PROOFS IN INTENSIONAL LOGIC

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Introduction

In a previous paper (Stirk, 1985), I presented a reasonably simple procedure for testing the validity of formulae in Montague's intensional logic. At that time it seemed intuitively clear that the method worked, but I doubted that anyone of my modest technical ability could possibly prove this. Since then, however, I have come to realize that a formal proof is guite easy, based simply on the recursive definition of intension and extension provided in Montague (1970).

I present this proof in what follows, and also include a brief section on extending the method to deal with the tense operators W and H, which appear in Montague (1970), but which I hesitated to discuss in my (1985).

It is appropriate to begin with a statement of Montague's recursive definition, for the original version contains an error which is corrected by a footnote under Thomason's editorship (Thomason, 1974, p259, footnote 10). The correction is incorporated into the definition below, as are clauses dealing with the possibility operator ◇ and the new notion of an *instance*. Otherwise the concepts and notation are

Montague's:

(1) If α is a constant, then $\alpha^{\mathfrak{A},i,j,g}$ is $F(\alpha)(\langle i,j \rangle)$.

(2) If α is a variable or an instance, then $\alpha^{\mathfrak{A},i,j,g}$ is $g(\alpha)$.

(3) If $\alpha \varepsilon M_a$ and u is a variable of type b, then $[\lambda u \alpha]^{\mathfrak{A},i,j,g}$ is that function h with domain $D_{b,A,I,J}$ such that whenever x is in that domain, h (x) is $\alpha^{\mathfrak{A},i,j,g'}$, where g' is the $\mathfrak{A}$-assignment just like g except for the possible difference that g'(u) is x.

(4) If $\alpha \varepsilon ME_{\langle a,b \rangle}$ and $\beta \varepsilon ME_a$, then $[\alpha(\beta)]^{\mathfrak{A},i,j,g}$ is $\alpha^{\mathfrak{A},i,j,g}(\beta^{\mathfrak{A},i,j,g})$ (that is, the value of the function $\alpha^{\mathfrak{A},i,j,g}$ for the argument $\beta^{\mathfrak{A},i,j,g}$).

(5) If α, $\beta \varepsilon ME_a$, then $[\alpha = \beta]^{\mathfrak{A},i,j,g}$ is 1 if and only if $\alpha^{\mathfrak{A},i,j,g}$ is $\beta^{\mathfrak{A},i,j,g}$

(6) If $\phi \varepsilon ME_t$, then $[\neg \phi]^{\mathfrak{A},i,j,g}$ is 1 if and only if $\phi^{a,i,j,g}$ is 0 ; and similarly for $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

(7) If $\phi \varepsilon ME_t$ and u is a variable of type a, then $[\vee u \phi]^{\mathfrak{A},i,j,g}$ is 1 if and only if there exists $x \varepsilon D_a$,A,I,J such that $\phi^{\mathfrak{A},i,j,g'}$ is 1, where g' is as in (3) ; and similarly for $\wedge u \phi$.

(8) If $\phi \varepsilon ME_t$, then $[\Box \phi]^{a,i,j,g}$ is 1 if and only if $\phi^{\mathfrak{A},i',j',g}$ is 1 for all i' ε I and j' ε J ; and similarly for $\Diamond \phi$. $[W \phi]^{\mathfrak{A},i,j,g}$ is 1 if and only if $\phi^{\mathfrak{A},i,j',g}$ is 1 for some j' such that $j \leqq j'$ and $j \neq j'$; and $[H \ \phi]^{\mathfrak{A},i,j,g}$ is 1 if and only if $\phi^{\mathfrak{A},i,j',g}$ is 1 for some $j' \leqq j$ and $j' \neq j$.

(9) If $\alpha \varepsilon ME_a$ then $[{}^\wedge \alpha]^{\mathfrak{A},i,j,g}$ is that function h with domain IxJ such that whenever $<i,j> \varepsilon$ IxJ, $h(<i,j>) = \alpha^{\mathfrak{A},i,j,g}$.

(10) If $\alpha \varepsilon ME_{\langle s,a \rangle}$, then $[{}^\vee \alpha]^{\mathfrak{A},i,j,g}$ is $\alpha^{\mathfrak{A},i,j,g}(<i,j>)$.

The term "instance", connected with "instantiation", is used to denote particular values of quantified variables. Thus the variable "x" in "$\wedge x\,\alpha(x)$" may be instantiated with the instance "k" to yield "$\alpha(k)$". The term "constant" might have been appropriate for this purpose, but of course it has already been usurped by Montague.

The test procedure is in fact a way of detecting inconsistency rather than validity, and has its origins in one called the "Main Method" by Quine in chapter 29 of his (1974). It is best to begin with an actual example of the method, and then to use this example to illustrate the proof that follows.

