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ON THE WELL POSEDNESS OF THE CAUCHY PROBLEM FOR A CLASS OF HYPERBOLIC OPERATORS WITH MULTIPLE INVOLUTIVE CHARACTERISTICS

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1. Introduction

Let $X \subset \boldsymbol{R}^{n+1} = \boldsymbol{R}_{x_0} \times \boldsymbol{R}^n_{x'}$, $x' = (x_1, x_2, \cdots, x_n)$ be an open set such that $0 \in X$ and let us consider a differential operator of order m with C^∞ coefficients:

$$P(x, D_x) = P_m(x, D_x) + P_{m-1}(x, D_x) + \cdots \tag{1.1}$$

where we denote by $P_{m-j}(x, D_x)$ the homogeneous part of order $m-j$ of P.

Let us suppose that:

(H_1) the hyperplane $x_0 = 0$ is non-characteristics for P and the principal symbol $p_m(x, \xi)$ is hyperbolic with respect to ξ_0.

In this paper we shall study the well posedness of the Cauchy problem in C^∞ for the operator P in some cases where $p_m(x, \xi)$ is not strictly hyperbolic but the set of multiple characteristics has a very special form, as we will specify further. (For a definition of correctly posed Cauchy problem in $X_0 = \{x \in X; x_0 < 0\}$ we refer to [5]).

We shall suppose that $p_m(x, \xi)$ vanishes exactly of order $m_1 \leq m$ on a smooth manifold Σ and that p_m is strictly hyperbolic outside Σ.

On Σ we make the following assumptions:

(H_2) for any point $\rho \in \Sigma$, there exists a conic neighborhood Ω of ρ and $d+1$ ($d < n$) smooth functions q_j, $j = 0, \cdots, d$, defined on $W =: \Omega \cup (-\Omega)$ and homogeous of degree one such that $\Sigma \cap W$ is given by

$$\{\rho \in W; q_0(\rho) = .. = q_d(\rho) = 0\} \tag{1.2}$$

with $\{q_i, q_j\}(\rho) = 0$ for any $\rho \in \Sigma \cap W$.

(Here we have set $-\Omega =: \{(x, \xi) \in T^*X \backslash 0; (x, -\xi) \in \Omega\}$).

Moreover, denoting by ω and $\sigma = d\omega$ the canonical 1 and 2 forms in T^*X we suppose that $dq_j(\rho)$ and $\omega(\rho)$ are linearly independent one forms and that $H_{x_0}(\rho)$

is transversal to Σ, for any $\rho \in \Sigma$.

This implies that Σ is a closed conic, non radial involutive submanifold of codimension $d+1$ in $T^*X\backslash 0$.

Hence, if $\rho \in \Sigma$, then $T_\rho(\Sigma)^\sigma \subset T_\rho(\Sigma)$. Here $T_\rho(\Sigma)^\sigma$ denotes the dual with respect to the bilinear form σ.

A consequence of (H_2) is that Σ is locally foliated of dimension $d+1$ by the flow out of the Hamiltonian fields of the q_j.

The leaf through $\rho \in \Sigma$, whose tangent space at ρ is $T_\rho(\Sigma)^\sigma$, will be denoted by F_ρ.

For any $\rho \in \Sigma$, the bilinear form σ induces an isomorphism

$$J_\rho : T_\rho(T^*X\backslash 0)\,/\,T_\rho(\Sigma) \rightarrow T_\rho^*(F_\rho).$$

Hence, for any $\rho \in \Sigma$, we can define the localization $p_{m,\rho}$ of the principal symbol p_m at ρ

$$p_{m,\rho}(v) = \lim_{t \to 0} t^{-m_1} p_m(\rho + tv) \qquad v \in T_\rho^*(F_\rho). \tag{1.3}$$

Clearly, $p_{m,\rho}(v)$ is hyperbolic with respect to $\tilde{H}_{x_0}(\rho) =: J_\rho(H_{x_0}(\rho))$. Let us assume that:

(H_3) $p_{m,\rho}$ is strictly hyperbolic with respect to $\tilde{H}_{x_0}(\rho)$, for any $\rho \in \Sigma$

It is well known that, under the assumptions (H_1), (H_2), the Cauchy problem for P cannot be correctly posed in C^∞ for arbitrary lower order terms.

In our case, the results of Ivrii-Petkov [7] give the following necessary condition for the well posedness of the Cauchy problem: the terms p_{m-j} must vanish of order $m-2j$ on Σ.

On the other hand, if this condition holds, it is possible to define the localization P_ρ of $P(x, D_x)$ at a point $\rho \in \Sigma$ (see: [4]).

A recent result of Nishitani [10] (see also [2]) states that, in order to have the well posedness of the Cauchy problem for P, it is necessary that $P_\rho = p_{m,\rho}$ but, it is clear that this kind of condition cannot be sufficient (even in the case of constant coefficients (see, for example, [3]).

Here we prove that if $P(x, D)$ satisfies (H_1), (H_2), (H_3) and the Cauchy problem for P is well posed in X_0 then the following Levi condition holds:

(H_4) in a conic neighborhood Ω of a point $\rho \in \Sigma$, P can be written in the form

$$P(x, D_x) = \sum_{|\alpha| \leq m_1} A_\alpha(x, D_x) Q_0^{\alpha_0}(x, D_x) \,..\, Q_d^{\alpha_d}(x, D_x)$$

for some $A_\alpha \in OPS^{m-m_1}(X)$ and $Q_j \in OPS^1(X)$ with principal symbol q_j.

More precisely, our result is the following:

Theorem 1.1. *Let $P(x, D_x)$ be a differential operator satisfying* (H_1), (H_2), (H_3). *The Cauchy Problem for P is well posed in X_0 iff* (H_4) *holds.*

The study of propagation of singularities for the operator P satisfying $(\mathrm{H}_1)--(\mathrm{H}_4)$ has been done by Melrose and Uhlmann [9] in the case $m_1=2$ and has been generalized by Bernardi [1] (see also [8] and [11]).

2. Reduction to a normal form

Let us consider the operator (1.1) satisfying (H_1), (H_2).

In this section we perform a canonical change of variables preserving the hyperplane $x_0=0$ and transforming, microlocally near the points of Σ, the manifold Σ into

$$\tilde{\Sigma}=\{(x,\xi);\xi_0=\xi_1=\cdots=\xi_d\}.$$

Let us fix a point $\rho_0\in\Sigma\cap\Omega$.

Since $H_{x_0}(\rho_0)$ is transversal to Σ, there exists $j\in\{0,\cdots,d\}$ such that

$$\{q_j,x_0\}(\rho_0)=\frac{\partial q_j}{\partial\xi_0}(\rho_0)\neq 0.$$

Without loss of generality, we can suppose that $\dfrac{\partial q_0}{\partial\xi_0}(\rho_0)\neq 0$.

