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A NECESSARY AND SUFFICIENT CONDITION FOR A 3-MANIFOLD TO HAVE GENUS g HEEGAARD SPLITTING (A PROOF OF HASS-THOMPSON CONJECTURE)

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1. Introduction

R.H. Bing had shown that a closed 3-manifold M is homeomorphic to S^3 if and only if every knot in M can be ambient isotoped to lie inside a 3-ball [1]. In [5], J. Hass and A. Thompson generalize this to show that M has a genus one Heegaard splitting if and only if there exists a genus one handlebody V embedded in M such that every knot in M can be ambient isotoped to lie inside V . Moreover, they conjectures that this can be naturally generalized for genus $g(>1)$. The purpose of this paper is to show that this is actually true. Namely we prove:

Main Theorem. *Let M be a closed 3-manifold. There exists a genus g handlebody V such that every knot in M can be ambient isotoped to lie inside V if and only if M has genus g Heegaard splitting.*

The proof of this goes as follows. First we generalize Myers' construction of hyperbolic knots in 3-manifolds [14] to show that, for each integer $g(\geq 1)$, every closed 3-manifold has a knot whose exterior contains no essential closed surfaces of genus less than or equal to g (Theorem 4.1). Knots with this property will be called g -characteristic knots. Then we show that, for each integer $h(\geq 1)$, there exists a knot K in M such that K cannot be ambient isotoped to a 'simple position' in any genus h handlebody which gives a Heegaard splitting of M . This is carried out by using good pencil argument of K. Johannson [9] (, and we note that this also can be proved by using inverse operation of type A isotopy argument of M. Ochiai [15]). By using this very complicated knot in M , we can show that if M contains a genus g handlebody as in Main Theorem, then M admits a Heegaard splitting of genus g .

This paper is organized as follows. In Section 2, we slightly generalize

results of Johansson in [8], which will be used in Sections 3 and 5. In Section 3, we generalize the concept of prime tangles [13] to 'height g ' tangles, and show that there are many height g tangles. In Section 4, we show that, by using these tangles, there are infinitely many g -characteristic knots in M . In Section 5, we show that there are non-simple position knots by using these g -characteristic knots. In Section 6, we prove Main Theorem.

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2. Preliminaries

Throughout this paper, we work in the piecewise linear category. All submanifolds are in general position unless otherwise specified. For a subcomplex H of a complex K , $N(H, K)$ denotes a regular neighborhood of H in K . When K is well understood, we often abbreviate $N(H, K)$ to $N(H)$. Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\text{Fr}_M N$ denotes the frontier of N in M . For the definitions of standard terms in 3-dimensional topology, we refer to [6], and [7].

An arc a properly embedded in a 2-manifold S is *inessential* if there exists an arc b in ∂S such that $a \cup b$ bounds a disk in S . We say that a is *essential* if it is not inessential. A *surface* is a connected 2-manifold. Let E be a 2-sided surface properly embedded in a 3-manifold M . We say that E is *essential* if E is incompressible and not parallel to a subsurface of ∂M . We say that E is ∂ -*compressible* if there is a disk Δ in M such that $\Delta \cap E = \partial \Delta \cap E = \alpha$ is an essential arc in E , and $\Delta \cap \partial M = \partial \Delta \cap \partial M = \beta$ is an arc such that $\alpha \cup \beta = \partial \Delta$. We say that E is ∂ -*incompressible* if it is not ∂ -compressible.

Let F be a closed surface of genus g . A *genus g compression body* W is a 3-manifold obtained from $F \times [0, 1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and attaching some 3-handles so that $\partial_- W = \partial W - \partial_+ W$ has no 2-sphere components, where $\partial_+ W$ is a component of ∂W which corresponds to $F \times \{0\}$. It is known that W is irreducible ([2, Lemma 2.3]). We note that W is a handlebody if $\partial_- W = \emptyset$.

A *complete disk system* D for a compression body W is a disjoint union of disks $(D, \partial D) \subset (W, \partial_+ W)$ such that W cut along D is homeomorphic to

$$\begin{cases} \partial_- W \times [0, 1], & \text{if } \partial_- W \neq \emptyset, \\ B^3, & \text{if } \partial_- W = \emptyset. \end{cases}$$

Note that for any handle decomposition of W as above, the union of the cores of the 2-handles extended vertically to $F \times [0, 1]$ contains a complete disk system for W .

Let M be a compact 3-manifold such that ∂M has no 2-sphere compon-

ents. A genus g Heegaard splitting of M is a pair (V, W) where V, W are genus g compression bodies such that $V \cup W = M$, $V \cap W = \partial_+ V = \partial_+ W$. Then the purpose of this section is to give a generalization of some results of Johannson [8] to the above Heegaard splittings.

The next lemma can be proved by using the above complete disk system, and the proof is left to the reader (cf. [2, Lemma 2.3]).

Lemma 2.1. *Let S be an incompressible and ∂ -incompressible surface properly embedded in a compression body W . Then S is either a closed surface parallel to a component of $\partial_- W$, disk D with $\partial D \subset \partial_+ W$, or an annulus A , where one component of ∂A lies in $\partial_+ W$ and the other in $\partial_- W$.*

The annulus A as in Lemma 2.1 is called *vertical*.

Let S be an essential surface in a 3-manifold M , and (W_1, W_2) a Heegaard splitting of M . We say that S is *normal* with respect to (W_1, W_2) if:

- (1) each component of $S \cap W_1$ is an essential disk or a vertical annulus, and
- (2) $S \cap W_2$ is an essential surface in W_2 .

By using the incompressibility of S and Lemma 2.1, we see that if M is irreducible then S is ambient isotopic to a normal surface. Suppose that S is normal. Let $S_2 = S \cap W_2$, and b an arc properly embedded in S_2 . We say that b is a *compression arc* (for S_2), if b is essential in S_2 , and there exists a disk Δ in W_2 such that $\partial \Delta = b \cup b'$, where $b' = \Delta \cap \partial_+ W_2$ (and, possibly, $\text{Int } \Delta \cap S_2 \neq \emptyset$). Let M , (W_1, W_2) , and S be as above. Let \mathcal{D} be a complete disk system for W_2 . We say that S is *strictly normal* (with respect to \mathcal{D}), if:

- (1) S is normal with respect to (W_1, W_2) , and
- (2) for each component D_i of \mathcal{D} , we have; (i) each component of $S_2 \cap D_i$ (if exists) is an essential arc in S_2 and (ii) if b is an arc of $S_2 \cap D_i$ such that ∂b is contained in mutually different components C_1, C_2 of ∂S_2 , and that C_1 or C_2 is a boundary of a disk component E of $S \cap W_1$, then for each (open arc) component $\partial D_i - \partial b$, say a_1, a_2 , we have $a_i \cap \partial E \neq \emptyset$.

Then the next proposition is a generalization of [8, 2.3].

Proposition 2.2. *Let $M, (W_1, W_2)$ be as above. Let S be an essential surface in M which is normal with respect to (W_1, W_2) . Then we have either:*

- (1) S is strictly normal, or
- (2) S is ambient isotopic to a surface S' in M such that; (i) S' is normal with respect to (W_1, W_2) , and (ii) $\#\{S' \cap W_1\} < \#\{S \cap W_1\}$.

The proof of this is essentially contained in [8, Sect. 2]. However, for the convenience of the reader, we give the proof here.

Lemma 2.3. *Let $M, (W_1, W_2)$, and S be as in Proposition 2.2. Let b be a*

compression arc for $S \cap W_2$, with a disk Δ in W_2 such that $\partial\Delta = b \cup b'$, where $b' = \Delta \cap \partial_+ W_2$ and $\partial b = \partial b'$. Suppose that there is a disk component E of $S \cap W_1$ such that $b' \cap E = \partial b' \cap \partial E$ consists of a point. Then S is ambient isotopic to a surface S' in M such that;

- (1) S' is normal with respect to (W_1, W_2) , and
- (2) $\# \{S' \cap W_1\} = \# \{S \cap W_1\} - 1$.

Proof. Note that b joins mutually different components of $S \cap W_1$, one of them is E and the other is D , say. Let E_+ be one of the components of $\text{Fr}_{W_1} N(E, W_1)$ which meets b' . We note that ∂E_+ meets b' in one point. Let $B = N(E_+, N(E, W_1)) \cup N(\Delta, W_2)$. Then B is a 3-ball in M since $\partial E_+ \cap \partial\Delta$ is a point. Move W_1 by an ambient isotopy along B so that the image W'_1 has the following form: $W'_1 = \text{cl}(W_1 - N(E_+, N(E, W_1))) \cup N(b, W_2)$.

Let $W'_2 = \text{cl}(M - W'_1)$. Then clearly (W'_1, W'_2) is a Heegaard splitting of M which is ambient isotopic to (W_1, W_2) . Note that $S \cap W'_1$ is a system of essential disks and vertical annuli which has the number of components one less than that of $S \cap W_1$, because E is connected with D by the band $S \cap N(b, W_2)$. Moreover, $S \cap W'_2$ is an essential surface since b is essential in S_2 . It follows that there exists an ambient isotopy of M which push S into S' so that S' is normal with respects to (W_1, W_2) and $\# \{S' \cap W_1\} = \# \{S \cap W_1\} - 1$. ■

Proof of Proposition 2.2. Let $\mathcal{D} = \cup D_i$ be a complete disk system for W_2 . Suppose that S is not strictly normal. Since S_2 is incompressible and W_2 is irreducible, by standard innermost disk argument, we may assume that $S_2 \cap D_i$ has no circle components. If there exists an inessential arc component b of $S_2 \cap D_i$ in S_2 , then without loss of generality, we may assume that there exists a disk Δ in S_2 such that $\Delta \cap \mathcal{D} = b$, and $\Delta \cap \partial_+ W_2$ is an arc b' such that $\partial b = \partial b'$, and $b \cup b' = \partial\Delta$. We note that $\text{Fr}_{W_2} N(D_i \cup \Delta, W_2)$ consists of three disks E_0, E_1, E_2 such that E_0 is parallel to D_i . Then it is easy to see that either $(\mathcal{D} - D_i) \cup E_1$ or $(\mathcal{D} - D_i) \cup E_2$ is a complete disk system for W_2 . Moreover this complete disk system intersects S_2 in less number of components. Continuing in this way, we can finally get the complete disk system for W_2 which intersects S_2 in all essential arcs.