Using the Method

In his (1970), Montague implies (p265) that the formula "$\Box[\delta(x) \longleftrightarrow \delta_*({}^{\vee}x)]$" is a logical consequence of the meaning postulate "$\vee M \wedge x \Box[\delta(x) \longleftrightarrow M({}^{\vee}x)]$". We can use the method to demonstrate this by showing that the conjunction of the meaning postulate and the negation of the formula is inconsistent. The negation of the formula is actually

$$\vee x \Diamond[\delta(x) \wedge \neg\delta_*({}^{\vee}x) . v . \neg\delta(x) \wedge \delta_*({}^{\vee}x)]$$

but in order to provide an example which illustrates almost every detail of the method, I shall reverse the order of the existential quantifier and the possibility operator, and use the expression:

$$\Diamond \vee x[\delta(x) \wedge \neg\delta_*({}^{\vee}x) . v . \neg\delta(x) \wedge \delta_*({}^{\vee}x)].$$

We start by assuming that the conjunction of the two formulae, or premises as we might call them, is true at some point of reference

$\langle 0, 0 \rangle$, that is , in world 0 at moment 0 :

$\langle 0, 0 \rangle$

1. $\vee M \wedge x \Box [\, \delta(x) \leftrightarrow M({}^{\vee}x)]$
2. $\Diamond \vee x [\, \delta(x) \wedge \neg\, \delta_{*}({}^{\vee}x) . v . \neg\, \delta(x) \wedge \delta_{*}({}^{\vee}x)]$

The $\langle 0, 0 \rangle$ label and the numbers help us to keep track, while the writing of one line below another signifies their conjunction in a more convenient way than by using "$\wedge$". Note also the use of Quine's dot notation to avoid excessive numbers of brackets. This seems preferable to Montague's rather haphazard system of just missing some out here and there.

The next steps involve the dropping of quantifiers and operators by processes of instantiation and the visiting of other worlds and times. The universal quantifier can be instantiated by whatever instance we like, provided only that it is of the same type as the variable bound by the quantifier. The process is called "*universal instantiation*". On the other hand, the existential quantifier must be instantiated by a *new* instance, that is, one that has not already been used in the course of the derivation. This is "*existential instantiation*". It is generally the best strategy to have as few instances in the derivation as possible, and that means instantiating existential quantifiers as soon as one can, so that the same instances may be used to replace universally quantified variables later.

In the present example, line (1) begins with an existential quantifier, so let us instantiate it with, say, "N", which is to be an instance of type $\langle s, \langle e, t \rangle \rangle$. This gives:

$\langle 0, 0 \rangle$

3. $\wedge x \Box [\delta(x) \longleftrightarrow N({}^{\vee}x)]$ 1, N

The number to the right shows which line this one has been derived from, while the "N" reminds us not to instantiate any other existentially quantified variable with N.

There is an existential quantifier in line (2) also, but there is a possibility operator in front of it. While formulae preceded by a necessity operator are to be true always and everywhere, those preceded by a possibility operator may only be true at some point or points of reference, not necessarily the $\langle 0, 0 \rangle$ of this example. So we go now to a point $\langle 1, 1 \rangle$, say, where

$\langle 1, 1 \rangle$

1. $\vee x[\delta(x)_\wedge \neg \delta_*({}^{\vee}x).v.\neg \delta(x)_\wedge \delta_*({}^{\vee}x)]$ $\langle 0, 0 \rangle 2$

Here $\langle 0, 0 \rangle$ 2 indicates that this line is derived from line (2) at point of reference $\langle 0, 0 \rangle$.

Line (1) in world 1 at time 1 begins with an existential quantifier, so we can instantiate it:

$\langle 1, 1 \rangle$

2. $\delta(k)_\wedge \neg \delta_*({}^{\vee}k).v.\neg \delta(k)_\wedge \delta_*({}^{\vee}k)]$ 1 ,k

with a reminder that k (an instance of type $\langle s,e \rangle$) is no longer new.

There is still a universally quantified variable of type $\langle s,e \rangle$ at point $\langle 0, 0 \rangle$, so we can jump back there and try instantiating it with k:

$\langle 0, 0 \rangle$

4. $\Box[\delta(k) \leftrightarrow N\{{}^{\vee}k\}]$ 3

Now back in $\langle 1, 1 \rangle$ we might try doing something about ${}^{\vee}k$. This should be something of type e. To give it a name, observe that according to Montague's definition, the formula $\bigvee x \bigwedge u\ [{}^{\vee}x = u]$ must be logically true. we incorporate it into the derivation as follows:

$\langle 1, 1 \rangle$

3. $\bigwedge x \bigvee u[{}^{\vee}x = u]$ Th

with "Th" to indicate it as a theorem.

The next couple of instantiations are obvious:

$\langle 1, 1 \rangle$

4. $\bigvee u[{}^{\vee}k = u]$ 3

5. ${}^{\vee}k = a$ 4, a

where a is an instance of type e.

