,\ )$ an invariant metric on the Lie algebra of G, and on $L(T), L(T)^*$. With respect to a certain linear order in $L(T)^*$ we have simple roots $\alpha_1, \cdots, \alpha_l$ and the corresponding fundamental weights $\omega_1, \cdots, \omega_l$ are given by the formula

$$2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$$

where δ_{ij} is the usual Kronecker symbol. As explained in [6] (or [18]), every weight can be regarded as an element of $H^2(BT; \boldsymbol{Z})$, and $H^*(BT; \boldsymbol{Z})$ is the polynomial algebra $\boldsymbol{Z}[\omega_1, \cdots, \omega_l]$. The Weyl group $W(G)=N(T)/T$ acts on T, and on BT, $H^*(BT; \boldsymbol{Z})$. The action of $W(G)$ on $H^*(BT; \boldsymbol{Z})$ is described as follows. For $1\leqq i\leqq l$ let R_i denote the reflection to the hyperplane $\{X\in L(T)\,|\, \alpha_i(X)=0\}$. Then $R_1, \cdots, R_l$ generate $W(G)$, and they act on $H^2(BT; \boldsymbol{Z})=\boldsymbol{Z}\{\omega_1, \cdots, \omega_l\}$ by

$$R_i(\omega_j) = \begin{cases} -\omega_i - \sum_{k\neq i} 2\dfrac{(\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)}\,\omega_k & \text{if } i=j \\ \omega_j & \text{if } i\neq j\,. \end{cases}$$

It is known that the representation ring $R(G)$ of G and the K-ring $K(X)$ of a space X have a λ-ring structure (see [11, Chapter 12]). A λ-ring is a commutative ring R together with functions $\Lambda^k: R \to R$ for $k \geqq 0$ satisfying the following properties:

$$(1.6)\qquad \begin{array}{lll} \text{(i)} & \Lambda^0(x) = 1 \quad \textit{and} \quad \Lambda^1(x) = x & \textit{for all} \quad x \in R; \\ \text{(ii)} & \Lambda^k(x+y) = \sum_{i+j=k} \Lambda^i(x) \cdot \Lambda^j(y) & \textit{for all} \quad x, y \in R. \end{array}$$

Furthermore, in the case of $R(G)$, if $\rho: G \to U(n)$ is a representation, then

$$\dim \Lambda^k \rho = \binom{n}{k}$$

and $\Lambda^k \rho = 0$ for $k > n$.

We now bring a famous result of Hodgkin on the K-theory of G in a form suitable for our use. According to the representation theory of compact Lie groups, there are l irreducible representations $\rho_1, \cdots, \rho_l$ of G which admit highest weights $\omega_1, \cdots, \omega_l$ respectively. Then $R(G)$ is the polynomial algebra $\boldsymbol{Z}[\rho_1, \cdots, \rho_l]$. Let $U = \varinjlim U(n)$ be the infinite unitary group and $\kappa_n: U(n) \to U$ the canonical inclusion. Let $\rho: G \to U(n)$ be a representation. Then the composite $\kappa_n \circ \rho$ gives rise to an element of $[G, U] = K^{-1}(G)$ which is denoted by $\beta(\rho)$. This correspondence $\rho \to \beta(\rho)$ extends to a map $\beta: R(G) \to K^{-1}(G)$, which is natural with respect to group homomorphisms, satisfying the following properties:

(i) $\beta(\rho+\sigma) = \beta(\rho) + \beta(\sigma)$ *for all* $\rho, \sigma \in R(G)$;

(ii) *If* ρ, σ *are representations of* G, *then*

$$(1.7)\qquad \beta(\rho\sigma) = m \cdot \beta(\rho) + n \cdot \beta(\sigma)$$

where $m = \dim \sigma$ and $n = \dim \rho$;

(iii) *For any* $k \in \boldsymbol{Z} \subset R(G)$, $\beta(k) = 0$.

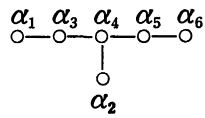
With the above notation, a reformulation of [10, Theorem A] is

Proposition 4. *Let G be a compact, 1-connected, simple Lie group of rank l. Then the $\boldsymbol{Z}/(2)$-graded K-theory $K^*(G)$ of G has no torsion and therefore it has a Hopf algebra structure. If some representations $\lambda_1, \cdots, \lambda_l$ form a base for the module of indecomposable elements in $R(G)$, i,e., $R(G) = \boldsymbol{Z}[\lambda_1, \cdots, \lambda_l]$, then*

$$K^*(G) = \Lambda_{\boldsymbol{Z}}(\beta(\lambda_1), \cdots, \beta(\lambda_l))$$

as a Hopf algebra, where each $\beta(\lambda_i)$ is primitive.

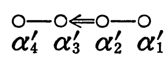
From now on, T will denote a maximal torus of E_6. Following [7], we have simple roots α_i, $1 \leqq i \leqq 6$, and the Dynkin diagram of E_6 is



where $(\alpha_i, \alpha_i)=2$ and for $i \neq j$,

$$(\alpha_i, \alpha_j) = \begin{cases} -1 & \text{if } \ (i,j) = (1,3), (2,4), (3,4), (4,5), (5,6) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the inclusion $j\colon F_4 \to E_6$. Choose a maximal torus T' of F_4 in such a way that $j(T') \subset T$. Similarly we have simple roots α'_i, $1 \leqq i \leqq 4$, and the Dynkin diagram of F_4 is



where $(\alpha'_i, \alpha'_i)=2$ if $i=1, 2$; $(\alpha'_i, \alpha'_i)=1$ if $i=3, 4$; and for $i \neq j$,

$$(\alpha'_i, \alpha'_j) = \begin{cases} -1 & \text{if } \ (i,j) = (1,2), (2,3) \\ -1/2 & \text{if } \ (i,j) = (3,4) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\omega_1, \cdots, \omega_6$ and $\omega'_1, \cdots, \omega'_4$ be the fundamental weights corresponding to $\alpha_1, \cdots, \alpha_6$ and $\alpha'_1, \cdots, \alpha'_4$ respectively. Then we have