Hence, in a neighborhood of ρ_0, we can write

$$q_0(x,\xi)=(\xi_0-\lambda(x,\xi'))r(x,\xi_0,\xi')$$

with $r(\rho_0)\neq 0$.

If we set $\bar{q}_j(x,\xi')=q_j(x,\lambda(x,\xi'),\xi')$, $j=1=\cdots=d$, the manifold Σ is defined, in a neighborhood of ρ_0, by the equations:

$$\xi_0-\lambda(x,\xi')=0,\ \bar{q}_1(x,\xi'),\cdots,\bar{q}_d(x,\xi')=0.$$

Let us consider the canonical map $\chi:T^*X\to T^*\boldsymbol{R}^{n+1}$, $\chi(x_0,x',\xi_0,\xi')=(y_0,y',\eta_0,\eta')$ with $y_0=x_0$ and $\eta_0=\xi_0-\lambda(x,\xi')$.

In a neighborhood of $\chi(\rho_0)=:\bar{\rho}=(\bar{y}_0,\bar{y}',\bar{\eta}_0,\bar{\eta}')$, we have

$$\chi(\Sigma)=:\bar{\Sigma}=\{(y,\eta);\eta_0=g_1(y,\eta')=..=g_d(y,\eta')\}$$

with $g_j(y,\eta')=\bar{q}_j(y_0,\chi^{-1}(y',\eta'))$, $j=1,\cdots,d$.

Since $\bar{\Sigma}$ is involutive, $\{\eta_0,g_j\}(y,\eta')=\dfrac{\partial g_j}{\partial y_0}(y,\eta')=0$ at any point $(y_0,y,\eta')\in\bar{\Sigma}$ close to $\bar{\rho}$.

Hence, in a neighborhood of $\bar{\rho}$ there exist smooth functions $b_{i,j}$, $i,j=1,\cdots,d$ such that:

$$\frac{\partial g_j}{\partial y_0}(y_0, y', \eta') = \sum_{j=1}^{d} b_{i,j}(y_0, y', \eta') g_j(y_0, y', \eta').$$

Let $B(y_0, y', \eta')$ be the $d \times d$ matrix with elements $b_{i,j}$ and let $G(y_0, y', \eta')$ be the vector with elements g_j. Then G satisfies the following first order system:

$$\begin{aligned} &\frac{dG}{dy_0}(y_0, y', \eta') = B(y_0, y', \eta') G(y_0, y', \eta') \\ &G_{|y_0 = \bar{y}_0} = G(\bar{y}_0, y', \eta'). \end{aligned} \tag{2.1}$$

If we denote by $C(y_0, y', \eta')$ the resolvent of the linear system (2.1), we have $G(y_0, y', \eta') = C(y_0, y', \eta') G(\bar{y}_0, y', \eta')$

Hence, in a neighborhood of $\bar{\rho}$, $\bar{\Sigma}$ is defined by the following equations:

$$\eta_0 = \bar{g}_1(y', \eta') = \cdots = \bar{g}_d(y', \eta') = 0$$

with $\bar{g}_j(y', \eta') = g_j(\bar{y}_0, y', \eta')$, $j = 1, \cdots, d$.

Let us define now the canonical map $\psi(y_0, y', \eta_0, \eta') = (x_0, x', \xi_0, \xi')$ with $x_0 = y_0$ and $\xi_0 = \eta_0$ such that $\bar{g}_j(\psi^{-1}(x, \xi)) = \xi_j$, for $j = 1, \cdots, d$.

Hence, microlocally near $\tilde{\rho}_0 = \psi(\bar{\rho})$, the manifold $\tilde{\Sigma} = \psi(\bar{\Sigma})$ is given, in the new coordinates, by the following equations

$$\xi_0 = \xi_1 = .. = \xi_d = 0.$$

Let us notice that since the q_j-s are positively homogeneous of degree one, the canonical change of variables can also be taken as positively homogeneus of degree one (see [6]).

Moreover, since the q_j-s are homogeneus of degree one, we can extend the positively homogeneous canonical change of coordinates $\chi : \Omega \to T^*\boldsymbol{R}^{n+1}$, $\chi(x, \xi) = (y(x, \xi), \eta(x, \xi))$ to a homogeneous canonical change of coordinates $\tilde{\chi} : W \to T^*\boldsymbol{R}^{n+1}$ setting $\tilde{\chi}(x, \xi) = (y(x, -\xi), -\eta(x, -\xi))$ for $(x, \xi) \in (-\Omega)$.

Notice that $\tilde{\chi}(-\rho) = -\tilde{\rho}$ and that $\tilde{\chi}$ maps $\Sigma \cap W$ into

$$\{(x, \xi) \in \tilde{W} =: \tilde{\Omega} \cup (-\tilde{\Omega}) ; \xi_0 = \xi_1 = .. = \xi_d = 0.\}$$

where $\tilde{\Omega}$ is a conic neighborhood of $\tilde{\rho}$.

3. Necessary conditions

In this section we show that, under assumptions (H_1) -- (H_3), the Levi conditions (H_4) are necessary for the well posedness of the Cauchy problem of P in X_0.

By using the results of Section 2, this fact will be a consequence of the following:

Proposition 3.1. *Let us consider the pseudodifferential operators*

$$\tilde{P}(x,D_x)=\tilde{P}_m(x,D_x)+\sum_{j=1}^{m}\sum_{k=0}^{m-j-\mu_j}\sum_{|\alpha|=m-j-k-\mu_j}A_{\alpha,k}^{(\mu_j)}(x,D_x)D_{x'}^{\alpha}D_{x_0}^{k}$$

with

$$\tilde{P}_m(x,D_x)=\sum_{k=0}^{m}\sum_{|\alpha|=m-k}A_{\alpha,k}^{(0)}(x,D_x)D_{x'}^{\alpha}D_{x_0}^{k}$$

with $A_{\alpha,k}^{(\mu_j)}(x,D_x)\in OPS^{\mu_j}(X)$, $0\le\mu_j\le m-j$, *having the principal symbol* $a_{\alpha,k}^{(\mu_j)}(x,\xi)$ *homogeneous of degree* μ_j.

Let us suppose that (H$_1$), (H$_2$), (H$_3$) *holds with* $\Sigma=\{(x,\xi);\ \xi_0=\xi'=0.\}$ *and* $\xi'=(\xi_1,\cdots,\xi_d)$.

If the Cauchy problem for P is well posed in X_0, *then* $a_{\alpha,k}^{(\mu_j)}(x,\xi)$ *must vanish at any point* $\rho\in\tilde{\Sigma}$ *if* $\mu_j\neq 0$.