Now we have a name for ${}^{\vee}k$, we can use it to replace ${}^{\vee}k$ in line $\langle 1, 1 \rangle 2$:

$\langle 1, 1 \rangle$

6. $\delta(k)_{\wedge} \neg\, \delta_{*}(a).v. \neg\, \delta(k)_{\wedge}\, \delta_{*}(a)$ 2, 5, Lz

The annotations on the right of this line show that it is derived from lines 2 and 5 by the use of Leibniz' law of identity.

δ and δ_{*} come together in line 6, but as yet we have not made use of the relation between them. A check on Montague's definition of δ_{*} (1970, p265) shows that we can incorporate this as a theorem:

$\langle 1,1 \rangle$

7. $\wedge u[\,\delta({}^\wedge u) \leftrightarrow \delta_*(u)$ Th

with the obvious instantiation we get:

$\langle 1,1 \rangle$

8. $\delta({}^\wedge a) \leftrightarrow \delta_*(a)$ 7

This seems to be making matters more complicated than before, since now $^\wedge a$ has entered the picture. How is $^\wedge a$ related to k? Line $\langle 0,0 \rangle$ 3 might provide an answer, since the universal quantifier there could be instantiated by $^\wedge a$ as well as by k, as they are both of the same type. In that case, the relation between the two might be mediated by N. Let us go back to $\langle 0,0 \rangle$ and try the instantiation:

$\langle 0,0 \rangle$

5. $\Box[\,\delta({}^\wedge a) \leftrightarrow N\{{}^\vee{}^\wedge a\}]$ 3

Lines $\langle 0,0 \rangle$ 4 and $\langle 0,0 \rangle$ 5 are universally quantified, so we can transport them to point $\langle 1,1 \rangle$, giving first

$\langle 1,1 \rangle$

9. $\delta(k) \leftrightarrow N\{{}^\vee k\}$ $\langle 0,0 \rangle$ 4

We might as well replace k right away:

$\langle 1,1 \rangle$

10. $\delta(k) \leftrightarrow N\{a\}$ 5, 9, Lz

Next we have

$\langle 1,1 \rangle$

11. $\delta({}^\wedge a) \leftrightarrow N\{{}^\vee{}^\wedge a\}$ $\langle 0,0 \rangle$ 5

One would suspect that $^{\vee\wedge}a$ would be the same as a itself, and in fact the following theorem is available from Montague's definition:

$\langle 1,1 \rangle$

12. $\wedge u[u = {}^{\vee\wedge}u]$ Th

This immediately gives the following:

$\langle 1,1 \rangle$

13. $a = {}^{\vee\wedge}a$ 12

14. $\delta({}^{\wedge}a) \longleftrightarrow N\{a\}$ 11,13,Lz

A comparison of lines 10 and 14 here shows that we can now immediately relate $^{\wedge}a$ and k:

$\langle 1,1 \rangle$

15. $\delta(k) \longleftrightarrow \delta({}^{\wedge}a)$ 10, 14

Line $\langle 1,1 \rangle 8$ enables δ_* to be brought in also:

$\langle 1,1 \rangle$

16. $\delta(k) \longleftrightarrow \delta_*(a)$ 8, 15

A comparison of 16 with $\langle 1,1 \rangle 6$ makes some inconsistency look likely, though it is still not immediate. Line 6 is an alternation, which complicates matters a little. A good strategy in such cases is to consider the two alternates separately, in two "branches". To make an unambiguous line numbering, it is best to prefix all the lines in one branch with "1." and those in the other with "2.". In this example, one branch will be

$\langle 1, 1 \rangle$

1.1	$\delta(k)$	6
1.2	$\neg\delta_*(a)$	6
1.3	$\delta_*(a)$	16, 1.1
1.4	⊠	1.2, 1.3

Lines 1.2 and 1.3 are obviously inconsistent, and this is indicated in line 1.4, by the sign "⊠". Somehow I previously had the impression that this symbol for inconsistency was used by Lewis Carroll in his "Symbolic Logic". It was a wrong impression: Carroll actually used a small circle for this purpose (see, for instance, Bartley, 1977, p282). Never mind: "⊠" is less ambiguous, so I will stick to it. Line 1.3 is derived from lines 16 and 1.1 by the process of *deduction.*

The other branch is very similar:

$\langle 1, 1 \rangle$

2.1	$\neg\delta(k)$	6
2.2	$\delta_*(a)$	6
2.3	$\delta(k)$	16, 2.2
2.4	⊠	2.1, 2.3

The demonstration of inconsistency is now complete. It will be convenient to have the whole thing set out as a whole without interruption:

