



Title	On the well posedness of the Cauchy problem for a class of hyperbolic operators with multiple involutive characteristics
Author(s)	Sordoni, Vania
Citation	Osaka Journal of Mathematics. 1996, 33(3), p. 589-605
Version Type	VoR
URL	https://doi.org/10.18910/9926
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

Sordoni, V.
 Osaka J. Math.
 33 (1996), 589–605

**ON THE WELL POSEDNESS
 OF THE CAUCHY PROBLEM
 FOR A CLASS OF HYPERBOLIC OPERATORS
 WITH MULTIPLE INVOLUTIVE CHARACTERISTICS**

VANIA SORDONI

(Received July 24, 1995)

1. Introduction

Let $X \subset \mathbf{R}^{n+1} = \mathbf{R}_{x_0} \times \mathbf{R}_x^n$, $x' = (x_1, x_2, \dots, x_n)$ be an open set such that $0 \in X$ and let us consider a differential operator of order m with C^∞ coefficients:

$$(1.1) \quad P(x, D_x) = P_m(x, D_x) + P_{m-1}(x, D_x) + \dots$$

where we denote by $P_{m-j}(x, D_x)$ the homogeneous part of order $m-j$ of P .

Let us suppose that:

(H₁) the hyperplane $x_0 = 0$ is non-characteristics for P and the principal symbol $p_m(x, \xi)$ is hyperbolic with respect to ξ_0 .

In this paper we shall study the well posedness of the Cauchy problem in C^∞ for the operator P in some cases where $p_m(x, \xi)$ is not strictly hyperbolic but the set of multiple characteristics has a very special form, as we will specify further. (For a definition of correctly posed Cauchy problem in $X_0 = \{x \in X; x_0 < 0\}$ we refer to [5]).

We shall suppose that $p_m(x, \xi)$ vanishes exactly of order $m_1 \leq m$ on a smooth manifold Σ and that p_m is strictly hyperbolic outside Σ .

On Σ we make the following assumptions:

(H₂) for any point $\rho \in \Sigma$, there exists a conic neighborhood Ω of ρ and $d+1$ ($d < n$) smooth functions q_j , $j = 0, \dots, d$, defined on $W = \Omega \cup (-\Omega)$ and homogeneous of degree one such that $\Sigma \cap W$ is given by

$$(1.2) \quad \{\rho \in W; q_0(\rho) = \dots = q_d(\rho) = 0\}$$

with $\{q_i, q_j\}(\rho) = 0$ for any $\rho \in \Sigma \cap W$.

(Here we have set $-\Omega = \{(x, \xi) \in T^*X \setminus 0; (x, -\xi) \in \Omega\}$).

Moreover, denoting by ω and $\sigma = d\omega$ the canonical 1 and 2 forms in T^*X we suppose that $dq_j(\rho)$ and $\omega(\rho)$ are linearly independent one forms and that $H_{x_0}(\rho)$

is transversal to Σ , for any $\rho \in \Sigma$.

This implies that Σ is a closed conic, non radial involutive submanifold of codimension $d+1$ in $T^*X \setminus 0$.

Hence, if $\rho \in \Sigma$, then $T_\rho(\Sigma)^\sigma \subset T_\rho(\Sigma)$. Here $T_\rho(\Sigma)^\sigma$ denotes the dual with respect to the bilinear form σ .

A consequence of (H_2) is that Σ is locally foliated of dimension $d+1$ by the flow out of the Hamiltonian fields of the q_j .

The leaf through $\rho \in \Sigma$, whose tangent space at ρ is $T_\rho(\Sigma)^\sigma$, will be denoted by F_ρ .

For any $\rho \in \Sigma$, the bilinear form σ induces an isomorphism

$$J_\rho : T_\rho(T^*X \setminus 0) / T_\rho(\Sigma) \rightarrow T_\rho^*(F_\rho).$$

Hence, for any $\rho \in \Sigma$, we can define the localization $p_{m,\rho}$ of the principal symbol p_m at ρ

$$(1.3) \quad p_{m,\rho}(v) = \lim_{t \rightarrow 0} t^{-m_1} p_m(\rho + tv) \quad v \in T_\rho^*(F_\rho).$$

Clearly, $p_{m,\rho}(v)$ is hyperbolic with respect to $\tilde{H}_{x_0}(\rho) =: J_\rho(H_{x_0}(\rho))$. Let us assume that:

(H_3) $p_{m,\rho}$ is strictly hyperbolic with respect to $\tilde{H}_{x_0}(\rho)$, for any $\rho \in \Sigma$

It is well known that, under the assumptions (H_1) , (H_2) , the Cauchy problem for P cannot be correctly posed in C^∞ for arbitrary lower order terms.

In our case, the results of Ivrii-Petkov [7] give the following necessary condition for the well posedness of the Cauchy problem: the terms p_{m-j} must vanish of order $m-2j$ on Σ .

On the other hand, if this condition holds, it is possible to define the localization P_ρ of $P(x, D_x)$ at a point $\rho \in \Sigma$ (see: [4]).

A recent result of Nishitani [10] (see also [2]) states that, in order to have the well posedness of the Cauchy problem for P , it is necessary that $P_\rho = p_{m,\rho}$ but, it is clear that this kind of condition cannot be sufficient (even in the case of constant coefficients (see, for example, [3]).

Here we prove that if $P(x, D)$ satisfies (H_1) , (H_2) , (H_3) and the Cauchy problem for P is well posed in X_0 then the following Levi condition holds:

(H_4) in a conic neighborhood Ω of a point $\rho \in \Sigma$, P can be written in the form

$$P(x, D_x) = \sum_{|\alpha| \leq m_1} A_\alpha(x, D_x) Q_0^{\alpha 0}(x, D_x) \dots Q_d^{\alpha d}(x, D_x)$$

for some $A_\alpha \in OPS^{m-m_1}(X)$ and $Q_j \in OPS^1(X)$ with principal symbol q_j .

More precisely, our result is the following:

Theorem 1.1. *Let $P(x, D_x)$ be a differential operator satisfying (H_1) , (H_2) , (H_3) . The Cauchy Problem for P is well posed in X_0 iff (H_4) holds.*

The study of propagation of singularities for the operator P satisfying $(H_1) \dots (H_4)$ has been done by Melrose and Uhlmann [9] in the case $m_1=2$ and has been generalized by Bernardi [1] (see also [8] and [11]).

2. Reduction to a normal form

Let us consider the operator (1.1) satisfying (H_1) , (H_2) .

In this section we perform a canonical change of variables preserving the hyperplane $x_0=0$ and transforming, microlocally near the points of Σ , the manifold Σ into

$$\tilde{\Sigma} = \{(x, \xi); \xi_0 = \xi_1 = \dots = \xi_d\}.$$

Let us fix a point $\rho_0 \in \Sigma \cap \Omega$.