Therefore, if S is not strictly normal, we may assume that it does not satisfy (ii) of the definition. Then, there exists an arc component b of $\mathcal{D} \cap S_2$ such that ∂b is contained in mutually different components C_1, C_2 of ∂S , and one of them, say C_2 , is a boundary of a disk component E of $S \cap W_1$, and for one of open arc components a of $\partial D_i - \partial b$, $a \cap \partial E = \emptyset$. Note that b is a compression arc for S_2 , and $b \cap E = \partial b \cap \partial E$ is a point. Hence by Lemma 2.3, S can be ambient isotoped to a 2-manifold S' which is normal with respects to (W_2, W_1) , and $\# \{S' \cap W_1\} < \# \{S \cap W_1\}$. ■

3. Height h tangles

An n -string tangle is a pair (B, t) , where B is a 3-ball, and t is a union of mutually disjoint n arcs properly embedded in B . We note that for each tangle (B, t) there is a (unique) 2-fold branched cover of B with branch set t . We say that a tangle (B, t) has height h if the 2-fold branched cover of B over t contains no essential surface S with $-\chi(S) \leq h$. We note that 2-string tangles with height -1 are called prime tangles in [13]. We say that a tangle (B, t) has *property I* if $X = \text{cl}(B - N(t, B))$ is ∂ -irreducible, i.e. ∂X is incompressible in X . The purpose of this section is to show that a height h tangle actually exists. Namely we prove:

Proposition 3.1. *For each even integer $g(\geq 2)$, and for each integer $m(\geq -1)$, there exists a g -string tangle (B, t) with height m . Moreover if we suppose that $2g-4 > m \geq 0$, then we can take (B, t) to have property I.*

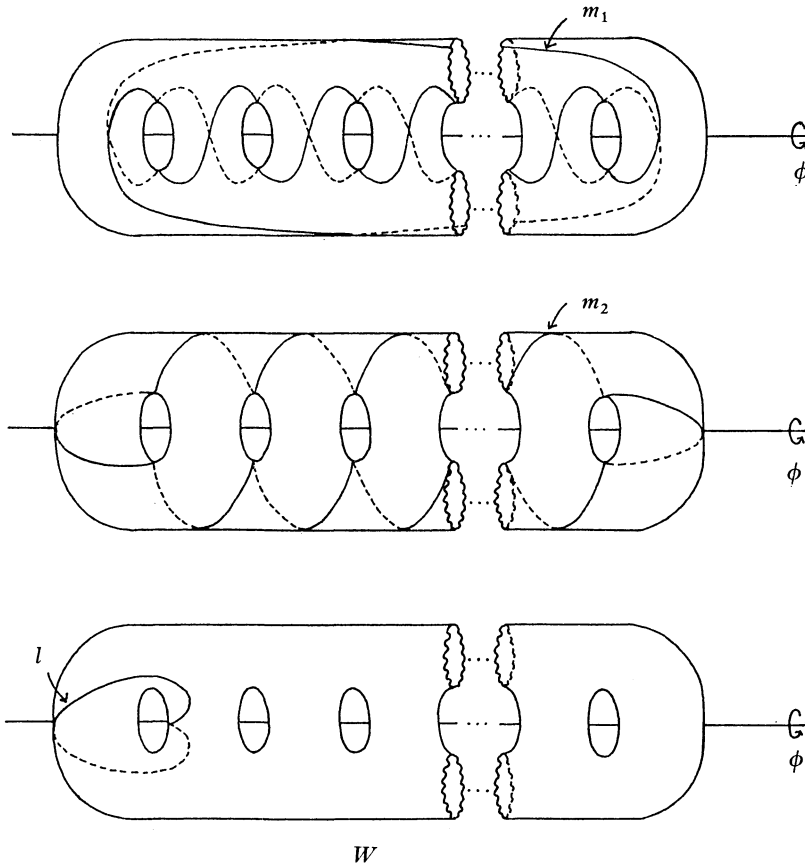


Figure 3.1

For the proof of Proposition 3.1, we recall some definitions and results from [12]. Let W be a compression body and $l(\subset \partial_+ W)$ a simple closed curve. Then the *height* of l for W , denoted by $h_W(l)$, is defined as follows [12].

$$h_W(l) = \min\{-\chi(S) \mid S \text{ is an essential surface in } W \text{ such that } \partial S \cap l = \emptyset\}.$$

Let W be a handlebody of genus $g(\geq 2)$, and m_1, m_2, l simple closed curves on ∂W as in Figure 3.1. Then for a sufficiently large integer q we let f be an automorphism of ∂W such that $f = T_{m_1} \circ T_{m_2}^{2q}$, where T_{m_i} denotes a right hand Dehn twist along the simple closed curve m_i . By sections 2, 3 of [12] we have:

Proposition 3.2. *For each $m(\geq -1)$, there exists a constant $N(m)$ such that if $p > N(m)$, then $h_W(\bar{l}) > m$ for each simple closed curve \bar{l} on ∂W which is disjoint from $f^p(l)$ and not contractible in ∂W .*

Let N be the 3-manifold obtained from W by attaching a 2-handle along the simple closed curve $f^{N(m)+1}(l)$. By Proposition 3.2 and the handle addition lemma (see, for example [3]), we see that N is irreducible. We note that W admits an orientation preserving involution ϕ as in Figure 3.1. Then we have:

Lemma 3.3. *The involution ϕ extends to an involution $\bar{\phi}$ of N . Moreover, the quotient space of N under $\bar{\phi}$ is a 3-ball B , and the singular set t in B consists of a union of g arcs properly embedded in B .*

Proof. We note that m_1, m_2 , and l are invariant under ϕ . Hence we may suppose that $f^{N(m)+1}(l)$ is invariant under ϕ . Hence the involution ϕ naturally extends to the 2-handle $D^2 \times [0, 1]$, where the quotient space of $D^2 \times [0, 1]$ is a 3-ball and the singular set in $D^2 \times [0, 1]$ is an arc α properly embedded in $D^2 \times \{1/2\}$. We note that W/ϕ is a 3-ball, the singular set consists of $g+1$ arcs s , and $N(f^{N(m)+1}(l), \partial W)/\phi$ is a 2-disk. Moreover it is easy to see that the components of $\partial\alpha$ are contained in mutually different components of s . Hence we see that B is a 3-ball and t consists of g arcs properly embedded in B . ■

Let B, t be as above, and we regard (B, t) as a g -string tangle. Then we show that (B, t) is a height m tangle (the first half of Proposition 3.1) by using good pencil argument of Johansson used in [9].

Lemma 3.4. *(B, t) has height m .*

Proof. Let $C = N(\partial W, W) \cup (\text{a 2-handle})$. Let E be a disk properly embedded in C , which is obtained by extending the core of the 2-handle vertically to $N(\partial W, W) (\cong \partial W \times [0, 1])$. Then C is a genus g compression body, and E is a complete disk system for C . We regard $\text{cl}(N - C)$ as W . Then we note that (C, W) is a Heegaard splitting of N .

Let $C' = \text{cl}(C - N(E, C))$, then C' is homeomorphic to $\partial_- C \times [0, 1]$, where $\partial_- C$ corresponds to $\partial_- C \times \{0\}$. Let E^+, E^- be the disks in $\partial_- C \times \{1\}$ corresponding to $\text{Fr}_C N(E, C)$.

Claim 1. Let D be an essential disk in C which is non-separating in C . Then D is ambient isotopic to E in C .

Proof. Since C is irreducible, by standard innermost disk argument, we may suppose that $D \cap E$ has no circle components. Suppose that $D \cap E = \emptyset$. Then ∂D bounds a disk D' in $\partial_- C \times \{1\}$ such that D is parallel to D' . Since ∂D is essential in $\partial_+ C$ and non-separating in $\partial_+ C$, we see that D' contains exactly one of E^+, E^- . Hence D is parallel to E in C . Suppose that $D \cap E \neq \emptyset$. Let Δ be an outermost disk in D , i.e. $\alpha = \Delta \cap E = \partial \Delta \cap E$ an arc, $\beta = \Delta \cap \partial D$ an arc such that $\alpha \cup \beta = \partial \Delta$ and $\alpha \cap \beta = \partial \alpha = \partial \beta$. Then we see that $\Delta \cap C'$ is a properly embedded disk in C' . Without loss of generality, we may suppose that $\partial(\Delta \cap C') \cap E^- = \emptyset$. Then there is a disk Δ' in $\partial_- C \times \{1\}$ such that $\partial \Delta' = \partial(\Delta \cap C')$. If Δ' does not contain E^- , then by moving D by an ambient isotopy, we can remove α from $D \cap E$. Suppose that Δ' contains E^- . Then, by tracing $\text{cl}(\partial D - \beta)$ from one endpoint to the other, we see that there exists a subarc β' in $\partial D - \beta$ such that $\beta' \cap E^+ = \emptyset$, $\beta' \subset \Delta'$, and $\partial \beta' \subset \partial E$. Hence, by moving D by an ambient isotopy, we can reduce the number of components of $D \cap E$. Then by the induction on $\# \{D \cap E\}$, we have the conclusion. ■

Claim 2. Let D be an essential disk in C which is separating in C . Then D can be ambient isotoped so that D is disjoint from E . Moreover, D splits C into a solid torus containing E , and a manifold homeomorphic to $\partial_- C \times [0, 1]$.

Proof. Since C is irreducible, by standard innermost disk argument, we may assume that $D \cap E$ has no circle components. Suppose that $D \cap E \neq \emptyset$. Let Δ be an outermost disk in D such that $\Delta \cap E = \alpha$ and $\beta = \Delta \cap \partial D$. Then $\Delta \cap C'$ is a properly embedded disk in C' . Without loss of generality, we may assume that $\partial(\Delta \cap C') \cap E^- = \emptyset$. Then there is a disk Δ' in $\partial_- C \times \{1\}$ such that $\partial \Delta' = \partial(\Delta \cap C')$. If Δ' does not contain E^- , then by moving D by an ambient isotopy, we can remove α from $D \cap E$. Suppose that Δ' contains E^- . Then, by tracing $\text{cl}(\partial D - \beta)$ from one endpoint to the other, we see that there exists a subarc β' in $\partial D - \beta$ such that $\beta' \cap E^+ = \emptyset$, $\beta' \subset \Delta'$, and $\partial \beta' \subset \partial E^-$. Hence, by moving D by an ambient isotopy, we can reduce the number of components of $D \cap E$. Then by the induction on $\# \{D \cap E\}$, we have the first conclusion of Claim 2. Hence we may assume that $D \cap E = \emptyset$.

Let T be the closure of the component of $C - D$ which contains E , and T' the closure of the other component. By [2, Corollary B.3], we see that T, T' are compression bodies. Since T contains a non-separating disk E , and $\partial_- C \subset T'$, we see that T is a handlebody. Then, by Claim 1, we see that

T' is a solid torus. This shows that $\partial_- T' (= \partial_- C)$ is homeomorphic to $\partial_+ T'$, so that T' is homeomorphic to $\partial_- C \times [0, 1]$. ■

By Claim 2, we immediately have:

Claim 3. Let D_1, D_2 be essential disks in C such that D_1 and D_2 are both separating, and mutually disjoint in C . Then D_1 is parallel to D_2 .

Next, we show:

Claim 4. Let A be a vertical annulus in C . Then A can be ambient isotoped so that it is disjoint from E .