Example 1

$\langle 0,0 \rangle$

1.	$\bigvee Mx\Box[\delta(x)\leftrightarrow M(^{\vee}x)]$	
2.	$\Diamond\bigvee x[\delta(x)\wedge\neg\delta_*(^{\vee}x).v.\neg\delta(x)\wedge\delta_*(^{\vee}x)]$	
3.	$\bigwedge x\Box[\delta(x)\leftrightarrow N\{^{\vee}x\}]$	1, N
	$\langle 1,1 \rangle 2$	
4.	$\Box[\delta(k)\leftrightarrow N\{^{\vee}k\}]$	3
	$\langle 1,1 \rangle 8$	
5.	$\Box[\delta(^{\wedge}a)\leftrightarrow N\{^{\vee\wedge}a\}]$	3

$\langle 1, 1 \rangle$

1.	$\bigvee x[\delta(x)\wedge\neg\delta_*(^{\vee}x).v.\neg\delta(x)\wedge\delta_*(^{\vee}x)]$	$\langle 0,0 \rangle 2$
2.	$\delta(k)\wedge\neg\delta_*(^{\vee}k).v.\neg\delta(k)\wedge\delta_*(^{\vee}k)$	1, k
3.	$\bigwedge x\bigvee u[^{\vee}x=u]$	Th
4.	$\bigvee u[^{\vee}k=u]$	3
5.	$^{\vee}k=a$	4, a
6.	$\delta(k)\wedge\neg\delta_*(a).v.\neg\delta(k)\wedge\delta_*(a)$	2, 5, Lz
7.	$\bigwedge u[\delta(^{\wedge}u)\leftrightarrow\delta_*(u)$	Th.
8.	$\delta(^{\wedge}a)\leftrightarrow\delta_*(a)$	7
9.	$\delta(k)\leftrightarrow N\{^{\vee}k\}$	$\langle 0,0 \rangle 4$
10.	$\delta(k)\leftrightarrow N\{a\}$	5, 9, Lz
11.	$\delta(^{\wedge}a)\leftrightarrow N\{^{\vee\wedge}a\}$	$\langle 0,0 \rangle 5$
12.	$\bigwedge u[u=^{\vee\wedge}u]$	Th
13.	$a=^{\vee\wedge}a$	12
14.	$\delta(^{\wedge}a)\leftrightarrow N\{a\}$	11, 13, Lz
15.	$\delta(k)\leftrightarrow\delta(^{\wedge}a)$	10, 14
16.	$\delta(k)\leftrightarrow\delta_*(a)$	8, 15

1.1	$\delta(k)$	6	2.1	$\neg\delta(k)$	6
1.2	$\neg\delta_*(a)$	6	2.2	$\delta_*(a)$	6
1.3	$\delta_*(a)$	16, 1.1	2.3	$\delta(k)$	16, 2.2
1.4	$\boxtimes$	1.2, 1.3	2.4	$\boxtimes$	2.1, 2.3

A convenient way of representing the various points of reference is shown, as well as branching. Just before lines 4 and 5 of point $\langle 0, 0 \rangle$, the indications $\langle 1, 1 \rangle 2$ and $\langle 1, 1 \rangle 8$, respectively, are given, to keep track of the order in which the various lines were derived.

Justifying the Method

What we want to prove is that the inconsistencies uncovered in each branch of example 1, and any other example derived in the same way, show that the premises are together inconsistent.

Before embarking on the proof, it is necessary to make a few remarks about the way derivations like example 1 are to be understood. All lines placed vertically are intended to be conjoined: furthermore, the numerical labels given to particular points of reference have no significance. Indeed, one might imagine the numbers x, y in $\langle x,y \rangle$ to disappear, so that the angular brackets come together to form a single possibility operator $\Diamond$. Example 1 should be thought of thus: "It is inconsistent to assert the possibility of the conjunction of lines $\langle 0, 0 \rangle 1$ to $\langle 0, 0 \rangle 5$, together with the possibility of the conjunction of lines $\langle 1, 1 \rangle 1$ to $\langle 1, 1 \rangle 16$ and either the conjunction of lines 1.1 to 1.4, or of lines 2.1 to 2.4".

It is simplest to approach the proof by a series of lemmas.

Lemma 1 The formula $p\wedge.\ q \vee r :\leftrightarrow: p\wedge q . \vee . p\wedge r$, where p, q and r are propositional variables, is true in any interpretation.

Clause (6) of Montague's definition shows that the tautologies of propositional calculus will have the value 1 under any assignment of values to the propositional variables. The formula of Lemma 1 is such a tautology, as the reader may verify by any of the standard methods (see for example Carnap, 1958, pp10-15, or Quine, 1974, Ch 5).

Lemma 1 justifies treating each branch in a derivation separately. Thus in Example 1, at the point of reference $\langle 1, 1 \rangle$, if we take "p" to be lines 1-16, "q" to be lines 1.1-1.4, and "r" to be lines 2.1-2.4, we can consider the conjunction 1-16 and 1.1-1.4 to be inconsistent independently of 1-16 and 2.1-2.4.

