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THE CHERN CHARACTER HOMOMORPHISM OF
THE COMPACT SIMPLY CONNECTED
EXCEPTIONAL GROUP E₆

Dedicated to Professor Shôrô Araki on his sixtieth birthday

TAKASHI WATANABE

(Received September 28, 1990)

0. Introduction

Let $F_4$ and $E_6$ be the compact, 1-connected representatives of the respective local classes. As in [22] there is an involutive automorphism $\theta$ of $E_6$ such that the subgroup consisting of fixed points of $\theta$ is $F_4$. Thus the quotient $E_6/F_4$ forms a compact symmetric space, which is denoted by $EIV$ in Cartan's notation. For brevity we shall write $EIV$ instead of $E_6/F_4$.

The ordinary cohomology and complex $K$-theory of three spaces $F_4$, $E_6$ and $EIV$ are well understood (see §1). Moreover, the Chern character homomorphism of $F_4$ was described explicitly in [20]. The purpose of this paper is to study those of $E_6$ and $EIV$. Our results are stated as follows (for notations used below, see §1):

**Theorem 1.** The Chern character homomorphism

\[ ch: K^*(E_6) = \Lambda^*_Q(\beta(p_1), \beta(p_2), \beta(\Lambda^3 p_1), \beta(\Lambda^3 p_2), \beta(\Lambda^5 p_5), \beta(p_6)) \]

\[ \rightarrow H^*(E_6; Q) = \Lambda_Q(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}) \]

is given by

\[ ch(\beta(p_1)) = 6x_3 + \frac{1}{2} x_9 + \frac{1}{20} x_{11} + \frac{1}{168} x_{15} + \frac{1}{480} x_{17} + \frac{1}{443520} x_{23} \]

\[ ch(\beta(p_2)) = 24x_3 - \frac{3}{10} x_{11} + \frac{3}{28} x_{15} - \frac{31}{221760} x_{23} \]

\[ ch(\beta(\Lambda^3 p_1)) = 150x_3 - \frac{11}{2} x_9 - \frac{1}{4} x_{11} - \frac{101}{168} x_{15} - \frac{229}{480} x_{17} - \frac{2021}{443520} x_{23} \]

\[ ch(\beta(\Lambda^3 p_2)) = 1800x_3 - \frac{27}{2} x_{11} - \frac{153}{28} x_{15} + \frac{6789}{24640} x_{23} \]

\[ ch(\beta(\Lambda^5 p_5)) = 150x_3 - \frac{11}{2} x_9 - \frac{1}{4} x_{11} - \frac{101}{168} x_{15} + \frac{229}{480} x_{17} - \frac{2021}{443520} x_{23} \]

\[ ch(\beta(p_6)) = 6x_3 - \frac{1}{2} x_9 + \frac{1}{20} x_{11} + \frac{1}{168} x_{15} - \frac{1}{480} x_{17} - \frac{1}{443520} x_{23} \]
Theorem 2. The Chern character homomorphism

\[ ch : K^*(EIV) = \Lambda_2 (\beta (\rho_1 - \rho_6), \beta (\Lambda^2 \rho_1 - \Lambda^2 \rho_6)) \]
\[ \to H^*(EIV ; \mathbb{Q}) = \Lambda_2 (x_9, x_{17}) \]

is given by

\[ ch(\beta (\rho_1 - \rho_6)) = x_9 + \frac{1}{240} x_{17} \]
\[ ch(\beta (\Lambda^2 \rho_1 - \Lambda^2 \rho_6)) = 11 x_9 + \frac{229}{240} x_{17} \]

The paper has the following organization. In §1 we collect some facts which we need. §2 contains various computations and consequently we obtain certain data. §3 is devoted to prove Theorems 1 and 2.

The author would like to thank Professor H. Minami for his kind advice.

1. Preliminaries

In this section we recollect some results on the cohomology and \( K \)-theory of our spaces \( F_4, E_6 \) and \( EIV \).

Let us begin with the cohomology of compact Lie groups. Throughout the paper \( G \) stands for a compact, 1-connected, simple Lie group of rank \( l \). Then the rational cohomology ring of \( G \) is an exterior algebra generated by primitive elements of degrees \( 2m_i - 1 \), \( 1 \leq i \leq l \), where the \( m_i \) are certain integers such that \( 2 = m_1 \leq m_2 \leq \cdots \leq m_l \) (see [5]). Since \( G \) is parallelizable, one can utilize the Poincaré duality theorem for choosing elements

\[ x_k \in H^k(G; \mathbb{Z}), k = 2m_i - 1, 1 \leq i \leq l, \]

which satisfy the following conditions:

(1.1) (i) \( x_k \) is not divisible in \( H^k(G; \mathbb{Z}) \);
(ii) The image of \( x_k \) under the coefficient group homomorphism \( H^k(G; \mathbb{Z}) \to H^k(G; \mathbb{Q}) \) induced by the natural inclusion \( \mathbb{Z} \to \mathbb{Q} \) belongs to \( PH^k(G; \mathbb{Q}) \), where \( P \) denotes the primitive module functor;
(iii) The cup product

\[ x_{2m_1 - 1} x_{2m_2 - 1} \cdots x_{2m_l - 1} \]

generates the infinite cyclic group \( H^\ast(G; \mathbb{Z}) \), where

\[ n = \sum_{i=1}^l (2m_i - 1) = \dim G. \]

We will use the same symbol \( x_k \) to denote the image of \( x_k \) under the homomorphism \( H^k(G; \mathbb{Z}) \to H^k(G; \mathbb{Q}) \).

As in [5],
If \( G = F_4 \), then \( l = 4 \) and \( (m_1, m_2, m_3, m_4) = (2, 6, 8, 12) \); if \( G = E_6 \), then \( l = 6 \) and \( (m_1, m_2, m_3, m_4, m_5, m_6) = (2, 5, 6, 8, 9, 12) \).

In this paper \( R \) stands for a commutative ring with a unit 1. If \( \Lambda_\mathfrak{g} \) denotes an exterior algebra over \( R \), then by (1.2)

\[
H^*(F_4; \mathbb{Q}) = \Lambda_\mathbb{Q}(x_3, x_{11}, x_{15}, x_{23})
\]

\[
H^*(E_6; \mathbb{Q}) = \Lambda_\mathbb{Q}(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}).
\]

Here we quote a result of Araki [1, Proposition 2.5]:

**Proposition 3.** \( H^*(EIV; \mathbb{Z}) \) has no torsion, and

\[
H^*(EIV; \mathbb{Z}) = \Lambda_\mathbb{Z}(x_9, x_{17})
\]

where \( x_k \in H^k(EIV; \mathbb{Z}) \) (\( k = 9, 17 \)) is primitive.