Proof. Since C is irreducible and A is incompressible in C , by standard innermost disk argument, we may suppose that $E \cap A$ has no circle components. Suppose that $A \cap E \neq \emptyset$. Then each component of $E \cap A$ is an arc whose end-points are contained in $\partial_+ C$. Let Δ be an outermost disk in A , such that $\Delta \cap E = \alpha$ an arc and $\beta = \Delta \cap \partial A$ an arc in $\partial A \cap \partial_+ C$. Then, $\Delta \cap C'$ is a properly embedded disk in C' . Without loss of generality, we may assume that $\partial(\Delta \cap C') \cap E^- = \emptyset$. Then there is a disk Δ' in $\partial_- C \times \{1\}$ such that $\partial \Delta' = \partial(\Delta \cap C')$. If Δ' does not contain E^- , then by moving A by an ambient isotopy, we can remove α from $A \cap E$. Suppose that Δ' contains E^- . Then, by tracing $\text{cl}(\partial A \cap \partial_+ C - \beta)$ from one endpoint to the other, we see that there exists a subarc β' in $(\partial A \cap \partial_+ C) - \beta$ such that $\beta' \cap E^+ = \emptyset$, $\beta' \subset \Delta'$, and $\partial \beta' \subset \partial E^-$. Hence, by moving A by an ambient isotopy, we can reduce the number of components of $A \cap E$. Then by the induction on $\#\{A \cap E\}$, we have the conclusion. ■

Let S be an essential surface properly embedded in N and chosen to minimize $-\chi(S)$. In the rest of this proof, we show that $-\chi(S) > m$. By moving S by an ambient isotopy, we may assume that S is normal with respect to (C, W) (Sect. 2). Then $S \cap C \neq \emptyset$, and each component of $S \cap C$ is an essential disk or a vertical annulus in C . Let p be the number of the disk components of $S \cap C$, and suppose that p is minimal among all the essential surfaces \bar{S} such that $-\chi(\bar{S}) = -\chi(S)$, and \bar{S} is normal with respect to (C, W) . Let $S^* = S \cap W$.

Suppose that $S \cap C$ has no disk components. Let A be any annulus component of $S \cap C$. Then, by Claim 4, we may assume that A is disjoint from E . Therefore $(S^*, \partial S^*) \subset (W, \partial W - \partial E) = (W, \partial W - f^{N(m)+1}(l))$. Since $f^{N(m)+1}(l)$ has height m , we have $-\chi(S) = -\chi(S^*) > m$.

Now suppose that $S \cap C$ has a disk component. By the argument of the proof of Proposition 2.2, there exists a complete disk system \mathcal{D} of W such that each component of $\mathcal{D} \cap S^*$ is an essential arc in S^* . Let α be an outermost arc component of $\mathcal{D} \cap S^*$, i.e. there exists a disk Δ in \mathcal{D} such that $\Delta \cap S^* = \partial \Delta \cap S^*$

$=\alpha$ an essential arc in S^* , and $\Delta \cap \partial W = \partial \Delta \cap \partial W = \beta$ an arc such that $\alpha \cup \beta = \partial \Delta$.

Assume that $\partial\beta$ is contained in mutually different components of ∂S^* , and one of which is a boundary of a disk component E^* of $S \cap C$. Then S is not strictly normal since $\text{Int} \beta \cap \partial E^* = \emptyset$. Hence, by Proposition 2.2, S is ambient isotopic to a normal surface S' with respect to (C, W) , and S' intersects W in less number of disk components than that of S , contradicting the minimality of p .

Therefore we have the following four cases.

Case 1. Both endpoints of β are contained in the boundaries of annulus components of $S \cap C$.

By Claims 1, and 2, we may suppose that $\beta \cap \partial E = \emptyset$. Let $\Delta_1 = \beta \times [0, 1] \subset C' (\cong \partial_- C \times [0, 1])$ be a disk in C such that $\beta \times \{1\}$ corresponds to β , and $\partial\beta \times [0, 1] = \Delta_1 \cap (S \cap C)$. Let $\tilde{\Delta} = \Delta \cup \Delta_1$. Let \tilde{S} be the 2-manifold obtained by ∂ -compressing S along $\tilde{\Delta}$. If \tilde{S} is disconnected, choose one essential component of \tilde{S} and we denote it by \tilde{S} again. Then \tilde{S} is an essential surface in N and $-\chi(\tilde{S}) \leq -\chi(S) - 1 < -\chi(S)$. This contradicts the minimality of $-\chi(S)$.

Case 2. Both endpoints of β are contained in the boundary of one non-separating disk component D of $S \cap C$.

Let S' be an essential surface obtained by moving S by an ambient isotopy along Δ . Then $S' \cap C$ has an annulus component A' , which is obtained from D by attaching a band produced 'along β '. Let $\partial A' = \{\alpha_1, \alpha_2\}$. By Claim 1, we may suppose that $\partial D \cap \partial E = \emptyset$, hence, that $\alpha_i \cap E = \emptyset$ ($i=1, 2$). Let $A_i = \alpha_i \times [0, 1] \subset \partial_- C \times [0, 1]$ be a vertical annulus in C . Let $\tilde{S} = (S' - A') \cup A_1 \cup A_2$. If \tilde{S} is disconnected, choose one essential component, and denote it by \tilde{S} again. Then \tilde{S} is an essential surface in N , and $-\chi(\tilde{S}) \leq -\chi(S)$. Moreover \tilde{S} is normal with respect to (C, W) , and the number of the disk components of $\tilde{S} \cap C$ is less than p . This contradicts the minimality of p .

Case 3. Both endpoints of β are contained in the boundary of one separating disk component D of $S \cap C$, and β does not lie in the solid torus T_0 splitted by D from C .

Let S' be as in Case 2. Then there exists an annulus A' in $S' \cap C$ such as in Case 2. Let $\partial A' = \{\alpha_1, \alpha_2\}$. Then, by Claim 2, we may assume that D is disjoint from E . Hence $\alpha_i \cap E = \emptyset$ ($i=1, 2$). Then, by the same argument as in Case 2, we have a contradiction.

Case 4. Both endpoints of β are contained in the boundary of one separating disk component D of $S \cap C$, and β lies in the solid torus T_0 splitted by

D from C .

Let S' , A' be as in Case 2.

Claim 5. A' is incompressible in C .

Proof. Assume that A' is compressible in C . Since S' is incompressible, the core curve of A' is contractible in S' . Hence there is a planar surface P in S^* such that $\partial P = l_0 \cup l_1 \cup \cdots \cup l_r$, where $r \geq 1$, $l_0 \cap D = l_0 \cap \partial D$ an arc, l_1, \dots, l_r are boundary of disk components of $S' \cap C$. See Figure 3.2. Since \mathcal{D} is a complete disk system for W , each component of $P - (\mathcal{D} \cap P)$ is simply connected. This shows that there is a component b of $\mathcal{D} \cap P (\subset \mathcal{D} \cap S^*)$ which satisfies the assumption of Lemma 2.3, contradicting the minimality of p . ■

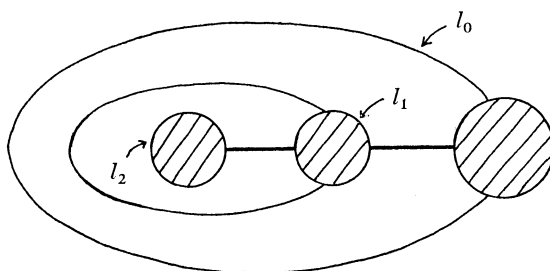


Figure 3.2

By Claims 1, and 5, we see that $S \cap C$ has no non-separating disk component. Let $\{D_1, D_2, \dots, D_q\}$ be the system of disk components of $S \cap C$ which lies in this order. Then, by Claim 3, these components are mutually parallel in C . Let A be an annulus in $\partial_+ C$ such that A contains $\partial D_1 \cup \cdots \cup \partial D_q$, and each ∂D_i is ambient isotopic in A to a core of A . We suppose that $\#\{\partial \mathcal{D} \cap \partial D_i\}$ is minimal in the ambient isotopy class of $\partial \mathcal{D}$ in $\partial W (= \partial_+ C)$, and hence, $I = \partial \mathcal{D} \cap A$ is a system of essential arcs in A . We label the points $\partial D_i \cap I$ by i , then in each component of I , they lie in this order.

Claim 6. There exists a subsystem P of $\mathcal{D} \cap S^*$ such that there exists a component I_0 of I which satisfies the following.

- (1) Every arc of P has one of its endpoints in I_0 .
- (2) Every arc of $\mathcal{D} \cap S^*$ which has one of its endpoints in I_0 belongs to P .
- (3) Every arc t of P joins I_0 with one of components of I which are neighbouring of I_0 in $\partial \mathcal{D}$, i.e. if s_1, s_2 are subarcs of $\partial \mathcal{D}$ such that $(\text{Int } s_i) \cap I = \emptyset$, and one of its endpoints lies in ∂I_0 and the other in the boundary of a component I_i of I , say, then one of the endpoints of t lies in $I_1 \cup s_1 \cup s_2 \cup I_2$ (Figure 3.3).

Proof. Let I_1 be a component of I . Suppose that I_1 does not satisfy the conclusions of Claim 6. Then there is an arc t_1 of $\mathcal{D} \cap S^*$ such that one of its

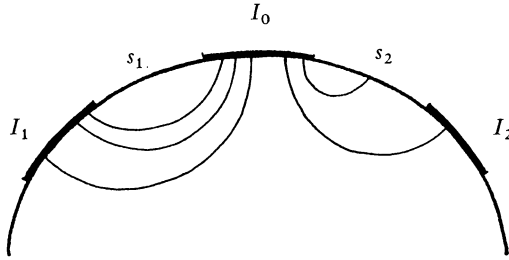


Figure 3.3

endpoints lies in I_1 and does not join two neighbouring components of I . Let E_1 be the closure of a component of $\mathcal{D} - t_1$, and I_2 a component of I contained in ∂E_1 . If I_2 does not satisfy the conclusions of Claim 6, then there is an arc t_2 of $E_1 \cap S^*$ such that one of its endpoints lies in I_2 and does not join two neighbouring of I . Let E_2 be the closure of the components of $\mathcal{D} - t_2$ such that $E_2 \subset E_1$. By continuing in this way, it is easy to see that we finally obtain a component of I satisfying the conclusion of Claim 6. ■

Claim 7. For each component of P in Claim 6, both of its endpoints are contained in I , and have the same label.

Proof. Assume that there exists an arc α such that it has one of its endpoints in I_0 and the other not in I . Then α satisfies the assumption of Lemma 2.3, contradicting the minimality of p . Let a_1, a_2 be the closures of the components of $\partial \mathcal{D} - \partial P$ which contains s_1, s_2 respectively. Since D_1, \dots, D_q are mutually parallel separating disks in C , we see that the points ∂a_i are contained in either ∂D_1 or ∂D_q . This immediately shows that, for each component α of P , the endpoints of α have the same label (Figure 3.4). ■

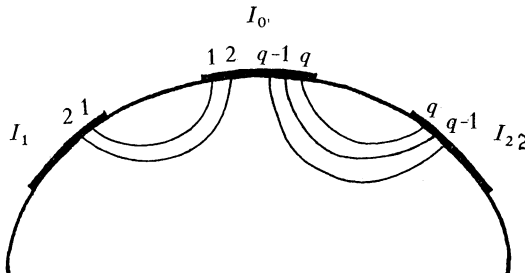


Figure 3.4

Claim 8. $\partial P \subset I_0 \cup I_1$, say (Figure 3.5).