$$(1.8) \qquad \begin{aligned} \omega_1 &= \frac{1}{3}(4\alpha_1+3\alpha_2+5\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6) \\ \omega_2 &= \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6 \\ \omega_3 &= \frac{1}{3}(5\alpha_1+6\alpha_2+10\alpha_3+12\alpha_4+8\alpha_5+4\alpha_6) \\ \omega_4 &= 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6 \\ \omega_5 &= \frac{1}{3}(4\alpha_1+6\alpha_2+8\alpha_3+12\alpha_4+10\alpha_5+5\alpha_6) \\ \omega_6 &= \frac{1}{3}(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6) \end{aligned}$$

and

$$(1.9) \qquad \begin{aligned} \omega'_1 &= 2\alpha'_1+3\alpha'_2+4\alpha'_3+2\alpha'_4 \\ \omega'_2 &= 3\alpha'_1+6\alpha'_2+8\alpha'_3+4\alpha'_4 \\ \omega'_3 &= 2\alpha'_1+4\alpha'_2+6\alpha'_3+3\alpha'_4 \\ \omega'_4 &= \alpha'_1+2\alpha'_2+3\alpha'_3+2\alpha'_4\,. \end{aligned}$$

Obviously there is a homomorphism $T' \to T$ which makes the diagram

$$\begin{array}{ccc} T' & \rightarrow & T \\ i' \downarrow & & \downarrow i \\ F_4 & \xrightarrow{j} & E_6 \end{array}$$

commute. We also denote it by j. Let us consider the behavior of the induced homomorphism $Bj^*: H^k(BT; \boldsymbol{Z}) \to H^k(BT'; \boldsymbol{Z})$. Since $H^*(BT; \boldsymbol{Z}) = \boldsymbol{Z}[\omega_1, \cdots, \omega_6]$ and ω_i is expressed as a linear combination of the α_i, it suffices to determine $Bj^*(\alpha_i)$, $1 \leqq i \leqq 6$. But as in [14, p. 130] they are given by

$$\begin{aligned} &Bj^*(\alpha_1) = \alpha'_4,\ Bj^*(\alpha_2) = \alpha'_1,\ Bj^*(\alpha_3) = \alpha'_3\,, \\ &Bj^*(\alpha_4) = \alpha'_2,\ Bj^*(\alpha_5) = \alpha'_3,\ Bj^*(\alpha_6) = \alpha'_4\,. \end{aligned}$$

From this, (1.8) and (1.9), it follows that

$$\begin{aligned} &Bj^*(\omega_1) = \omega'_4,\ Bj^*(\omega_2) = \omega'_1,\ Bj^*(\omega_3) = \omega'_3\,, \\ &Bj^*(\omega_4) = \omega'_2,\ Bj^*(\omega_5) = \omega'_3,\ Bj^*(\omega_6) = \omega'_4\,. \end{aligned} \tag{1.10}$$

Consider the automorphism $\theta: E_6 \to E_6$. There is an automorphism $T \to T$ which makes the diagram

$$\begin{array}{ccc} T & \rightarrow & T \\ i \downarrow & & \downarrow i \\ E_6 & \xrightarrow{\theta} & E_6 \end{array}$$

commute. We also denote it by θ. Let us describe the behavior of the induced automorphism $B\theta^*: H^k(BT; \boldsymbol{Z}) \to H^k(BT; \boldsymbol{Z})$. To do so it suffices to determine $B\theta^*(\alpha_i)$, $1 \leqq i \leqq 6$. But as in [14, p. 130] they are given by

$$\begin{aligned} &B\theta^*(\alpha_1) = \alpha_6,\ B\theta^*(\alpha_2) = \alpha_2,\ B\theta^*(\alpha_3) = \alpha_5\,, \\ &B\theta^*(\alpha_4) = \alpha_4,\ B\theta^*(\alpha_5) = \alpha_3,\ B\theta^*(\alpha_6) = \alpha_1\,. \end{aligned}$$

From this and (1.8), it follows that

$$\begin{aligned} &B\theta^*(\omega_1) = \omega_6,\ B\theta^*(\omega_2) = \omega_2,\ B\theta^*(\omega_3) = \omega_5\,, \\ &B\theta^*(\omega_4) = \omega_4,\ B\theta^*(\omega_5) = \omega_3,\ B\theta^*(\omega_6) = \omega_1\,. \end{aligned} \tag{1.11}$$

Let $\rho_1, \cdots, \rho_6$ be the irreducible representations of E_6 whose highest weights are $\omega_1, \cdots, \omega_6$ respectively. Then by [9],

(1.12) $\quad R(E_6) = \boldsymbol{Z}[\rho_1, \rho_2, \Lambda^2\rho_1, \Lambda^3\rho_1, \Lambda^2\rho_6, \rho_6]$

where $\dim \rho_1 = \dim \rho_6 = 27$, $\dim \rho_2 = 78$ (*in fact*, ρ_2 *is the adjoint representation of* E_6) *and the relation* $\Lambda^3\rho_6 = \Lambda^3\rho_1$ *holds.*

On the other hand, let $\rho'_1, \cdots, \rho'_4$ be the irreducible representations of F_4 whose highest weights are $\omega_1', \cdots, \omega_4'$ respectively. Then

(1.13) $$R(F_4) = \boldsymbol{Z}[\rho_4', \Lambda^2\rho_4', \Lambda^3\rho_4', \rho_1']$$

where $\dim \rho_4' = 26$ *and* $\dim \rho_1' = 52$ (*in fact*, ρ_1' *is the adjoint representation of* F_4).

Combining Proposition 4 with (1.12) (resp. (1.13)), we have a description of $K^*(E_6)$ (resp. $K^*(F_4)$), which is exhibited in Theorem 1.