Proof. Let us fix $\rho\in\Sigma$. Without loss of generality, we can take $\rho=(0,e_n)\in\tilde{\Sigma}$. The proof is done by induction.

Let us suppose that $a_{\alpha,k}^{(\mu_j)}(\rho)=0$, $1\le j<p<m$ for $|\alpha|+k=m-j-\mu_j$ with $\mu_j\neq 0$ and let us prove that if $a_{\alpha,k}^{(\mu_p)}(\rho)\neq 0$ for some α,k, $|\alpha|+k=m-p-\mu_p$ then we must have $\mu_p=0$.

Let us set

$$t=:\sup\left\{\frac{\mu_j}{\mu_j+j};\ a_{\alpha,k}^{(\mu_p)}(\rho)\neq 0 \text{ for some } |\alpha|+k=m-j-\mu_j,\ j=p,\cdots,m-1\right\}. \tag{3.1}$$

We have $t\ge\dfrac{\mu_p}{\mu_p+p}$ and $0\le t<1$.

Notice that the results of [7] and [10] implies that t must be strictly less than $\dfrac{1}{2}$ in order to have the well-posedness of the Cauchy problem for $\tilde{P}$.

On the other hand, in our situation, the cases $t<\dfrac{1}{2}$ and $t\ge\dfrac{1}{2}$ can be treated in the same way and we prefer consider both the case and find directly the Levi condition of [7] and [10], in our particular setting.

Suppose that $\mu_p\neq 0$ and then $t>0$ and let us show that this fact contradicts the assumptions on the well posedness of the Cauchy problem.

Let $j_1<j_2<..<j_r$ $(1\le p\le j_1<j_2<..<j_r\le m-1,\ \ 1\le r\le m-1-p)$ such that $\dfrac{\mu_{j_i}}{\mu_{j_i}+j_i}=t$ for $i=1,\cdots,r$.

If s_n is a positive real number, let us take $s=(s''',s_n)=(s_0,\cdots,s_{n-1},s_n)$ with $s_j=ts_n$, for $j=0,\cdots,n-1$ and let us consider the change of variables $y=\rho^{-s}x$.

Denoting by P_ρ the operator $P_\rho(x,D_x)=\rho^{-ts_nm}\tilde{P}(\rho^{-s}x,\rho^s D_x)$ we have:

$$
\begin{aligned}
P_\rho(x, D_x) &= \rho^{-ts_n m}\Bigg\{\sum_{k=0}^{m} \sum_{|\alpha|=m-k} A_{\alpha,k}^{(0)}(\rho^{-s}x, \rho^s D_x)\rho^{ts_n(|\alpha|+k)} D_{x'}^\alpha D_{x_0}^k \\
&\quad + \sum_{j=1}^{m} \sum_{k=0}^{m-j-\mu_j} \sum_{|\alpha|=m-j-k-\mu_j} A_{\alpha,k}^{(\mu_j)}(\rho^{-s}x, \rho^s D_x)\rho^{ts_n(|\alpha|+k)} D_{x'}^\alpha D_{x_0}^k\Bigg\} \\
&= \rho^{-ts_n m}\Bigg\{\sum_{k=0}^{m} \sum_{|\alpha|=m-k} A_{\alpha,k}^{(0)}\Big(\rho^{-s}x, \rho^{(t-1)s_n}\frac{D_{x'''}}{D_{x_n}}, 1\Big)\rho^{ts_n(|\alpha|+k)} D_{x'}^\alpha D_{x_0}^k \\
&\quad + \sum_{j=1}^{m} \sum_{k=0}^{m-j-\mu_j} \sum_{|\alpha|=m-j-k-\mu_j} A_{\alpha,k}^{(\mu_j)}\Big(\rho^{-s}x, \rho^{(t-1)s_n}\frac{D_{x'''}}{D_{x_n}}, 1\Big) \\
&\qquad \times \rho^{ts_n(|\alpha|+k)+s_n\mu_j} D_{x_n}^{\mu_j} D_{x'}^\alpha D_{x_0}^k\Bigg\}.
\end{aligned} \tag{3.2}
$$

Applying the Taylor formula, we get

$$
A_{\alpha,k}^{(\mu_j)}\Big(\rho^{-s}x, \rho^{(t-1)s_n}\frac{D_{x'''}}{D_{x_n}}, 1\Big) = a_{\alpha,k}^{(\mu_j)}(0, e_n) + O(\rho^{-ts_n}) + O(\rho^{(t-1)s_n}).
$$

Hence

$$
\begin{aligned}
P_\rho(x, D_x) &= \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) D_{x'}^\alpha D_{x_0}^k \\
&\quad + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j-k-\mu_{j_i}} a_{\alpha,k}^{(\mu_{j_i})}(0, e_n) D_{x_n}^{\mu_{j_i}} D_{x'}^\alpha D_{x_0}^k \\
&\quad + O(\rho^{-ts_n}) + O(\rho^{(t-1)s_n}) + \sum_{j\neq j_1,\cdots,j_r, j\in\{p,\cdots,m-1\}} O(\rho^{s_n(\mu_j - t(\mu_j+j))}) \\
&\quad + \sum_{j=1}^{m} O(\rho^{s_n(\mu_j - t(\mu_j+j)-t)}) + \sum_{j=1}^{m} O(\rho^{s_n(\mu_j - t(\mu_j+j)+(t-1))}).
\end{aligned} \tag{3.3}
$$

Since all the powers of ρ in the remainder terms of (3.3) are negative, if we choose s_n sufficiently large we get

$$
\begin{aligned}
P_\rho(x, D_x) &= \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0, e_n) D_{x'}^\alpha D_{x_0}^k \\
&\quad + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j-k-\mu_{j_i}} a_{\alpha,k}^{(\mu_{j_i})}(0, e_n) D_{x_n}^{\mu_{j_i}} D_{x'}^\alpha D_{x_0}^k \\
&\quad + O(\rho^{-N})
\end{aligned} \tag{3.4}
$$

for any $N \in \boldsymbol{N}$.

Let us consider the simplectic dilatation $S_\rho(x_0, \cdots, x_n) = (\rho^{-2}x_0, x_1, \cdots, x_{n-1}, \rho^{-2/t}x_n)$.