Since $H_{x_0}(\rho_0)$ is transversal to Σ , there exists $j \in \{0, \dots, d\}$ such that

$$\{q_j, x_0\}(\rho_0) = \frac{\partial q_j}{\partial \xi_0}(\rho_0) \neq 0.$$

Without loss of generality, we can suppose that $\frac{\partial q_0}{\partial \xi_0}(\rho_0) \neq 0$.

Hence, in a neighborhood of ρ_0 , we can write

$$q_0(x, \xi) = (\xi_0 - \lambda(x, \xi')) r(x, \xi_0, \xi')$$

with $r(\rho_0) \neq 0$.

If we set $\bar{q}_j(x, \xi') = q_j(x, \lambda(x, \xi'), \xi')$, $j = 1 = \dots = d$, the manifold Σ is defined, in a neighborhood of ρ_0 , by the equations:

$$\xi_0 - \lambda(x, \xi') = 0, \bar{q}_1(x, \xi'), \dots, \bar{q}_d(x, \xi') = 0.$$

Let us consider the canonical map $\chi: T^*X \rightarrow T^*\mathbb{R}^{n+1}$, $\chi(x_0, x', \xi_0, \xi') = (y_0, y', \eta_0, \eta')$ with $y_0 = x_0$ and $\eta_0 = \xi_0 - \lambda(x, \xi')$.

In a neighborhood of $\chi(\rho_0) =: \bar{\rho} = (\bar{y}_0, \bar{y}', \bar{\eta}_0, \bar{\eta}')$, we have

$$\chi(\Sigma) =: \bar{\Sigma} = \{(y, \eta); \eta_0 = g_1(y, \eta') = \dots = g_d(y, \eta')\}$$

with $g_j(y, \eta') = \bar{q}_j(y_0, \chi^{-1}(y', \eta'))$, $j = 1, \dots, d$.

Since $\bar{\Sigma}$ is involutive, $\{\eta_0, g_j\}(y, \eta') = \frac{\partial g_j}{\partial y_0}(y, \eta') = 0$ at any point $(y_0, y, \eta') \in \bar{\Sigma}$ close to $\bar{\rho}$.

Hence, in a neighborhood of $\bar{\rho}$ there exist smooth functions $b_{i,j}$, $i, j = 1, \dots, d$ such that:

$$\frac{\partial g_j}{\partial y_0}(y_0, y', \eta') = \sum_{i=1}^d b_{i,j}(y_0, y', \eta') g_i(y_0, y', \eta').$$

Let $B(y_0, y', \eta')$ be the $d \times d$ matrix with elements $b_{i,j}$ and let $G(y_0, y', \eta')$ be the vector with elements g_j . Then G satisfies the following first order system:

$$(2.1) \quad \begin{aligned} \frac{dG}{dy_0}(y_0, y', \eta') &= B(y_0, y', \eta')G(y_0, y', \eta') \\ G|_{y_0=\bar{y}_0} &= G(\bar{y}_0, y', \eta'). \end{aligned}$$

If we denote by $C(y_0, y', \eta')$ the resolvent of the linear system (2.1), we have $G(y_0, y', \eta') = C(y_0, y', \eta')G(\bar{y}_0, y', \eta')$.

Hence, in a neighborhood of $\bar{\rho}$, $\bar{\Sigma}$ is defined by the following equations:

$$\eta_0 = \bar{g}_1(y', \eta') = \dots = \bar{g}_d(y', \eta') = 0$$

with $\bar{g}_j(y', \eta') = g_j(\bar{y}_0, y', \eta')$, $j = 1, \dots, d$.

Let us define now the canonical map $\psi(y_0, y', \eta_0, \eta') = (x_0, x', \xi_0, \xi')$ with $x_0 = y_0$ and $\xi_0 = \eta_0$ such that $\bar{g}_j(\psi^{-1}(x, \xi)) = \xi_j$, for $j = 1, \dots, d$.

Hence, microlocally near $\bar{\rho}_0 = \psi(\bar{\rho})$, the manifold $\tilde{\Sigma} = \psi(\bar{\Sigma})$ is given, in the new coordinates, by the following equations

$$\xi_0 = \xi_1 = \dots = \xi_d = 0.$$

Let us notice that since the g_j -s are positively homogeneous of degree one, the canonical change of variables can also be taken as positively homogeneous of degree one (see [6]).

Moreover, since the g_j -s are homogeneous of degree one, we can extend the positively homogeneous canonical change of coordinates $\chi: \Omega \rightarrow T^*R^{n+1}$, $\chi(x, \xi) = (y(x, \xi), \eta(x, \xi))$ to a homogeneous canonical change of coordinates $\tilde{\chi}: \tilde{W} \rightarrow T^*R^{n+1}$ setting $\tilde{\chi}(x, \xi) = (y(x, -\xi), -\eta(x, -\xi))$ for $(x, \xi) \in (-\Omega)$.

Notice that $\tilde{\chi}(-\rho) = -\tilde{\rho}$ and that $\tilde{\chi}$ maps $\Sigma \cap W$ into

$$\{(x, \xi) \in \tilde{W} =: \tilde{\Omega} \cup (-\tilde{\Omega}); \xi_0 = \xi_1 = \dots = \xi_d = 0\}$$

where $\tilde{\Omega}$ is a conic neighborhood of $\tilde{\rho}$.

3. Necessary conditions

In this section we show that, under assumptions $(H_1) - (H_3)$, the Levi conditions (H_4) are necessary for the well posedness of the Cauchy problem of P in X_0 .

By using the results of Section 2, this fact will be a consequence of the following:

Proposition 3.1. *Let us consider the pseudodifferential operators*

$$\tilde{P}(x, D_x) = \tilde{P}_m(x, D_x) + \sum_{j=1}^m \sum_{k=0}^{m-j-\mu_j} \sum_{|\alpha|=m-j-k-\mu_j} A_{\alpha, k}^{(\mu_j)}(x, D_x) D_x^\alpha D_{x_0}^k$$

with

$$\tilde{P}_m(x, D_x) = \sum_{k=0}^m \sum_{|\alpha|=m-k} A_{\alpha, k}^{(0)}(x, D_x) D_x^\alpha D_{x_0}^k$$

with $A_{\alpha, k}^{(\mu_j)}(x, D_x) \in OPS^{\mu_j}(X)$, $0 \leq \mu_j \leq m-j$, having the principal symbol $a_{\alpha, k}^{(\mu_j)}(x, \xi)$ homogeneous of degree μ_j .

Let us suppose that (H_1) , (H_2) , (H_3) holds with $\Sigma = \{(x, \xi); \xi_0 = \xi' = 0\}$ and $\xi' = (\xi_1, \dots, \xi_d)$.