Consider now the λ-ring homomorphism $j^*: R(E_6) \to R(F_4)$. Its behavior is given by

(1.14)
(i) $j^*(\rho_1) = j^*(\rho_6) = \rho_4' + 1$;
(ii) $j^*(\rho_2) = \rho_4' + \rho_1'$;
(iii) $j^*(\Lambda^2\rho_1) = j^*(\Lambda^2\rho_6) = \Lambda^2\rho_4' + \rho_4'$;
(iv) $j^*(\Lambda^3\rho_1) = \Lambda^3\rho_4' + \Lambda^2\rho_4'$.

This follows from [15, (6.7) and (6.8)] and (1.6). Consider next the λ-ring automorphism $\theta^*: R(E_6) \to R(E_6)$. Its behavior is given by

(1.15)
(i) $\theta^*(\Lambda^k\rho_1) = \Lambda^k\rho_6 \ (k = 1, 2, 3)$;
(ii) $\theta^*(\rho_2) = \rho_2$;
(iii) $\theta^*(\Lambda^k\rho_6) = \Lambda^k\rho_1 \ (k = 1, 2)$.

This follows from [15, (6.6)] and $\theta^2 = 1$.

In order to describe the K-theory of EIV, we need one more notation. Generally, let G be as before and H a closed subgroup of G. When two representations $\rho, \rho': G \to U(n)$ agree on H, we have a map $f: G/H \to U(n)$ defined by $f(gH) = \rho(g) \cdot \rho'(g)^{-1}$ for $gH \in G/H$. Then the composite $\kappa_n \circ f$ gives rise to an element of $[G/H, U] = K^{-1}(G/H)$ which is denoted by $\beta(\rho - \rho')$. Let $q: G \to G/H$ be the natural projection. It follows from [10, p. 8] that

(1.16) $$q^*(\beta(\rho - \rho')) = \beta(\rho) - \beta(\rho').$$

By (1.12) and (1.14), two elements $\beta(\rho_1 - \rho_6)$, $\beta(\Lambda^2\rho_1 - \Lambda^2\rho_6)$ of $K^{-1}(EIV)$ can be considered. Then Minami [15, Proposition 2.8] showed

Proposition 5. $K^*(EIV)$ *has no torsion, and*

$$K^*(EIV) = \Lambda_Z(\beta(\rho_1 - \rho_6), \beta(\Lambda^2\rho_1 - \Lambda^2\rho_6)).$$

2. Computations

The target of this section is to compute a part of $ch(\beta(\rho_1))$, where ρ_1: $E_6 \to U(27)$ is the irreducible representation whose highest weight is ω_1.

We first review the argument of [20, pp. 464–466]. Let ρ be an (indecomposable) element of $R(G)$. According to [10, Theorem 2.1], $\beta(\rho)$ is primitive in the Hopf algebra $K^*(G)$. Since $ch\colon K^*(G)\to H^*(G;\boldsymbol{Q})$ is a homomorphism of Hopf algebras, so is $ch(\beta(\rho))$. Therefore, by the aid of (1.1) (ii), it can be written as a linear combination of the x_{2m_i-1}:

$$(2.1) \qquad ch(\beta(\rho))=\sum_{i=1}^{l} a(\rho,i)\,x_{2m_i-1} \quad in \quad PH^*(G;\boldsymbol{Q})$$

for some $a(\rho,i)\in\boldsymbol{Q}$. By virtue of (1.1) (i), this equality determines $a(\rho,i)$ up to sign.

Let us recall some facts about the rational cohomology of a classifying space BG for G (see [4]). $H^*(BG;\boldsymbol{Q})$ is a polynomial algebra generated by elements of degrees $2m_i$, $1\leqq i\leqq l$. The induced homomorphism $Bi^*\colon H^*(BG;\boldsymbol{Q})\to H^*(BT;\boldsymbol{Q})$ maps $H^*(BG;\boldsymbol{Q})$ isomorphically onto $H^*(BT;\boldsymbol{Q})^{W(G)}$, the subalgebra of invariants under the action of $W(G)$. Hence

$$H^*(BG;\boldsymbol{Q})=\boldsymbol{Q}[y_{2m_1},\cdots,y_{2m_l}]$$
$$H^*(BT;\boldsymbol{Q})^{W(G)}=\boldsymbol{Q}[f_{2m_1},\cdots,f_{2m_l}]$$

where generators y_{2m_i} and f_{2m_i} are chosen to be integral and not divisible by other integral generators. Therefore we may set

$$(2.2) \qquad Bi^*(y_{2m_i})=c(m_i)f_{2m_i} \quad in \quad QH^{2m_i}(BT;\boldsymbol{Q})$$

for some $c(m_i)\in\boldsymbol{Z}$, where Q denotes the indecomposable module functor.

Let $\sigma\colon H^k(BG;\boldsymbol{Q})\to H^{k-1}(G;\boldsymbol{Q})$ be the cohomology suspension (see [21, Chapter VIII]). Since it induces a map $QH^k(BG;\boldsymbol{Q})\to PH^{k-1}(G;\boldsymbol{Q})$, we may set

$$(2.3) \qquad \sigma(y_{2m_i})=b(m_i)\,x_{2m_i-1} \quad in \quad PH^{2m_i-1}(G;\boldsymbol{Q})$$

for some $b(m_i)\in\boldsymbol{Z}$.

Consider the composition

$$R(G)\xrightarrow{i^*}R(T)\xrightarrow{\alpha}K^*(BT)\xrightarrow{ch}H^*(BT;\boldsymbol{Q})$$

where α is the λ-ring homomorphism of [3, §4]. Let $\rho\colon G\to U(n)$ be a representation with weights $\mu_1,\cdots,\mu_n\in H^2(BT;\boldsymbol{Z})$. Then we have

$$ch\alpha i^*(\rho)=\sum_{j=1}^{n}\exp(\mu_j)=\sum_{j=1}^{n}\Big(\sum_{k\geqq 0}\mu_j^k/k!\Big)=\sum_{k\geqq 0}\Big(\sum_{j=1}^{n}\mu_j^k\Big)/k!\,.$$

Since the set $\{\mu_1,\cdots,\mu_n\}$ is invariant under the action of $W(G)$, $ch\alpha i^*(\rho)$ belongs to $H^*(BT;\boldsymbol{Q})^{W(G)}$. So we may write

$$(2.4) \qquad ch\alpha i^*(\rho)=\sum_{i=1}^{l} f(\rho,i)f_{2m_i} \quad in \quad QH^*(BT;\boldsymbol{Q})^{W(G)}$$

for some $f(\rho, i)\in \boldsymbol{Q}$.