Then

$$
\begin{aligned}
P'_\rho(x, D_x) &=: \rho^{-2m} P(\rho^{-2}x_0, x_1, \cdots, x_{n-1}, \rho^{-2/t}x_n, \rho^2 D_{x_0}, D_{x_1}, \cdots, D_{x_{n-1}}, \rho^{2/t} D_{x_n}) \\
&= \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(0, e_n)\left(\frac{D_{x'}}{\rho^2}\right)^{\alpha} D^k_{x_0} \\
&\quad + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A^{(\mu_{j_i})}_{\alpha,k}(0, e_n) D^{\mu_{j_i}}_{x_n}\left(\frac{D_{x'}}{\rho^2}\right)^{\alpha} D^k_{x_0} \\
&\quad + O(\rho^{-N}).
\end{aligned} \tag{3.5}
$$

Set $E_\rho = e^{i\psi_\rho}$ with

$$
\psi_\rho(x) = \rho^{1/t} x_n \xi_n + \rho^3 \langle x', \xi' \rangle + \rho \gamma x_0 + i\rho |x'''|^2/2 + i\rho^{-1+1/t} x_n^2/2.
$$

(Here $(x_0, \cdots, x_n) =: (x_0, x', x''', x_n)$).

We have

$$
\begin{aligned}
&E_\rho^{-1} D^k_{x_0} E_\rho = \rho^k \gamma^k + k\rho^{k-1}\gamma^{k-1} D_{x_0} + O(\rho^{k-2}) \\
&E_\rho^{-1} D^{\mu_{j_i}}_{x_n} E_\rho = \rho^{\mu_{j_i}/t} \xi^{\mu_{j_i}} + i\mu_{j_i} \rho^{-1+\mu_{j_i}/t} \xi^{\mu_{j_i}-1} x_n + O(\rho^{(\mu_{j_i}-1)/t}) + O(\rho^{-1+(-1+\mu_{j_i})/t}) \\
&E_\rho^{-1} D^{\alpha_j}_{x_j} E_\rho = \rho^{3\alpha_j} \xi'^{\alpha_j}_j + O(\rho^{3\alpha_j - 3}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
E_\rho^{-1} P'_\rho E_\rho &= \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(0, e_n) \rho^{|\alpha|+k} \xi'^{\alpha} \gamma^k \\
&\quad + \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(0, e_n) k \rho^{|\alpha|+k-1} \xi'^{\alpha} \gamma^{k-1} D_{x_0} \\
&\quad + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A^{(\mu_{j_i})}_{\alpha,k}(0, e_n) \rho^{|\alpha|+k+\mu_{j_i}/t} \xi'^{\alpha} \xi_n^{\mu_{j_i}} \gamma^k \\
&\quad + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A^{(\mu_{j_i})}_{\alpha,k}(0, e_n) \rho^{|\alpha|+k-1+\mu_{j_i}/t} \\
&\qquad\qquad \times (k \xi'^{\alpha} \xi_n^{\mu_{j_i}} \gamma^{k-1} D_{x_0} + i\mu_{j_i} \xi'^{\alpha} \xi_n^{\mu_{j_i}-1} \gamma^k x_n) \\
&\quad + \sum_{k=0}^{m} \sum_{|\alpha|=m-k} O(\rho^{|\alpha|-2+k}) \\
&\quad + \sum_{i=1}^{r} \sum_{|\alpha|+k=m-j_i-\mu_{j_i}} (O(\rho^{|\alpha|+k-2+\mu_{j_i}/t}) + O(\rho^{|\alpha|+k-1+(\mu_{j_i}-1)/t})) \\
&\quad + O(\rho^{-N}).
\end{aligned} \tag{3.6}
$$

Notice that, if $|\alpha| = m - k - \mu_{j_i} - j_i$, then $|\alpha| + k + \mu_{j_i}/t = m - (\mu_{j_i} + j_i) + \mu_{j_i}/t = m$.

Hence

$$
\begin{aligned}
E_\rho^{-1}P'_\rho E_\rho = \rho^m\Bigg(& \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0,e_n)\xi'^\alpha\gamma^k \\
& + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0,e_n)\xi'^\alpha\xi_n^{\mu_{j_i}}\gamma^k\Bigg) \\
& + L_\rho
\end{aligned} \tag{3.7}
$$

with

$$L_\rho = \rho^{m-1}L_0 + \rho^{m-1/t}\tilde{L}_1 + \rho^{m-2}\tilde{L}_2 + \rho^{m-1-1/t}\tilde{L}_3 + \cdots \tag{3.8}$$

and

$$
\begin{aligned}
L_0 = \Bigg(& \sum_{k=1}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0,e_n)k\xi'^\alpha\gamma^{k-1} \\
& + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0,e_n)k\xi'^\alpha\xi_n^{\mu_{j_i}}\gamma^{k-1}\Bigg)D_{x_0} \\
& + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} A_{\alpha,k}^{(\mu_{j_i})}(0,e_n)i\mu_{j_i}\xi'^\alpha\xi_n^{\mu_{j_i}-1}\gamma^k x_n.
\end{aligned} \tag{3.9}
$$

Set now

$$
\begin{aligned}
\tilde{p}_m(\gamma,\xi',\xi_n) = & \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}^{(0)}(0,e_n)\xi'^\alpha\gamma^k \\
& + \sum_{i=1}^{r} \sum_{k=0}^{m-j_i-\mu_{j_i}} \sum_{|\alpha|=m-j_i-k-\mu_{j_i}} a_{\alpha,k}^{(\mu_{j_i})}(0,e_n)\xi'^\alpha\gamma^k\xi_n^{\mu_{j_i}}.
\end{aligned} \tag{3.10}
$$

Let us suppose that there exists j_i, $p \le j_i \le m-1$, and $a_{\alpha,k}^{(\mu_{j_i})}(0,e_n) \ne 0$ with $\mu_{j_i} > 0$ for some α, k, $|\alpha|+k = m-j_i-\mu_{j_i}$, $i=1,\cdots,r$. We show that, in this case, the equation

$$\tilde{p}_m(\gamma,\xi',\xi_n) = 0$$

has at least a root γ with $\operatorname{Im}\gamma < 0$ for a suitable choice of ξ', ξ_n and moreover that it is possible to find an asymptotic solution u_ρ of $L_\rho u_\rho = 0$.

This will imply that there exists a solution of $P'_\rho v_\rho = 0$ of the form $v_\rho = e^{i\psi_\rho}u_\rho$ such that $\operatorname{Im}(\psi_\rho) > \rho^\varepsilon|x|$, if $x_0 < 0$, for some $\varepsilon > 0$, that is in contradiction with the assumption of the well posedness of the Cauchy problem (see [5]).

Notice that, since $j_i + \mu_{j_i} \ge 2$, the coefficient of γ^{m-1} in (3.10), given by $\sum_{|\alpha|=1} a_{\alpha,m-1}^{(0)}(0,e_n)\xi'^\alpha$, is real. Hence, it is sufficient to prove the existence of a root γ of $\tilde{p}_m(\gamma,\xi',\xi_n) = 0$ with $\operatorname{Im}\gamma \ne 0$ for some ξ', ξ_n.