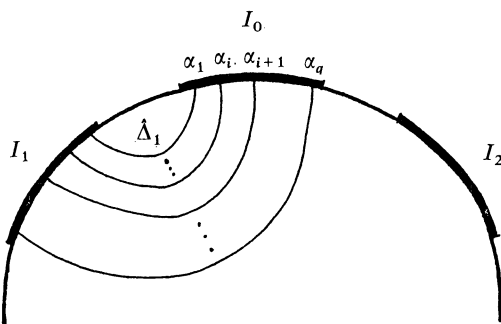


Figure 3.5

Proof. Let α_i be the component of P such that one of its endpoints contained in I_0 is labelled by i . Assume that one endpoint of α_1 is contained in I_1 , and that there exists α_i such that one endpoint of α_i is contained in I_2 . Then by Claim 7, $\partial\alpha_q$ is contained in ∂D_q , and one endpoint of α_q is contained in I_2 . Let Δ be a disk in \mathcal{D} which is splitted by α_q and does not contain $\alpha_1 \cup \cdots \cup \alpha_{q-1}$. We may suppose that $\Delta \cap \partial_+ C$ is not contained in the solid torus splitted by D_q from W . Assume that there exists a component α of $\mathcal{D} \cap S^*$ in $\Delta - \alpha_q$. Then $\partial\alpha$ is contained in annulus components of $S \cap C$. Hence it reduces to Case 1, and we have a contradiction. Therefore $\Delta \cap S^* = \alpha_q$. Let $\beta_q = \Delta \cap \partial\mathcal{D}$. Since β_q cannot lie in the solid torus T_0 , it reduces to Case 3, a contradiction. ■

Let $P = \{\alpha_1, \dots, \alpha_q\}$ be as above. Let Δ_1 be the disk in \mathcal{D} splitted by α_1 and does not contain $\alpha_2 \cup \cdots \cup \alpha_q$, and $\Delta_i (2 \leq i \leq q)$ the closure of the component of $\mathcal{D} - \alpha_i$ such that $\Delta_i \supset \Delta_1$. By moving S by an ambient isotopy along Δ_i successively, we obtain a surface S'' which intersects C in annuli, and in particular, there exist q annuli which are mutually parallel in C . Let \bar{l} be one of the components of ∂A . Then \bar{l} is a simple closed curve in ∂W , and by Claim 2, we may assume that \bar{l} is disjoint from $f^{N(m)+1}(l) (= \partial E)$. Let \tilde{S} be an essential component of $S'' \cap W$. Then $(\tilde{S}, \partial\tilde{S}) \subset (W, \partial W - \bar{l})$. By Proposition 3.2, we see that $-\chi(S) \geq -\chi(\tilde{S}) > m$. This completes the proof. ■

Now we give the proof of the latter half of Proposition 3.1. Let W' be a genus g compression body with $\partial_- W'$ a genus $g-1$ closed surface, m'_1, m'_2, l' simple closed curves on $\partial_+ W'$ as in Figure 3.1. Then by applying the above argument to W' and $f' = T_{m'_2} \circ T_{m'_2}^{2q'}$ together with Sect. 6 of [12] we have:

Proposition 3.2'. *For each $m (\geq -1)$, there exists a constant $N'(m)$ such that if $p > N'(m)$, then $h_{W'}(\bar{l}) > m$ for each simple closed curve \bar{l} on $\partial_+ W'$ which is disjoint from $f^p(l')$ and not contractible in $\partial^+ W'$.*

Let ϕ' be the involution on W' as in Figure 3.6. Let N' be a 3-manifold

obtained from W' by attaching a 2-handle along $f^{N'(m)+1}(l')$. Then we have:

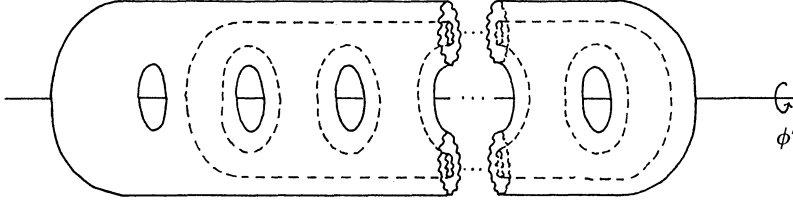


Figure 3.6

Lemma 3.3'. *The involution ϕ' extends to the involution $\bar{\phi}'$ of N' . Moreover, the quotient space of N' under $\bar{\phi}'$, denoted by B' , is homeomorphic to $(2\text{-sphere}) \times [0, 1]$, and the singular set t' in B' consists of a union of $2g$ arcs such that the endpoints of each component of t' are contained in pairwise different components of $\partial B'$.*

Moreover, by applying the argument of the proof of Lemma 3.4 to N' , we have:

Lemma 3.4'. *Let S be an essential surface in N' . Then we have $-\chi(S) > m$.*

The proofs of these are essentially the same as above, and we omit them.

Proof of the latter half of Proposition 3.1. Let (\tilde{B}, \tilde{t}) be a tangle which is obtained from (B, t) by capping off (B', t') so that ∂t is joined with $\partial t'$ in a component of $\partial B'$. Then the 2-fold branched cover \tilde{N} of \tilde{B} branched over \tilde{t} is regarded as a union of N and N' . Let $F = N \cap N'$, then F is a closed orientable surface of genus $g-1$.

Claim. \tilde{N} is irreducible and F is incompressible in \tilde{N} .

Proof. Since $h_W(f^{N(m)+1}(l)) > m$, $\partial_+ W - f^{N(m)+1}(l)$ is incompressible in W . We note that W is irreducible. Then by the handle addition lemma, we see that N is irreducible and ∂N is incompressible in N . Similarly, N' is irreducible and $\partial N'$ is incompressible in N' . Hence \tilde{N} is irreducible and F is incompressible in \tilde{N} . ■

First we show that (\tilde{B}, \tilde{t}) has height m . Let S be an essential surface in \tilde{N} , chosen to minimize $-\chi(S)$. Suppose that $S \cap F = \emptyset$. If S is boundary-parallel in N or N' , then $-\chi(S) = 2g - 4 > m$. If S is not boundary-parallel (hence, essential) in N , then by Lemma 3.4, $-\chi(S) > m$. If S is not boundary-parallel (hence, essential) in N' , then by Lemma 3.4', we see that $-\chi(S) > m$.

Suppose that $S \cap F \neq \emptyset$ and $S \cap F$ has the minimal number of the components among all the essential surfaces in \tilde{N} ambient isotopic to S . Then,

by the irreducibility of N , we see that each component of $S \cap N$ is incompressible in N . Moreover, by using the minimality of $\#\{S \cap F\}$ again, we see that each component of $S \cap N$ is an essential surface in N . Hence we have $-\chi(S \cap N) > m$, by Lemma 3.4. On the other hand, since F is incompressible in N' , $S \cap N'$ has no disk components. Therefore $\chi(S \cap N') \leq 0$, and, hence, $-\chi(S) = -(\chi(S \cap N) + \chi(S \cap N')) \geq -\chi(S \cap N) > m$.

Next, we show that (\tilde{B}, \tilde{t}) has Property I. Let $\tilde{X} = \text{cl}(\tilde{B} - N(\tilde{t}, \tilde{B}))$ be the tangle space and $X = \tilde{X} \cap B$, $X' = \tilde{X} \cap B'$. Let $P = X \cap X'$. Then P is a planar surface properly embedded in \tilde{X} . By Propositions 3.2 and 3.2', it is easy to see that P is incompressible in X and X' . Suppose that there exists a compressing disk D for $\partial\tilde{X}$, and $\#\{D \cap P\}$ is minimal among all the compressing disks for $\partial\tilde{X}$.

If $D \cap P = \emptyset$, then $D \subset X'$ and $\partial D \subset \partial X' - P$. Hence by moving D by a rel P ambient isotopy of X' , we may suppose that $\partial D \subset \partial X' \cap \partial\tilde{B}$. Since $\partial X' \cap \partial\tilde{B}$ is incompressible in X' , we see that ∂D bounds a disk in $\partial X' \cap \partial\tilde{B}$, a contradiction.

Suppose that $D \cap P \neq \emptyset$. Since P is incompressible in \tilde{X} , and \tilde{X} is irreducible, by standard innermost disk argument, we may suppose that $D \cap P$ has no circle components. Moreover, by the minimality of $\#\{D \cap P\}$, we see that $D \cap P$ has no inessential components in P . Let α be an outermost arc component of $D \cap P$ in D , i.e. there exists a disk Δ in D such that $\Delta \cap P = \alpha$, $\Delta \cap \partial D = \beta$ an arc such that $\partial\Delta = \alpha \cup \beta$ and $\partial\alpha = \partial\beta$. Then Δ is properly embedded in either X or X' . The first case contradicts the incompressibility of P in X . Then we consider the second case. Suppose that the endpoints of α are contained in different boundary components of P , say d_1, d_2 . Let t'_1, t'_2 be the components of t' such that $N(t'_i, B') \cap P = d_i$ ($i=1, 2$). Let $A = \text{Fr}_{X'} N(N(t'_1, B') \cup \Delta \cup N(t'_2, B'), X')$. Recall that $N' \rightarrow B'$ is the 2-fold branched cover with $\tilde{\phi}'$ generating the group of covering translation. Let \tilde{A} be the lift of A in N' . Then \tilde{A} consists of two annuli. If \tilde{A} is compressible in N' , then by equivariant loop theorem ([10]), there exists a compressing disk \tilde{D} such that $\phi(\tilde{D}) \cap \tilde{D} = \emptyset$ or $\phi(\tilde{D}) = \tilde{D}$. The first case contradicts the incompressibility of A . Since ϕ exchanges the components of \tilde{A} , the second case does not occur. Therefore \tilde{A} is incompressible in N' . Since \tilde{A} is not boundary parallel, \tilde{A} is essential in N' with $\chi(\tilde{A}) = 0$. This contradicts Lemma 3.4.' Suppose that $\partial\alpha$ lies in one component of ∂P , say α_0 . Let t'_0 be the component of t' such that $N(t'_0, B') \cap P = \alpha_0$. Let A be the component of $\text{Fr}_{X'} N(N(t'_0, B') \cup \Delta)$ such that each component of $P - (A \cap P)$ contains even components of ∂P . Then we have a contradiction as above, completing the proof. ■

4. Characteristic knots

Let M be a closed 3-manifold throughout this section.

Two knots K_0 and K_1 in M are *equivalent* if there exists an ambient isotopy h_t ($0 \leq t \leq 1$) of M such that $h_0 = \text{id}$, and $h_1(K_0) = K_1$. We say that K_0 and K_1 are *inequivalent* if they are not equivalent. Let g be an integer such that $g \geq 1$. A knot K in M is a *g -characteristic knot* if the exterior of K has no 2-sided closed incompressible surfaces of genus less than or equal to g except for boundary-parallel tori.