Now the conclusion of [20, Method I] is

Proposition 6. *For* $1\leqq i\leqq l$,

$$a(\rho, i) = b(m_i)f(\rho, i)/c(m_i)$$

up to sign.

In what follows we shall compute a part of $ch\alpha i^*(\rho_1)$ explicitly. Although $\{\omega_1, \cdots, \omega_6\}$ is a base for $H^2(BT; \boldsymbol{Z})$, we use the base of [19, p. 266] as a matter of convenience:

$$\begin{aligned} t_6 &= \omega_6 \\ t_5 &= \omega_5-\omega_6 \\ t_4 &= \omega_4-\omega_5 \\ t_3 &= \omega_2+\omega_3-\omega_4 \\ t_2 &= \omega_1+\omega_2-\omega_3 \\ t_1 &= -\omega_1+\omega_2 \\ x &= \omega_2\,. \end{aligned} \tag{2.5}$$

Then we have

$$H^*(BT; \boldsymbol{Z}) = \boldsymbol{Z}[t_1, \cdots, t_6, x]/(c_1-3x) \quad \textit{where} \quad c_1 = t_1+\cdots+t_6\,. \tag{2.6}$$

The action of $W(E_6)$ on this base is given by the upper table of [19, p. 267]. Using it, we can determine the $W(E_6)$-orbit of ω_1 as follows. First we apply R_i to $\omega_1=x-t_1$ and get $x-t_i(1\leqq i\leqq 6)$. Applying R_2 to $x-t_6$, we get $-x+t_4+t_5$. Applying R_i to it, we get $-x+t_i+t_j(1\leqq i<j\leqq 6)$. Applying R_2 to $-x+t_1+t_2$, we get $-t_3$. Finally we apply R_i to it and get $-t_i(1\leqq i\leqq 6)$. Let

$$\Omega = \{x-t_i,\ -x+t_i+t_j,\ -t_i\,|\,1\leqq i<j\leqq 6\}\ .$$

Then it is easy to see that Ω is invariant under the action of the R_i. Since Ω consists of 27 elements and dim $\rho_1=27$, Ω is just the set of weights of ρ_1 (cf. [16, p. 176]). Therefore, if we put

$$F_k = \sum_{\omega\in\Omega} \omega^k \in H^{2k}(BT; \boldsymbol{Z})$$

for $k\geqq 0$, we have

$$ch\alpha i^*(\rho_1) = \sum_{k\geqq 0} F_k/k!\ . \tag{2.7}$$

Let us compute F_k. For $i\geqq 1$ let $c_i=\sigma_i(t_1, \cdots, t_6)$ be the i-th elementary symmetric polynomial in $t_1, \cdots, t_6$, where $c_i=0$ if $i>6$. For $n\geqq 0$ let $s_n=t_1^n+\cdots+t_6^n$, where $s_0=6$. Then the Newton formulas express s_n in terms of the c_i:

$$s_n = \sum_{i=1}^{n-1}(-1)^{i-1} s_{n-i}\, c_i+(-1)^{n-1} nc_n \tag{2.8}$$

(cf. [19, (5.8)] in which there is a misprint). In particular, $s_1=c_1=3x$ by (2.6). For $k\geqq 0$ let F'_k be the polynomial of degree $2k$ in $t_1, \cdots, t_6$ such that

$$\sum_{i<j}\exp(t_i+t_j) = \sum_{k\geqq 0} F'_k/k!$$

where we assign 2 for the degree of t_i. Since

$$\sum_{i=1}^{6}\exp(t_i) = \sum_{k\geqq 0} s_k/k!\,,$$

we have

$$\begin{aligned}\sum_{i<j}\exp(t_i+t_j) &= \frac{1}{2}\{(\sum_{i=1}^{6}\exp(t_i))^2-\sum_{i=1}^{6}\exp(2t_i)\}\\ &= \frac{1}{2}(\sum_{m\geqq 0} s_m/m!)(\sum_{n\geqq 0} s_n/n!)-\frac{1}{2}\sum_{k\geqq 0} 2^k s_k/k!\\ &= \frac{1}{2}\sum_{k\geqq 0}\ \sum_{m+n=k} s_m s_n/m!\,n!-\sum_{k\geqq 0} 2^{k-1} s_k/k!\,.\end{aligned}$$

Hence

$$F'_k = \frac{1}{2}\sum_{m+n=k}\binom{k}{m} s_m\, s_n-2^{k-1} s_k\,. \tag{2.9}$$

Similarly,

$$\begin{aligned}ch\alpha i^*(\rho_1) &= \sum_{i=1}^{6}\exp(x-t_i)+\sum_{i<j}\exp(-x+t_i+t_j)+\sum_{i=1}^{6}\exp(-t_i)\\ &= \exp(x)\sum_{i=1}^{6}\exp(-t_i)+\exp(-x)\sum_{i>j}\exp(t_i+t_j)+\sum_{i=1}^{6}\exp(-t_i)\\ &= (\sum_{n\geqq 0} x^n/n!)(\sum_{m\geqq 0}(-1)^m s_m/m!)+(\sum_{n\geqq 0}(-1)^n x^n/n!)(\sum_{m\geqq 0} F'_m/m!)\\ &\quad +\sum_{k\geqq 0}(-1)^k s_k/k!\\ &= \sum_{k\geqq 0}\ \sum_{m+n=k}(-1)^m s_m x^n/m!\,n!+\sum_{k\geqq 0}\ \sum_{m+n=k}(-1)^n F'_m x^n/m!\,n!\\ &\quad +\sum_{k\geqq 0}(-1)^k s_k/k!\,.\end{aligned}$$