As mentioned in §0, there is a fibration

\[
F_4 \xrightarrow{j} E_6 \xrightarrow{q} EIV.
\]

Consider the Leray-Serre spectral sequence for the cohomology with coefficients in \( R \) of this fibration. When \( R = \mathbb{Q} \), it follows from (1.3) and Proposition 3 that the spectral sequence collapses. When \( R = \mathbb{Z}/(p) \), by [1, Proposition 2.8] the spectral sequence collapses for every prime \( p \). Therefore, when \( R = \mathbb{Z} \), the spectral sequence collapses. This implies that

\[
(1.4) \quad \begin{cases} 
\text{(i) The induced homomorphism } j^*: H^k(E_6; \mathbb{Z}) \to H^k(F_4; \mathbb{Z}) \text{ satisfies} \\
& \quad j^*(x_k) = \begin{cases} x_k & \text{for } k = 3, 11, 15, 23 \\
& \quad 0 & \text{for } k = 9, 17; \end{cases} \\
\text{(ii) The induced homomorphism } q^*: H^k(EIV; \mathbb{Z}) \to H^k(E_6; \mathbb{Z}) \text{ satisfies} \\
& \quad q^*(x_k) = x_k & \text{for } k = 9, 17. 
\end{cases}
\]

In view of these circumstances, it seems natural to assert that

\[
(1.5) \quad \begin{cases} 
\text{The induced homomorphism } \theta^*: H^k(E_6; \mathbb{Z}) \to H^k(E_6; \mathbb{Z}) \text{ satisfies} \\
& \quad \theta^*(x_k) = \begin{cases} x_k & \text{for } k = 3, 11, 15, 23 \\
& \quad -x_k & \text{for } k = 9, 17. \end{cases} 
\end{cases}
\]

This will be verified at the end of the next section.

Let \( T \) be a maximal torus of \( G \). Consider a complex representation \( \rho \) of \( G \), i.e., \( \rho: G \to U(n) \), a continuous homomorphism of \( G \) into the unitary group, where \( n \) is the dimension of \( \rho \). Since \( \rho(T) \) is a torus subgroup of \( U(n) \), there
exists a maximal torus \( T' \) of \( U(n) \) with \( \rho(T) \subset T' \). Let \( T' \) be the standard maximal torus of \( U(n) \), i.e., the group of diagonal matrices in \( U(n) \). Since any two maximal tori are conjugate, we have a commutative diagram

\[
\begin{align*}
T & \to T' \to T' \\
\downarrow i & \downarrow i' \downarrow i_n \\
G & \xrightarrow{\rho} U(n) \to U(n)
\end{align*}
\]

of continuous homomorphisms, where \( i, i', i_n \) are the inclusions and the lower right horizontal map is an inner automorphism of \( U(n) \). For simplicity we denote by \( \rho \) the composite of the lower horizontal maps and also that of the upper horizontal maps. Since any inner automorphism of a connected compact Lie group induces a self-map of its classifying space which is homotopic to the identity, we have a homotopy commutative diagram

\[
\begin{array}{ccc}
BT & \xrightarrow{B\rho} & BT' \\
\downarrow Bi & & \downarrow Bi' \\
BG & \xrightarrow{B\rho} & BU(n)
\end{array}
\]

Let \( t_1, \ldots, t_n \) be the standard base for \( H^2(BT'; \mathbb{Z}) \). Then the elements

\[
\mu_j = B\rho^*(t_j) \in H^2(BT'; \mathbb{Z}), \quad 1 \leq j \leq n,
\]

are called the weights of \( \rho \).

Let \( L(T) \) be the Lie algebra of \( T \) and \( L(T)^* = \text{Hom}(L(T), \mathbb{R}) \) the dual of \( L(T) \). Denote by \( (\ , \ ) \) an invariant metric on the Lie algebra of \( G \), and on \( L(T), L(T)^* \). With respect to a certain linear order in \( L(T)^* \) we have simple roots \( \alpha_1, \ldots, \alpha_l \) and the corresponding fundamental weights \( \omega_1, \ldots, \omega_l \) are given by the formula

\[
2(\omega_i, \alpha_j)(\alpha_j, \alpha_j) = \delta_{ij}
\]

where \( \delta_{ij} \) is the usual Kronecker symbol. As explained in [6] (or [18]), every weight can be regarded as an element of \( H^2(BT; \mathbb{Z}) \), and \( H^*(BT; \mathbb{Z}) \) is the polynomial algebra \( \mathbb{Z}[\omega_1, \ldots, \omega_l] \). The Weyl group \( W(G) = N(T)/T \) acts on \( T \), and on \( BT, H^*(BT; \mathbb{Z}) \). The action of \( W(G) \) on \( H^*(BT; \mathbb{Z}) \) is described as follows. For \( 1 \leq i \leq l \) let \( R_i \) denote the reflection to the hyperplane \( \{x \in L(T) | \alpha_i(x) = 0\} \). Then \( R_1, \ldots, R_l \) generate \( W(G) \), and they act on \( H^2(BT; \mathbb{Z}) = \mathbb{Z}\{\omega_1, \ldots, \omega_l\} \) by

\[
R_i(\omega_j) = \begin{cases} 
-\omega_i - \sum_{k \neq i} 2(\alpha_i, \alpha_k) \omega_k & \text{if } i = j \\
\omega_j & \text{if } i \neq j.
\end{cases}
\]

It is known that the representation ring \( R(G) \) of \( G \) and the \( K \)-ring \( K(X) \) of
a space $X$ have a $\lambda$-ring structure (see [11, Chapter 12]). A $\lambda$-ring is a commutative ring $R$ together with functions $\Lambda^k: R \to R$ for $k \geq 0$ satisfying the following properties:

\begin{align}
(1.6) \quad & (i) \quad \Lambda^0(x) = 1 \quad \text{and} \quad \Lambda^1(x) = x \quad \text{for all} \quad x \in R; \\
& (ii) \quad \Lambda^k(x+y) = \sum_{i+j=k} \Lambda^i(x) \cdot \Lambda^j(y) \quad \text{for all} \quad x, y \in R.
\end{align}

Furthermore, in the case of $R(G)$, if $\rho: G \to U(n)$ is a representation, then

$$\dim \Lambda^k \rho = \binom{n}{k}$$

and $\Lambda^k \rho = 0$ for $k > n$.

We now bring a famous result of Hodgkin on the $K$-theory of $G$ in a form suitable for our use. According to the representation theory of compact Lie groups, there are $l$ irreducible representations $\rho_1, \ldots, \rho_l$ of $G$ which admit highest weights $\omega_1, \ldots, \omega_l$ respectively. Then $R(G)$ is the polynomial algebra $\mathbb{Z}[\rho_1, \ldots, \rho_l]$. Let $U = \lim_{\to} U(n)$ be the infinite unitary group and $\kappa_n: U(n) \to U$ the canonical inclusion. Let $\rho: G \to U(n)$ be a representation. Then the composite $\kappa_n \circ \rho$ gives rise to an element of $[G, U] = K^{-1}(G)$ which is denoted by $\beta(\rho)$. This correspondence $\rho \to \beta(\rho)$ extends to a map $\beta: R(G) \to K^{-1}(G)$, which is natural with respect to group homomorphisms, satisfying the following properties:

\begin{align}
(1.7) \quad & (i) \quad \beta(\rho + \sigma) = \beta(\rho) + \beta(\sigma) \quad \text{for all} \quad \rho, \sigma \in R(G); \\
& (ii) \quad \text{If} \quad \rho, \sigma \quad \text{are representations of} \quad G, \text{then} \\
& \quad \beta(\rho \sigma) = m \cdot \beta(\rho) + n \cdot \beta(\sigma) \\
& \quad \text{where} \quad m = \dim \sigma \quad \text{and} \quad n = \dim \rho; \\
& (iii) \quad \text{For any} \quad k \in \mathbb{Z} \subset R(G), \quad \beta(k) = 0.
\end{align}