If the Cauchy problem for P is well posed in X_0 , then $a_{\alpha, k}^{(\mu_j)}(x, \xi)$ must vanish at any point $\rho \in \tilde{\Sigma}$ if $\mu_j \neq 0$.

Proof. Let us fix $\rho \in \Sigma$. Without loss of generality, we can take $\rho = (0, e_n) \in \tilde{\Sigma}$. The proof is done by induction.

Let us suppose that $a_{\alpha, k}^{(\mu_j)}(\rho) = 0$, $1 \leq j < p < m$ for $|\alpha| + k = m - j - \mu_j$ with $\mu_j \neq 0$ and let us prove that if $a_{\alpha, k}^{(\mu_p)}(\rho) \neq 0$ for some α, k , $|\alpha| + k = m - p - \mu_p$ then we must have $\mu_p = 0$.

Let us set

$$(3.1) \quad t = \sup \left\{ \frac{\mu_j}{\mu_j + j}; a_{\alpha, k}^{(\mu_p)}(\rho) \neq 0 \text{ for some } |\alpha| + k = m - j - \mu_j, j = p, \dots, m-1 \right\}.$$

We have $t \geq \frac{\mu_p}{\mu_p + p}$ and $0 \leq t < 1$.

Notice that the results of [7] and [10] implies that t must be strictly less than $\frac{1}{2}$ in order to have the well-posedness of the Cauchy problem for \tilde{P} .

On the other hand, in our situation, the cases $t < \frac{1}{2}$ and $t \geq \frac{1}{2}$ can be treated in the same way and we prefer consider both the case and find directly the Levi condition of [7] and [10], in our particular setting.

Suppose that $\mu_p \neq 0$ and then $t > 0$ and let us show that this fact contradicts the assumptions on the well posedness of the Cauchy problem.

Let $j_1 < j_2 < \dots < j_r$ ($1 \leq p \leq j_1 < j_2 < \dots < j_r \leq m-1$, $1 \leq r \leq m-1-p$) such that $\frac{\mu_{j_i}}{\mu_{j_i} + j_i} = t$ for $i = 1, \dots, r$.

If s_n is a positive real number, let us take $s = (s'', s_n) = (s_0, \dots, s_{n-1}, s_n)$ with $s_j = ts_n$, for $j = 0, \dots, n-1$ and let us consider the change of variables $y = \rho^{-s}x$.

Denoting by P_ρ the operator $P_\rho(x, D_x) = \rho^{-ts_n m} \tilde{P}(\rho^{-s}x, \rho^s D_x)$ we have:

$$\begin{aligned}
P_\rho(x, D_x) &= \rho^{-ts_n m} \left\{ \sum_{k=0}^m \sum_{|\alpha|=m-k} A_{\alpha, k}^{(0)}(\rho^{-s} x, \rho^s D_x) \rho^{ts_n(|\alpha|+k)} D_x^\alpha D_{x_0}^k \right. \\
&\quad + \sum_{j=1}^m \sum_{k=0}^{m-j-\mu_j} \sum_{|\alpha|=m-j-k-\mu_j} A_{\alpha, k}^{(\mu_j)}(\rho^{-s} x, \rho^s D_x) \rho^{ts_n(|\alpha|+k)} D_x^\alpha D_{x_0}^k \Big\} \\
(3.2) \quad &= \rho^{-ts_n m} \left\{ \sum_{k=0}^m \sum_{|\alpha|=m-k} A_{\alpha, k}^{(0)}(\rho^{-s} x, \rho^{(t-1)s_n} \frac{D_{x''}}{D_{x_n}}, 1) \rho^{ts_n(|\alpha|+k)} D_x^\alpha D_{x_0}^k \right. \\
&\quad + \sum_{j=1}^m \sum_{k=0}^{m-j-\mu_j} \sum_{|\alpha|=m-j-k-\mu_j} A_{\alpha, k}^{(\mu_j)}(\rho^{-s} x, \rho^{(t-1)s_n} \frac{D_{x''}}{D_{x_n}}, 1) \\
&\quad \quad \quad \times \rho^{ts_n(|\alpha|+k)+s_n \mu_j} D_{x_n}^{\mu_j} D_x^\alpha D_{x_0}^k \Big\}.
\end{aligned}$$

Applying the Taylor formula, we get

$$A_{\alpha, k}^{(\mu_j)}(\rho^{-s} x, \rho^{(t-1)s_n} \frac{D_{x''}}{D_{x_n}}, 1) = a_{\alpha, k}^{(\mu_j)}(0, e_n) + O(\rho^{-ts_n}) + O(\rho^{(t-1)s_n}).$$

Hence

$$\begin{aligned}
P_\rho(x, D_x) &= \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha, k}^{(0)}(0, e_n) D_x^\alpha D_{x_0}^k \\
&\quad + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j-k-\mu_{j_i}} a_{\alpha, k}^{(\mu_{j_i})}(0, e_n) D_{x_n}^{\mu_{j_i}} D_x^\alpha D_{x_0}^k \\
(3.3) \quad &\quad + O(\rho^{-ts_n}) + O(\rho^{(t-1)s_n}) + \sum_{j \neq j_1, \dots, j_r, j \in \{p, \dots, m-1\}} O(\rho^{s_n(\mu_j - t(\mu_j + j))}) \\
&\quad + \sum_{j=1}^m O(\rho^{s_n(\mu_j - t(\mu_j + j) - t)}) + \sum_{j=1}^m O(\rho^{s_n(\mu_j - t(\mu_j + j) + (t-1))}).
\end{aligned}$$

Since all the powers of ρ in the remainder terms of (3.3) are negative, if we choose s_n sufficiently large we get

$$\begin{aligned}
P_\rho(x, D_x) &= \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha, k}^{(0)}(0, e_n) D_x^\alpha D_{x_0}^k \\
(3.4) \quad &\quad + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j-k-\mu_{j_i}} a_{\alpha, k}^{(\mu_{j_i})}(0, e_n) D_{x_n}^{\mu_{j_i}} D_x^\alpha D_{x_0}^k \\
&\quad + O(\rho^{-N})
\end{aligned}$$

for any $N \in \mathbb{N}$.

Let us consider the symplectic dilatation $S_\rho(x_0, \dots, x_n) = (\rho^{-2} x_0, x_1, \dots, x_{n-1}, \rho^{-2/t} x_n)$.

Then

$$\begin{aligned}
 P'_\rho(x, D_x) &=: \rho^{-2m} P(\rho^{-2} x_0, x_1, \dots, x_{n-1}, \rho^{-2/t} x_n, \rho^2 D_{x_0}, D_{x_1}, \dots, D_{x_{n-1}}, \rho^{2/t} D_{x_n}) \\
 &= \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) \left(\frac{D_{x'}}{\rho^2} \right)^\alpha D_{x_0}^k \\
 (3.5) \quad &+ \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) D_{x_n}^{\mu_{j_i}} \left(\frac{D_{x'}}{\rho^2} \right)^\alpha D_{x_0}^k \\
 &+ O(\rho^{-N}).
 \end{aligned}$$

Set $E_\rho = e^{i\psi_\rho}$ with

$$\psi_\rho(x) = \rho^{1/t} x_n \xi_n + \rho^3 \langle x', \xi' \rangle + \rho \gamma x_0 + i\rho |x'''|^2/2 + i\rho^{-1+1/t} x_n^2/2.$$

(Here $(x_0, \dots, x_n) =: (x_0, x', x'', x_n)$).