Therefore

$$F_k = \sum_{m=0}^{k}\binom{k}{m}((-1)^m s_m+(-1)^{k-m} F'_m)\, x^{k-m}+(-1)^k s_k\,.$$

Combining this, (2.8) and (2.9), one can compute F_k, and our final results are

$$
(2.10)\quad \begin{aligned} &F_0 = 27,\ F_1 = 0,\ F_2 = -2^2\cdot 3\,(c_2-4x^2),\ F_3 = 0,\ F_4 = 2^2\cdot 3(c_2-4x^2)^2, \\ &F_5 = -2^2\cdot 3\cdot 5\,(c_5-c_4\,x+c_3\,x^2-c_2\,x^3+2x^5)\ \textit{and so on.} \end{aligned}
$$

REMARK. Watching [19, pp. 271–275], we find that the set S of [19, p. 272] is equal to $\{2\omega\,|\,\omega\in\Omega\}$ and that for $n\geqq 0$ the element

$$I_n = \sum_{y\in S} y^n$$

is expressed as a polynomial in the c_i and x modulo $(I_m\,|\,m<n)$. In the above paragraph we have mimicked the computation of $I_n(=2^n F_n)$ developed there.

It follows from (2.6), (2.10) and [19, Lemma 5.2] that the elements

$$
\begin{aligned} &c_2-4x^2\in H^4(BT;\,\boldsymbol{Z})\,, \\ &c_5-c_4\,x+c_3\,x^2-c_2\,x^3+2x^5\in H^{10}(BT;\,\boldsymbol{Z}) \end{aligned}
$$

are indivisible and give the first two generators of the polynomial ring $H^*(BT;\,\boldsymbol{Q})^{W(E_6)}$. Thus we may take

$$
\begin{aligned} f_4 &= -(c_2-4x^2)\,, \\ f_{10} &= -(c_5-c_4\,x+c_3\,x^2-c_2\,x^3+2\,x^5) \end{aligned}
$$

(for details see [20, Remark in p. 466]). Simultaneously we deduce from this, (2.7) and (2.10) that

$$
(2.11)\quad \begin{aligned} f(\rho_1,\,1) &= 2^2\cdot 3/2! = 6\,, \\ f(\rho_1,\,2) &= 2^2\cdot 3\cdot 5/5! = 1/2\,. \end{aligned}
$$

By (1.2) we note that (2.3) and (2.2) give

$$
\begin{aligned} &\sigma(y_4) = b(2)\,x_3,\ \sigma(y_{10}) = b(5)\,x_9,\ \cdots \\ &Bi^*(y_4) = c(2)\,f_4,\ Bi^*(y_{10}) = c(5)\,f_{10},\ \cdots \end{aligned}
$$

where $b(2),\,b(5),\,\cdots,\,c(2),\,c(5),\,\cdots\in\boldsymbol{Z}$.

Proposition 7. *We have, up to sign,*

(i) $b(2) = 1$ *and* $b(5) = 1$;
(ii) $c(2) = 1$ *and* $c(5) = 1$.

Proof. Our argument will be based on the fact that $H^*(E_6;\,\boldsymbol{Z})$ has p-torsion if and only if $p=2,\,3$.

We first show (i). Consider the Leray-Serre spectral sequence $\{E_r(\boldsymbol{Z})\}$ for the integral cohomology of a universal E_6-bundle $E_6\to EE_6\to BE_6$. To investigate it, we use the Leray-Serre spectral sequence $\{E_r(\boldsymbol{Z}/(p))\}$ for the mod p

cohomology of the same bundle, where p runs over all primes. As seen in [12] and [13], for degrees $\leqq 9$

$$H^*(E_6; \boldsymbol{Z}/(p)) = \begin{cases} \boldsymbol{Z}/(2)\ \{1, \bar{x}_3, \bar{x}_5, \bar{x}_3^2, \bar{x}_3\bar{x}_5, \bar{x}_3^3, \bar{x}_9\} & \text{if } p=2 \\ \boldsymbol{Z}/(3)\ \{1, \bar{x}_3, \bar{x}_7, \bar{x}_8, \bar{x}_9\} & \text{if } p=3 \\ \boldsymbol{Z}/(p)\ \{1, \bar{x}_3, \bar{x}_9\} & \text{if } p \geqq 5, \end{cases}$$

and for degrees $\leqq 10$

$$H^*(BE_6; \boldsymbol{Z}/(p)) = \begin{cases} \boldsymbol{Z}/(2)\ \{1, \bar{y}_4, \bar{y}_6, \bar{y}_7, \bar{y}_4^2, \bar{y}_4\bar{y}_6, \bar{y}_{10}\} & \text{if } p=2 \\ \boldsymbol{Z}/(3)\ \{1, \bar{y}_4, \bar{y}_4^2, \bar{y}_8, \bar{y}_9, \bar{y}_{10}\} & \text{if } p=3 \\ \boldsymbol{Z}/(p)\ \{1, \bar{y}_4, \bar{y}_4^2, \bar{y}_{10}\} & \text{if } p \geqq 5 \end{cases}$$

where for each prescribed k $\bar{x}_k \in H^k(E_6; \boldsymbol{Z}/(p))$ transgresses to $\bar{y}_{k+1} \in H^{k+1}(BE_6; \boldsymbol{Z}/(p))$ in $E_{k+1}(\boldsymbol{Z}/(p))$, and if β_p: $H^k(\ ; \boldsymbol{Z}/(p)) \to H^{k+1}(\ ; \boldsymbol{Z}/(p))$ is the mod p Bockstein operator, then

$$\begin{aligned} &\beta_2(\bar{x}_5) = \bar{x}_3^2 \quad \textit{and} \quad \beta_2(\bar{y}_6) = \bar{y}_7 \quad \textit{for} \quad p=2; \\ &\beta_3(\bar{x}_7) = \bar{x}_8 \quad \textit{and} \quad \beta_3(\bar{y}_8) = \bar{y}_9 \quad \textit{for} \quad p=3. \end{aligned} \tag{2.12}$$