With the above notation, a reformulation of [10, Theorem A] is

**Proposition 4.** Let $G$ be a compact, 1-connected, simple Lie group of rank 1. Then the $\mathbb{Z}(2)$-graded $K$-theory $K^*(G)$ of $G$ has no torsion and therefore it has a Hopf algebra structure. If some representations $\lambda_1, \ldots, \lambda_l$ form a base for the module of indecomposable elements in $R(G)$, i.e., $R(G) = \mathbb{Z}[\lambda_1, \ldots, \lambda_l]$, then

$$K^*(G) = \Lambda_\mathbb{Z}(\beta(\lambda_1), \ldots, \beta(\lambda_l))$$

as a Hopf algebra, where each $\beta(\lambda_i)$ is primitive.

From now on, $T$ will denote a maximal torus of $E_6$. Following [7], we have simple roots $\alpha_i$, $1 \leq i \leq 6$, and the Dynkin diagram of $E_6$ is
Consider the inclusion $j: F_4 \hookrightarrow E_6$. Choose a maximal torus $T'$ of $F_4$ in such a way that $j(T') \subset T$. Similarly we have simple roots $\alpha'_i$, $1 \leq i \leq 4$, and the Dynkin diagram of $F_4$ is

\[
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

where $(\alpha_i, \alpha_j) = 2$ and for $i \neq j$,

\[
(\alpha_i, \alpha_j) = \begin{cases} 
-1 & \text{if } (i, j) = (1, 3), (2, 4), (3, 4), (4, 5), (5, 6) \\
0 & \text{otherwise.}
\end{cases}
\]

Let $\omega_1, \ldots, \omega_6$ and $\omega'_1, \ldots, \omega'_4$ be the fundamental weights corresponding to $\alpha_1, \ldots, \alpha_6$ and $\alpha'_1, \ldots, \alpha'_4$ respectively. Then we have

\[
\begin{align*}
\omega_1 &= \frac{1}{3} (4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\
\omega_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
\omega_3 &= \frac{1}{3} (5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6) \\
\omega_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \\
\omega_5 &= \frac{1}{3} (4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6) \\
\omega_6 &= \frac{1}{3} (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)
\end{align*}
\]

and

\[
\begin{align*}
\omega'_1 &= 2\alpha'_1 + 3\alpha'_2 + 4\alpha'_3 + 2\alpha'_4 \\
\omega'_2 &= 3\alpha'_1 + 6\alpha'_2 + 8\alpha'_3 + 4\alpha'_4 \\
\omega'_3 &= 2\alpha'_1 + 4\alpha'_2 + 6\alpha'_3 + 3\alpha'_4 \\
\omega'_4 &= \alpha'_1 + 2\alpha'_2 + 3\alpha'_3 + 2\alpha'_4
\end{align*}
\]

Obviously there is a homomorphism $T' \to T$ which makes the diagram
commute. We also denote it by $j$. Let us consider the behavior of the induced homomorphism $Bj^*: H^*(BT; \mathbb{Z}) \to H^*(BT'; \mathbb{Z})$. Since $H^*(BT; \mathbb{Z}) = \mathbb{Z}[\omega_1, \ldots, \omega_6]$ and $\omega_i$ is expressed as a linear combination of the $\alpha_i$, it suffices to determine $Bj^*(\alpha_i)$, $1 \leq i \leq 6$. But as in [14, p. 130] they are given by

$$Bj^*(\alpha_1) = \alpha'_1, \quad Bj^*(\alpha_2) = \alpha'_1, \quad Bj^*(\alpha_3) = \alpha'_3,$$

$$Bj^*(\alpha_4) = \alpha'_5, \quad Bj^*(\alpha_5) = \alpha'_5, \quad Bj^*(\alpha_6) = \alpha'_3.$$

From this, (1.8) and (1.9), it follows that

$$(1.10) \quad Bj^*(\omega_1) = \omega'_1, \quad Bj^*(\omega_2) = \omega'_1, \quad Bj^*(\omega_3) = \omega'_3,$$

$$Bj^*(\omega_4) = \omega'_3, \quad Bj^*(\omega_5) = \omega'_5, \quad Bj^*(\omega_6) = \omega'_5.$$

Consider the automorphism $\theta: E_6 \to E_6$. There is an automorphism $T \to T$ which makes the diagram

$$\begin{array}{ccc}
T' & \to & T \\
\downarrow & \downarrow & i' \\
F_4 & \to & E_6
\end{array}$$

commute. We also denote it by $\theta$. Let us describe the behavior of the induced automorphism $B\theta^*: H^*(BT; \mathbb{Z}) \to H^*(BT; \mathbb{Z})$. To do so it suffices to determine $B\theta^*(\alpha_i)$, $1 \leq i \leq 6$. But as in [14, p. 130] they are given by

$$B\theta^*(\alpha_1) = \alpha_6, \quad B\theta^*(\alpha_2) = \alpha_2, \quad B\theta^*(\alpha_3) = \alpha_5,$$

$$B\theta^*(\alpha_4) = \alpha_5, \quad B\theta^*(\alpha_5) = \alpha_3, \quad B\theta^*(\alpha_6) = \alpha_1.$$

From this and (1.8), it follows that

$$(1.11) \quad B\theta^*(\omega_1) = \omega_6, \quad B\theta^*(\omega_2) = \omega_2, \quad B\theta^*(\omega_3) = \omega_5,$$

$$B\theta^*(\omega_4) = \omega_5, \quad B\theta^*(\omega_5) = \omega_3, \quad B\theta^*(\omega_6) = \omega_1.$$

Let $\rho_1, \ldots, \rho_6$ be the irreducible representations of $E_6$ whose highest weights are $\omega_1, \ldots, \omega_6$ respectively. Then by [9],

$$(1.12) \quad R(E_6) = \mathbb{Z}[\rho_1, \rho_2, \Lambda^2 \rho_1, \Lambda^3 \rho_1, \Lambda^2 \rho_6, \rho_6]$$

where $\dim \rho_1 = \dim \rho_6 = 27$, $\dim \rho_2 = 78$ (in fact, $\rho_2$ is the adjoint representation of $E_6$) and the relation $\Lambda^3 \rho_6 = \Lambda^3 \rho_1$ holds.

On the other hand, let $\rho'_1, \ldots, \rho'_4$ be the irreducible representations of $F_4$ whose
highest weights are $\omega_1, \cdots, \omega_i$ respectively. Then

\begin{equation}
R(F_4) = \mathbb{Z}[\rho'_i, \Lambda^2 \rho'_i, \Lambda^3 \rho'_i, \rho'_i]
\end{equation}

where $\dim \rho'_i = 26$ and $\dim \rho'_i = 52$ (in fact, $\rho'_i$ is the adjoint representation of $F_4$).

