THE JOYS OF NATURAL DEDUCTION

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Introduction

In previous papers (Stirk 1985, 1994), I have pointed out the value and soundness of a certain *reductio ad absurdum* method of proving the logical truth of formulae in the version of intensional logic described in Montague (1970). The method was based on the "Main Method" of Quine (1974).

The method is usually employed in showing that a certain proposition q, say, follows from another, say p, by proving that the conjunction of p and not-q is inconsistent. Both p and q must be provided at the start.

There are many occasions, however, when we know a proposition p and would like to experiment with it and find out what follows from it. *Reductio ad absurdum* is of no help in this. The best we can do is to guess at some proposition q and test whether or not it follows from p. If the method does not turn up an inconsistency, we also have to decide whether q really does not follow from p, or whether we have just missed some subtle instantiation.

There is a method called *Natural Deduction* which should enable us to start out from some proposition p and see where it might lead us. Quine attributes the origins of natural deduction to two logicians, Gerhard Gentzen and Stanislaw Jaskowski, working independently and publishing in 1934 (Quine, 1974, p107).

For a long time I thought that natural deduction would be far too cumbersome a method to employ in the stratospheric realm of intensional logic. It can be tedious enough in the humble propositional calculus. Examples of this can be found in McCawley, 1981, p24 et seq. Even when rescued from McCawley's abominable notation, the various rules of exploitation and introduction are not very intuitive, to say the least. Hughes and Cresswell (1968) offer a short appendix on extending this same method to modal propositional calculus (pp 331-4). Exploitation and introduction of the modal operators are just added to the previous difficulties, so I imagined that the presence of quantification would make the whole system quite unmanageable.

This view has turned out to be quite wrong. Quine's version of natural deduction, which he used extensively in earlier editions of his (1974), avoids the awkwardness of introducing and exploiting truth functional connectives, and deals only with the removal and addition of quantifiers. It turns out to be quite a simple matter to extend this version to modal calculi and thence to intensional logic.

I will set out the various steps in what follows, beginning with a description of Quinean natural deduction.

Quine's Natural Deduction

In his (1974) p201, Quine describes natural deduction as a "split and partly inverted" Main Method. This can best be illustrated by using an example of Quine's, also from his (1974) p201. We are required to show that $(x)(\ni y)(Gx \supset .Fy.Hxy)$ follows from $(\ni y)(x)(Fy.Gx \supset Hxy)$. The Main Method would have it as follows:

1.	$(\ni y)(x)(Fy.Gx \supset Hxy)$	
2.	$(\ni x)(y)(Gx.-Fy \vee -Hxy)$	
3.	$(x)(Fa.Gx \supset Hxa)$	1,a
4.	$(y)(Gb.-Fy \vee -Hby)$	2,b
5.	$Fa.Gb \supset Hba$	3
6.	$Gb.-Fa \vee -Hba$	4
7.	Hba	5,6
8.	$-Fa$	6,7
9.	Fa	5
10.	$\boxtimes$	8,9

Ignoring the leftmost column of numbers for a moment, the Natural Deduction derivation would look like this:

1	1.	$(\ni y)(x)(Fy.Gx \supset Hxy)$	
3	2.	$(x)(Fa.Gx \supset Hxa)$	1,a
5	3.	$Fa.Gb \supset Hba$	2
9	4.	Fa	3
-6	5.	$Gb \supset .Fa.Hba$	3,4
-4	6.	$(\ni y)(Gb \supset .Fy.Hby)$	5

-2 7. $(\exists x)(\supset y)(Gx \supset .Fy.Hxy)$ 6,b

The leftmost column refers to lines in the Main Method demonstration, with negation signs preceding the numbers of lines which appear negated in the Natural Deduction. The numbers make the "inversion" clear. Line 8 of the Main Method derivation does not appear, because of course its negation would merely be a repetition of "Fa", which is line 4 of the Natural Deduction. Line 7 disappears also, but this is a consequence of the way of dealing with propositional calculus matters. In both the Main Method and this version of Natural Deduction, tautologies are assumed and exploited. For the Main Method, the tautology "p.p ⊃ q. ⊃ q" was employed to move from lines 5 and 6 to line 7, and "p.-pv-q. ⊃ -q" was used derive line 8 from lines 6 and 7. For the Natural Deduction, one slightly more complicated tautology, "p.q ⊃ r. ⊃ .q ⊃ pr", suffices to derive line 5 from lines 3 and 4.