Therefore, for $k=3, 9$ the mod p reduction homomorphism $H^k(\ ; \boldsymbol{Z}) \to H^k(\ ; \boldsymbol{Z}/(p))$ sends x_k (resp. y_{k+1}) to $\bar{x}_k$ (resp. $\bar{y}_{k+1}$) for every prime p. Thus we see that for $k=3, 9$ x_k transgresses to y_{k+1} in $E_{k+1}(\boldsymbol{Z})$. Since the cohomology suspension and cohomology transgression are inverse, it follows that $\sigma(y_{k+1}) = x_k$ for $k=3, 9$. This proves (i).

We next show (ii). Consider the Leray-Serre spectral sequence $\{E_r\}$ for the integral cohomology of the fibration

$$E_6/T \to BT \xrightarrow{Bi} BE_6 .$$

Then $E_2^{s,t} \cong H^s(BE_6; H^t(E_6/T; \boldsymbol{Z}))$. For all $t \geqq 0$ $H^t(E_6/T; \boldsymbol{Z})$ is a free abelian group whose rank is known (see [19]), while it follows from (2.12) that for $0 \leqq s \leqq 10$

s	0	1	2	3	4	5	6	7	8	9	10
$H^s(BE_6; \boldsymbol{Z})$	$\boldsymbol{Z}$	0	0	0	$\boldsymbol{Z}$	0	0	$\boldsymbol{Z}/(2)$	$\boldsymbol{Z}$	$\boldsymbol{Z}/(3)$	$\boldsymbol{Z}$

Therefore, it is easy to check that if $k=4, 10$ $E_2^{s,k-s}$ has no torsion for all s and hence so does $E_\infty^{s,k-s}$. By the interpretation of Bi^*: $H^k(BE_6; \boldsymbol{Z}) \to H^k(BT; \boldsymbol{Z})$ as an edge homomorphism in $\{E_r\}$, this implies (ii).

Now apply Proposition 6 with $\rho = \rho_1$. Then by (2.11) and Proposition 7 we have

Lemma 8. $a(\rho_1, 1)=6$ *and* $a(\rho_1, 2)=1/2$.

We conclude this section by verifying (1.5).

Proof of (1.5).
Consider the commutative diagram

$$\begin{array}{ccc} H^k(BT;\boldsymbol{Z}) & \xrightarrow{B\theta^*} & H^k(BT;\boldsymbol{Z}) \\ Bi^*\uparrow & & \uparrow Bi^* \\ H^k(BE_6;\boldsymbol{Z}) & \xrightarrow{B\theta^*} & H^k(BE_6;\boldsymbol{Z}) \\ \sigma\downarrow & & \downarrow\sigma \\ H^{k-1}(E_6;\boldsymbol{Z}) & \xrightarrow{\theta^*} & H^{k-1}(E_6;\boldsymbol{Z}) . \end{array}$$

It follows from (1.11) and (2.5) that

$$B\theta^*(t_i) = x - t_{7-i} \qquad (1 \leqq i \leqq 6) \text{ and } B\theta^*(x) = x .$$

From this we deduce that

$$\{B\theta^*(\omega) \mid \omega \in \Omega\} = \{-\omega \mid \omega \in \Omega\} .$$

Therefore

$$\begin{aligned} B\theta^*(F_k) &= B\theta^*(\sum_{\omega\in\Omega} \omega^k) = \sum_{\omega\in\Omega} B\theta^*(\omega)^k \\ &= \sum_{\omega\in\Omega} (-\omega)^k = (-1)^k \sum_{\omega\in\Omega} \omega^k = (-1)^k F_k . \end{aligned}$$

Suppose for a moment that $k\,(=m_i)=2, 5, 6, 8, 9, 12$. Then we observe from [19, Lemma 5.2] and [20, Remark in p. 466] that F_k gives rise to f_{2k}. Hence

$$B\theta^*(f_{2k}) = \begin{cases} f_{2k} & \text{for } k = 2, 6, 8, 12 \\ -f_{2k} & \text{for } k = 5, 9 . \end{cases}$$

Because of (2.2), (2.3) and the commutativity of the above diagram, this implies (1.5).

3. Proof of the main results

In this section we complete the proof of Theorems 1 and 2.
By (1.2), (2.1) and (1.7), if ρ is a representation of E_6, then

$$\begin{aligned} ch(\beta(\rho)) = a(\rho, 1)\, x_3 + a(\rho, 2)\, x_9 + a(\rho, 3)\, x_{11} \\ + a(\rho, 4)\, x_{15} + a(\rho, 5)\, x_{17} + a(\rho, 6)\, x_{23} , \end{aligned} \tag{3.1}$$

and if ρ' is a representation of F_4, then

$$ch(\beta(\rho')) = a(\rho', 1)\, x_3 + a(\rho', 2)\, x_{11} + a(\rho', 3)\, x_{15} + a(\rho', 4)\, x_{23} .$$

By Propositions 3 and 5, we may write

$$\begin{aligned}ch(\beta(\rho_1-\rho_6)) &= a\cdot x_9+b\cdot x_{17}\,,\\ ch(\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) &= c\cdot x_9+d\cdot x_{17}\end{aligned}\tag{3.2}$$

for some $a, b, c, d\in \boldsymbol{Q}$.

Proposition 9. $\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\pm 1$.