Set

$$A^{(0)}_{m-k}(\xi')=\sum_{|\alpha|=m-k} a^{(0)}_{\alpha,k}(0,e_n)\xi'^{\alpha}$$

$$A^{(j_i+\mu_{j_i})}_{m-j_i-\mu_{j_i}-k}(\xi')=\sum_{|\alpha|=m-j_i-\mu_{j_i}-k} a^{(\mu_{j_i})}_{\alpha,k}(0,e_n)\xi'^{\alpha}$$

$$q_m(\gamma,\xi')=\sum_{k=0}^{m} A^{(0)}_{m-k}(\xi')\gamma^k$$

$$q_{m-j_i-\mu_{j_i}}(\gamma,\xi')=\sum_{k=0}^{m-j_i-\mu_{j_i}} A^{(j_i+\mu_{j_i})}_{m-j_i-\mu_{j_i}-k}(\xi')\gamma^k.$$

Then

$$\tilde{p}_m(\gamma,\xi',\xi_n)=q_m(\gamma,\xi')+\sum_{i=1}^{r} q_{m-j_i-\mu_{j_i}}(\gamma,\xi')\xi_n^{\mu_{j_i}}. \tag{3.11}$$

Notice that

$$\tilde{p}_m(\gamma,\xi',\xi_n)=|\xi'|^m\tilde{p}_m\left(\frac{\gamma}{|\xi'|},\frac{\xi'}{|\xi'|},\frac{\xi_n}{|\xi'|^t}\right). \tag{3.12}$$

We have the following.

Lemma 3.2. *Let $q_m(\gamma)=\sum_{k=0}^{m} A^{(0)}_{m-k}\gamma^k$ be a real polynomial of degree m in the variable γ and $q_{m-s}(\gamma)=\sum_{k=0}^{m-s} A^{(s)}_{m-k}\gamma^k$, $s=2,\cdots,m$ polynomials of degree $m-s$ in the variable γ. Let $\delta_s\in\boldsymbol{N}$ with $1\le\delta_s\le s-1$, $s=2,\cdots,m$.*

If $q_m(\gamma)$ has m real roots then $\tilde{p}_m(\gamma,\lambda)=q_m(\gamma)+\sum_{s=2}^{m} q_{m-s}(\gamma)\lambda^{\delta_s}$ has still m real roots for any $\lambda\in\boldsymbol{R}$ iff $q_{m-s}(\gamma)$ is identically zero for $s=2,\cdots,m$.

Proof. Let us prove the statement by induction on the degree m of $\tilde{p}_m$.

Notice that, if $\tilde{p}_m(\gamma,\lambda)$ has m real roots for any $\lambda\in\boldsymbol{R}$ then $q_{m-s}(\gamma)$ must be a real polynomial in the variable γ.

The statement is clearly obvious for $m=2$.

Suppose that the statement is true for a polynomial $\tilde{p}_m(\gamma,\lambda)$ of degree m and let us prove it for $\tilde{p}_{m+1}(\gamma,\lambda)$.

Suppose that $\tilde{p}_{m+1}(\gamma,\lambda)=q_{m+1}(\gamma)+\sum_{s=2}^{m+1} q_{m+1-s}(\gamma)\lambda^{\delta_s}$, with $1\le\delta_s\le s-1$, has $m+1$ real roots in the variable γ.

As a consequence

$$\frac{d}{d\gamma}\tilde{p}_{m+1}(\gamma,\lambda)=\frac{d}{d\gamma}q_{m+1}(\gamma)+\sum_{s=2}^{m}\frac{d}{d\gamma}q_{m+1-s}(\gamma)\lambda^{\delta_s}$$

has m real roots. By induction, this implies that $\frac{d}{d\gamma}q_{m+1-s}(\gamma)$ is identically zero i.e. $A^{(s)}_{m+1-k}=0$ for any $k=1,\cdots,m+1-s$, $s=2,\cdots,m$.

Hence

$$\tilde{p}_{m+1}(\gamma,\lambda)=q_{m+1}(\gamma)+\sum_{s=2}^{m+1}A^{(s)}_{m+1}\lambda^{\delta_s}$$

and it is easy to check that $\tilde{p}_{m+1}(\gamma,\lambda)$ has only real roots, for any $\lambda\in\boldsymbol{R}$ iff $A^{(s)}_{m+1}=0$ for any $s=2,\cdots,m+1$. □

End of the proof of Proposition 3.1. Applying Lemma 3.2 to the equation (3.11) with $\lambda=\xi_n$ we can conclude that $\tilde{p}_m(\gamma,v,\xi_n)=0$ must have a root $\gamma(v,\xi_n)$ with $\operatorname{Im}\gamma\neq 0$ for some ξ_n and $v\in\boldsymbol{R}^d$ with $|v|=1$.

This root is simple. Actually, by (3.12), $\gamma(\mu v,\mu^t\xi_n)=\mu\gamma(v,\xi_n)$ for any $\mu\in\boldsymbol{R}^+$ and (H_3) implies that $\gamma(\mu v,\mu^t\xi_n)$ is simple for small μ.

Writing $t=p/q$, with $p,q\in\boldsymbol{N}$ from (3.7) we get

$$L_\rho=\rho^{m-1}L_0+\rho^{m-q/p}\tilde{L}_1+\rho^{m-2}\tilde{L}_2+\rho^{m-(q+p)/p}\tilde{L}_3+\cdots$$

Eventually by adding some $L_j=0$ we can write

$$L_\rho=\sum_{j=0}^{+\infty}\rho^{m-(p+j)/p}L_j.$$

Following the arguments of [5], we can find an asymptotic solution u_ρ of $L_\rho u^\rho=0$ in the form

$$u_\rho=\sum_{k=0}^{+\infty}\rho^{-k/p}u_k$$

and this fact contradicts the assumption on the well posedness of the Cauchy problem for P.

Hence $a^{(\mu_{j_p})}_{\alpha,k}(0,e_n)=0$ if $\mu_{j_p}>0$ for any α, k.

Repeating these arguments a finite number of times we can conclude that $a^{(\mu_{j_i})}_{\alpha,k}(0,e_n)=0$ if $\mu_{j_i}>0$ for any α, k and end the proof of the proposition. □

Proof of Theorem 1.1 (Necessary conditions). Let $P(x,D)=P_m(x,D_x)+P_{m-1}(x,D_x)+\cdots$ be a differential operator satisfying (H_1), (H_2), (H_3) and Ω be a neighboorhood of a point $\bar{\rho}\in\Sigma$.

Without loss of generality we can suppose $m_1=m$

Since p_m vanishes of order m on $\Sigma\cap\Omega$, the principal symbol $p_m(x,\xi)$ can be written at a point $(x,\xi)\in\Omega$ as

$$p_m(x,\xi)=\sum_{|\alpha|=m} a_\alpha^{(0)}(x,\xi)q(x,\xi)^\alpha \tag{3.13}$$

for some symbol $a_\alpha^{(0)}(x,\xi)$ positively homogeneous of degree zero.