It is this use of tautologies that enables Quine's method of Natural Deduction to be so much easier to handle. Inferences are made with tautologies just as they are in the Main Method, and quantifiers are removed under the same conditions as they are in the Main Method. The only remaining problem is that quantifiers must also be inserted.

There are two examples of this in the derivation above. Line 6 of the Natural Deduction comes from line 5 by adding an existential quantifier. Quine calls this process "Existential Generalization", or

"EG". It poses no difficulties, for a glance at the parallel Main Method derivation shows that it fulfils the same role as the universal instantiation which got line 6 from line 4.

The other case is not quite so easy. Corresponding to the existential instantiation that takes us from line 2 to line 4 in the Main Method example, we find what Quine calls "Universal Generalization", or "UG", in going from line 6 to line 7 of the Natural Deduction. The correspondence suggests that a new letter must be involved in UG, as it must be in existential instantiation. The letter is shown, or "flagged", as Quine has it, to the right of the line, so that its newness will be apparent. The rule then is that no letter may be flagged twice in a derivation.

Unfortunately that rule does not go far enough. Here is an innocuous Natural Deduction to prove the familiar logical truth "$(\exists y)(x)Fxy \supset (x)(\exists y)Fxy$":

1.	$(\exists y)(x)Fxy$	
2.	$(x)Fxa$	1,a
3.	Fba	2
4.	$(\exists y)Fby$	3
5.	$(x)(\exists y)Fxy$	4,b

The trouble is that the next deduction looks equally innocuous:

1.	$(x)(\exists y)Fxy$	
2.	$(\exists y)Fay$	1

3.	Fab	2,b
4.	(x)Fxb	3,a
5.	(∃y)(x)Fxy	4

Yet we know that "(x)(∃y)Fxy ⊃ (∃y)(x)Fxy" is not logically true. The error that has crept in can easily be seen if we try to reconstruct the Main Method derivation corresponding to that last deduction:

1.	(x)(∃y)Fxy	
-5.	(y)(∃x)-Fxy	
2.	(∃y)Fay	1
3.	Fab	2,b
-4.	(∃x)-Fxb	-5
-3.	-Fab	-4,a

The line numbering system is self explanatory.

A blunder is immediately apparent in this application of the Main Method. The instantiation in line -3 uses a letter which is not new, although it had not been previously flagged, as it appeared in a universal instantiation. It is sometimes necessary, as in this case, to instantiate an existential quantifier *after* a universal one. using a new letter.

It is much easier in the Main Method to see which letters have already been used. The order in which Natural Deduction proceeds makes this more difficult. Quine makes sure that letters are not misused by following a rule that *flagged letters must not only be* *different from each other, but also a flagged letter must be alphabetically later than any other free letter in the line where it is flagged.* (See Quine, 1974, p204).

In the example above, to obey this rule would prevent the passage from line 3 to line 4.

In his version of the Main Method, Quine (1974) does not employ any branching scheme to deal with alternation. I used it extensively in my (1985), however, so perhaps we should investigate how it relates to Natural Deduction. The following Main Method demonstration makes a good, if rather contrived, example:

1.	(x)(Fx v Gx)	
2.	(x)(-Fx v Hx)	
3.	(x)(-Gx v Hx)	
4.	(∃x)-Hx	
5.	-Ha	4,a
6.	Fa v Ga	1

1.1.	Fa	6	2.1.	Ga	6
1.2.	-Fa v Ha	2	2.2.	-Ga v Ha	3
1.3.	Ha	1.1,1.2	2.3	Ha	2.1,2.2
1.4	⊠	1.3,5	2.4	⊠	2.3,5

An equivalent Natural Deduction would be as follows:

1. (x)(Fx v Gx)
2. (x)(-Fx v Hx)
3. (x)(-Gx v Hx)
4. Fa v Ga 1

1.1.	Fa	4	2.1.	Ga	4
1.2.	-Fa v Ha	2	2.2.	-Ga v Ha	3
1.3.	Ha	1.1,1.2	2.3	Ha	2.1,2.2

5. Ha 1.3,2.3
6. (x)Hx 5,a

Corresponding lines should be clear without special numbering.