In this section, we prove the following theorem. The proof of this is a generalization of a construction of simple knots in [14] (see also [5]).

Theorem 4.1. *For each integer $g(\geq 1)$, every closed orientable 3-manifold M contains infinitely many, mutually inequivalent g -characteristic knots.*

REMARK. We note that if $\text{rank } H_1(M; \mathbb{Q}) \geq 2$, then, for each knot K in M , there exists a non-separating closed incompressible surface in $E(K)$.

Proof. First we recall a *special handle decomposition* of M from [14]. A handle decomposition $\{h_i^k\}$ of M is *special* if;

- (1) The intersection of any handle with any other handle is either empty or connected.
- (2) Each 0-handle meets exactly four 1-handles and six 2-handles.
- (3) Each 1-handle meets exactly two 0-handles and three 2-handles.
- (4) Each pair of 2-handles either
 - (a) meets no common 0-handle or 1-handle, or
 - (b) meets exactly one common 0-handle and no common 1-handle, or
 - (c) meets exactly one common 1-handle and two common 0-handles.
- (5) The complement of any 0-handles in H is connected, where H is the union of the 0-handles and the 1-handles.
- (6) The union of any 0-handle with H' is a handlebody, where H' is the union of the 2-handles and the 3-handles.

Note that every closed orientable 3-manifold has a special handle decomposition [14, Lemma 5.1].

Now we fix a special handle decomposition $\{h_i^k\}$ of M . For each 1-handle h_j^1 , we identify h_j^1 with $D \times [0, 1]$, where D is a disk and $D \times [0, 1]$ meets 0-handles in $D \times \{0, 1\}$. Let g be an integer such that $g \geq 1$. Let α_j be a system of $2g+2$ arcs properly embedded in h_j^1 such that each arc is identified with $\{\text{one point}\} \times [0, 1] (\subset D^2 \times [0, 1])$. Let $\tau_i = (B_i, t_i)$ be a copy of $(4g+4)$ -string tangle with height $4g-4$ and Property *I* (Proposition 3.1). Identify each 0-handle h_i^0 with B_i in a way that ∂t_i is joined with the boundary of the arcs $\alpha_{j_i(1)}, \alpha_{j_i(2)}, \alpha_{j_i(3)}, \alpha_{j_i(4)}$, where $h_{j_i(1)}^1, \dots, h_{j_i(4)}^1$ are the four 1-handles which meet the 0-handle h_i^0 , and $(\cup_i t_i) \cup (\cup_j \alpha_j)$ becomes a knot K where the unions are taken over all the 0-handles and 1-handles of the handle decomposition.

Let $V = (\cup_i h_i^0) \cup (\cup_j h_j^1)$ and $V' = M - \text{Int } V$. Then we note that (V, V') is a Heegaard splitting of M .

Assertion 1. *The above knot K in M is a g -characteristic knot.*

Proof. Let $V_1 = \text{cl}(V - N(K))$, $V_2 = V'$, $X_i^0 = V_1 \cap h_i^0$, and $X_i^1 = V_1 \cap h_i^1$. Then $X_i^1 \cap (\cup X_i^1)$ consists of four disk-with- $(2g+2)$ -holes properly embedded in V_1 , say $P_{i1}, P_{i2}, P_{i3}, P_{i4}$.

Claim 1. Each P_{ij} is incompressible in V_1 , and V_1 is irreducible.

Proof. Suppose that $X_k^0 \cap X_l^1 = P_{kj}$. Since the height of τ_i is greater than -1 , we see that P_{kj} is incompressible in X_k^0 . Since (X_l^1, P_{kj}) is homeomorphic to $(P_{kj} \times [0, 1], P_{kj} \times \{0\})$, we see that P_{kj} is incompressible in X_l^1 . From these facts, it is easy to see that each P_{kj} is incompressible in V_1 . Then the irreducibility of each X_k^0, X_l^1 , and the incompressibility of each P_{ij} imply that V_1 is irreducible. ■

Let $Q_i = \partial X_i^0 \cap \partial B_i$. Then Q_i is an $(8g+8)$ -punctured sphere properly embedded in $E(K)$.

Claim 2. Each Q_i is incompressible in $E(K)$, and $E(K)$ is irreducible.

Proof. Let $W = \text{cl}(V - \cup_j X_j^1)$ and $W' = V' \cup (\cup_j X_j^1)$ (Figure 4.1). Then we note that W, W' are handlebodies.

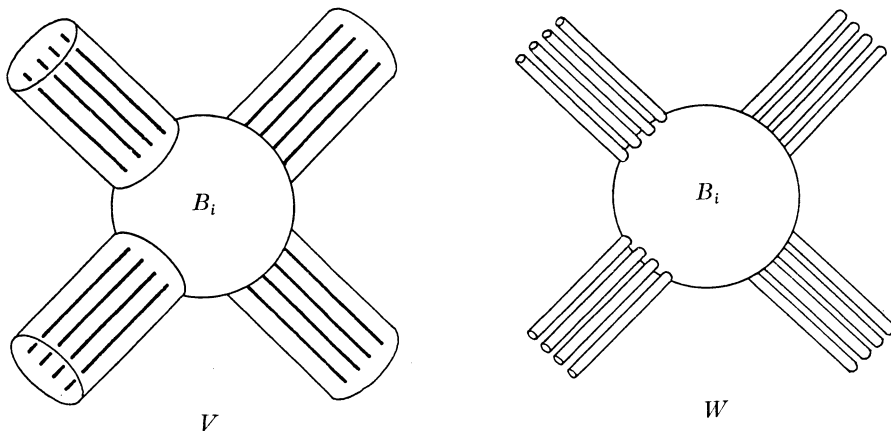


Figure 4.1

Suppose that there exists a compressing disk D for Q_i in $E(K)$. Since (B_i, t_i) has height $4g-4$, we see that $\text{Int } D$ is not contained in h_i^0 . Let D' be a disk in ∂h_i^0 such that $\partial D' = \partial D$. We note that $V' \cup h_i^0$ is a handlebody by the definition of a special handle decomposition (6). Then it is easy to see that $W' \cup h_i^0$ is a

handlebody. Hence $W' \cup h_i^0$ is irreducible, and the 2-sphere $D \cup D'$ bounds a 3-ball B in $W' \cup h_i^0$. Since $V - h_i^0$ is connected by the definition of a special handle decomposition (5), we see that $W - h_i^0$ is connected. Since $\partial D = \partial D' \subset Q_i$, and $W - h_i^0$ is not contained in B , this implies that ∂D bounds a disk in Q_i . Hence Q_i is incompressible. Since $E(K) = W' \cup (\cup_i X_i^0)$, $W' \cap X_i^0 = Q_i$, by the irreducibility of W' , X_i^0 , and the incompressibility of Q_i , we see that $E(K)$ is irreducible. ■

Let S be a closed incompressible surface in the exterior $E(K)$ of K in M which is not a boundary parallel torus in $E(K)$. Then S must intersect V_1 since V_2 is a handlebody. We suppose that $\# \{S \cap \partial V_1\}$ is minimal among all surfaces which is ambient isotopic to S in $E(K)$.

Claim 3. $S \cap V_1$ is incompressible in V_1 , and there exists X_i^0 such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$.

Proof. By the irreducibility of $E(K)$ (Claim 2), and the minimality of $\# \{S \cap \partial V_1\}$, we see that $S \cap V_1$ is incompressible in V_1 . Assume that $X_i^0 \cap (S \cap V_1) = \emptyset$ for each i , i.e. $S \cap V_1 \subset \cup X_j^1$. Suppose that $X_j^1 \cap (S \cap V_1) \neq \emptyset$. Let $S_j = X_j^1 \cap (S \cap V_1)$. Then, by [4, Sect.8 Lemma], we see that each component of S_j is an annulus which is parallel to an annulus in $X_j^1 \cap \partial V_2$, contradicting the minimality of $\# \{S \cap \partial V_1\}$. ■

Now we suppose that $\# \{(S \cap V_1) \cap (\cup_i Q_i)\}$ is minimal among the ambient isotopy class of $S \cap V_1$ in V_1 . Let X_i^0 be the tangle space in a 0-handle h_i^0 such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$, and $S_i = X_i^0 \cap (S \cap V_1)$. Let $p: N \rightarrow B_i$ be the 2-fold branched cover of B_i over t_i with ϕ generating the group of the covering translation. Let $\tilde{S}_i = p^{-1}(S_i)$. If \tilde{S}_i is compressible in N , there exists a compressing disk D for \tilde{S}_i in N such that either $\phi(D) \cap D = \emptyset$ or $\phi(D) = D$ [10]. However the first case contradicts the incompressibility of S_i . Hence $\phi(D) = D$ and $p(D)$ is a disk in B_i meeting t_i in one point. Then compress S_i by $p(D)$ (hence, the surface intersects K in two points). By repeating this step finitely many times for all i such that $X_i^0 \cap (S \cap V_1) \neq \emptyset$, we finally get a 2-manifold S' in M such that each component of $\tilde{S}'_i = p^{-1}(S'_i)$ is incompressible in N , where $S'_i = B_i \cap (S' \cap V_1)$. Then we have the following two cases.

Case 1. There exists i such that \tilde{S}'_i has a non-boundary-parallel component.

Then \tilde{S}'_i has an essential component F in N . Since (B_i, t_i) has height $4g-4$, $-\chi(F) > 4g-4$. Suppose that $p(F)$ does not intersect with the singular set. Then either $p(F)$ is homeomorphic to F , or $p: F \rightarrow p(F)$ is a regular covering, and, hence, we have either $\chi(F) = \chi(p(F))$, or $\chi(p(F)) = \chi(F)/2$. By the minimality of $\# \{(S \cap V_1) \cap (\cup_i Q_i)\}$, incompressibility of Q_i , and Claim 2, we see that each component of $\partial p(F)$ is essential in S . Hence we have $-\chi(S) \geq$

$-\chi(F) > 2g - 2$, and the genus of S is greater than g . Suppose that F intersects the singular set in $q (\geq 1)$ points. Then we have $\chi(p(F) - K) = (\chi(F) - q)/2 < (\chi(F))/2 < 2 - 2g$. By the same reason as above, we see that each component of $\partial p(F)$ is essential in S . Hence we see that $-\chi(S) = -\chi(S' - K) \geq -\chi(p(F) - K) > 2g - 2$. Hence the genus of S is greater than g .

Case 2. For every i , each component of \tilde{S}'_i is boundary-parallel in N .