Proof. As is well known [11], if g_{26} (resp. u_{26}) is a generator of $\tilde{K}(S^{26})$ (resp. $\tilde{H}^{26}(S^{26}; \boldsymbol{Z})$), then $ch(g_{26})=\pm u_{26}$ in $\tilde{H}^{26}(S^{26}; \boldsymbol{Q})$. According to [8], EIV has a cell decomposition $S^9\cup e^{17}\cup e^{26}$. Consider the cofibration

$$S^9\cup e^{17} \rightarrow EIV \xrightarrow{f} S^{26}\,.$$

By Propositions 5 and 3 it is easy to see that $f^*(g_{26})=\beta(\rho_1-\rho_6)\cdot\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)$ in $\tilde{K}(EIV)=\boldsymbol{Z}$ and $f^*(u_{26})=x_9\, x_{17}$ in $\tilde{H}^{26}(EIV; \boldsymbol{Z})=\boldsymbol{Z}$. Then it follows from the naturality of ch that

$$ch(\beta(\rho_1-\rho_6)\cdot\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) = \pm x_9\, x_{17}$$

in $\tilde{H}^{26}(EIV; \boldsymbol{Q})$. Since ch is a ring homomorphism, the result follows from this equality, (3.2) and Propositions 3 and 5.

This proposition can be viewed as a variant of [2, Proposition 1] (cf. [17, p. 156] and [20, p. 463]).

Proof of Theorem 2.

Since ch is a multiplicative natural transformation, we have

$$\begin{aligned}q^*ch(\beta(\rho_1-\rho_6)) &= ch(q^*(\beta(\rho_1-\rho_6)))\\ &= ch(\beta(\rho_1)-\beta(\rho_6)) \quad \text{by (1.16)}\\ &= ch(\beta(\rho_1))-ch(\beta(\rho_6))\end{aligned}$$

and similarly

$$q^*ch(\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) = ch(\beta(\Lambda^2\rho_1))-ch(\beta(\Lambda^2\rho_6))\,.$$

Therefore, it follows from (3.1), (3.2) and (1.4) (ii) that for $i=1, 3, 4, 6$ $a(\rho_1, i)=a(\rho_6, i)$ and $a(\Lambda^2\rho_1, i)=a(\Lambda^2\rho_6, i)$, and that

$$\begin{aligned}a &= a(\rho_1, 2)-a(\rho_6, 2)\,, & b &= a(\rho_1, 5)-a(\rho_6, 5)\,,\\ c &= a(\Lambda^2\rho_1, 2)-a(\Lambda^2\rho_6, 2)\,, & d &= a(\Lambda^2\rho_1, 5)-a(\Lambda^2\rho_6, 5)\,.\end{aligned}\tag{3.3}$$

Applying [20, Lemma 1] to $\rho_j (j=1, 6)$, we have

$$a(\Lambda^2\rho_j, 2) = \varphi(27, 2, 5)\cdot a(\rho_j, 2)\,,$$
$$a(\Lambda^2\rho_j, 5) = \varphi(27, 2, 9)\cdot a(\rho_j, 5)$$

where ($27=\dim \rho_j$, $5=m_2$, $9=m_5$ and) $\varphi(n, k, m)$ is the integer defined for three positive integers n, k, m by

$$\varphi(n, k, m) = \sum_{i=1}^{k} (-1)^{i-1} \binom{n}{k-i} i^{m-1}\,.$$

A direct calculation gives $\varphi(27, 2, 5)=11$ and $\varphi(27, 2, 9)=-229$. It follows from these and (3.3) that

(3.4) $$c = 11a \quad \text{and} \quad d = -229b\,.$$

Substituting these relations in the equality

$$-1 = ad-bc$$

of Proposition 9, we have

$$-1 = a(-229b)-b(11a) = -240ab$$

and hence

(3.5) $$ab = 1/240\,.$$

Let us apply j^* to (3.1) with $\rho=\rho_j (j=1, 6)$. Then the left hand side becomes

$$\begin{aligned} j^*ch(\beta(\rho_j)) &= ch(\beta(j^*(\rho_j))) = ch(\beta(\rho_4'+1)) && \text{by (1.14) (i)} \\ &= ch(\beta(\rho_4')+\beta(1)) && \text{by (1.7) (i)} \\ &= ch(\beta(\rho_4')) && \text{by (1.7) (iii)} \end{aligned}$$

and the right hand side becomes

$$\begin{aligned} & j^*(a(\rho_j, 1)\, x_3+a(\rho_j, 2)\, x_9+a(\rho_j, 3)\, x_{11} \\ & \qquad +a(\rho_j, 4)\, x_{15}+a(\rho_j, 5)\, x_{17}+a(\rho_j, 6)\, x_{23}) \\ & \quad = a(\rho_j, 1)\, x_3+a(\rho_j, 3)\, x_{11}+a(\rho_j, 4)\, x_{15}+a(\rho_j, 6)\, x_{23} \end{aligned}$$

by (1.4) (i). Here we quote from [20, p. 486] that

(3.6) $$ch(\beta(\rho_4')) = 6x_3+(1/20)\, x_{11}+(1/168)\, x_{15}+(1/443520)\, x_{23}\,.$$

Hence

(3.7) $a(\rho_j, 1) = 6$, $a(\rho_j, 3) = 1/20$, $a(\rho_j, 4) = 1/168$ *and* $a(\rho_j, 6) = 1/443520$ *where* $j = 1, 6$.