We have

$$\begin{aligned}
 E_\rho^{-1} D_{x_0}^k E_\rho &= \rho^k \gamma^k + k \rho^{k-1} \gamma^{k-1} D_{x_0} + O(\rho^{k-2}) \\
 E_\rho^{-1} D_{x_n}^{\mu_{j_i}} E_\rho &= \rho^{\mu_{j_i}/t} \xi_n^{\mu_{j_i}} + i \mu_{j_i} \rho^{-1+\mu_{j_i}/t} \xi_n^{\mu_{j_i}-1} x_n + O(\rho^{(\mu_{j_i}-1)/t}) + O(\rho^{-1+(-1+\mu_{j_i})/t}) \\
 E_\rho^{-1} D_{x'}^{\alpha_j} E_\rho &= \rho^{3\alpha_j} \xi_j^{\alpha_j} + O(\rho^{3\alpha_j-3}).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (3.6) \quad E_\rho^{-1} P'_\rho E_\rho &= \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) \rho^{|\alpha|+k} \xi'^\alpha \gamma^k \\
 &+ \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) k \rho^{|\alpha|+k-1} \xi'^\alpha \gamma^{k-1} D_{x_0} \\
 &+ \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \rho^{|\alpha|+k+\mu_{j_i}/t} \xi_n^{\alpha} \xi_n^{\mu_{j_i}} \gamma^k \\
 &+ \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \rho^{|\alpha|+k-1+\mu_{j_i}/t} \\
 &\quad \times (k \xi'^\alpha \xi_n^{\mu_{j_i}} \gamma^{k-1} D_{x_0} + i \mu_{j_i} \xi'^\alpha \xi_n^{\mu_{j_i}-1} \gamma^k x_n) \\
 &+ \sum_{k=0}^m \sum_{|\alpha|=m-k} O(\rho^{|\alpha|-2+k}) \\
 &+ \sum_{i=1}^r \sum_{|\alpha|+k=m-j_i-\mu_{j_i}} (O(\rho^{|\alpha|+k-2+\mu_{j_i}/t}) + O(\rho^{|\alpha|+k-1+(\mu_{j_i}-1)/t})) \\
 &+ O(\rho^{-N}).
 \end{aligned}$$

Notice that, if $|\alpha|=m-k-\mu_{j_i}-j_i$, then $|\alpha|+k+\mu_{j_i}/t=m-(\mu_{j_i}+j_i)+\mu_{j_i}/t=m$.

Hence

$$\begin{aligned}
(3.7) \quad E_\rho^{-1} P'_\rho E_\rho &= \rho^m \left(\sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) \xi'^\alpha \gamma^k \right. \\
&\quad \left. + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \xi'^\alpha \xi_n^{\mu_{j_i}} \gamma^k \right) \\
&\quad + L_\rho
\end{aligned}$$

with

$$(3.8) \quad L_\rho = \rho^{m-1} L_0 + \rho^{m-1/t} \tilde{L}_1 + \rho^{m-2} \tilde{L}_2 + \rho^{m-1-1/t} \tilde{L}_3 + \dots$$

and

$$\begin{aligned}
(3.9) \quad L_0 &= \left(\sum_{k=1}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) k \xi'^\alpha \gamma^{k-1} \right. \\
&\quad \left. + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) k \xi'^\alpha \xi_n^{\mu_{j_i}} \gamma^{k-1} \right) D_{x_0} \\
&\quad + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0, e_n) i \mu_{j_i} \xi'^\alpha \xi_n^{\mu_{j_i}-1} \gamma^k x_n.
\end{aligned}$$

Set now

$$\begin{aligned}
(3.10) \quad \tilde{p}_m(\gamma, \xi', \xi_n) &= \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) \xi'^\alpha \gamma^k \\
&\quad + \sum_{i=1}^r \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} a_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \xi'^\alpha \gamma^k \xi_n^{\mu_{j_i}}.
\end{aligned}$$

Let us suppose that there exists j_i , $p \leq j_i \leq m-1$, and $a_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \neq 0$ with $\mu_{j_i} > 0$ for some α , k , $|\alpha|+k=m-j_i-\mu_{j_i}$, $i=1, \dots, r$. We show that, in this case, the equation

$$\tilde{p}_m(\gamma, \xi', \xi_n) = 0$$

has at least a root γ with $\operatorname{Im} \gamma < 0$ for a suitable choice of ξ' , ξ_n and moreover that it is possible to find an asymptotic solution u_ρ of $L_\rho u_\rho = 0$.

This will imply that there exists a solution of $P'_\rho v_\rho = 0$ of the form $v_\rho = e^{i\psi_\rho} u_\rho$ such that $\operatorname{Im}(\psi_\rho) > \rho^\varepsilon |x|$, if $x_0 < 0$, for some $\varepsilon > 0$, that is in contradiction with the assumption of the well posedness of the Cauchy problem (see [5]).

Notice that, since $j_i + \mu_{j_i} \geq 2$, the coefficient of γ^{m-1} in (3.10), given by

$\sum_{|\alpha|=1} a_{\alpha,m-1}^{(0)}(0, e_n) \xi'^\alpha$, is real. Hence, it is sufficient to prove the existence of a root γ of $\tilde{p}_m(\gamma, \xi', \xi_n) = 0$ with $\operatorname{Im} \gamma \neq 0$ for some ξ' , ξ_n .

Set

$$\begin{aligned}
A_{m-k}^{(0)}(\xi') &= \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) \xi'^\alpha \\
A_{m-j_i-\mu_{j_i}-k}^{(j_i+\mu_{j_i})}(\xi') &= \sum_{|\alpha|=m-j_i-\mu_{j_i}-k} a_{\alpha,k}^{(\mu_{j_i})}(0, e_n) \xi'^\alpha \\
q_m(\gamma, \xi') &= \sum_{k=0}^m A_{m-k}^{(0)}(\xi') \gamma^k \\
q_{m-j_i-\mu_{j_i}}(\gamma, \xi') &= \sum_{k=0}^{m-j_i-\mu_{j_i}} A_{m-j_i-\mu_{j_i}-k}^{(j_i+\mu_{j_i})}(\xi') \gamma^k.
\end{aligned}$$

Then

$$(3.11) \quad \tilde{p}_m(\gamma, \xi', \xi_n) = q_m(\gamma, \xi') + \sum_{i=1}^r q_{m-j_i-\mu_{j_i}}(\gamma, \xi') \xi_n^{\mu_{j_i}}.$$

Notice that

$$(3.12) \quad \tilde{p}_m(\gamma, \xi', \xi_n) = |\xi'|^m \tilde{p}_m\left(\frac{\gamma}{|\xi'|}, \frac{\xi'}{|\xi'|}, \frac{\xi_n}{|\xi'|}\right).$$

We have the following.