Combining Proposition 4 with (1.12) (resp. (1.13)), we have a description of $K^*(E_6)$ (resp. $K^*(F_4)$), which is exhibited in Theorem 1.

Consider now the $\lambda$-ring homomorphism $j^*: R(E_6) \to R(F_4)$. Its behavior is given by

\begin{enumerate}
  \item $j^*(\rho_1) = j^*(\rho_6) = \rho'_i + 1$;
  \item $j^*(\rho_2) = \rho'_i + \rho'_i$;
  \item $j^*(\Lambda^2 \rho_1) = j^*(\Lambda^2 \rho_6) = \Lambda^2 \rho'_i + \rho'_i$;
  \item $j^*(\Lambda^3 \rho_1) = \Lambda^3 \rho'_i + \Lambda^2 \rho'_i$.
\end{enumerate}

This follows from [15, (6.7) and (6.8)] and (1.6). Consider next the $\lambda$-ring automorphism $\theta^*: R(E_6) \to R(E_6)$. Its behavior is given by

\begin{enumerate}
  \item $\theta^*(\Lambda^k \rho_1) = \Lambda^k \rho_6 (k = 1, 2, 3)$;
  \item $\theta^*(\rho_2) = \rho_2$;
  \item $\theta^*(\Lambda^k \rho_6) = \Lambda^k \rho_1 (k = 1, 2)$.
\end{enumerate}

This follows from [15, (6.6)] and $\theta^2 = 1$.

In order to describe the $K$-theory of $EIV$, we need one more notation. Generally, let $G$ be as before and $H$ a closed subgroup of $G$. When two representations $\rho, \rho': G \to U(n)$ agree on $H$, we have a map $f: G/H \to U(n)$ defined by $f(gH) = \rho(g) \cdot \rho'(g)^{-1}$ for $gH \subseteq G/H$. Then the composite $\kappa$ of gives rise to an element of $[G/H, U] = K^{-1}(G/H)$ which is denoted by $\beta(\rho - \rho')$. Let $q: G \to G/H$ be the natural projection. It follows from [10, p. 8] that

\begin{equation}
q^*(\beta(\rho - \rho')) = \beta(\rho) - \beta(\rho').
\end{equation}

By (1.12) and (1.14), two elements $\beta(\rho_1 - \rho_6), \beta(\Lambda^2 \rho_1 - \Lambda^2 \rho_6)$ of $K^{-1}(EIV)$ can be considered. Then Minami [15, Proposition 2.8] showed

**Proposition 5.** $K^*(EIV)$ has no torsion, and

$$K^*(EIV) = \Lambda_{EIV}^*(\beta(\rho_1 - \rho_6), \beta(\Lambda^2 \rho_1 - \Lambda^2 \rho_6)).$$

**2. Computations**

The target of this section is to compute a part of $ch(\beta(\rho_1))$, where $\rho_1: E_6 \to U(27)$ is the irreducible representation whose highest weight is $\omega_1$.

We first review the argument of [20, pp. 464–466]. Let $\rho$ be an (indecom-
posable) element of $R(G)$. According to [10, Theorem 2.1], $\beta(\rho)$ is primitive in the Hopf algebra $K^*(G)$. Since $ch: K^*(G) \rightarrow H^*(G; \mathbb{Q})$ is a homomorphism of Hopf algebras, so is $ch(\beta(\rho))$. Therefore, by the aid of (1.1) (ii), it can be written as a linear combination of the $x_{2m_i-1}$:

\begin{equation}
ch(\beta(\rho)) = \sum_{i=1}^{l} a(\rho, i) x_{2m_i-1} \quad \text{in} \quad PH^*(G; \mathbb{Q})
\end{equation}

for some $a(\rho, i) \in \mathbb{Q}$. By virtue of (1.1) (i), this equality determines $a(\rho, i)$ up to sign.

Let us recall some facts about the rational cohomology of a classifying space $BG$ for $G$ (see [4]). $H^*(BG; \mathbb{Q})$ is a polynomial algebra generated by elements of degrees $2m_i$, $1 \leq i \leq l$. The induced homomorphism $Bi^*: H^*(BG; \mathbb{Q}) \rightarrow H^*(BT; \mathbb{Q})$ maps $H^*(BG; \mathbb{Q})$ isomorphically onto $H^*(BT; \mathbb{Q})^{W(G)}$, the subalgebra of invariants under the action of $W(G)$. Hence

\[ H^*(BG; \mathbb{Q}) = \mathbb{Q}[y_{2m_1}, \ldots, y_{2m_l}] \]
\[ H^*(BT; \mathbb{Q})^{W(G)} = \mathbb{Q}[f_{2m_1}, \ldots, f_{2m_l}] \]

where generators $y_{2m_i}$ and $f_{2m_i}$ are chosen to be integral and not divisible by other integral generators. Therefore we may set

\begin{equation}
Bi^*(y_{2m_i}) = c(m_i) f_{2m_i} \quad \text{in} \quad QH^{2m_i}(BT; \mathbb{Q})
\end{equation}

for some $c(m_i) \in \mathbb{Z}$, where $Q$ denotes the indecomposable module functor.

Let $\sigma: H^*(BG; \mathbb{Q}) \rightarrow H^{*-1}(G; \mathbb{Q})$ be the cohomology suspension (see [21, Chapter VIII]). Since it induces a map $QH^*(BG; \mathbb{Q}) \rightarrow PH^{*+1}(G; \mathbb{Q})$, we may set

\begin{equation}
\sigma(y_{2m_i}) = b(m_i) x_{2m_i-1} \quad \text{in} \quad PH^{2m_i+1}(G; \mathbb{Q})
\end{equation}

for some $b(m_i) \in \mathbb{Z}$.

Consider the composition

\[ R(G) \xrightarrow{i^*} R(T) \xrightarrow{\alpha} K^*(BT) \xrightarrow{ch} H^*(BT; \mathbb{Q}) \]

where $\alpha$ is the $\lambda$-ring homomorphism of [3, §4]. Let $\rho: G \rightarrow U(n)$ be a representation with weights $\mu_1, \ldots, \mu_n \in \mathbb{H}^2(BT; \mathbb{Z})$. Then we have

\[ ch\alpha^*(\rho) = \sum_{j=1}^{\lambda} \exp(\mu_j) = \sum_{j=1}^{\lambda} \left( \sum_{k=0}^{n} \mu_j^k \right) = \sum_{j=1}^{\lambda} \left( \sum_{k=1}^{\lambda} \mu_j^k \right) |k|! . \]

Since the set $\{\mu_1, \ldots, \mu_n\}$ is invariant under the action of $W(G)$, $ch\alpha^*(\rho)$ belongs to $H^*(BT; \mathbb{Q})^{W(G)}$. So we may write

\begin{equation}
ch\alpha^*(\rho) = \sum_{i=1}^{l} f(\rho, i) f_{2m_i} \quad \text{in} \quad QH^*(BT; \mathbb{Q})^{W(G)}
\end{equation}
for some \( f(\rho, i) \in Q \).