By taking, for $(x,\xi)\in(-\Omega)$, $a_\alpha^{(0)}(x,\xi)=:a_\alpha^{(0)}(x,-\xi)$, (3.13) holds for $(x,\xi)\in W=:\Omega\cup(-\Omega)$.

Let $A_\alpha^{(0)}(x,D_x)$ and $Q_j(x,D_x)$ be pseudodifferential operators with principal symbols $a_\alpha^{(0)}(x,\xi)$ and $q_j(x,\xi)$ respectively.

Hence, in W, we can write

$$P(x,D_x)=\sum_{|\alpha|=m} A_\alpha^{(0)}(x,D_x)Q(x,D_x)^\alpha+\tilde{P}_{m-1}(x,D_x)+\cdots \tag{3.14}$$

Let $\tilde{\chi}(x,\xi)=(y,\eta)$ be the canonical change of variables of Section 2 and let F be the elliptic Fourier integral operator associated to $\tilde{\chi}$.

$$\begin{aligned}\tilde{P}(y,D_y)=:FP(x,D_x)F^{-1}&=\sum_{|\alpha|=m}\tilde{A}_\alpha^{(0)}(y,D_y)(D_{y_0}+R_0)^{\alpha_0}(D_{y_d}+R_d)^{\alpha_d}\\&\quad+\tilde{G}_{m-1}(y,D_y)+\tilde{G}_{m-2}(y,D_y)+\cdots\end{aligned}$$

for some pseudodifferential operator R_j of order 0 and $\tilde{G}_{m-j}$ of order $m-j$

Applying Proposition 3.1 to $\tilde{P}$ and then coming back to P we can conclude than, if P satisfies (H_1), (H_2), (H_3) and the Cauchy problem for P is well posed in X_0, then (H_4) holds. □

4. Sufficient conditions: the energy estimates

In this section we prove the well posedness of the Cauchy problem for P in X_0, under assumptions (H_1)−−(H_4), by using the method of energy estimates (see [5]).

Taking into account that the principal symbol of P is strictly hyperbolic outside Σ we can assume, without loss of generality, that $m_1=m$.

Since all the canonical transformations we made in Section 2 preserve the hyperplane $x_0=0$, then it will be enough to establish some suitable energy estimate for the operator

$$\tilde{P}(x,D_x)=\tilde{P}_m(x,D_x)+\sum_{j=1}^{m}\sum_{k=0}^{m-j}\sum_{|\alpha|=m-j-k} A_{\alpha,k}(x,D_x)D_{x'}^\alpha D_{x_0}^k \tag{4.1}$$

where

(4.2) $$\tilde{P}_m(x, D_x) = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} A_{\alpha,k}(x, D_{x'}, D_{x''}) D_{x'}^{\alpha} D_{x_0}^{k}$$

with $A_{\alpha,k}(x, D_x) \in OPS^0(X)$ and $A_{0,m} = I$.

Here we have set $x' = (x_1, \cdots, x_d)$ and $x'' = (x_{d+1}, \cdots, x_n)$.

Moreover we may assume that the symbol of P is supported in a conic neighborhood of $\tilde{\rho} = (\tilde{x}, \tilde{\xi}_0 = 0, \tilde{\xi}' = 0, \tilde{\xi}'') \in \tilde{\Sigma}$, $\tilde{\xi}'' \neq 0$, of the form

$$\Gamma_\varepsilon = \{(x, \xi); |x - \tilde{x}| < \varepsilon, |\xi'| < \varepsilon|\xi''|, \left|\frac{\xi''}{|\xi''|} - \frac{\tilde{\xi}''}{|\tilde{\xi}''|}\right| < \varepsilon\}.$$

Let us start by introducing a suitable class of symbols of pseudodifferential operators.

DEFINITION 4.1. Let X be an open set of $\boldsymbol{R}_x^n = \boldsymbol{R}_{x'}^d \times \boldsymbol{R}_{x''}^{n-d}$. We say that $a \in S^{m,p}(X \times \boldsymbol{R}^n)$ iff $a \in C^\infty(X \times \boldsymbol{R}^n)$ and for any compact $K \subset\subset X$, for any $\alpha \in \boldsymbol{Z}^n$, $\beta' \in \boldsymbol{Z}^d$, $\beta'' \in \boldsymbol{Z}^{n-d}$ there exists a positive constant $C_{\alpha,\beta',\beta'',K}$ such that:

(4.3) $$|D_x^{\alpha} D_{\xi'}^{\beta'} D_{\xi''}^{\beta''} a(x, \xi', \xi'')| \leq C_{\alpha,\beta',\beta'',K} \langle \xi' \rangle^{m-|\beta'|} \langle \xi', \xi'' \rangle^{p-|\beta''|}$$

where $\langle \xi' \rangle =: (1 + |\xi'|^2)^{1/2}$ and $\langle \xi', \xi'' \rangle =: (1 + |\xi'|^2 + |\xi''|^2)^{1/2}$.
We denote by $OPS^{m,p}(X)$ the class of pseudodifferential operators associated with $S^{m,p}(X \times \boldsymbol{R}^n)$ and we set:

$$H^{m,p}(\boldsymbol{R}^n) = \{v \in L^2(\boldsymbol{R}^n); \|v\|_{m,p}^2 = \int (1 + |\xi'|^2)^m (1 + |\xi''|^2)^p |\hat{v}(\xi', \xi'')|^2 d\xi' d\xi'' < +\infty\}.$$

In the following we denote simply by $\|\cdot\|$ the norm in $L^2(\boldsymbol{R}^n)$.

REMARK 4.2. It is easy to check that:

1. If $a \in S^{m,p}(X \times \boldsymbol{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi'| \leq c|\xi''|\}$ then for any compact $K \subset\subset X$, for any $\alpha \in \boldsymbol{Z}^n$, $\beta' \in \boldsymbol{Z}^d$, $\beta'' \in \boldsymbol{Z}^{n-d}$ there exists a positive constant $C_{\alpha,\beta',\beta'',K}$ such that:

$$|D_x^{\alpha} D_{\xi'}^{\beta'} D_{\xi''}^{\beta''} a(x, \xi', \xi'')| \leq C_{\alpha,\beta',\beta'',K} \langle \xi' \rangle^{m-|\beta'|} \langle \xi'' \rangle^{p-|\beta''|}$$

where $\langle \xi'' \rangle =: (1 + |\xi''|^2)^{1/2}$.