Lemma 3.2. *Let $q_m(\gamma) = \sum_{k=0}^m A_{m-k}^{(0)} \gamma^k$ be a real polynomial of degree m in the variable γ and $q_{m-s}(\gamma) = \sum_{k=0}^{m-s} A_{m-k}^{(s)} \gamma^k$, $s=2, \dots, m$ polynomials of degree $m-s$ in the variable γ . Let $\delta_s \in \mathbb{N}$ with $1 \leq \delta_s \leq s-1$, $s=2, \dots, m$. If $q_m(\gamma)$ has m real roots then $\tilde{p}_m(\gamma, \lambda) = q_m(\gamma) + \sum_{s=2}^m q_{m-s}(\gamma) \lambda^{\delta_s}$ has still m real roots for any $\lambda \in \mathbb{R}$ iff $q_{m-s}(\gamma)$ is identically zero for $s=2, \dots, m$.*

Proof. Let us prove the statement by induction on the degree m of \tilde{p}_m . Notice that, if $\tilde{p}_m(\gamma, \lambda)$ has m real roots for any $\lambda \in \mathbb{R}$ then $q_{m-s}(\gamma)$ must be a real polynomial in the variable γ .

The statement is clearly obvious for $m=2$. Suppose that the statement is true for a polynomial $\tilde{p}_m(\gamma, \lambda)$ of degree m and let us prove it for $\tilde{p}_{m+1}(\gamma, \lambda)$.

Suppose that $\tilde{p}_{m+1}(\gamma, \lambda) = q_{m+1}(\gamma) + \sum_{s=2}^{m+1} q_{m+1-s}(\gamma) \lambda^{\delta_s}$, with $1 \leq \delta_s \leq s-1$, has $m+1$ real roots in the variable γ .

As a consequence

$$\frac{d}{d\gamma} \tilde{p}_{m+1}(\gamma, \lambda) = \frac{d}{d\gamma} q_{m+1}(\gamma) + \sum_{s=2}^m \frac{d}{d\gamma} q_{m+1-s}(\gamma) \lambda^{\delta_s}$$

has m real roots. By induction, this implies that $\frac{d}{d\gamma} q_{m+1-s}(\gamma)$ is identically zero i.e. $A_{m+1-s}^{(s)} = 0$ for any $k = 1, \dots, m+1-s$, $s = 2, \dots, m$.

Hence

$$\tilde{p}_{m+1}(\gamma, \lambda) = q_{m+1}(\gamma) + \sum_{s=2}^{m+1} A_{m+1}^{(s)} \lambda^{\delta_s}$$

and it is easy to check that $\tilde{p}_{m+1}(\gamma, \lambda)$ has only real roots, for any $\lambda \in \mathbf{R}$ iff $A_{m+1}^{(s)} = 0$ for any $s = 2, \dots, m+1$. \square

End of the proof of Proposition 3.1. Applying Lemma 3.2 to the equation (3.11) with $\lambda = \xi_n$ we can conclude that $\tilde{p}_m(\gamma, v, \xi_n) = 0$ must have a root $\gamma(v, \xi_n)$ with $\operatorname{Im} \gamma \neq 0$ for some ξ_n and $v \in \mathbf{R}^d$ with $|v| = 1$.

This root is simple. Actually, by (3.12), $\gamma(\mu v, \mu^t \xi_n) = \mu \gamma(v, \xi_n)$ for any $\mu \in \mathbf{R}^+$ and (H₃) implies that $\gamma(\mu v, \mu^t \xi_n)$ is simple for small μ .

Writing $t = p/q$, with $p, q \in \mathbf{N}$ from (3.7) we get

$$L_\rho = \rho^{m-1} L_0 + \rho^{m-q/p} \tilde{L}_1 + \rho^{m-2} \tilde{L}_2 + \rho^{m-(q+p)/p} \tilde{L}_3 + \dots$$

Eventually by adding some $L_j = 0$ we can write

$$L_\rho = \sum_{j=0}^{+\infty} \rho^{m-(p+j)/p} L_j.$$

Following the arguments of [5], we can find an asymptotic solution u_ρ of $L_\rho u^\rho = 0$ in the form

$$u_\rho = \sum_{k=0}^{+\infty} \rho^{-k/p} u_k$$

and this fact contradicts the assumption on the well posedness of the Cauchy problem for P .

Hence $a_{\alpha, k}^{(\mu, p)}(0, e_n) = 0$ if $\mu_{j_p} > 0$ for any α, k .

Repeating these arguments a finite number of times we can conclude that $a_{\alpha, k}^{(\mu, p)}(0, e_n) = 0$ if $\mu_{j_i} > 0$ for any α, k and end the proof of the proposition. \square

Proof of Theorem 1.1 (Necessary conditions). Let $P(x, D) = P_m(x, D_x) + P_{m-1}(x, D_x) + \dots$ be a differential operator satisfying (H₁), (H₂), (H₃) and Ω be a neighbourhood of a point $\bar{p} \in \Sigma$.

Without loss of generality we can suppose $m_1=m$

Since p_m vanishes of order m on $\Sigma \cap \Omega$, the principal symbol $p_m(x, \xi)$ can be written at a point $(x, \xi) \in \Omega$ as

$$(3.13) \quad p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha^{(0)}(x, \xi) q(x, \xi)^\alpha$$

for some symbol $a_\alpha^{(0)}(x, \xi)$ positively homogeneous of degree zero.

By taking, for $(x, \xi) \in (-\Omega)$, $a_\alpha^{(0)}(x, \xi) =: a_\alpha^{(0)}(x, -\xi)$, (3.13) holds for $(x, \xi) \in W =: \Omega \cup (-\Omega)$.

Let $A_\alpha^{(0)}(x, D_x)$ and $Q(x, D_x)$ be pseudodifferential operators with principal symbols $a_\alpha^{(0)}(x, \xi)$ and $q(x, \xi)$ respectively.