It is plain that in general, diverging branches will need to be reunited to a main "trunk" in Natural Deduction. The complications can be avoided by employing a method of *hypothesis*, illustrated below:

1.	(x)(Fx v Gx)	
2.	(x)(-Fx v Hx)	
3.	(x)(-Gx v Hx)	
4.	Fa v Ga	1
*5.	Fa	
*6.	-Fa v Ha	2
*7.	Ha	5,6

8.	Fa ⊃ Ha	*7
*9.	Ga	
*10.	-Ga v Ha	3
*11.	Ha	9,10
12.	Ga ⊃ Ha	*11
13.	Fa v Ga. ⊃ Ha	8,12
14.	Ha	4,13
15.	(x)Hx	14,a

In line 5, "Fa" is assumed to be true, and we go on to explore the consequences in lines 6 and 7. These lines are preceded by an asterisk to show that they are part of the hypothesis. In line 8, we "jump out" of the hypothesis by using a conditional. This is justified, because line 8 would be true regardless of the truth value of "Fa". Similar hypothesis making gets us to line 12, after which we obtain line 13 by using a familiar tautology.

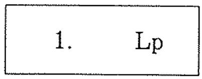
More details of the hypothesis procedure are to be found in Quine (1974). It is clear that branching can be avoided in Natural Deduction by this convenient method of forming hypotheses.

That completes a description of Natural Deduction as it applies in first order predicate calculus. A proof of its soundness can be found in the earlier editions of Quine's (1974).

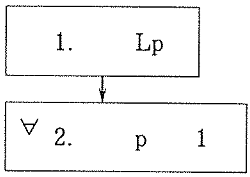
Modal Predicate Calculus

The next step is to find a way to bring in the modal operators, while avoiding the complications of the method described by Hughes and Cresswell (1968, appendix one).

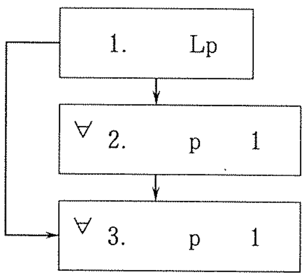
The modal operators fit easily into the Main Method, as I showed in my (1985) and (1994). With Natural Deduction there is bound to be the extra complication of introducing operators, as well as eliminating them. For instance, suppose we wish to show that in S4 the formula "Lp ⊃ LLp" is logically true. We inevitably begin:



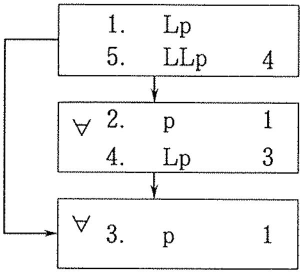
So, "Lp" is true in some world, but where to go next? Perhaps this is a likely procedure for eliminating the operator:



The sign "∀" is used to represent "any world", that is, any world accessible from the starting world. Remembering that in S4 the accessibility relation is transitive, we can continue the diagram thus:

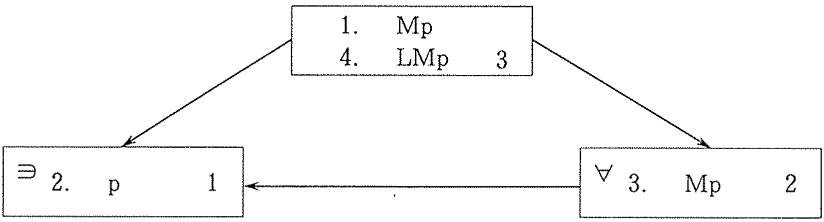


Since the worlds we have added are any worlds, we can add necessity operators as follows:



reaching the conclusion "LLp" in line 5. Notice that I am adopting a different numbering system, and a system of not numbering worlds, which are different from those in my (1985) and (1994). The present system is much less cumbersome.