2. If $a \in S^{m,p}(X \times \boldsymbol{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi''| \leq c|\xi'|\}$ then $a \in S^{m+p}(X \times \boldsymbol{R}^n)$.
3. If $a \in S^{m,p}(X \times \boldsymbol{R}^n)$, $\text{supp}(a) \subset \{(x, \xi); |\xi'| \leq c\}$ then $a \in S^{0,p}(X \times \boldsymbol{R}^n)$.
4. If X' is an open set of $\boldsymbol{R}^d$ and $a \in S^m(X' \times \boldsymbol{R}^d)$ then $a \in S^{m,0}(X \times \boldsymbol{R}^n)$ with $X = X' \times \boldsymbol{R}^{n-p}$
5. For any $j \geq 0$, $S^{m,p}(X \times \boldsymbol{R}^n) \subset S^{m-j,p+j}(X \times \boldsymbol{R}^n)$.
6. If $A(x, D_x) \in OPS^{m,p}(X)$ and $\sigma(A)(x, \xi', \xi'') = 0$ for $|x| > R$, for some $R > 0$ then $A(x, D_x)$ is continuous from $H^{m,p}(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$ i.e

$$\|A(x, D_x)u\| \leq C\|u\|_{m,p}, \qquad \forall u \in L^2(\boldsymbol{R}^n).$$

We can prove the following energy estimates.

Proposition 4.3. *For any $K \subset\subset X$ there exist a constant $C = C_k > 0$ and a real number $\tau_K > 0$ such that for any $u \in C_0^\infty(K)$ and any $\tau > \tau_k$ the following inequality holds:*

$$C\int_{x_0<0} \|\tilde{P}(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0} dx_0 \geq \sum_{j=1}^{m} \tau^{2j-1} \sum_{k=0}^{m-j} \|D_{x_0}^k u(0, \cdot)\|_{m-j-k,0}^2$$

$$(4.4) \qquad + \sum_{j=1}^{m} \tau^{2j} \sum_{k=0}^{m-j} \int_{x_0<0} \|D_{x_0}^k u(x_0, \cdot)\|_{m-j-k,0}^2 e^{-2\tau x_0} dx_0.$$

Proof. The proof is done along the same lines of the proof of the well posedness of the Cauchy problem in the strictly hyperbolic case.

Let

$$\tilde{p}_m(x, \xi) = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}(x, \xi', \xi'')\xi'^\alpha \xi_0^k$$

be the principal symbol of $\tilde{P}$.

If $\tilde{\rho} = (\tilde{x}, \tilde{\xi}_0 = 0, \tilde{\xi}' = 0, \tilde{\xi}'') \in \tilde{\Sigma}$, $\tilde{\xi}'' \neq 0$, the assumption (H_3) guarantees that the localization of $\tilde{p}_m$ at $\tilde{\rho}$:

$$\tilde{p}_{m,\tilde{\rho}}(y_0, y', \eta_0, \eta') = \sum_{k=0}^{m} \sum_{|\alpha|=m-k} a_{\alpha,k}(y_0, y', \tilde{x}'', 0, \frac{\tilde{\xi}''}{|\tilde{\xi}''|})\eta'^\alpha \eta_0^k$$

has m distinct real roots in η_0, for any $y_0, y', \eta' \neq 0$.

Hence, for $(x, \xi', \xi'') \in \Gamma_\varepsilon$ with ε sufficiently small and $\xi' \neq 0$, $\tilde{p}_m$ has m distinct real roots $\lambda_j(x, \xi', \xi'') = |\xi'|\lambda_j(x, \frac{\xi'}{|\xi'|}, \frac{\xi''}{|\xi''|})$, $j = 1, \cdots, m$,

$$\lambda_1(x, \xi', \xi'') \leq \lambda_2(x, \xi', \xi'') \leq \cdots \leq \lambda_m(x, \xi', \xi'').$$

Moreover, the strict hyperbolicity of p_m outside Σ, implies that, for $(x, \xi', \xi'') \in \Gamma_\varepsilon$ and ε small, there exist some positive constants c, C such that:

$$c|\xi'| \leq |\lambda_i(x, \xi', \xi'') - \lambda_j(x, \xi', \xi'')| \leq C|\xi'|, \qquad \text{for } i \neq j$$
$$|\lambda_i(x, \xi', \xi'')| \leq C|\xi'|, \qquad \text{for any } i.$$

Let us take now a cutoff function $\chi \in C_0^\infty(\boldsymbol{R}^k)$ with $\chi(\xi') = 1$ if $|\xi'| \leq 1$ and $\chi(\xi') = 0$ if $|\xi'| \geq 2$ and set $\tilde{\lambda}_j(x, \xi', \xi'') = (1 - \chi(\xi'))\lambda_j(x, \xi', \xi'')$.

It is easy to check that $\tilde{\lambda}_j \in S^{1,0}(X \times \boldsymbol{R}^n)$.

If $\Lambda_j \in OPS^{1,0}$ is a pseudodifferential operator with principal symbol $\tilde{\lambda}_j$ we have

$$\tilde{P}_m(x, D_x) = (D_{x_0} - \Lambda_j(x, D_{x'}, D_{x''}))Q_j(x, D_{x_0}, D_{x'}, D_{x''}) + S_j(x, D_{x_0}, D_{x'}, D_{x''})$$

where

$$Q_j(x, D_{x_0}, D_{x'}, D_{x''}) = \sum_{k=0}^{m-1} C_{j,k}(x, D_{x'}, D_{x''})D_{x_0}^{m-1-k},$$

with $C_{j,k} \in OPS^{k,0}(X \times \boldsymbol{R}^n)$, $\mathrm{supp}(c_{j,k}) \subset \{(x, \xi); |\xi'| > 1\}$ and

$$S_j(x, D_{x_0}, D_{x'}, D_{x''}) = \sum_{k=0}^{m-1} S_{j,k}(x, D_{x'}, D_{x''})D_{x_0}^{m-1-k},$$

with $S_{j,k} \in OPS^{k+1,0}(X \times \boldsymbol{R}^n)$ and $\mathrm{supp}(s_{j,k}) \subset \{(x, \xi); |\xi'| \le 2\}$.

Notice that, thanks to 3) of Remark 4.2, $S_{j,k} \in OPS^{0,0}(X \times \boldsymbol{R}^n)$.

Let us calculate $2i\,\mathrm{Im}\langle \tilde{P}(x, D_x)u,\ Q_j(x, D_x)u\rangle$, for $u \in C_0^\infty(K)$, $K \subset\subset X$.

