Title	Naturally ad Absurdum
Author(s)	Stirk, C. Ian
Citation	大阪外国語大学英米研究. 2004, 28, p. 99-110
Version Type	VoR
URL	https://hdl.handle.net/11094/99281
rights	
Note	

Naturally ad Absurdum

Ian C. Stirk

W.V. Quine introduced an elegant method of Natural Deduction (ND) and a straightforward method of Reductio ad Absurdum (RaA) in various editions of his "Methods of Logic" (Quine 1974). There he used them only for first order predicate calculus, but they can both be easily adapted to solve problems in higher order calculi, as I have pointed out on various occasions (Stirk 1985, 1994, 1995).

However, I must now admit to brushing some difficulties under the carpet and concealing them from my dear reader. These difficulties all involved proofs which were plain sailing using RaA, but which had no obvious ND equivalent. This was disturbing, because anything that can be proved using one of the two methods should have a proof using the other, even if it is a less simple proof.

I think I have now solved these problems, and the solutions all have something in common. They are presented below with a sense of relief.

Here is a proof that the plausible looking formula $(\exists F)(x)Fx$ is indeed logically true :

1.	(F)(∃x)-Fx	
2.	(∃x)-(λy)(Gy ∨ -Gy)x	1
3.	-(λy)(Gy ∨ -Gy)a	2
4.	-(Ga ∨ -Ga)	3
5.	-Ga . Ga	4
6.	⊗	5

This is a proof by RaA, so the first line is the negation of the formula we are trying to prove. It begins with a universal quantifier over predicates, so any predicate may be chosen to instantiate it. We try to find a predicate that will result in an inconsistency, and the one chosen, (λy)(Gy ∨ -Gy), does this admirably. Line 5 is obviously inconsistent, and this is indicated by the sign ⊗ in line 6.

But with ND, how could we begin? These proofs generally start with some fully quantified formula p, from which another fully quantified formula q is deduced, thus proving that p ⊃ q is logically true. What could p be in this case? If p were a tautology, and p ⊃ q logically true, then q itself would have to be logically true. For instance, if we started with

1. Fa ∨ -Fa tautology

then "a" could be the name of any individual : it does not need to be flagged, and could be universally generalized :

2. (x)(Fx ∨ -Fx) 1 a

Now "a" is flagged for the first time, quite legitimately. We have a ND proof in first order predicate calculus that (x) (Fx ∨ -Fx) is logically true. Obviously the proof corresponds to the RaA demonstration

1. $(\exists x)(-Fx \,.\, Fx)$
2. $-Fa \,.\, Fa$ 1
3. $\otimes$

With that working, let us go back to $(\exists F)(x)Fx$. Beginning with a tautology

1. $Ga \vee -Ga$ tautology

we can go on to form a predicate by λ-conversion :

2. $(\lambda y)(Gy \vee -Gy)a$ 1 $(\lambda y)(Gy \vee -Gy)$

This predicate has been flagged to the right, because although we began with a tautology, the predicate of line 2 is a particular kind, not any predicate, so it cannot undergo universal generalisation. The letter "a" has not been flagged, however, so we can continue :

3. $(x)(\lambda y)(Gy \vee -Gy)x$ 2 a

flagging "a", and finish up with

4. $(\exists F)(x)Fx$ 3

just as we wanted.

The simple tautology we have been employing is an alternation, so we might wonder what the effect of branching would be. Starting like this :

1. $-Ga \lor Ga$ tautology

2. | Ga a,G

We want to consider the right hand branch. The idea of branching is to consider what-would happen if the material in that branch were true : in this case, what would happen if some individual called "a" had a property whose name is "G". We certainly could not universally generalise either "a" or "G". In fact, the effect of considering only one branch is to flag the letters occurring in it, as has been done with "a" and "G" above. We could continue with existential generalisation :

3. | $(\exists x)Gx$ 2

and then bring the branches back together :

4. $Ga \supset (\exists x)Gx$ 1,3 {a,G}

The alternation has been replaced with a conditional, taking advantage of the tautology

$$-p \lor q \,.\equiv.\, p \supset q$$

The process of rejoining the branches is itself justified by the tautology

$$p \lor q \,.\, p \supset r \,.\, q \supset s \,.\, \supset .\, r \lor s$$

What we have ended up with is a proof that $Ga \supset (\exists x)Gx$ is logically true. But there is more. Since we have got something logically true, "G" could now be the name of any predicate, and "a" that of any individual. We could say that they have both been "deflagged", and the curly bracket notation to the right of line 4 is meant to illustrate this. Instead of stopping at line 4, we could continue :

5.	$(y)(Gy \supset (\exists x)Gx)$	4 a
6.	$(F)(y)(Fy \supset (\exists x)Fx)$	5 G

ending up with another logically true formula.