Hence, in W , we can write

$$(3.14) \quad P(x, D_x) = \sum_{|\alpha|=m} A_\alpha^{(0)}(x, D_x) Q(x, D_x)^\alpha + \tilde{P}_{m-1}(x, D_x) + \dots$$

Let $\tilde{\chi}(x, \xi) = (y, \eta)$ be the canonical change of variables of Section 2 and let F be the elliptic Fourier integral operator associated to $\tilde{\chi}$.

$$\begin{aligned} \tilde{P}(y, D_y) &=: FP(x, D_x)F^{-1} = \sum_{|\alpha|=m} \tilde{A}_\alpha^{(0)}(y, D_y) (D_{y_0} + R_0)^{\alpha_0} (D_{y_d} + R_d)^{\alpha_d} \\ &\quad + \tilde{G}_{m-1}(y, D_y) + \tilde{G}_{m-2}(y, D_y) + \dots \end{aligned}$$

for some pseudodifferential operator R_j of order 0 and \tilde{G}_{m-j} of order $m-j$

Applying Proposition 3.1 to \tilde{P} and then coming back to P we can conclude than, if P satisfies (H_1) , (H_2) , (H_3) and the Cauchy problem for P is well posed in X_0 , then (H_4) holds. \square

4. Sufficient conditions: the energy estimates

In this section we prove the well posedness of the Cauchy problem for P in X_0 , under assumptions $(H_1) - (H_4)$, by using the method of energy estimates (see [5]).

Taking into account that the principal symbol of P is strictly hyperbolic outside Σ we can assume, without loss of generality, that $m_1=m$.

Since all the canonical transformations we made in Section 2 preserve the hyperplane $x_0=0$, then it will be enough to establish some suitable energy estimate for the operator

$$(4.1) \quad \tilde{P}(x, D_x) = \tilde{P}_m(x, D_x) + \sum_{j=1}^m \sum_{k=0}^{m-j} \sum_{|\alpha|=m-j-k} A_{\alpha,k}(x, D_x) D_x^\alpha D_{x_0}^k$$

where

$$(4.2) \quad \tilde{P}_m(x, D_x) = \sum_{k=0}^m \sum_{|\alpha|=m-k} A_{\alpha,k}(x, D_{x'}, D_{x''}) D_{x'}^\alpha D_{x''}^k$$

with $A_{\alpha,k}(x, D_x) \in OPS^0(X)$ and $A_{0,m} = I$.

Here we have set $x' = (x_1, \dots, x_d)$ and $x'' = (x_{d+1}, \dots, x_n)$.

Moreover we may assume that the symbol of P is supported in a conic neighborhood of $\tilde{\rho} = (\tilde{x}, \tilde{\xi}_0 = 0, \tilde{\xi}' = 0, \tilde{\xi}'') \in \tilde{\Sigma}$, $\tilde{\xi}'' \neq 0$, of the form

$$\Gamma_\varepsilon = \{(x, \xi); |x - \tilde{x}| < \varepsilon, |\xi'| < \varepsilon |\xi''|, \left| \frac{\xi''}{|\xi''|} - \frac{\tilde{\xi}''}{|\tilde{\xi}''|} \right| < \varepsilon\}.$$

Let us start by introducing a suitable class of symbols of pseudodifferential operators.

DEFINITION 4.1. Let X be an open set of $\mathbf{R}_x^n = \mathbf{R}_{x'}^d \times \mathbf{R}_{x''}^{n-d}$. We say that $a \in S^{m,p}(X \times \mathbf{R}^n)$ iff $a \in C^\infty(X \times \mathbf{R}^n)$ and for any compact $K \subset \subset X$, for any $\alpha \in \mathbf{Z}^n$, $\beta' \in \mathbf{Z}^d$, $\beta'' \in \mathbf{Z}^{n-d}$ there exists a positive constant $C_{\alpha, \beta', \beta'', K}$ such that:

$$(4.3) \quad |D_x^\alpha D_{\xi'}^{\beta'} D_{\xi''}^{\beta''} a(x, \xi', \xi'')| \leq C_{\alpha, \beta', \beta'', K} \langle \xi' \rangle^{m-|\beta'|} \langle \xi', \xi'' \rangle^{p-|\beta''|}$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$ and $\langle \xi', \xi'' \rangle = (1 + |\xi'|^2 + |\xi''|^2)^{1/2}$.

We denote by $OPS^{m,p}(X)$ the class of pseudodifferential operators associated with $S^{m,p}(X \times \mathbf{R}^n)$ and we set:

$$H^{m,p}(\mathbf{R}^n) = \{v \in L^2(\mathbf{R}^n); \|v\|_{m,p}^2 = \int (1 + |\xi'|^2)^m (1 + |\xi''|^2)^p |\hat{v}(\xi', \xi'')|^2 d\xi' d\xi'' < +\infty\}.$$

In the following we denote simply by $\|\cdot\|$ the norm in $L^2(\mathbf{R}^n)$.

REMARK 4.2. It is easy to check that:

1. If $a \in S^{m,p}(X \times \mathbf{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi'| \leq c|\xi''|\}$ then for any compact $K \subset \subset X$, for any $\alpha \in \mathbf{Z}^n$, $\beta' \in \mathbf{Z}^d$, $\beta'' \in \mathbf{Z}^{n-d}$ there exists a positive constant $C_{\alpha, \beta', \beta'', K}$ such that:

$$|D_x^\alpha D_{\xi'}^{\beta'} D_{\xi''}^{\beta''} a(x, \xi', \xi'')| \leq C_{\alpha, \beta', \beta'', K} \langle \xi' \rangle^{m-|\beta'|} \langle \xi', \xi'' \rangle^{p-|\beta''|}$$

where $\langle \xi'' \rangle = (1 + |\xi''|^2)^{1/2}$.

2. If $a \in S^{m,p}(X \times \mathbf{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi'| \leq c|\xi'|\}$ then $a \in S^{m+p}(X \times \mathbf{R}^n)$.
3. If $a \in S^{m,p}(X \times \mathbf{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi'| \leq c\}$ then $a \in S^{0,p}(X \times \mathbf{R}^n)$.
4. If X' is an open set of \mathbf{R}^d and $a \in S^m(X' \times \mathbf{R}^d)$ then $a \in S^{m,0}(X \times \mathbf{R}^n)$ with $X = X' \times \mathbf{R}^{n-d}$
5. For any $j \geq 0$, $S^{m,p}(X \times \mathbf{R}^n) \subset S^{m-j,p+j}(X \times \mathbf{R}^n)$.
6. If $A(x, D_x) \in OPS^{m,p}(X)$ and $\sigma(A)(x, \xi', \xi'') = 0$ for $|x| > R$, for some $R > 0$ then $A(x, D_x)$ is continuous from $H^{m,p}(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ i.e.

$$\|A(x, D_x)u\| \leq C\|u\|_{m,p}, \quad \forall u \in L^2(\mathbf{R}^n).$$

We can prove the following energy estimates.