We have:

$$\begin{aligned}
&2i\,\mathrm{Im}\langle \tilde{P}(x, D_x)u,\ Q_j(x, D_x)u\rangle \\
&= 2i\,\mathrm{Im}\langle \tilde{P}_m(x, D_x)u,\ Q_j(x, D_x)u\rangle + 2i\,\mathrm{Im}\langle (\tilde{P} - \tilde{P}_m)(x, D_x)u,\ Q_j(x, D_x)u\rangle \\
&= 2i\,\mathrm{Im}\langle (D_{x_0} - \Lambda_j(x, D_{x'}, D_{x''}))Q_j(x, D_x)u,\ Q_j(x, D_x)u\rangle \qquad (4.5)\\
&+ 2i\,\mathrm{Im}\langle S_j(x, D_x)u,\ Q_j(x, D_x)u\rangle \\
&+ 2i\,\mathrm{Im}\langle (\tilde{P} - \tilde{P}_m)(x, D_x)u,\ Q_j(x, D_x)u\rangle
\end{aligned}$$

Hence, multiplying the above identity by $i\tau e^{-2\tau x_0}$ and integrating it for $x_0 < 0$, we have, for τ sufficiently large:

$$\begin{aligned}
&C\int_{x_0<0} \|\tilde{P}(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0}dx_0 \\
&\ge \tau \sum_{j=1}^m \|Q_j(x, D_x)u(0, \cdot)\|^2 \\
&+ \tau^2 \sum_{j=1}^m \int_{x_0<0} \|Q_j(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0}dx_0 \qquad (4.6)\\
&- \tau \int_{x_0<0} \|(\tilde{P} - \tilde{P}_m)(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0}dx_0 \\
&- \tau \sum_{j=1}^m \int_{x_0<0} \|S_j(x, D_x)u(x_0, \cdot)\|^2 e^{-2\tau x_0}dx_0.
\end{aligned}$$

Now, we can estimate the last two terms in (4.6) by

$$\tau\int_{x_0<0}\|(\tilde{P}-\tilde{P}_m)(x,D_x)u(x_0,\cdot)\|^2 e^{-2\tau x_0}dx_0$$
$$+\tau\sum_{j=1}^{m}\int_{x_0<0}\|S_j(x,D_x)u(x_0,\cdot)\|^2 e^{-2\tau x_0}dx_0$$
$$\leq \tau C\sum_{j=1}^{m}\sum_{k=0}^{m-j}\int_{x_0<0}\|D_{x_0}^k u(x_0,\cdot)\|_{m-j-k,0}^2 e^{-2\tau x_0}dx_0.$$

On the other hand, using the Lagrange interpolation formula, we have, for $k=0,\cdots,m-1$

$$\xi_0^k=\sum_{j=1}^{m}\frac{q_j(x,\xi)\tilde{\lambda}_j(x,\xi',\xi'')^k}{\Pi_{i\neq j}(\tilde{\lambda}_j(x,\xi',\xi'')-\tilde{\lambda}_i(x,\xi',\xi''))}\qquad \text{if } |\xi'|\geq 2.$$

Take now a cutoff function $\chi\in C_0^\infty(\boldsymbol{R}^k)$ with $\chi(\xi')=1$ if $|\xi'|\leq 5/2$ and $\chi(\xi')=0$ if $|\xi'|\geq 3$.

Hence

$$(1-\chi'(\xi'))\langle\xi'\rangle^{m-1-k}\xi_0^k=\sum_{j=1}^{m}q_j(x,\xi)\frac{(1-\chi'(\xi'))\langle\xi'\rangle^{m-1-k}\tilde{\lambda}_j(x,\xi',\xi'')^k}{\Pi_{i\neq j}(\tilde{\lambda}_j(x,\xi',\xi'')-\tilde{\lambda}_i(x,\xi',\xi''))}\qquad \text{if } |\xi'|\geq 2.$$

Since

$$m_{j,k}=:\frac{(1-\chi'(\xi'))\langle\xi'\rangle^{m-1-k}\tilde{\lambda}_j(x,\xi',\xi'')^k}{\Pi_{i\neq j}(\tilde{\lambda}_j(x,\xi',\xi'')-\tilde{\lambda}_i(x,\xi',\xi''))}$$

belongs to S^0, we have:

$$\begin{aligned}\|(1-\chi'(D_{x'}))D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2 &\leq \sum_{j=1}^{m}\|Q_j(x,D_x)u(x_0,\cdot)\|^2\\ &\quad + C\sum_{j=2}^{m}\sum_{k=0}^{m-j}\|D_{x_0}^k u(x_0,\cdot)\|_{m-j-k,0}^2\end{aligned}\tag{4.7}$$

On the other hand, if $k\leq m-2$

$$\|\chi'(D_{x'})D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2\leq C\|D_{x_0}^k u(x_0,\cdot)\|^2.\tag{4.8}$$

Hence (4.7) and (4.8) give, for $k\leq m-2$:

$$\begin{aligned}\|D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2 &\leq \sum_{j=1}^{m}\|Q_j(x,D_x)u(x_0,\cdot)\|^2\\ &\quad + C\sum_{j=2}^{m}\sum_{k=0}^{m-j}\|D_{x_0}^k u(x_0,\cdot)\|_{m-j-k,0}^2.\end{aligned}\tag{4.9}$$

Moreover, since $D_{x_0}^{m-1}=Q_j-\sum_{k=1}^{m-1}C_{j,k}D_{x_0}^{m-j-k}$ we have

$$\|D_{x_0}^{m-1}u(x_0,\cdot)\|^2\leq\|Q_j(x,D_x)u(x_0,\cdot)\|^2+C\sum_{k=0}^{m-2}\|D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2. \tag{4.10}$$

From (4.6), (4.9) and (4.10) we get, for large τ

$$\begin{aligned}C\int_{x_0<0}\|\tilde{P}(x,D_x)u(x_0,\cdot)\|^2e^{-2\tau x_0}dx_0&\geq\tau\sum_{k=0}^{m-1}\|D_{x_0}^k u(0,\cdot)\|_{m-1-k,0}^2\\&+\tau^2\sum_{k=0}^{m-1}\int_{x_0<0}\|D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2e^{-2\tau x_0}dx_0\\&-\tau^2\sum_{j=2}^{m}\sum_{k=0}^{m-j}\int_{x_0<0}\|D_{x_0}^k u(x_0,\cdot)\|_{m-j-k}e^{-2\tau x_0}dx_0\end{aligned}\tag{4.11}$$

and using classical estimates for the terms

$$\int_{x_0<0}\|D_{x_0}^k u(x_0,\cdot)\|_{m-1-k,0}^2e^{-2\tau x_0}dx_0,\qquad k=0,\cdots,m-1,\tag{4.12}$$

we get (4.4) and we end the proof of the proposition.

Proof of Theorem 1.1 (Sufficient conditions). The proof of the theorem follows easily from Proposition 4.3.

Actually, we remark that $P(x,D_x)$ is a hyperbolic differential operator with simple characteristics outside Σ.

Hence, by using classical estimates for strictly hyperbolic operator, Proposition 4.3 and a microlocal partition of the unity, the proof of Theorem 1.1 can be completed by following the arguments of [5]. □

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