Lemma 2 If $p\wedge q\wedge r$ is inconsistent, and $q\rightarrow r$ is logically true, then $p\wedge q$ alone is inconsistent.

"Logically true" has the same meaning as "true in any interpretation". Now in a case where r is true. then $p\wedge q$ must be false to make $p\wedge q\wedge r$ inconsistent. In a case where r is false, then since $q\rightarrow r$ is true, q, and therefore $p\wedge q$, must also be false. Thus under any interpretation of r, $p\wedge q$ works out false. This is enough to prove the lemma.

At point $\langle 1, 1 \rangle$ in Example 1 again, take "p" to be lines 1 through 15 and line 1.2, "q" to be lines 16 and 1.1, and "r" to be line 1.3. Now $p \wedge q \wedge r$ is inconsistent, while $q \rightarrow r$ is logically true. According to Lemma 2, the inconsistency will remain if line 1.3 is struck out of the derivation. In a similar way, line 2.3 may be struck out, as may lines 15 and 16. In general, Lemma 2 permits us to strike out lines in a derivation obtained through *deduction,* and to be sure that the remaining lines will still be inconsistent, if the previous ones were.

Lines 1.4 and 2.4 in Example 1 merely serve to show where the inconsistency is located, and do not contribute to it themselves, so they too may be struck out. The branched part of the derivation now contains just lines 1.1 and 1.2 in the left hand part, and lines 2.1 and 2.2 on the right. Remembering that vertical positioning of lines indicates conjunction, and branching disjunction, we see that the four lines are a notational variant of

$$\delta(k) \wedge \neg \delta_*(a).v.\neg \delta(k) \wedge \delta_*(a)$$

which is the same as $\langle 1, 1 \rangle 6$. Merely repeating a line is a peculiar instance of deduction, so the repeated line can be struck out as a special case of Lemma 2. We now find that in $\langle 1, 1 \rangle$, lines 1-14 are inconsistent.

By the end of this proof it will become clear that in any derivation involving branching, the branches can finally be struck out by changing the notation and using Lemma 2.

Lemma 3 If α and β are of type a, while ϕ is of type $\langle a,t \rangle$, then $\alpha = \beta \wedge \phi(\alpha). \rightarrow \phi(\beta)$ is logically true.

It is sufficient to show that any interpretation which makes the antecedent of the conditional true makes the consequent true also. So consider an interpretation where $\alpha = \beta$ and $\phi(\alpha)$ are true. Then, according to clause (5) of Montague's definition, the extension of α is the same as that of β. Thus, according to clause (4), $[\phi(\alpha)]^{\mathfrak{A},i,j,g}$ must be the same as $[\phi(\beta)]^{\mathfrak{A},i,j,g}$, namely 1 in this case. This proves the lemma.

The formula of the lemma is essentially *Leibniz' law of identity.* If one line has been derived from another by the use of Leibniz' law, the lemma provides the logically true connection between the lines which enables the derived line to be struck out according to Lemma 2. In $\langle 1, 1 \rangle$ of Example 1, lines 6, 10 and 14 may be struck out for this reason, leaving a conjunction of lines which is still inconsistent.

Lemma 4 If α is a variable and β an instance of type a, while ϕ is of type $\langle a,t \rangle$, and if $p_\wedge \wedge \alpha[\phi(\alpha)]_\wedge \phi(\beta)$ is inconsistent, then $p_\wedge \wedge \alpha[\phi(\alpha)]$ is inconsistent.

According to the hypothesis, $p_\wedge \wedge \alpha[\phi(\alpha)]_\wedge \phi(\beta)$ is false under any interpretation. Consider any interpretation which makes $\phi(\beta)$ come out true. In that case, $\phi(\beta)$ does not contribute to the falsity of the whole, so the remaining part $p_\wedge \wedge \alpha[\phi(\alpha)]$ must be false. Now consider any interpretation where $\phi(\beta)$ comes out false. According to clause (8) of Montague's definition, $\wedge \alpha[\phi(\alpha)]$ will be false, since $\phi(\alpha)$ is false at least in the case where $g(\alpha)=g(\beta)$. This means that $p_\wedge \wedge \alpha[\phi(\alpha)]$ will also be false, and once again the truth value of $\phi(\beta)$ does not contribute to the falsity of the whole.

This is sufficient to prove the lemma.

Lemma 4 may be employed to justify the striking out of any lines obtained from others by means of *universal instantiation.* In $\langle 1, 1 \rangle$ of Example 1, then, lines 4, 8, and 13 can disappear. However just for now, let us strike out only 8 and 13 — line 4 will need to be around until after the next lemma has been used.

Lemma 5 If α is a variable and β an instance of type a, while ϕ is of type $\langle a,t \rangle$, and if $p \wedge \vee \alpha[\phi(\alpha)] \wedge \phi(\beta)$ is inconsistent, then $p \wedge \vee \alpha[\phi(\alpha)]$ is inconsistent.