Move \tilde{S}'_i by an equivariant ambient isotopy along those parallelisms so that S'_i is pushed off B_i . By Claim 3, we see that S' meets K . Let $A_j = \partial h_j^1 - (\cup_i \partial h_i^0)$. Assume that $S' \cap (\cup_j A_j) = \emptyset$. Then $S' \subset \text{Int}(\cup_j h_j^1)$. Then, by [4, Sect. 8 Lemma], we see that each component of S' is a 2-sphere intersecting exactly one component of α_j in two points. This implies that S is a boundary-parallel torus, contradicting our assumption. Therefore $S' \cap (\cup_j A_j) \neq \emptyset$. Since S is incompressible in $E(K)$, and $E(K)$ is irreducible (Claim 2), the minimality of $\#\{S \cap \partial V_i\}$ implies that $S' \cap (\cup_j A_j)$ has no inessential components in $\cup_j A_j$. Hence, by [4, Sect. 8 Lemma], we see that each component of $S' \cap h_j^1$ is a horizontal disk in $h_j^1 \cong D \times [0, 1]$. It follows that S' meets all the components of α_j . Since α_j consists of $2g + 2$ arcs, this shows that for each component F' of S' , we have $\chi(F' - K) \leq 2 - (2g + 2) = -2g$. Hence $\chi(S) = \chi(S' - K) \leq -2g$. Then we conclude that the genus of S is greater than g . ■

Let n be the number of 0-handles of $\{h_i^1\}$. Let $F_i (i = 1, \dots, n)$ be a closed surface of genus $4g + 4$ in $E(K)$ obtained by pushing ∂X_i^0 slightly into $\text{Int } E(K)$.

Assertion 2. F_1, \dots, F_n are incompressible in $E(K)$ and F_i is not parallel to F_j for each $i \neq j$.

Proof. Assume that there is a compressing disk D for F_i in $E(K)$. Since the tangle τ_i has Property I, D lies in $\text{cl}(E(K) - X_i)$. Let \mathcal{A} be the union of $4g + 4$ annuli in $\text{cl}(E(K) - X_i)$ such that one boundary component of each annulus is contained in F_i and the other boundary component is a union of core curves of the annuli in $\partial E(K)$ corresponding to $\text{Fr}_{B_i} N(t_i, B_i)$ (Figure 4.2).

If $D \cap \mathcal{A} = \emptyset$, by moving D by an ambient isotopy of $E(K)$, we may assume that ∂D lies in $Q_i = \partial B_i \cap X_i$. This contradicts the incompressibility of Q_i in $E(K)$ (Claim 2 in the proof of Theorem 4.1). Hence we have $D \cap \mathcal{A} \neq \emptyset$. Then we suppose that $\#\{D \cap \mathcal{A}\}$ is minimal among all compressing disks for F_i . Since $\text{cl}(E(K) - X_i)$ is irreducible, we see that $D \cap \mathcal{A}$ has no circle components, by standard innermost disk argument. Let α be an outermost arc component of $D \cap \mathcal{A}$ in \mathcal{A} , i.e. there exists a disk Δ in \mathcal{A} such that $\Delta \cap D = \alpha$, $\Delta \cap \partial \mathcal{A} = \beta$ an arc such that $\partial \Delta = \alpha \cup \beta$ and $\partial \alpha = \partial \beta$. Then by compressing D along Δ toward F_i we have two disks D', D'' such that $\partial D' \subset F_i$, $\partial D'' \subset F_i$. Since D is a compressing disk for F_i , we see that one of D', D'' is a compressing disk for F_i , contradicting the minimality of $\#\{D \cap \mathcal{A}\}$. Hence F_i is incompressible in $E(K)$.

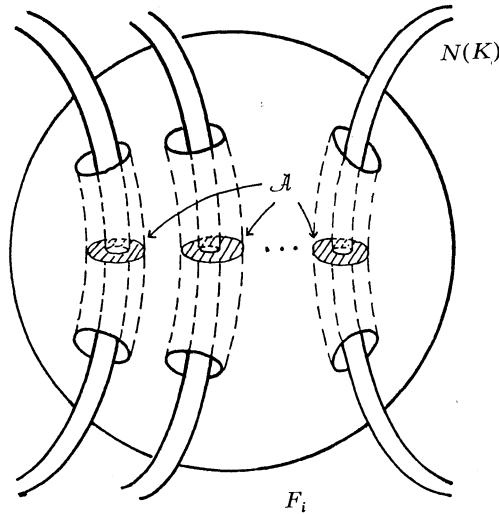


Figure 4.2

Next suppose that F_i and F_j are parallel in $E(K)$ for some $i \neq j$. Then $n=2$, and contradicting the fact that $\{h_i^j\}$ is special (cf. [5, Fact 1 of Proposition 3]). ■

For the proof of Theorem 4.1, we need the following theorem which is due to Haken.

Theorem 4.2. ([4], [6]). *Let M be a compact, orientable 3-manifold. There is an integer $n(M)$ such that if $\{F_1, \dots, F_k\}$ is any collection of mutually disjoint incompressible closed surfaces in M , then either $k < n(M)$, or for some $i \neq j$, F_i is parallel to F_j in M .*

Completion of the Proof of Theorem 4.1. First we note that for every non-negative integer h , there exists a special handle decomposition of M with more than h 0-handles [5, Fact 2 of Proposition 3].

Let $K_0 = K$ be a g -characteristic knot in M obtained by the above construction (Assertion 1). Let $M_0 = M - \text{Int } N(K_0)$. Then we find a special handle decomposition of M with h 0-handles, where $h > n(M_0)$. Let K_1 be a g -characteristic knot constructed as above by using this handle decomposition. Then $M_1 = M - \text{Int } N(K_1)$ contains h incompressible, mutually non-parallel closed surfaces (Assertion 2). Then, by Theorem 4.2, we see that M_1 is not homeomorphic to M_0 . Hence K_0 and K_1 are inequivalent. Continuing in this way, we obtain infinitely many inequivalent g -characteristic knots in M . ■

5. Existence of a non-simple position knot

Let H be a handlebody, and k a knot in H . We say that k is in a *simple*

position in H if there exists a disk D properly embedded in H such that $D \cap k = \emptyset$, and D splits a solid torus V from H such that $k \subset V$ and k is a core curve of V (Figure 5.1). We note that k is in a simple position in H if and only if $\text{cl}(H - N(k))$ is a compression body.

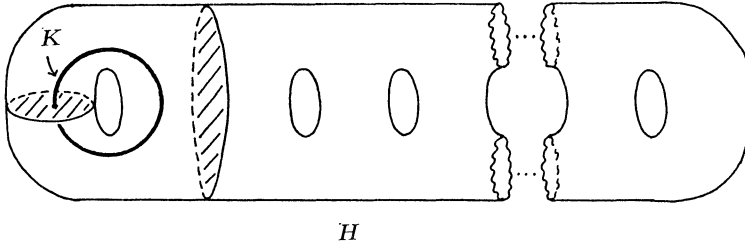


Figure 5.1

Then the purpose of this section is to prove:

Theorem 5.1. *Suppose that a closed, orientable 3-manifold M admits a Heegaard splitting of genus h . Then for each integer $g \geq 1$, there exists a g -characteristic knot K in M such that, for any genus h Heegaard splitting (V, W) of M , K is not ambient isotopic in M to a simple position knot in V .*

Proof. Let $\{h_i\}$ be a special handle decomposition of M with n 0-handles, where $n \geq 8(3h-3)+1$. By applying the argument of Sect. 4 to this handle decomposition, we get a g -characteristic knot K whose complement contains a system of mutually disjoint, non-parallel incompressible closed surfaces of genus $4g+4$, denoted by $\mathcal{F} = \{F_1, \dots, F_n\}$ (Sect. 4 Assertion 2).

We show that this knot K satisfies the conclusion of Theorem 5.1.

Assume that there is a genus h Heegaard splitting (V, W) of M such that K is in a simple position in V . Let $V_1 = \text{cl}(V - N(K))$ and $V_2 = W$. Then V_1 is a genus h compression body with $\partial_- V_1$ a torus. We note that (V_1, V_2) is a Heegaard splitting of $E(K)$. Then, by the irreducibility of $E(K)$, \mathcal{F} can be ambient isotoped to be normal with respect to (V_1, V_2) (see Sect. 2). We suppose that $\#\{\mathcal{F} \cap V_1\}$ is minimal in the ambient isotopy class of \mathcal{F} in $E(K)$.

First we show that there exists a system \mathcal{F}' of surfaces which is ambient isotopic to \mathcal{F} in $E(K)$ and $\mathcal{F}' \cap V_1$ has at least five annulus components A_1, \dots, A_5 which are mutually parallel in V' , and essential in \mathcal{F}' .

Let $\mathcal{F}_i = \mathcal{F} \cap V_i$ ($i=1, 2$). Then we note that since ∂V_i can contain at most $3h-3$ parallel classes of mutually disjoint essential simple closed curves, there exists a system of mutually parallel disk components $\{D_1, \dots, D_q\}$ of \mathcal{F}_1 which lies in this order in V_1 , where $q \geq 9$.

By the argument of the proof of Proposition 2.2, there exists a complete disk system \mathcal{D} for V_2 such that each component of $\mathcal{D} \cap \mathcal{F}_2$ is an essential arc in \mathcal{F}_2 . Let A be an annulus in $\partial_+ V_1$ such that A contains $\partial D_1 \cup \dots \cup \partial D_q$, and each

∂D_i is isotopic to a core of A . We suppose that $\#\{\partial \mathcal{D} \cup \partial D_i\}$ is minimal in the ambient isotopy class of $\partial \mathcal{D}$ in $\partial V_2 (= \partial_+ V_1)$, and hence, $I = \partial \mathcal{D} \cap A$ is a system of essential arcs in A . We label the points $\partial D_i \cap I$ by i , then, in each component of I , they lie in this order. Let D be a component of \mathcal{D} such that $D \cap A \neq \emptyset$. Then by applying the argument of Claim 6 of Lemma 3.4, we see that there exists a subsystem P of $D \cap \mathcal{F}_2$ such that there exists a component I_0 of I which satisfies the following.

- (1) Every arc of P has one end-point in I_0 .
- (2) Every arc of $D \cap \mathcal{F}_2$ which has one end point in I_0 belongs to P .
- (3) Every arc t of P joints I_0 with one of components of I which are neighbouring of I_0 in ∂D .

Moreover, by the argument of Claim 7 of Lemma 3.4, for each component of P , both of its endpoints are contained in I . Then, by using Lemma 2.3, we see that the endpoints of each component of P have the same label. Hence P consists of at most two subsystems each of which contains all arcs of P joining two components of I . Therefore by labelling “1, 2, ..., q ” instead of “ $q, q-1, \dots, 1$ ” if necessary, we may assume that there exists a subsystem of at least five arcs $\{\alpha_1, \dots, \alpha_p\}$ ($p \geq 5$) of $D \cap \mathcal{F}_2$ such that α_i joints two points in I_0 and I_1 , say. Let Δ_1 be the disk in D splitted by α_1 and does not contain $\alpha_2 \cup \dots \cup \alpha_p$, and Δ_i ($2 \leq i \leq p$) the closure of the component of $D - \alpha_i$ such that $\Delta_i \supset \Delta_1$. Move \mathcal{F} by an ambient isotopy along Δ_i successively, and denote the image by \mathcal{F}' . Then we see that $\mathcal{F}' \cap V_1$ has p mutually parallel annuli $\{A_1, \dots, A_p\}$ in V_1 . By the argument of the proof of Claim 5 of Lemma 3.4, we see that A_i is incompressible, hence essential in V_1 .