As a further example, let us try showing that "Mp ⊃ LMp" is logically true in S5. The complete proof works out as follows:



The sign "∋" is used for "some world". The step from line 2 to line 3 is justified because in S5 *any* world will be accessible to the one where line 2 is true. We can go from line 3 to line 4 since "Mp" is true in any world in the derivation.

Armed with these devices, we can go on to a more substantial example, that of showing that in T the following is logically true:

$$L(p \supset L(q \supset r)) \supset M(q \supset (Lp \supset Mr))$$

That is a formula of Hughes and Cresswell (1968, p83). The proof runs like this:

1.	$L(p \supset L(q \supset r))$	
12.	$M(q \supset (Lp \supset Mr))$	11

↓

∀ 2.	$p \supset L(q \supset r)$	1
*3.	q	
**4.	Lp	
**5.	p	4
**6.	$L(q \supset r)$	2,5
**7.	$q \supset r$	6
**8.	r	3,7
**9.	Mr	8
*10.	$Lp \supset Mr$	*9
11.	$q \supset (Lp \supset Mr)$	*10

This application of Natural Deduction shows clearly the contrived nature of this example. The conclusion is too weak: "L" could perfectly well have been prefixed. This does not become apparent with the Main Method treatment of the formula, as readers may verify for themselves.

This example shows also that "L" may be eliminated without going to another world (line 7), and that "M" may similarly be inserted (line 9).

Modal Predicate Calculi

By now we should be ready to tackle problems in modal predicate calculi, but first it is a good idea to try something from a higher order predicate calculus. Let us try deriving the second order calculus definition of identity from the axioms of identity:

1.	$(x)(x=x)$	
2.	$(F)(x)(y)(x=y.Fx. \supset Fy)$	

Substituting a suitable predicate for "F" gives us

3.	$(x)(y)(x=y.(\lambda z)(z=a)x. \supset (\lambda z)(z=a)y)$	2
4.	$a=b.a=a. \supset .b=a$	3
5.	$a=a$	1
6.	$a=b. \supset .b=a$	4,5
7.	$a=b.Ga. \supset Gb$	2
8.	$b=a.Gb. \supset Ga$	2
*9.	$a=b$	
*10.	$Ga \supset Gb$	7,9
*11.	$Gb \supset Ga$	6,8,9
*12.	$Ga \equiv Gb$	10,11
*13.	$(F)(Fa \equiv Fb)$	12,G
14.	$a=b. \supset (F)(Fa \equiv Fb)$	*13
*15.	$(F)(Fa \equiv Fb)$	
*16.	$(\lambda x)(a=x)a \equiv (\lambda x)(a=x)b$	15
*17.	$a=a. \equiv .a=b$	16
*18.	$a=b$	5,17

19.	$(F)(Fa \equiv Fb) \supset .a=b$	$*18$
20.	$a=b. \equiv (F)(Fa \equiv Fb)$	14,19
21.	$(y)(a=y. \equiv (F)(Fa \equiv Fy))$	20,b
22.	$(x)(y)(x=y. \equiv (F)(Fx \equiv Fy))$	21,a

The paradox of this definition of identity, that things that are identical turn out to be necessarily identical, makes a good example for a first excursion into a higher order calculus with modality:

1.	$(x)(y)[x=y. \equiv (F)(Fx \equiv Fy)]$	
2.	$a=b. \equiv (F)(Fa \equiv Fb)$	1
$*$3.	$a=b$	
$*$4.	$(F)(Fa \equiv Fb)$	2,3
$*$5.	$(\lambda x)L(a=x)a \equiv (\lambda x)L(a=x)b$	4
$*$6.	$L(a=a) \equiv L(a=b)$	5
14.	$L(a=a)$	13
$*$15.	$L(a=b)$	6,14
16.	$a=b. \supset L(a=b)$	$*15$
17.	$(y)[a=y. \supset L(a=y)]$	16,b
18.	$(x)(y)[x=y. \supset L(x=y)]$	17,a