Now the conclusion of [20, Method I] is

**Proposition 6.** For \( 1 \leq i \leq l \),

\[
a(\rho, i) = b(m_i) f(\rho, i) c(m_i)
\]

up to sign.

In what follows we shall compute a part of \( \text{char}^*(\rho_i) \) explicitly. Although \( \{\omega_1, \cdots, \omega_6\} \) is a base for \( H^*(BT; \mathbb{Z}) \), we use the base of [19, p. 266] as a matter of convenience:

\[
\begin{align*}
t_6 &= \omega_6 \\
t_5 &= \omega_5 - \omega_6 \\
t_4 &= \omega_4 - \omega_5 \\
t_3 &= \omega_3 + \omega_4 - \omega_5 \\
t_2 &= \omega_2 + \omega_3 - \omega_4 \\
t_1 &= -\omega_1 + \omega_2 \\
x &= \omega_2.
\end{align*}
\]

Then we have

\[
H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, \cdots, t_6, x]/(c_1 - 3x)
\]

where \( c_1 = t_1 + \cdots + t_6 \).

The action of \( W(E_6) \) on this base is given by the upper table of [19, p. 267]. Using it, we can determine the \( W(E_6) \)-orbit of \( \omega_1 \) as follows. First we apply \( R_i \) to \( \omega_1 = x - t_1 \) and get \( x - t_1 (1 \leq i \leq 6) \). Applying \( R_2 \) to \( x - t_6 \), we get \(-x + t_4 + t_5 \). Applying \( R_i \) to it, we get \(-x + t_4 + t_5 (1 \leq i < j \leq 6) \). Applying \( R_2 \) to \(-x + t_4 + t_5 \), we get \(-t_3 \). Finally we apply \( R_i \) to it and get \(-t_4 (1 \leq i \leq 6) \). Let

\[
\Omega = \{x - t_i, -x + t_i + t_j, -t_i | 1 \leq i < j \leq 6\}.
\]

Then it is easy to see that \( \Omega \) is invariant under the action of the \( R_i \). Since \( \Omega \) consists of 27 elements and \( \dim \rho_1 = 27 \), \( \Omega \) is just the set of weights of \( \rho_1 \) (cf. [16, p. 176]). Therefore, if we put

\[
F_k = \sum_{\omega \in \Omega} \omega^k \in H^k(BT; \mathbb{Z})
\]

for \( k \geq 0 \), we have

\[
\text{char}^*(\rho_1) = \sum_{k \geq 0} F_k / k!.
\]

Let us compute \( F_k \). For \( i \geq 1 \) let \( c_i = \sigma_i(t_1, \cdots, t_6) \) be the \( i \)-th elementary sym-
metric polynomial in \( t_1, \ldots, t_6 \), where \( c_i = 0 \) if \( i > 6 \). For \( n \geq 0 \) let \( s_n = t_1^i + \cdots + t_6^i \), where \( s_6 = 6 \). Then the Newton formulas express \( s_n \) in terms of the \( c_i \):

\[
(2.8) \quad s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} c_i + (-1)^{n-1} nc_n
\]

(cf. [19, (5.8)] in which there is a misprint). In particular, \( s_1 = c_1 = 3x \) by (2.6).

For \( k \geq 0 \) let \( F'_k \) be the polynomial of degree \( 2k \) in \( t_1, \ldots, t_6 \) such that

\[
\sum_{i<j} \exp(t_i + t_j) = \sum_{k=0}^{\infty} F'_k/k!
\]

where we assign 2 for the degree of \( t_i \). Since

\[
\sum_{i=1}^{6} \exp(t_i) = \sum_{k=0}^{\infty} s_j/k!,
\]

we have

\[
\sum_{i<j} \exp(t_i + t_j) = \frac{1}{2} \left\{ \sum_{i=1}^{6} \exp(t_i) \right\}^2 - \sum_{i=1}^{6} \exp(2t_i)
\]

\[
= \frac{1}{2} \left( \sum_{m=0}^{\infty} s_m/m! \right) \left( \sum_{n=0}^{\infty} s_n/n! \right) - \frac{1}{2} \sum_{k=0}^{\infty} 2^k s_k/k!
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{m=n-k} s_m s_n/m! n! - \sum_{k=0}^{\infty} 2^{k-1} s_k/k!.
\]

Hence

\[
(2.9) \quad F'_k = \frac{1}{2} \sum_{m=n-k}^{\infty} \binom{k}{m} s_m s_n - 2^{k-1} s_k.
\]

Similarly,

\[
\text{char}^* (p_i) = \sum_{i=1}^{6} \exp(x - t_i) + \sum_{i<j} \exp(-x + t_i + t_j) + \sum_{i=1}^{6} \exp(-t_i)
\]

\[
= \exp(x) \sum_{i=1}^{6} \exp(-t_i) + \exp(-x) \sum_{i<j} \exp(t_i + t_j) + \sum_{i=1}^{6} \exp(-t_i)
\]

\[
= \left( \sum_{m=0}^{\infty} x^m/m! \right) \left( \sum_{n=0}^{\infty} (-1)^m s_m/m! \right) + \left( \sum_{n=0}^{\infty} (-1)^n x^n/n! \right) \left( \sum_{m=0}^{\infty} F'_m/m! \right)
\]

\[
+ \sum_{k=0}^{\infty} (-1)^k s_k/k!
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m s_m x^n/m! n! + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n F'_m x^n/m! n!
\]

\[
+ \sum_{k=0}^{\infty} (-1)^k s_k/k!.
\]

Therefore

\[
F_k = \sum_{m=0}^{6} \binom{k}{m} \left( (-1)^m s_m + (-1)^{k-m} F'_m \right) x^{k-n} + (-1)^k s_k.
\]

Combining this, (2.8) and (2.9), one can compute \( F_k \), and our final results are
\[ F_0 = 27, F_1 = 0, F_2 = -2^2 \cdot 3 (c_2 - 4x^2), F_3 = 0, F_4 = 2^2 \cdot 3 (c_2 - 4x^2), F_5 = -2^2 \cdot 3 \cdot 5 (c_2 - c_4 x + c_3 x^2 - c_2 x^3 + 2x^5) \text{ and so on.} \]

**Remark.** Watching [19, pp. 271–275], we find that the set \( S \) of [19, p. 272] is equal to \( \{ 2\omega | \omega \in \Omega \} \) and that for \( n \geq 0 \) the element

\[ I_n = \sum y^s \]

is expressed as a polynomial in the \( c_i \) and \( x \) modulo \( (I_m | m < n) \). In the above paragraph we have mimicked the computation of \( I_n(=2^n \cdot F_n) \) developed there.