Except for a change of quantifier, the formulae are just the same as those of Lemma 4. In cases where $\phi(\beta)$ is true, the argument is just the same too: the inconsistency must lie in the $p \wedge \vee \alpha[\phi(\alpha)]$ part. But when $\phi(\beta)$ is false, the picture is not so simple, for $\vee \alpha \, \phi(\alpha)$ may yet be true. If $\vee \alpha \, \phi(\alpha)$ is true, however, there must be another interpretation of β which makes $\phi(\beta)$ come out true also. Since we were careful to be sure that β is not free in any other line of the derivation, everything other than the truth value of $\phi(\beta)$ remains the same. But since "inconsistent" means "false in every interpretation", it is false in this one too. The inconsistency must be somewhere in p, in this case.

It is interesting to compare this *semantic* proof of Lemma 5 with the *syntactic* one in Quine, 1974, Ch 29. Lemma 5 justifies our use of *existential instantiation.* In $\langle 1, 1 \rangle$ of the example, lines 2 and 5 can be struck out as a result of the lemma. Line 4 can also go, as permitted by Lemma 4, for it is no longer required as a source of line 5.

Lemma 6 If $p \wedge q$ is inconsistent, and q is a theorem, then p itself is inconsistent.

The proof could hardly be more obvious: theorems are clearly logically true, and thus true under every interpertation. They cannot contribute to the inconsistency of any conjunction.

Some theorems may of course be proved by the very method being described and justified here. I used to think that all theorems were best proved by this method, but after further experience, I have changed my mind. Quine is right, I think, to warn against monolithic methods of proof. For instance, the theorem of line $\langle 1, 1 \rangle 3$ in the example, $\bigwedge x \bigvee u\ [{}^{\vee}x = u]$, is often useful, and easy to prove directly from Montague's definition. For the value given to x under any interpretation will be some function from points of reference to entities. According to clause (10), ${}^{\vee}x$ will be the value of that function at a certain point of reference, that is, an entity. The value given to u will be some entity, which under some interpretation could be the same entity as that denoted by ${}^{\vee}x$. This is sufficient to prove the theorem.

The theorem of $\langle 1, 1 \rangle 7$ is even more straightforward, since it follows directly from Montague's definition of δ_* on page 265 of his (1970).

The last theorem used in Example 1 is the one in line $\langle 1, 1 \rangle$12, $\wedge u[u = {}^{\vee\wedge}u]$, also a very useful one. Again in any interpretation, u will denote some entity, so that, according to Montague's definition clause (9), $^{\wedge}u$ will be a constant function from points of reference to that entity. At any point, therefore, $^{\vee\wedge}u$ will denote that entity, just like u. This proves the theorem. Such a straightforward proof may be compared with the very clumsy one I provided in my (1985).

With Lemma 6 at our disposal, we are entitled to strike out lines 3, 7 and 12 from $\langle 1, 1 \rangle$ in Example 1.

Lemma 7 If $\Diamond[p \wedge \Box q] \wedge \Diamond[q \wedge r]$ is inconsistent, then $\Diamond[p \wedge \Box q] \wedge \Diamond r$ is also inconsistent.

In interpretations where $\Diamond[p \wedge \Box q]$ or $\Diamond r$ come out false, then clearly both formulae come out false. The only problematical ones would be any interpretations in which $\Diamond[p \wedge \Box q]$ and $\Diamond r$ are true, but in which $\Diamond[q \wedge r]$ is false. We show that there can be no such interpretation. For since $\Diamond[p \wedge \Box q]$ is true, then $\Box q$ is true at some point of reference, meaning that q is true at all points of reference. Thus $\Diamond[q \wedge r]$ cannot be false unless r is everywhere false, meaning that $\Diamond r$ is false.

Notice that since Montague's definition implies that the modal system is S5 (see Hughes and Cresswell, 1968, p115 et seq.), we do not need to mention any accessibility relation between points of reference: they are all mutually accessible.

Lemma 7 allows us to strike out lines 9 and 11 from $\langle 1, 1 \rangle$ in the example.

Nothing now remains of $\langle 1, 1 \rangle$ except its first line. Recalling how derivations are to be read, $\langle 1, 1 \rangle$ is now reduced to this:

$$\Diamond \vee x[\,\delta(x)_{\wedge} \neg\, \delta_{*}(\vee x).\mathrm{v}.\neg\, \delta(x)_{\wedge}\, \delta_{*}(\vee x)]$$

which is a mere repetition of $\langle 0, 0 \rangle 2$, and may be struck out .

We have now found that lines 1 to 5 of $\langle 0, 0 \rangle$ are together inconsistent. Lines 4 and 5 were obtained by universal instantiation, so they may be struck out, as may line 3, which arose as a case of existential instantiation.

Only lines 1 and 2, the original premises, are left, and we have proved that they are together inconsistent, as required.