↓

$\forall$ 7.	$Hc \equiv Hc$	(tautology)
8.	$(F)(Fc \equiv Fc)$	7,F
9.	$(x)(F)(Fx \equiv Fx)$	8,c
10.	$(F)(Fa \equiv Fa)$	9
11.	$(x)(y)[x=y. \equiv (F)(Fx \equiv Fy)]$	Defn.
12.	$a=a. \equiv (F)(Fa \equiv Fa)$	11
13.	$a=a$	10,12

If higher order modal predicate calculi are going to work, we can turn straight to

Intensional Logic

Perhaps the least perspicuous feature of intensional logic is that the formula "$\Box[\delta(x) \leftrightarrow \delta_*({}^{\vee}x)]$" is implied by "$\vee M \wedge x \Box [\delta(x) \leftrightarrow M\{{}^{\vee}x\}]$". The Main Method shows this to be so, but does not clearly point out the reason.

Natural Deduction will proceed like this:

1. $\vee M \wedge x \Box [\delta(x) \leftrightarrow M\{{}^{\vee}x\}]$

We also need the relation between "δ" and "δ_*":

2. $\vee u \Box [\delta_*(u) \leftrightarrow \delta({}^{\wedge}u)]$

From there we can continue:

3.	$\wedge x \Box [\delta(x) \leftrightarrow N\{{}^{\vee}x\}]$	1,N
4.	$\Box [\delta(k) \leftrightarrow N\{{}^{\vee}k\}]$	3
5.	$\Box [\delta({}^{\vee}k) \leftrightarrow ({}^{\wedge\vee}k)]$	2
8.	$\Box [\delta_*({}^{\wedge\vee}k) \leftrightarrow N\{{}^{\vee}k\}]$	3
17.	$\Box [\delta(k) \leftrightarrow \delta_*({}^{\vee}k)]$	16
18.	$\wedge x \Box [\delta(x) \leftrightarrow \delta_*({}^{\vee}x)]$	17,k

And the other world:

6.	$\delta(k) \leftrightarrow N\{{}^{\vee}k\}$	4
7.	$\delta_*({}^{\vee}k) \leftrightarrow \delta({}^{\wedge\vee}k)$	5

9.	$\delta({}^{\wedge\vee}k) \longleftrightarrow N\{{}^{\vee}k\}$	8
*10.	$\delta(k)$	
*11.	$\delta_*({}^{\vee}k)$	6,9,7,10
12.	$\delta(k) \rightarrow \delta_*({}^{\vee}k)$	*11
*13.	$\delta_*({}^{\vee}k)$	
*14	$\delta(k)$	7,9,6,13
15.	$\delta_*({}^{\vee}k) \rightarrow \delta(k)$	*14
16.	$\delta(k) \longleftrightarrow \delta_*({}^{\vee}k)$	12,15

The way in which the previously rather mysterious predicate "N" is eliminated becomes much more clear when we look at lines 6,7 and 9.

The converse implication also goes through quite simply with the Main Method, as I showed in my (1985). But observe how elegantly it can be done with Natural Deduction:

1.	$\wedge x \Box [\delta(x) \longleftrightarrow \delta_*({}^{\vee}x)]$	
2.	$\wedge x \Box [\delta(x) \longleftrightarrow {}^{\wedge}\delta_*\{{}^{\vee}x\}]$	1
3.	$\vee M \wedge x \Box [\delta(x) \longleftrightarrow M\{{}^{\vee}x\}]$	2

Threè lines only! Another advantage of Natural Deduction is that not every modality or quantifier needs to be instantiated in every derivation.

Conclusion

I think I have managed to show some of the benefits of Natural Deduction methods in intensional logic. A proof of soundness has not been given. Quine, naturally enough, gives a *syntactic* proof of the soundness of the method in earlier editions of his (1974), but a *semantic* one is necessary if the proof is to be extended to higher order calculi with modality. I hope to present such a proof in a later paper.

Meanwhile, of course, if users have any doubt of the correctness of some Natural Deduction, it is only necessary to check it using the Main Method!

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