It follows from (2.6), (2.10) and [19, Lemma 5.2] that the elements

\[ c_2 - 4x^2 \in H^4(\ BT; \ Z), \]
\[ c_5 - c_4 x + c_3 x^2 - c_2 x^3 + 2x^5 \in H^{10}(\ BT; \ Z) \]

are indivisible and give the first two generators of the polynomial ring \( H^* (\ BT; \ Q)^{\text{mod}(x^5)} \). Thus we may take

\[ f_4 = -(c_2 - 4x^2), \]
\[ f_{10} = -(c_5 - c_4 x + c_3 x^2 - c_2 x^3 + 2x^5) \]

(for details see [20, Remark in p. 466]). Simultaneously we deduce from this, (2.7) and (2.10) that

\[ f(\rho_1, 1) = 2^2 \cdot 3 / 2! = 6, \]
\[ f(\rho_1, 2) = 2^2 \cdot 3 \cdot 5 / 5! = 1 / 2. \]

By (1.2) we note that (2.3) and (2.2) give

\[ \sigma(y_4) = b(2) x_3, \sigma(y_{10}) = b(5) x_5, \ldots \]
\[ Bi^*(y_4) = c(2) f_4, Bi^*(y_{10}) = c(5) f_{10}, \ldots \]

where \( b(2), b(5), \ldots, c(2), c(5), \ldots \in \mathbb{Z} \).

**Proposition 7.** *We have, up to sign,*

(i) \( b(2) = 1 \) and \( b(5) = 1 \);

(ii) \( c(2) = 1 \) and \( c(5) = 1. \)

**Proof.** Our argument will be based on the fact that \( H^*(E_6; \ Z) \) has \( p \)-torsion if and only if \( p = 2, 3 \).

We first show (i). Consider the Leray-Serre spectral sequence \( \{ E_\ast (Z) \} \) for the integral cohomology of a universal \( E_6 \)-bundle \( E_6 \to EE_6 \to BE_6 \). To investigate it, we use the Leray-Serre spectral sequence \( \{ E_\ast (Z/(p)) \} \) for the mod \( p \)
cohomology of the same bundle, where \( p \) runs over all primes. As seen in [12] and [13], for degrees \( \leq 9 \)

\[
H^*(E_6; \mathbb{Z}/(p)) = \begin{cases} 
\mathbb{Z}/(2) \{1, x_2, x_5, x_7, x_9, x_{12}\} & \text{if } p = 2 \\
\mathbb{Z}/(3) \{1, x_2, x_7, x_{12}\} & \text{if } p = 3 \\
\mathbb{Z}/(p) \{1, x_9\} & \text{if } p \geq 5 \end{cases}
\]

and for degrees \( \leq 10 \)

\[
H^*(BE_6; \mathbb{Z}/(p)) = \begin{cases} 
\mathbb{Z}/(2) \{1, \bar{y}_4, \bar{y}_9, \bar{y}_{12}, \bar{y}_6, \bar{y}_{10}\} & \text{if } p = 2 \\
\mathbb{Z}/(3) \{1, \bar{y}_4, \bar{y}_9, \bar{y}_{10}\} & \text{if } p = 3 \\
\mathbb{Z}/(p) \{1, \bar{y}_9\} & \text{if } p \geq 5 
\end{cases}
\]

where for each prescribed \( k \) \( x_k \in H^k(E_6; \mathbb{Z}/(p)) \) transgresses to \( \bar{y}_{k+1} \in H^{k+1} \)

\( (BE_6; \mathbb{Z}/(p)) \) in \( E_{k+1}(\mathbb{Z}/(p)) \), and if \( \beta_p: H^*(\ ; \mathbb{Z}/(p)) \to H^{k+1}(\ ; \mathbb{Z}/(p)) \) is the mod \( p \) Bockstein operator, then

\[
\beta_p(x_k) = x_3 \quad \text{and} \quad \beta_p(\bar{y}_k) = \bar{y}_7 \quad \text{for} \quad p = 2; \\
\beta_p(x_k) = x_9 \quad \text{and} \quad \beta_p(\bar{y}_k) = \bar{y}_9 \quad \text{for} \quad p = 3.
\]

Therefore, for \( k=3, 9 \) the mod \( p \) reduction homomorphism \( H^k(\ ; \mathbb{Z}) \to H^k \)

\( (\ ; \mathbb{Z}/(p)) \) sends \( x_k \) (resp. \( y_{k+1} \)) to \( x_k \) (resp. \( \bar{y}_{k+1} \)) for every prime \( p \). Thus we see that for \( k=3, 9 \) \( x_k \) transgresses to \( y_{k+1} \) in \( E_{k+1}(\mathbb{Z}) \). Since the cohomology suspension and cohomology transgression are inverse, it follows that \( \sigma(y_{k+1})=x_k \) for \( k=3, 9 \). This proves (i).

We next show (ii). Consider the Leray-Serre spectral sequence \( \{E_s\} \) for the integral cohomology of the fibration

\[
E_6/T \to BT \xrightarrow{Bi} BE_6.
\]

Then \( E_s^{k,t}=H^t(BE_6; H^k(E_6/T; \mathbb{Z})) \). For all \( t \geq 0 \) \( H^t(E_6/T; \mathbb{Z}) \) is a free abelian group whose rank is known (see [19]), while it follows from (2.12) that for \( 0 \leq s \leq 10 \)

<table>
<thead>
<tr>
<th>( s )</th>
<th>0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H^t(BE_6; \mathbb{Z}) )</td>
<td>( \mathbb{Z}, 0, 0, 0, \mathbb{Z}, 0, 0, \mathbb{Z}/(2), \mathbb{Z}, \mathbb{Z}/(3), \mathbb{Z} ).</td>
</tr>
</tbody>
</table>

Therefore, it is easy to check that if \( k=4, 10 \) \( E_s^{k,s} \) has no torsion for all \( s \) and hence so does \( E_s^{k,s} \). By the interpretation of \( Bi^*: H^k(BE_6; \mathbb{Z}) \to H^k(BT; \mathbb{Z}) \) as an edge homomorphism in \( \{E_s\} \), this implies (ii).

Now apply Proposition 6 with \( \rho=\rho_1 \). Then by (2.11) and Proposition 7 we have
Lemma 8. \( a(\rho_1, 1) = 6 \) and \( a(\rho_1, 2) = 1/2 \).

We conclude this section by verifying (1.5).

Proof of (1.5).
Consider the commutative diagram

\[
\begin{array}{ccc}
H^*(BT; \mathbb{Z}) & \xrightarrow{B\theta^*} & H^*(BT; \mathbb{Z}) \\
B\iota^* \uparrow & & \uparrow B\iota^* \\
H^*(BE_6; \mathbb{Z}) & \xrightarrow{B\theta^*} & H^*(BE_6; \mathbb{Z}) \\
\sigma \downarrow & & \downarrow \sigma \\
H^{*-1}(E_6; \mathbb{Z}) & \xrightarrow{\theta^*} & H^{*-1}(E_6; \mathbb{Z}).
\end{array}
\]

It follows from (1.11) and (2.5) that

\[
B\theta^*(t_i) = x - t_{7-i} \quad (1 \leq i \leq 6) \quad \text{and} \quad B\theta^*(x) = x.
\]

From this we deduce that

\[
\{B\theta^*(\omega) | \omega \in \Omega\} = \{-\omega | \omega \in \Omega\}.
\]

Therefore

\[
B\theta^*(F_k) = B\theta^*(\sum_{\omega \in \Omega} \omega^k) = \sum_{\omega \in \Omega} B\theta^*(\omega)^k
\]

\[
= \sum_{\omega \in \Omega} (-\omega)^k = (-1)^k \sum_{\omega \in \Omega} \omega^k = (-1)^k F_k.
\]

Suppose for a moment that \( k (= m_i) = 2, 5, 6, 8, 9, 12 \). Then we observe from [19, Lemma 5.2] and [20, Remark in p. 466] that \( F_k \) gives rise to \( f_{2k} \). Hence

\[
B\theta^*(f_{2k}) = \begin{cases} f_{2k} & \text{for } k = 2, 6, 8, 12 \\ -f_{2k} & \text{for } k = 5, 9. \end{cases}
\]

Because of (2.2), (2.3) and the commutativity of the above diagram, this implies (1.5).