Now in these parallelisms $A_i \times [0, 1]$ in V_1 where $A_i \times \{0\} = A_i$, $A_i \times \{1\} = A_{i+1}$ ($1 \leq i \leq p-1$), there exist annuli Λ_i such that each Λ_i corresponds to $C_i \times [0, 1]$ where C_i is a core curve of A_i ($i=1, \dots, p-1$) (Figure 5.2).

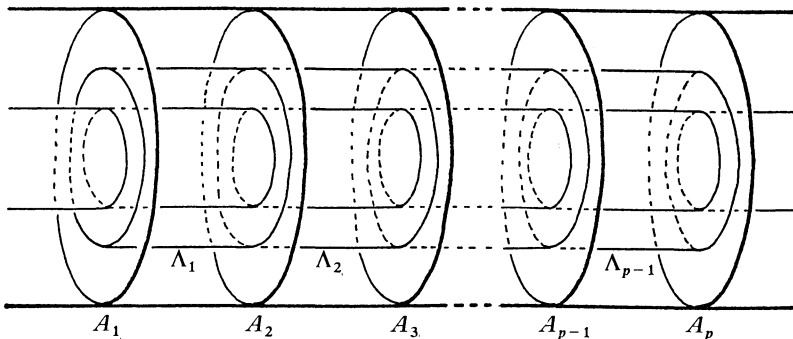


Figure 5.2

Let $E(K) = X_0 \cup X_1 \cup \dots \cup X_n$ where X_j corresponds to the ‘inside’ of F_j

(hence $X_0 \cap X_j = F_j$, $j=1, \dots, n$). Then Λ_i is an annulus properly embedded in X_k , for some k . Assume that there exists a compressing disk D for Λ_i in X_j . Let Λ be a subannulus in Λ_i cobounded by ∂D and C_i . Move the disk $D \cup \Lambda$ slightly by an ambient isotopy so that $D \cup \Lambda$ becomes a properly embedded disk in X_k . This contradicts the incompressibility of \mathcal{F} in $E(K)$. Hence, Λ_i is incompressible in X_k . We have either $\Lambda_1 \subset X_0$ or $\Lambda_2 \subset X_0$. If $\Lambda_1 \subset X_0$, then we have $\Lambda_3 \subset X_0$, and if $\Lambda_2 \subset X_0$, then we have $\Lambda_4 \subset X_0$. Now we suppose that $\Delta_1 \subset X_0$, $\Lambda_2 \subset X_1$, and $\Lambda_3 \subset X_0$. (The case of $\Lambda_2, \Lambda_4 \subset X_0$ is essentially the same.)

Claim. We have either one of:

- (1) Λ_1 is boundary-parallel in X_0 , or
- (2) Λ_2 is boundary-parallel in X_1 , or
- (3) Λ_3 is boundary-parallel in X_0 .

Proof. Recall that Q_i is a planar surface in ∂X_i , which corresponds to $\partial X_i \cap \partial B_i$ (Sect. 4). Let \mathcal{A} be a disjoint union of annuli properly embedded in X_0 , which is defined in the proof of Assertion 2 of Sect. 4 (Figure 4.2). We suppose that $\#\{\Lambda_1 \cap \mathcal{A}\}$ is minimal among the ambient isotopy class of Λ_1 in X_0 . Suppose that $\Lambda_1 \cap \mathcal{A} \neq \emptyset$. If there are inessential arc components of $\Lambda_1 \cap \mathcal{A}$ in Λ_1 , let α be the outermost arc component of $\Lambda_1 \cap \mathcal{A}$ in Λ_1 , i.e. there exists a disk Δ in Λ_1 such that $\Delta \cap \mathcal{A} = \alpha$, $\Delta \cap \partial \Lambda_1 = \alpha$ an arc in $\partial \Lambda_1$ such that $\partial \Delta = \alpha \cup \beta$ and $\partial \alpha = \partial \beta = \alpha \cap \beta$. Let Δ' be the disk in \mathcal{A} such that $\text{Fr}_{\mathcal{A}} \Delta' = \alpha$. Then, by moving $\Delta \cup \Delta'$ in a neighborhood of \mathcal{A} by an ambient isotopy of X_0 , we get a disk properly embedded in X_0 , whose boundary contained in Q_1 . Since Q_1 is incompressible in $E(K)$ and X_0 is irreducible, we see that this disk is parallel to a disk in Q_1 . This shows that $\alpha \cap Q_1$ is an inessential arc in Q_1 . Therefore there is an ambient isotopy which removes α from $\Lambda_1 \cap \mathcal{A}$, contradicting the minimality of $\#\{\Lambda_1 \cap \mathcal{A}\}$. Suppose that every component of $\Lambda_1 \cap \mathcal{A}$ is an essential arc in Λ_1 . Let Π be a disk in Λ_1 which is bounded by two arcs $a_1 a_2$, of $\Lambda_1 \cap \mathcal{A}$ and two arcs in $\partial \Lambda_1$ such that $\text{Int } \Pi \cap \mathcal{A} = \emptyset$. Let Δ_i be a disk in \mathcal{A} such that a_i bounds Δ_i with an arc in $\partial \mathcal{A}$ ($i=1, 2$). Assume that one of Δ_i is contained in the other. Without loss of generality, we may assume that $\Delta_1 \subset \Delta_2$. Then by moving $\Pi \cup \Delta_1$ by rel a_2 isotopy, we get a disk Π' in X_0 such that $\Pi' \cap \mathcal{A} = a_2$, $\Pi' \cap \partial X_0 = \text{cl}(\partial \Pi' - a_2)$, and $(\Pi' \cap \partial X_0) \cap Q_1 = \beta'$ an arc. By the above argument, we see that β' is an inessential arc in Q_1 (i.e. there is a disk Δ^* in Q_1 such that $\text{Fr}_{Q_1} \Delta^* = \beta'$). Since Π is reproduced by adding a band to Π' along an arc γ such that $\gamma \cap \Delta^* \neq \emptyset$, we see that $\Pi \cap Q_1$ consists of two inessential arcs in Q_1 , contradicting the minimality of $\#\{\Lambda_1 \cap \mathcal{A}\}$. Hence $\Delta_1 \cap \Delta_2 = \emptyset$. Let $E = \Pi \cup \Delta_1 \cup \Delta_2$. Then, by moving the disk E in a neighborhood of \mathcal{A} by an ambient isotopy of X_0 , we may assume that E is a disk properly embedded in X_0 and ∂E in Q_1 . Then by the above argument we see that E is parallel to a

disk in Q_1 . The same is hold for any pair of neighbouring arcs of $\Lambda_1 \cap \mathcal{A}$. Then we conclude that Λ_1 is boundary parallel in X_0 . Similarly, if every component of $\Lambda_3 \cap \mathcal{A}$ is an essential arc in Λ_3 , Λ_3 is boundary-parallel in X_0 .

Now suppose that $\partial\Lambda_i \cap \partial\mathcal{A} = \emptyset$ ($i=1, 3$) (hence $\Lambda_i \cap \mathcal{A} = \emptyset$ or each component of $\Lambda_i \cap \mathcal{A}$ is an essential circle in Λ_i). Then $\partial\Lambda_2 \cap \partial\mathcal{A} = \emptyset$. Assume that Λ_2 is not boundary-parallel in X_1 . Let $p: N \rightarrow B_1$ be the 2-fold branched cover over $t_1 = K \cap B_1$ with ϕ generating the group of covering translation. Let $\tilde{\Lambda}_2 = p^{-1}(\Lambda_2)$. Since the tangle (B_1, t_1) has height $4g-4$, $\tilde{\Lambda}_2$ is compressible in N . Then there exists a compressing disk \tilde{D} for $\tilde{\Lambda}_2$ in N such that $\phi(\tilde{D}) \cap \tilde{D} = \emptyset$ or $\phi(\tilde{D}) = \tilde{D}$ ([10]). The first case contradicts the incompressibility of Λ_2 in X_1 . In the second case, $D = p(\tilde{D})$ meets t_1 in one point. Let D_1 and D_2 be disks obtained by compressing Λ_2 by D . Since the height of (B_1, t_1) is greater than -1 , there is a closure of a component of $B_1 - D_i$, say B^i , such that $(B^i, B^i \cap t_1)$ is a 1-string trivial tangle. Then we have either $B^1 \cap B^2 = \emptyset$, or one of B^1, B^2 is contained in the other (Figure 5.3). In the first case, we see that Λ_2 is parallel to an annulus in ∂X_0 corresponding to a component of $\text{Fr}_{B_1} N(t_1, B_1)$. In the second case, we see that Λ_2 is parallel to an annulus in Q_1 . Hence we have the conclusion (2) of Claim. ■

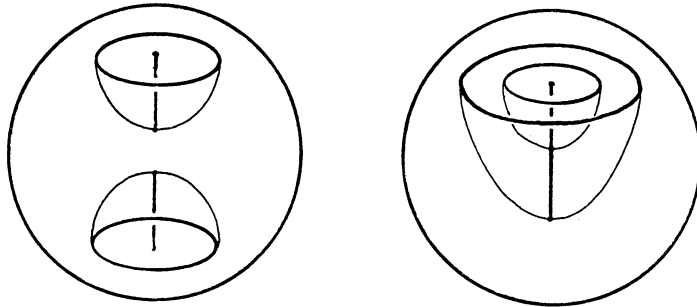


Figure 5.3

Now we may assume that Λ_i is boundary-parallel in X_j for some i and j . By extending the ambient isotopy along this parallelism, we can remove two annuli A_i and A_{i+1} from $\mathcal{F}' \cap V_1$. Denote this image by \mathcal{F}'' . Then moving \mathcal{F}'' by an ambient isotopy, which corresponds to the reverse that of \mathcal{F} to \mathcal{F}' , we obtained a system of surfaces \mathcal{F}''' which intersects V_1 in essential disks and the number of the components of $\mathcal{F}''' \cap V_1$ is less than that of $\mathcal{F} \cap V_1$. This contradicts the minimality of the number of the components of $\mathcal{F} \cap V_1$, completing the proof. ■

6. Proof of Main Theorem

In this section, we give a proof of Hass-Thompson conjecture. First we prepare the following lemma.