On the other hand, let us apply θ^* to (3.1) with $\rho=\rho_1$. Then the left hand side becomes

$$\begin{aligned}\theta^* ch(\beta(\rho_1)) = ch(\beta(\theta^*(\rho_1))) &= ch(\beta(\rho_6)) \quad \text{by (1.15) (i)}\\ &= a(\rho_6, 1)\, x_3 + a(\rho_6, 2)\, x_9 + a(\rho_6, 3)\, x_{11}\\ &\quad + a(\rho_6, 4)\, x_{15} + a(\rho_6, 5)\, x_{17} + a(\rho_6, 6)\, x_{23}\end{aligned}$$

by (3.1) with $\rho=\rho_6$, and the right hand side becomes

$$\begin{aligned}&\theta^*(a(\rho_1, 1)\, x_3 + a(\rho_1, 2)\, x_9 + a(\rho_1, 3)\, x_{11}\\ &\qquad + a(\rho_1, 4)\, x_{15} + a(\rho_1, 5)\, x_{17} + a(\rho_1, 6)\, x_{23})\\ &= a(\rho_1, 1)\, x_3 - a(\rho_1, 2)\, x_9 + a(\rho_1, 3)\, x_{11}\\ &\qquad + a(\rho_1, 4)\, x_{15} - a(\rho_1, 5)\, x_{17} + a(\rho_1, 6)\, x_{23}\end{aligned}$$

by (1.5). Hence

$$a(\rho_6, 2) = -a(\rho_1, 2) \quad \textit{and} \quad a(\rho_6, 5) = -a(\rho_1, 5)\,. \tag{3.8}$$

Combining these and (3.3), we have

$$a = 2\cdot a(\rho_1, 2) \quad \textit{and} \quad b = 2\cdot a(\rho_1, 5)\,. \tag{3.9}$$

Since $a(\rho_1, 2)=1/2$ by Lemma 8, it follows that $a=1$. Substituting this in (3.5) gives $b=1/240$. Therefore, by (3.4), $c=11$ and $d=-229/240$. Thus Theorem 2 is proved.

Proof of Theorem 1.

By (1.12), Proposition 4 and (3.1), it suffices to compute the numbers $a(\rho_1, i)$, $a(\rho_2, i)$, $a(\Lambda^2\rho_1, i)$, $a(\Lambda^3\rho_1, i)$, $a(\Lambda^2\rho_6, i)$ and $a(\rho_6, i)$ for $i=1, 2, \cdots, 6$.

Every $a(\rho_1, i)$ has been found in Lemma 8 and (3.7) except $i=5$. But, since $b=1/240$, it follows from (3.9) that $a(\rho_1, 5)=1/480$. Thus we know all of the $a(\rho_1, i)$.

For $i=1, 3, 4, 6$ $a(\rho_6, i)$ has been found in (3.7). For $i=2, 5$ $a(\rho_6, i)$ is determined by $a(\rho_1, i)$ through (3.8). Thus we know all of the $a(\rho_6, i)$.

Applying [20, Lemma 1] to $\rho_j (j=1, 6)$, we have

$$a(\Lambda^k\rho_j, i) = \varphi(27, k, m_i)\cdot a(\rho_j, i)$$

for all $k\geqq 1$ and $1\leqq i\leqq 6$. It follows from the definition of $\varphi(n, k, m)$ that $\varphi(27, 2, 2)=25$, $(\varphi(27, 2, 5)=11,)$ $\varphi(27, 2, 6)=-5$, $\varphi(27, 2, 8)=-101$, $(\varphi(27, 2, 9) =-229,)$ $\varphi(27, 2, 12)=-2021$, $\varphi(27, 3, 2)=300$, $\varphi(27, 3, 5)=0$, $\varphi(27, 3, 6)= -270$, $\varphi(27, 3, 8)=-918$, $\varphi(27, 3, 9)=0$ and $\varphi(27, 3, 12)=122202$. Thus $a(\Lambda^2\rho_1, i)$, $a(\Lambda^2\rho_6, i)$ and $a(\Lambda^3\rho_1, i)$ can be computed from $a(\rho_1, i)$ and $a(\rho_6, i)$.

It remains to compute $a(\rho_2, i)$. Let us apply j^* to (3.1) with $\rho=\rho_2$. Then the left hand side becomes

$$\begin{aligned} j^*ch(\beta(\rho_2)) &= ch(\beta(j^*(\rho_2))) \\ &= ch(\beta(\rho_4'+\rho_1')) && \text{by (1.14) (ii)} \\ &= ch(\beta(\rho_4')+\beta(\rho_1')) && \text{by (1.7) (i)} \\ &= ch(\beta(\rho_4'))+ch(\beta(\rho_1')) \end{aligned}$$

and the right hand side becomes

$$a(\rho_2, 1)\, x_3+a(\rho_2, 3)\, x_{11}+a(\rho_2, 4)\, x_{15}+a(\rho_2, 6)\, x_{23}$$

by (1.4) (i). Here we quote from [20, p. 386] that

$$ch(\beta(\rho_1')) = 18\, x_3-(7/20)\, x_{11}+(17/168)\, x_{15}-(1/7040)\, x_{23}\,.$$

Adding this to (3.6) gives

$$ch(\beta(\rho_4'))+ch(\beta(\rho_1')) = 24\, x_3-(3/10)\, x_{11}+(3/28)\, x_{15}-(31/221760)\, x_{23}\,.$$

Hence $a(\rho_2, 1)=24$, $a(\rho_2, 3)=-3/10$, $a(\rho_2, 4)=3/28$ and $a(\rho_2, 6)=-31/221760$. On the other hand, let us apply θ^* to (3.1) with $\rho=\rho_2$. Then the left hand side becomes

$$\begin{aligned} \theta^*ch(\beta(\rho_2)) &= ch(\beta(\theta^*(\rho_2))) = ch(\beta(\rho_2)) \quad \text{by (1.15) (ii)} \\ &= a(\rho_2, 1)\, x_3+a(\rho_2, 2)\, x_9+a(\rho_2, 3)\, x_{11} \\ &\quad +a(\rho_2, 4)\, x_{15}+a(\rho_2, 5)\, x_{17}+a(\rho_2, 6)\, x_{23} \end{aligned}$$

by (3.1) with $\rho=\rho_2$, and the right hand side becomes

$$\begin{aligned} &a(\rho_2, 1)\, x_3-a(\rho_2, 2)\, x_9+a(\rho_2, 3)\, x_{11} \\ &\quad +a(\rho_2, 4)\, x_{15}-a(\rho_2, 5)\, x_{17}+a(\rho_2, 6)\, x_{23} \end{aligned}$$

by (1.5). Hence $a(\rho_2, 2)=0$ and $a(\rho_2, 5)=0$. Thus we know all of the $a(\rho_2, i)$, and Theorem 1 is proved.

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