3. Proof of the main results
In this section we complete the proof of Theorems 1 and 2.

By (1.2), (2.1) and (1.7), if \( \rho \) is a representation of \( E_6 \), then

\[
\text{ch}(\beta(\rho)) = a(\rho, 1) x_3 + a(\rho, 2) x_5 + a(\rho, 3) x_{11}
\]

\[
+ a(\rho, 4) x_{15} + a(\rho, 5) x_{17} + a(\rho, 6) x_{23},
\]

and if \( \rho' \) is a representation of \( F_4 \), then

\[
\text{ch}(\beta(\rho')) = a(\rho', 1) x_3 + a(\rho', 2) x_{11} + a(\rho', 3) x_{15} + a(\rho', 4) x_{23}.
\]
By Propositions 3 and 5, we may write

\begin{equation}
\begin{align*}
\text{ch}(\beta(\rho_1-\rho_6)) &= a \cdot x_9 + b \cdot x_{17}, \\
\text{ch}(\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) &= c \cdot x_9 + d \cdot x_{17}
\end{align*}
\end{equation}

for some \(a, b, c, d \in \mathbb{Q}\).

**Proposition 9.** \(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1\).

**Proof.** As is well known [11], if \(g_{26}\) (resp. \(u_{26}\)) is a generator of \(K(S^{26})\) (resp. \(H^{26}(S^{26}; \mathbb{Z})\)), then \(\text{ch}(g_{26}) = \pm u_{26}\) in \(H^{26}(S^{26}; \mathbb{Q})\). According to [8], \(EIV\) has a cell decomposition \(S^9 \cup e^{17} \cup e^{26}\). Consider the cofibration

\[ S^9 \cup e^{17} \to EIV \xrightarrow{f} S^{26}. \]

By Propositions 5 and 3 it is easy to see that \(f^*(g_{26}) = \beta(\rho_1-\rho_6) \cdot \beta(\Lambda^2\rho_1-\Lambda^2\rho_6)\) in \(K(EIV) = \mathbb{Z}\) and \(f^*(u_{26}) = x_9 x_{17}\) in \(H^{26}(EIV; \mathbb{Z}) = \mathbb{Z}\). Then it follows from the naturality of \(\text{ch}\) that

\[ \text{ch}(\beta(\rho_1-\rho_6) \cdot \beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) = \pm x_9 x_{17} \]

in \(H^{26}(EIV; \mathbb{Q})\). Since \(\text{ch}\) is a ring homomorphism, the result follows from this equality, (3.2) and Propositions 3 and 5.

This proposition can be viewed as a variant of [2, Proposition 1] (cf. [17, p. 156] and [20, p. 463]).

**Proof of Theorem 2.**

Since \(\text{ch}\) is a multiplicative natural transformation, we have

\begin{align*}
q^* \text{ch}(\beta(\rho_1-\rho_6)) &= q^*(q^*(\beta(\rho_1-\rho_6))) \\
&= \text{ch}(q^*(\beta(\rho_1-\rho_6))) \\
&= \text{ch}(\beta(\rho_1)-\beta(\rho_6)) \quad \text{by (1.16)} \\
&= \text{ch}(\beta(\rho_1))-\text{ch}(\beta(\rho_6))
\end{align*}

and similarly

\[ q^* \text{ch}(\beta(\Lambda^2\rho_1-\Lambda^2\rho_6)) = \text{ch}(\beta(\Lambda^2\rho_1))-\text{ch}(\beta(\Lambda^2\rho_6)). \]

Therefore, it follows from (3.1), (3.2) and (1.4) (ii) that for \(i=1, 3, 4, 6\) \(a(\rho_1, i) = a(\rho_6, i)\) and \(a(\Lambda^2\rho_1, i) = a(\Lambda^2\rho_6, i)\), and that

\begin{align*}
(3.3) \quad a &= a(\rho_1, 2) - a(\rho_6, 2), \\
&= a(\rho_1, 5) - a(\rho_6, 5), \\
&= a(\Lambda^2\rho_1, 2) - a(\Lambda^2\rho_6, 2), \\
&= a(\Lambda^2\rho_1, 5) - a(\Lambda^2\rho_6, 5).
\end{align*}

Applying [20, Lemma 1] to \(\rho_j (j=1, 6)\), we have
\[ a(\Lambda^2 \rho_j, 2) = \varphi(27, 2, 5) \cdot a(\rho_j, 2), \]
\[ a(\Lambda^2 \rho_j, 5) = \varphi(27, 2, 9) \cdot a(\rho_j, 5) \]

where \((27=\dim \rho_j, 5=m_2, 9=m_3 \text{ and}) \varphi(n, k, m)\) is the integer defined for three positive integers \(n, k, m\) by
\[ \varphi(n, k, m) = \sum_{i=1}^{k} (-1)^{i-1} \binom{n}{k-i} i^{n-i}. \]

A direct calculation gives \(\varphi(27, 2, 5)=11\) and \(\varphi(27, 2, 9)=-229\). It follows from these and (3.3) that
\[ (3.4) \quad c = 11a \quad \text{and} \quad d = -229b. \]

Substituting these relations in the equality
\[ -1 = ad-bc \]

of Proposition 9, we have
\[ -1 = a(-229b)-b(11a) = -240ab \]
and hence
\[ (3.5) \quad ab = 1/240. \]

Let us apply \(j^*\) to (3.1) with \(\rho=\rho_j(j=1, 6)\). Then the left hand side becomes
\[ j^*c_h(\beta(\rho_j)) = c_h(\beta(j^*(\rho_j))) = c_h(\beta(\rho_j)+1) \quad \text{by (1.14) (i)} \]
\[ = c_h(\beta(\rho_j)+\beta(1)) \quad \text{by (1.7) (i)} \]
\[ = c_h(\beta(\rho_j)) \quad \text{by (1.7) (iii)} \]

and the right hand side becomes
\[ j^*(a(\rho_j, 1) x_3+a(\rho_j, 2) x_5+a(\rho_j, 3) x_{11} +a(\rho_j, 4) x_{15}+a(\rho_j, 5) x_{17}+a(\rho_j, 6) x_{25}) \]
\[ = a(\rho_j, 1) x_3+a(\rho_j, 2) x_5+a(\rho_j, 3) x_{11}+a(\rho_j, 4) x_{15}+a(\rho_j, 5) x_{25} \]