Proposition 4.3. *For any $K \subset \subset X$ there exist a constant $C = C_k > 0$ and a real number $\tau_K > 0$ such that for any $u \in C_0^\infty(K)$ and any $\tau > \tau_k$ the following inequality holds:*

$$(4.4) \quad \begin{aligned} C \int_{x_0 < 0} \|\tilde{P}(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 &\geq \sum_{j=1}^m \tau^{2j-1} \sum_{k=0}^{m-j} \|D_{x_0}^k u(0, \cdot)\|_{m-j-k,0}^2 \\ &+ \sum_{j=1}^m \tau^{2j} \sum_{k=0}^{m-j} \int_{x_0 < 0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-j-k,0}^2 e^{-2\tau x_0} dx_0. \end{aligned}$$

Proof. The proof is done along the same lines of the proof of the well posedness of the Cauchy problem in the strictly hyperbolic case.

Let

$$\tilde{p}_m(x, \xi) = \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}(x, \xi', \xi'') \xi'^\alpha \xi''^k$$

be the principal symbol of \tilde{P} .

If $\tilde{\rho} = (\tilde{x}, \tilde{\xi}_0 = 0, \tilde{\xi}' = 0, \tilde{\xi}'') \in \tilde{\Sigma}$, $\tilde{\xi}'' \neq 0$, the assumption (H₃) guarantees that the localization of \tilde{p}_m at $\tilde{\rho}$:

$$\tilde{p}_{m,\tilde{\rho}}(y_0, y', \eta_0, \eta') = \sum_{k=0}^m \sum_{|\alpha|=m-k} a_{\alpha,k}(y_0, y', \tilde{x}'', 0, \frac{\tilde{\xi}''}{|\tilde{\xi}''|}) \eta'^\alpha \eta_0^k$$

has m distinct real roots in η_0 , for any $y_0, y', \eta' \neq 0$.

Hence, for $(x, \xi', \xi'') \in \Gamma_\varepsilon$ with ε sufficiently small and $\xi' \neq 0$, \tilde{p}_m has m distinct real roots $\lambda_j(x, \xi', \xi'') = |\xi'| \lambda_j(x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi''|})$, $j = 1, \dots, m$,

$$\lambda_1(x, \xi', \xi'') \leq \lambda_2(x, \xi', \xi'') \leq \dots \leq \lambda_m(x, \xi', \xi'').$$

Moreover, the strict hyperbolicity of p_m outside Σ , implies that, for $(x, \xi', \xi'') \in \Gamma_\varepsilon$ and ε small, there exist some positive constants c, C such that:

$$\begin{aligned} c|\xi'| &\leq |\lambda_i(x, \xi', \xi'') - \lambda_j(x, \xi', \xi'')| \leq C|\xi'|, \quad \text{for } i \neq j \\ |\lambda_i(x, \xi', \xi'')| &\leq C|\xi'|, \quad \text{for any } i. \end{aligned}$$

Let us take now a cutoff function $\chi \in C_0^\infty(\mathbf{R}^k)$ with $\chi(\xi') = 1$ if $|\xi'| \leq 1$ and $\chi(\xi') = 0$ if $|\xi'| \geq 2$ and set $\tilde{\lambda}_j(x, \xi', \xi'') = (1 - \chi(\xi')) \lambda_j(x, \xi', \xi'')$.

It is easy to check that $\tilde{\lambda}_j \in S^{1,0}(X \times \mathbf{R}^n)$.

If $\Lambda_j \in OPS^{1,0}$ is a pseudodifferential operator with principal symbol $\tilde{\lambda}_j$ we have

$$\tilde{P}_m(x, D_x) = (D_{x_0} - \Lambda_j(x, D_{x'}, D_{x''})) Q_j(x, D_{x_0}, D_{x'}, D_{x''}) + S_j(x, D_{x_0}, D_{x'}, D_{x''})$$

where

$$Q_j(x, D_{x_0}, D_{x'}, D_{x''}) = \sum_{k=0}^{m-1} C_{j,k}(x, D_{x'}, D_{x''}) D_{x_0}^{m-1-k},$$

with $C_{j,k} \in OPS^{k,0}(X \times \mathbb{R}^n)$, $\text{supp}(c_{j,k}) \subset \{(x, \xi); |\xi'| > 1\}$ and

$$S_j(x, D_{x_0}, D_{x'}, D_{x''}) = \sum_{k=0}^{m-1} S_{j,k}(x, D_{x'}, D_{x''}) D_{x_0}^{m-1-k},$$

with $S_{j,k} \in OPS^{k+1,0}(X \times \mathbb{R}^n)$ and $\text{supp}(s_{j,k}) \subset \{(x, \xi); |\xi'| \leq 2\}$.

Notice that, thanks to 3) of Remark 4.2, $S_{j,k} \in OPS^{0,0}(X \times \mathbb{R}^n)$.

Let us calculate $2i \text{Im} \langle \tilde{P}(x, D_x)u, Q_j(x, D_x)u \rangle$, for $u \in C_0^\infty(K)$, $K \subset \subset X$.

We have:

$$\begin{aligned} (4.5) \quad & 2i \text{Im} \langle \tilde{P}(x, D_x)u, Q_j(x, D_x)u \rangle \\ & = 2i \text{Im} \langle \tilde{P}_m(x, D_x)u, Q_j(x, D_x)u \rangle + 2i \text{Im} \langle (\tilde{P} - \tilde{P}_m)(x, D_x)u, Q_j(x, D_x)u \rangle \\ & = 2i \text{Im} \langle (D_{x_0} - \Lambda_j(x, D_{x'}, D_{x''})) Q_j(x, D_x)u, Q_j(x, D_x)u \rangle \\ & \quad + 2i \text{Im} \langle S_j(x, D_x)u, Q_j(x, D_x)u \rangle \\ & \quad + 2i \text{Im} \langle (\tilde{P} - \tilde{P}_m)(x, D_x)u, Q_j(x, D_x)u \rangle \end{aligned}$$

Hence, multiplying the above identity by $i\tau e^{-2\tau x_0}$ and integrating it for $x_0 < 0$, we have, for τ sufficiently large:

$$\begin{aligned} (4.6) \quad & C \int_{x_0 < 0} \|\tilde{P}(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \\ & \geq \tau \sum_{j=1}^m \|Q_j(x, D_x)u(0, \cdot)\|^2 \\ & \quad + \tau^2 \sum_{j=1}^m \int_{x_0 < 0} \|Q_j(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \\ & \quad - \tau \int_{x_0 < 0} \|(\tilde{P} - \tilde{P}_m)(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \\ & \quad - \tau \sum_{j=1}^m \int_{x_0 < 0} \|S_j(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0. \end{aligned}$$

Now, we can estimate the last two terms in (4.6) by

$$\begin{aligned}
& \tau \int_{x_0 < 0} \|(\tilde{P} - \tilde{P}_m)(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \\
& + \tau \sum_{j=1}^m \int_{x_0 < 0} \|S_j(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \\
& \leq \tau C \sum_{j=1}^m \sum_{k=0}^{m-j} \int_{x_0 < 0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-j-k, 0}^2 e^{-2\tau x_0} dx_0.
\end{aligned}$$

On the other hand, using the Lagrange interpolation formula, we have, for $k=0, \dots, m-1$

$$\xi_0^k = \sum_{j=1}^m \frac{q_j(x, \xi) \tilde{\lambda}_j(x, \xi', \xi'')^k}{\prod_{i \neq j} (\tilde{\lambda}_j(x, \xi', \xi'') - \tilde{\lambda}_i(x, \xi', \xi''))} \quad \text{if } |\xi'| \geq 2.$$

Take now a cutoff function $\chi \in C_0^\infty(\mathbf{R}^k)$ with $\chi(\xi') = 1$ if $|\xi'| \leq 5/2$ and $\chi(\xi') = 0$ if $|\xi'| \geq 3$.