Lemma 6.1. ([3]). *Let (W_1, W_2) be a Heegaard splitting of a 3-manifold M . Let S be a disjoint union of essential 2-spheres and disks in M . Then, there exists a disjoint union of essential 2-spheres and disks S' in M such that*

- (1) *S' is obtained from S by ambient 1-surgery and isotopy,*
- (2) *each component of S' meets $\partial_+ W_1 - \partial_+ W_2$ in a circle,*
- (3) *there exists complete disk systems \mathcal{D}_i for W_i , such that $\mathcal{D}_i \cap S' = \emptyset$ ($i=1, 2$).*
- (4) *if M is irreducible, then S' is actually isotopic to S .*

Let M be a compact, orientable 3-manifold such that ∂M has no 2-sphere components. A Heegaard splitting (V, W) of M is of type $T(\text{unnel})$, if W is a handlebody (hence V is a compression body with $\partial_- V = \partial M$). Then we define the T -Heegaard genus of M , denoted by $g^T(M)$, as the minimal genus of the type T Heegaard splittings. Then for the proof of Main Theorem, we first show:

Proposition 6.2. *Let M be a connected 3-manifold such that ∂M has no 2-sphere components. Suppose that there exists a compressing disk for ∂M in M . Let \bar{M} be a 3-manifold obtained by cutting M along D . Then*

$$g^T(\bar{M}) = \begin{cases} g^T(M), & \text{if } \bar{M} \text{ is disconnected,} \\ g^T(M) - 1, & \text{if } \bar{M} \text{ connected} \end{cases}$$

Proof. First we note that the T -Heegaard genus is additive under connected sum [3]. Let S be a system of 2-spheres which gives a prime decomposition of M . By standard innermost disk argument, we may assume that D is disjoint from S . Therefore we may assume, without loss of generality, that M is irreducible.

Case 1. D is separating in M .

Let $\bar{M} = M_1 \cup M_2$ where M_i ($i=1, 2$) is a connected component of \bar{M} . Then M is a boundary connected sum of M_1 and M_2 , i.e. $M = M_1 \natural M_2$. Hence, the fact that $g^T(\bar{M}) = g^T(M)$ follows from Lemma 6.1 (for the detailed argument, see [3]).

Case 2. D is non-separating in M .

Let (V, W) be a minimal genus type T Heegaard splitting of M . Then, by Lemma 6.1, we may assume that D meets ∂W in a circle. Let $\bar{D} = D \cap W$ and $\bar{A} = D \cap V$. Then \bar{D} is an essential disk in W and \bar{A} is an essential annulus in V . Let $\bar{W} = \text{cl}(W - N(\bar{D}, W))$, and N a sufficiently small regular neighborhood of D in M such that $N \cap \bar{W} = \emptyset$. We identify \bar{M} to $\text{cl}(M - N)$, and let $\bar{V} = \text{cl}(\bar{M} - \bar{W})$. Then we see that (\bar{V}, \bar{W}) is a type T Heegaard splitting of \bar{M} . Hence $g^T(\bar{M}) \leq g(\partial \bar{W}) = g^T(M) - 1$.

Next suppose that (\bar{V}, \bar{W}) is a type T Heegaard splitting of \bar{M} which realizes T-Heegaard genus of \bar{M} . By considering dual picture, we identify \bar{V} to $\partial_- \bar{V} \times I \cup (1\text{-handles})$. We identify $N(D, M)$ as $D \times [0, 1]$, then $M = \bar{M} \cup (D \times [0, 1])$. Let α be an arc obtained by extending the core of $D \times [0, 1]$ vertically to $\partial_- \bar{V} \times [0, 1]$. By general position argument, we may suppose that $\alpha \cap (1\text{-handles}) = \emptyset$ (hence, α is properly embedded in $\text{cl}(M - \bar{W})$). Let N' be a regular neighborhood of α in $\text{cl}(M - \bar{W})$, $W = \bar{W} \cup N'$, and $V = \text{cl}(M - W)$. Then it is easy to see that W is a handlebody in $\text{Int } M$, and V is a compression body in M . Therefore (V, W) is a type T Heegaard splitting of M . Hence $g^T(M) \leq g(\partial W) = g(\partial \bar{W}) + 1 = g^T(\bar{M}) + 1$. Therefore $g^T(\bar{M}) = g^T(M) - 1$. ■

Proof of Main Theorem. The 'if' part of Main Theorem is clear. Hence we give a proof of 'only if' part. Let M, V be as in Main Theorem. Let $E = \text{cl}(M - V)$. If E is a handlebody, then we are done. Hence we suppose that E is not a handlebody. Let \bar{g} be an integer such that V can be extended to a genus \bar{g} Heegaard splitting of M (\bar{V}, \bar{W}), i.e. there exists a system of mutually disjoint $\bar{g} - g$ arcs \mathcal{A} properly embedded in E such that $\bar{V} = V \cup N(\mathcal{A}, E)$, $\bar{W} = \text{cl}(M - \bar{V})$ are handlebodies. Let K be a g -characteristic knot in M which is not ambient isotopic to a simple position in any genus \bar{g} handlebody giving Heegaard splittings of M (Theorem 5.1). Then take a handlebody V_* in M with the following properties; (i) V_* contains K , (ii) V_* can be extended to a genus \bar{g} Heegaard splitting, and (iii) the genus of V_* , denoted by g_* , is minimal among all the handlebodies in M satisfying the above conditions (i), and (ii). We note that V satisfies the above conditions (i), and (ii), and, hence, $g_* \leq g$. Let $E_* = \text{cl}(M - V_*)$. Then in the rest of this section, we show that E_* is a handlebody, which completes the proof of Main Theorem.

Now assume that E_* is not a handlebody. Since $E(K)$ is irreducible and $E_* \subset E(K)$, E_* is irreducible. Hence there exists a maximal compression body W_* for ∂E_* in E_* unique up to ambient isotopy [2]. Since E_* is not a handlebody, $\partial_- W_* \neq \emptyset$. Let $Y = V_* \cup W_*$, then (V_*, W_*) is a Heegaard splitting of Y . We note that $\partial_- W_*$ lies in $E(K)$, and the sum of the genus of components of $\partial_- W_*$ is less than or equal to g_* . Then, by the property of g -characteristic knot K , each component of $\partial_- W_*$ is a boundary-parallel torus or a compressible closed surface in $E(K)$. Hence we have the following two cases.

Case 1. Each component of $\partial_- W_*$ is a boundary-parallel torus in $E(K)$.

Assume that $\partial_- W_*$ has more than one components T_1, \dots, T_n ($n \geq 2$). Let P_i ($i = 1, \dots, n$) be the parallelisms between T_i and $\partial E(K)$. By exchanging the suffix if necessary, we may suppose that $P_i \subset P_j$ if $i < j$. Then we have $P_1 \supset W_*$. On the other hand, we have $\partial W_* = \partial V_* \cup \partial_- W_* = \partial V_* \cup T_1 \cup T_2 \cdots \cup T_n$. Hence $P_1 \supset T_2, \dots, T_n$, a contradiction.

Therefore $\partial_- W_*$ consists of one boundary-parallel torus in $E(K)$. Then

we see that $Y = V_* \cup W_*$ is a solid torus. Let D be a meridian disk of Y . Since Y is irreducible, by moving D by an ambient isotopy, we may suppose that D meets ∂V_* in a circle (Lemma 6.1). By considering dual picture, we identify W_* to $\partial_- W_* \times [0, 1] \cup (1\text{-handles})$. Then, by Lemma 6.1 (3), we may suppose that $D \cap W_*$ is disjoint from the 1-handles. Let $\alpha_1, \dots, \alpha_{g_*-1}$ be arcs properly embedded in W_* obtained by extending the cores of the 1-handles vertically to $\partial_- W_* \times [0, 1]$ (hence $\partial_- W_* \cup \alpha_1 \cup \dots \cup \alpha_{g_*-1}$ is a deformation retract of W_*). Let $Q = N(Y, M)$. Then, move K by an ambient isotopy in Q so that $K \subset \partial Y$, $N(K, Q) \cap N(\alpha_i, Y) = \emptyset$, and $K \cap D = K \cap \partial D$ consists of one point. Let $Y^* = Y \cup N(K, Q) (\cong Y)$, and identify $\text{cl}(Q - Y^*)$ with the product of a torus $T (= \partial Y^*)$ and an interval $T \times [0, 1]$. Then, we may view W_* , V_* as follows: $W_* = (T \times [0, 1]) \cup (\cup_i N(\alpha_i, Y))$, $V_* = \text{cl}(Y^* - (\cup_i N(\alpha_i, Y)))$.

Let $\Delta = \text{Fr}_{Y^*}(N(K, Q) \cup N(D, Y))$ be a disk properly embedded in V_* . Then Δ splits a solid torus $N(K, Q) \cup N(D, Y)$ from V_* , and K lies in it as a core curve. This implies that K is in a simple position in V_* . Since V_* can be extended to a genus \bar{g} Heegaard splitting, which is ambient isotopic to (\bar{V}, \bar{W}) , we see that K is ambient isotopic to a simple position in \bar{V} , a contradiction.

Case 2. There exists a component of $\partial_- W_*$ which is compressible in $E(K)$.

Let D be a compressing disk for $\partial_- W_*$. Since W_* is a maximal compression body for ∂E_* in E_* , we see that $D \subset Y$. Let \bar{Y} be the 3-manifold obtained by cutting Y along D . Then, by the proof of Proposition 6.2, there exists a minimal genus Heegaard splitting (V^*, W^*) of Y such that $V^* \cap D$ is an essential disk in V^* . We note that since $D \subset E(K)$, K is disjoint from D . Moreover, by moving K by an ambient isotopy in \bar{Y} , we may suppose that $K \subset V^* - (D \cap V^*)$. If $g(V^*) < g_*$, attach $g_* - g(V^*)$ trivial 1-handles in W^* disjoint from D to V^* . We denote the new genus g_* Heegaard splitting of Y by (V^*, W^*) , again. Then (V^*, W^*) is a genus g_* Heegaard splitting of Y such that V^* contains K and there exists an essential disk $D^* = V^* \cap D$ in V^* which is disjoint from K .

Let $E^* = \text{cl}(M - Y) \cup W^*$. Since W_* and W^* are compression bodies such that $\partial_- W_* = \partial_- W^* = \partial Y$, and $\partial_+ W_* \cong \partial_+ W^*$ a genus g_* closed surface, W_* is homeomorphic to W^* . Hence $E_* = \text{cl}(M - V_*) = \text{cl}(M - Y) \cup W_* \cong \text{cl}(M - Y) \cup W^* = E^*$ i.e., E_* is homeomorphic to E^* .

By the assumption, V_* can be extended to a genus \bar{g} Heegaard splitting (\bar{V}_*, \bar{W}_*) of M . Let $V'_* = \text{cl}(N(\bar{V}_*, M) - V_*)$, and $W'_* = \text{cl}(E_* - V'_*)$. Then (V'_*, W'_*) is a genus \bar{g} type T Heegaard splitting of E_* . Since E^* is homeomorphic to E_* , there is a genus \bar{g} type T Heegaard splitting (V'^*, W'^*) of E^* corresponding to (V'_*, W'_*) . We note that since $\partial V'^* \cap V^* = \partial_- V'^* = \partial V^*$,

$V^{*'} \cup V^*$ is a handlebody in M . Hence $(V^{*'} \cup V^*, W^{*'})$ is a genus \bar{g} Heegaard splitting of M . Let \tilde{V} be a component of $V^* - N(D^*)$ which contains K inside. Then \tilde{V} is a handlebody of genus less than g_* and it can be extended to a genus \bar{g} Heegaard splitting $(V^{*'} \cup V^*, W^{*'})$ of M . This contradicts the minimality of g_* .

This completes the proof of Main Theorem. ■

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