by (1.4) (i). Here we quote from [20, p. 486] that
\[ (3.6) \quad c_h(\beta(\rho_i)) = 6x_3+(1/20) x_{11}+(1/168) x_{15}+(1/443520) x_{25}. \]
Hence
\[ (3.7) \quad a(\rho_j, 1) = 6, a(\rho_j, 3) = 1/20, a(\rho_j, 4) = 1/168 \quad \text{and} \]
\[ a(\rho_j, 6) = 1/443520 \quad \text{where} \quad j = 1, 6. \]
On the other hand, let us apply $\theta^*$ to (3.1) with $\rho=\rho_1$. Then the left hand side becomes
\[
\theta^*ch(\beta(\rho_1)) = ch(\beta(\theta^*(\rho_1))) = ch(\beta(\rho_0)) \quad \text{by (1.15) (i)}
\]
\[
= a(\rho_0, 1) x_3 + a(\rho_0, 2) x_9 + a(\rho_0, 3) x_{11}
+ a(\rho_0, 4) x_{15} + a(\rho_0, 5) x_{17} + a(\rho_0, 6) x_{23}
\]
by (3.1) with $\rho=\rho_0$, and the right hand side becomes
\[
\theta^*(a(\rho_1, 1) x_3 + a(\rho_1, 2) x_9 + a(\rho_1, 3) x_{11}
+ a(\rho_1, 4) x_{15} + a(\rho_1, 5) x_{17} + a(\rho_1, 6) x_{23})
\]
\[
= a(\rho_1, 1) x_3 - a(\rho_1, 2) x_9 + a(\rho_1, 3) x_{11}
+ a(\rho_1, 4) x_{15} - a(\rho_1, 5) x_{17} + a(\rho_1, 6) x_{23}
\]
by (1.5). Hence

\[
(3.8) \quad a(\rho_0, 2) = -a(\rho_1, 2) \quad \text{and} \quad a(\rho_0, 5) = -a(\rho_1, 5).
\]

Combining these and (3.3), we have

\[
(3.9) \quad a = 2 \cdot a(\rho_1, 2) \quad \text{and} \quad b = 2 \cdot a(\rho_1, 5).
\]

Since $a(\rho_1, 2)=1/2$ by Lemma 8, it follows that $a=1$. Substituting this in (3.5) gives $b=1/240$. Therefore, by (3.4), $c=11$ and $d=-229/240$. Thus Theorem 2 is proved.

Proof of Theorem 1.

By (1.12), Proposition 4 and (3.1), it suffices to compute the numbers $a(\rho_1, i)$, $a(\rho_2, i)$, $a(\Lambda^2 \rho_1, i)$, $a(\Lambda^3 \rho_1, i)$, $a(\Lambda^2 \rho_0, i)$ and $a(\rho_0, i)$ for $i=1, 2, \ldots, 6$.

Every $a(\rho_1, i)$ has been found in Lemma 8 and (3.7) except $i=5$. But, since $b=1/240$, it follows from (3.9) that $a(\rho_1, 5)=1/480$. Thus we know all of the $a(\rho_1, i)$.

For $i=1, 3, 4, 6$ $a(\rho_0, i)$ has been found in (3.7). For $i=2, 5$ $a(\rho_0, i)$ is determined by $a(\rho_1, i)$ through (3.8). Thus we know all of the $a(\rho_0, i)$.

Applying [20, Lemma 1] to $\rho_j(j=1, 6)$, we have

\[
a(\Lambda^k \rho_j, i) = \varphi(27, k, m_i) \cdot a(\rho_j, i)
\]

for all $k \geq 1$ and $1 \leq i \leq 6$. It follows from the definition of $\varphi(n, k, m)$ that $\varphi(27, 2, 2)=25$, $\varphi(27, 2, 5)=11$, $\varphi(27, 2, 6)=-5$, $\varphi(27, 2, 8)=-101$, $\varphi(27, 2, 9)=-229$, $\varphi(27, 2, 12)=-2021$, $\varphi(27, 3, 2)=300$, $\varphi(27, 3, 5)=0$, $\varphi(27, 3, 6)=-270$, $\varphi(27, 3, 8)=-918$, $\varphi(27, 3, 9)=0$ and $\varphi(27, 3, 12)=122202$. Thus $a(\Lambda^2 \rho_1, i)$, $a(\Lambda^3 \rho_0, i)$ and $a(\Lambda^3 \rho_1, i)$ can be computed from $a(\rho_1, i)$ and $a(\rho_0, i)$.

It remains to compute $a(\rho_2, i)$. Let us apply $j^*$ to (3.1) with $\rho=\rho_2$. Then
the left hand side becomes
\[ j^* \text{ch}(\beta(\rho_2)) = \text{ch}(\beta(j^*(\rho_2))) \]
\[ = \text{ch}(\beta(\rho_1') + \rho_1') \quad \text{by (1.14) (ii)} \]
\[ = \text{ch}(\beta(\rho_1')) + \beta(\rho_1') \quad \text{by (1.7) (i)} \]
\[ = \text{ch}(\beta(\rho_1')) + \text{ch}(\beta(\rho_1')) \]
and the right hand side becomes
\[ a(p_2, 1) x_3 + a(p_2, 3) x_{11} + a(p_2, 4) x_{15} + a(p_2, 6) x_{23} \]
by (1.4) (i). Here we quote from [20, p. 386] that
\[ \text{ch}(\beta(\rho_1')) = 18 x_3 - (7/20) x_{11} + (17/168) x_{15} - (1/7040) x_{23}. \]
Adding this to (3.6) gives
\[ \text{ch}(\beta(\rho_1')) + \text{ch}(\beta(\rho_1')) = 24 x_3 - (3/10) x_{11} + (3/28) x_{15} - (31/221760) x_{23}. \]
Hence \( a(p_2, 1) = 24, a(p_2, 3) = -3/10, a(p_2, 4) = 3/28 \) and \( a(p_2, 6) = -31/221760. \)

On the other hand, let us apply \( \theta^* \) to (3.1) with \( \rho = \rho_2 \). Then the left hand side becomes
\[ \theta^* \text{ch}(\beta(\rho_2)) = \text{ch}(\beta(\theta^*(\rho_2))) = \text{ch}(\beta(\rho_2)) \quad \text{by (1.15) (ii)} \]
\[ = a(p_2, 1) x_3 + a(p_2, 2) x_9 + a(p_2, 3) x_{11} \]
\[ + a(p_2, 4) x_{15} + a(p_2, 5) x_{17} + a(p_2, 6) x_{23} \]
by (3.1) with \( \rho = \rho_2 \), and the right hand side becomes
\[ a(p_2, 1) x_3 - a(p_2, 2) x_9 + a(p_2, 3) x_{11} \]
\[ + a(p_2, 4) x_{15} - a(p_2, 5) x_{17} + a(p_2, 6) x_{23} \]
by (1.5). Hence \( a(p_2, 2) = 0 \) and \( a(p_2, 5) = 0 \). Thus we know all of the \( a(p_2, i) \), and Theorem 1 is proved.

References

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Department of Applied Mathematics
Osaka Women's University
Daisen-cho, Sakai
Osaka 590, Japan