Hence

$$(1 - \chi'(\xi')) \langle \xi' \rangle^{m-1-k} \xi_0^k = \sum_{j=1}^m q_j(x, \xi) \frac{(1 - \chi'(\xi')) \langle \xi' \rangle^{m-1-k} \tilde{\lambda}_j(x, \xi', \xi'')^k}{\prod_{i \neq j} (\tilde{\lambda}_j(x, \xi', \xi'') - \tilde{\lambda}_i(x, \xi', \xi''))} \quad \text{if } |\xi'| \geq 2.$$

Since

$$m_{j,k} := \frac{(1 - \chi'(\xi')) \langle \xi' \rangle^{m-1-k} \tilde{\lambda}_j(x, \xi', \xi'')^k}{\prod_{i \neq j} (\tilde{\lambda}_j(x, \xi', \xi'') - \tilde{\lambda}_i(x, \xi', \xi''))}$$

belongs to S^0 , we have:

$$\begin{aligned}
(4.7) \quad & \| (1 - \chi'(D_{x'})) D_{x_0}^k u(x_0, \cdot) \|_{m-1-k, 0}^2 \leq \sum_{j=1}^m \| Q_j(x, D_x) u(x_0, \cdot) \|^2 \\
& + C \sum_{j=2}^m \sum_{k=0}^{m-j} \| D_{x_0}^k u(x_0, \cdot) \|_{m-j-k, 0}^2
\end{aligned}$$

On the other hand, if $k \leq m-2$

$$(4.8) \quad \| \chi'(D_{x'}) D_{x_0}^k u(x_0, \cdot) \|_{m-1-k, 0}^2 \leq C \| D_{x_0}^k u(x_0, \cdot) \|^2.$$

Hence (4.7) and (4.8) give, for $k \leq m-2$:

$$\begin{aligned}
(4.9) \quad & \| D_{x_0}^k u(x_0, \cdot) \|_{m-1-k, 0}^2 \leq \sum_{j=1}^m \| Q_j(x, D_x) u(x_0, \cdot) \|^2 \\
& + C \sum_{j=2}^m \sum_{k=0}^{m-j} \| D_{x_0}^k u(x_0, \cdot) \|_{m-j-k, 0}^2.
\end{aligned}$$

Moreover, since $D_{x_0}^{m-1} = Q_j - \sum_{k=1}^{m-1} C_{j,k} D_{x_0}^{m-j-k}$ we have

$$(4.10) \quad \|D_{x_0}^{m-1} u(x_0, \cdot)\|^2 \leq \|Q_j(x, D_x)u(x_0, \cdot)\|^2 + C \sum_{k=0}^{m-2} \|D_{x_0}^k u(x_0, \cdot)\|_{m-1-k,0}^2.$$

From (4.6), (4.9) and (4.10) we get, for large τ

$$(4.11) \quad \begin{aligned} C \int_{x_0 < 0} \|\tilde{P}(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 &\geq \tau \sum_{k=0}^{m-1} \|D_{x_0}^k u(0, \cdot)\|_{m-1-k,0}^2 \\ &+ \tau^2 \sum_{k=0}^{m-1} \int_{x_0 < 0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-1-k,0}^2 e^{-2\tau x_0} dx_0 \\ &- \tau^2 \sum_{j=2}^m \sum_{k=0}^{m-j} \int_{x_0 < 0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-j-k} e^{-2\tau x_0} dx_0 \end{aligned}$$

and using classical estimates for the terms

$$(4.12) \quad \int_{x_0 < 0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-1-k,0}^2 e^{-2\tau x_0} dx_0, \quad k=0, \dots, m-1,$$

we get (4.4) and we end the proof of the proposition.

Proof of Theorem 1.1 (Sufficient conditions). The proof of the theorem follows easily from Proposition 4.3.

Actually, we remark that $P(x, D_x)$ is a hyperbolic differential operator with simple characteristics outside Σ .

Hence, by using classical estimates for strictly hyperbolic operator, Proposition 4.3 and a microlocal partition of the unity, the proof of Theorem 1.1 can be completed by following the arguments of [5]. \square

References

- [1] E. Bernardi: *Propagation of singularities for hyperbolic operators with multiple involutive characteristics*, Osaka J. Math., **25** (1988), 19–31.
- [2] E. Bernardi, A. Bove and T. Nishitani: *Well posedness of the Cauchy problem for a class of hyperbolic operators with a stratified multiple variety: necessary conditions*, (preprint).
- [3] L. Garding: *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. **85** (1951), 1–62.
- [4] B. Helffer: *Invariants associés à une classe d'opérateurs pseudodifférentiels et applications à l'hypoellipticité*, Ann. Inst. Fourier **26**, 2 (1976), 55–70.
- [5] L. Hörmander: *The Cauchy problem for differential equations with double characteristics*, J. d'Analyse Math. **32** (1977), 118–196.
- [6] L. Hörmander: *The analysis of linear partial differential operators I-IV*, Springer-Verlag (1985)

Berlin.

- [7] V. Ja. Ivrii, V.M. Petkov: *Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed*, Uspehi Mat. Nauk. **29** (1974), 3–70.
- [8] R. Lascar: *Propagation des singularités des solutions d'équations pseudo-differentielles a caractéristiques de multiplicité variables*, Lecture Notes in Math. 856, Springer-Verlag, 1981.
- [9] R.B. Melrose, G.A. Uhlmann: *Microlocal Structure of Involutive Conical Refraction*, Duke Math. Journal **46** (1979), 571–582.
- [10] T. Nishitani: *Hyperbolicity of localization*, Ark. Mat. **31** (1993), 377–393.
- [11] M. Petroni, V. Sordoni: *Propagation of singularities for hyperbolic operators with multiple involutive characteristics*, Osaka J. Math. **28** (1991), 911–933.
- [12] J. Sjöstrand: *Propagation of singularities for operators with multiple involutive characteristics*, Ann. Inst. Fourier **26** (1) (1976), 141–155.

Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato,
40127 Bologna, Italy
email: sordoni@dm.unibo.it