The efficacy of the method is now proved, since clearly the same process may be applied to any other example, provided that the derivation was made only in the ways sanctioned by Lemmas 1 to 7. Only the premises will remain as inconsistent.

It is important to have obtained such a formal proof, for it leaves open the possibility of discovering other ways of developing derivations. Provided these ways can be justified by proving suitable extra lemmas about them, they may be incorporated into the method, to generalise it.

Example 1 was developed in rather an unnecessarily complicated way, just in order that it would be an example of everything. The derivation could have been simpler, for instance by forgetting about a and sticking to ${}^{\vee}k$: the same inconsistency would have been reached. There can only be an aesthetic justification for short and neat derivations, though. Their value is like the value of Theseus' ball of thread: they save you from becoming lost as you explore among the possible worlds. No matter how far you wander, if an inconsistency arises, you can be certain that it is due to the original premises.

Yet just for the sake of a neat example, let me present one.

A Second Example

Among the examples at the end of his (1970), Montague (p266) states that the *de dicto* reading of "John seeks a unicorn" is the following, in intensional logic translation:

$$seek'({}^{\wedge}j,\ \hat{P}\vee u[unicorn'_{*}(U)\wedge P\{\wedge u\ \}])$$

However, the syntactic rules and the rules of translation come to a stop with this version:

$$seek'({}^{\wedge}j,\ \hat{P}\vee x[unicorn'(x)\wedge P\{x\}])$$

Now Leibniz' law would make these forms equivalent if only we could show that

$$\hat{P}\vee u[unicorn'_{*}(u)\wedge P\{{}^{\wedge}u\}]=\hat{P}\vee x[unicorn'(x)\wedge P\{x\}]$$

The items being equated here are the intensions of sets, so the formula is equivalent to

$$\Box[\hat{P}\bigvee u[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]=\hat{P}\bigvee x[\text{unicorn}'(x)\wedge P\{x\}]]$$

Since sets are identical if and only if they have the same members, the previous formula is itself equivalent to

$$\Box\bigwedge P[\bigvee u[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]\longleftrightarrow\bigvee x[\text{unicorn}'(x)\wedge P\{x\}]]$$

If we can show that this is logically true, the trick is done. This means showing that the negation of the last formula is inconsistent, so that this will be the single premise:

$$\Diamond\bigvee P[[\bigvee u[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]\wedge\bigwedge x[\neg[\text{unicorn}'(x)\wedge P\{x\}]]$$
$$.v.\bigwedge u[\neg[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]]\wedge\bigvee x[\text{unicorn}'(x)\wedge P\{x\}]]$$

The derivation will look like this:

Example 2

$\langle 0,\ 0\rangle$

1. $\bigvee P[\bigvee u[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]\wedge\bigwedge x[\neg[\text{unicorn}'(x)\wedge P\{x\}]]$
   $.v.\bigwedge u[\neg[\text{unicorn}'_{*}(u)\wedge P\{{}^{\wedge}u\}]]\wedge\bigvee x[\text{unicorn}'(x)\wedge P\{x\}]]$
2. $\bigvee u[\text{unicorn}'_{*}(u)\wedge Q\{{}^{\wedge}u\}]\wedge\bigwedge x[\neg[\text{unicorn}'(x)\wedge Q\{x\}]]$
   $.v.\bigwedge u[\neg[\text{unicorn}'_{*}(u)\wedge Q\{{}^{\wedge}u\}]]\wedge\bigvee x[\text{unicorn}'(x)\wedge Q\{x\}]]$ 1, Q
3. $\bigwedge u[\text{unicorn}'({}^{\wedge}u)\longleftrightarrow\text{unicorn}'_{*}(u)]$ Th

1. 1 $\bigvee u[\text{unicorn}'_{*}(u)\wedge Q\{{}^{\wedge}u\}]$ 2

1. 2 $\bigvee x[\neg[\text{unicorn}'(x)\wedge Q\{x\}]]$ 2

1.3	$\text{unicorn}'_*(a) \wedge Q\{^\wedge a\}$	1.1, a
1.4	$\text{unicorn}'\{^\wedge a) \leftrightarrow \text{unicorn}'_*(a)$	3
1.5	$\text{unicorn}'(^\wedge a) \wedge Q\{^\wedge a\}$	1.3, 1.4
1.6	$\neg[\text{unicorn}'(^\wedge a) \wedge Q\{^\wedge a\}]$	1.2
1.7	$\boxtimes$	1.5, 1.6
2.1	$\bigwedge u[\neg[\text{unicorn}'_*(u) \wedge Q\{^\wedge u\}]]$	2
2.2	$\bigvee x[\text{unicorn}'(x) \wedge Q\{x\}]$	2
2.3	$\text{unicorn}'(k) \wedge Q\{k\}$	2.2, k
2.4	$\bigwedge x \Box[\text{unicorn}'(x) \rightarrow \bigvee u[x = {^\wedge u}]]$	Th
2.5	$\Box[\text{unicorn}'(k) \rightarrow \bigvee u[k = {^\wedge u}]]$	2.4
2.6	$\text{unicorn}'(k) \rightarrow \bigvee u[k = {^\wedge u}]$	2.5
2.7	$\bigvee u[k = {^\wedge u}]$	2.3., 2.6
2.8	$k = {^\wedge a}$	2.7, a
2.9	$\text{unicorn}'(^\wedge a) \wedge Q\{\bigwedge a\}$	2.3, 2.8, Lz
2.10	$\text{unicorn}'(^\wedge a) \leftrightarrow \text{unicorn}'_*(a)$	3
2.11	$\text{unicorn}'_*(a) \wedge Q\{^\wedge a\}$	2.9, 2.10
2.12	$\neg \text{unicorn}'_*(a) \wedge Q\{^\wedge a\}$	2.1
2.13	$\boxtimes$	2.11, 2.12

I hope this derivation is self explanatory, despite the topological awkwardness caused by the length of the lines in each branch. Notice that the theorem of line 2.4 is Montague's meaning postulate (2) (1970, p263). I have used this, together with the definition of δ_* in line 3, instead of Montague's formula on 1970, p265. I pointed out before that this formula is in fact false for common nouns (Stirk, 1985, p101).

There are two new, though minor, features in this example which should be mentioned. The first is that the same instance "a" has been employed in both branches in cases of existential instantiation. This does not matter, as the branches are derived independently: purists have only to replace "a" by "b" in one of the branches.

The other new feature is slightly more significant. In line 2.6, a necessity operator is removed. This is essentially a special case of Lemma 7, but I have not been able to work out a formulation or a proof which covers this case too. Perhaps then we need:

Lemma 8 If $\Diamond[p \wedge \Box q \wedge q]$ is inconsistent, then $\Diamond[p \wedge \Box q]$ is inconsistent.

I leave the obvious proof to the reader.

Visiting the Past and the Future

The ability to prove the validity of our method of detecting inconisistency gives us the courage to add other features to the system. As long as we can prove additional lemmas to cover the new cases, we can be confident of obtaining proper results. Montague's operators W and H should be the next items to incorporate.

The results of one or two preliminary experiments, however, reveal some oddities in Montague's definition as it stands. The obvious first step is to use the numbering system we have for points of reference to keep track of positions in time. Thus if we start from point $\langle 0, 0 \rangle$, W would be instantiated with $\langle 0, 1 \rangle$, and H with

$<0, -1>$. There could be intermediate points, $<0, 0.5>$, $<0, -0.5>$ and so on, if necessary. A few experiments should suggest just what lemmas are needed.

Let us try first to prove some really obvious formulae. It is clear from Montague's definition that $W\phi \rightarrow \Diamond\phi$ should be valid. And indeed we find this:

$<0, 0>$

1. $W\phi$
2. $\Box\neg\phi$

$<0, 1>$

1. ϕ $\quad$ $<0, 0>1$
2. $\neg\phi$ $\quad$ $<0, 0>2$
3. $\boxtimes$ $\quad$ 1, 2

That looks fine. Lemmas must clearly involve the numbers we are allowed to use for points of reference. Testing slightly more complicated formulae, though, shows that items like $W\Box\phi \rightarrow \Box\phi$ are valid, which is a little odd. Surely future necessity does not imply present necessity? Yet:

$<0, 0>$

1. $W\Box\phi$
2. $\Diamond\neg\phi$

$<0, 1>$

1. $\Box\phi$ $\quad$ $<0, 0>1$

$\langle 1,\ 0 \rangle$

1. $\neg \phi$ $\langle 0,\ 0 \rangle 2$
2. ϕ $\langle 0,\ 1 \rangle 1$
3. $\otimes$ 1, 2

There seems no way out of that, short of giving up S5.

Another oddity is this. One might expect that $W\phi \rightarrow \Diamond H\phi$ would be valid, for surely the future will one day become the past? But this happens:

$\langle 0,\ 0 \rangle$

1. $W\phi$
2. $\Box \neg H\phi$

$\langle 0,\ 1 \rangle$

1. ϕ $\langle 0,\ 0 \rangle 1$
2. $\neg H\phi$ $\langle 0,\ 0 \rangle 2$

No inconsistency looms, for ϕ could well be false at $\langle 0, 0 \rangle$. The problem here is the number of points of reference - the derivation seems to suggest that $\langle 0, 1 \rangle$ is the end of world 0: it has no future! There must be some mechanism to add further future points of reference. In that case, we could add a point $\langle 0, 2 \rangle$, where $H\phi$ would be true, and cause an inconsistency at $\langle 0, 1 \rangle$.

I hope to investigate these points further in a future paper.

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