Here are another couple of examples :

1. $-Ga \vee Ga$ tautology

2.	$-Ga$	a,G	5.	Ga	a,G
3.	$(\exists x)\text{-}Gx$	2	6.	$(\exists x)Gx$	5
4.	$(\exists F)(\exists x)\text{-}Fx$	3	7.	$(\exists F)(\exists x)Fx$	6

8. $(\exists F)(\exists x)\text{-}Fx \vee (\exists F)(\exists x)Fx$

1. $-Ga \vee Ga$ tautology

2.	$-Ga$	a,G	4.	Ga	a,G

3. $(\exists x)\text{-}Gx$ 2 5. $(\exists x)Gx$ 4

6. $(\exists x)\text{-}Gx \lor (\exists)Gx$ 3,5 {G}

7. $(F)[(\exists x)\text{-}Fx \lor (\exists x)Fx]$ 6 G

Trivial examples, maybe, but this technique of starting with a tautology may have some value in more complex situations – I just haven't thought of any yet! Notice that if we start with a tautology of the form $p \supset p$, and deduce q from p in the right hand branch, to prove the logical truth of $p \supset q$, we are performing the same task as Quine does with his method of conditionalisation in Natural Deduction (Quine, 1974, p205). I think the method of splitting and rejoining branches illustrated above is easier to handle than Quine's system of asterisks, and shows clearly the relation with branching in RaA.

Other problems involving modal prepositional calculus (MPC) seemed quite impossible to solve. It is easy to adapt RaA to MPC, as the following example shows. We want to prove the logical truth of the formula $Lp \,.\, M(p \supset q) \,.\, \supset Mq$. The necessary premises will be

1. Lp
2. M $(p \supset q)$
3. L-q

There is of course an analogy between the necessity operator L and universal quantification, and between the possibility operator M and existential quantification.
This suggests that we "instantiate" M with a different possible world, represented by a rectangle as follows :

4.	p ⊃ q	2
5.	p	1
6.	-q	3
7.	q	4,5
8.	⊗	6,7

This is a world where p ⊃ q is true. But L may be instantiated in any accessible world, including this one. An inconsistency is soon reached, completing the proof.

Next we try to show that Lp ⊃ LLp :

1. Lp
2. MM-p

3.	M-p	2

4.	-p	3
5.	**p**	**1**
6.	**⊗**	**4,5**

In order to reach a world where –p is true, we have to go via the world containing M-p. Now if the world containing –p is accessible from the starting world, there will be an inconsistency, as shown by the lines in bold type. This means that the accessibility relation has to be transitive. If it is not transitive, there is no inconsistency.

Now we try to adapt ND to these proofs. The first one is straightforward :

1.	Lp	
2.	M $(p \supset q)$	
6.	Mq	5

3.	$p \supset q$	2	
4.	p	1	*
5.	q	3,4	

After line 2, we go to a world where $p \supset q$ is true. The asterisk to the right of it is to remind us of flagging : this world may not be "universally generalised". After line 5, we can go back to the original world and write the deduction Mq as line 6. This is the analogue of existential generalisation.

The other proof does not go so smoothly, however. Off we go with

1. Lp

2.	p	1

This time there is no asterisk to the right of the new world, as we were "instantiating" L. The analogue of universal generalisation can be applied here – but to what? It seems impossible to get any further, certainly nowhere in the direction of the LLp we want.

It finally dawned on me that introducing a tautology can do the trick in this case as well as in the previous ones. Here is the whole proof :

1. Lp

9. LLp 8

2.	p	1	
3.	-Lp ∨ Lp	tautology	
4.	M-p ∨	Lp	3
5.	-p	**8. Lp**	**4**
6.	**p**	**1**	
7.	**⊗**	**5,6**	

Again bold type shows steps that are only possible if the accessibility relation is transitive. In that case, the left hand branch after line 4 ends up in an inconsistency, leaving only the right branch, so Lp has to be true. In that case, we obtain LLp in line 9 by "universal generalisation". There is a curious admixture of RaA in this proof.

Finally let us try out the formula Mp ⊃ LMp. With RaA we get this :

1. Mp
2. ML-p

3.	p	1
5.	**-p**	**4**
6.	**⊗**	**3,5**

4. L-p

Here we "instantiate" two different worlds, one for the M in line 1, the other for the M in line 2. If the world containing line 3 is accessible from the world containing line 4, then an inconsistency arises as shown in the bold lines. This will be the case if the accessibility relation is symmetric as well as transitive, for then all the worlds will be accessible to each other.

Now let us see if introducing a tautology into a ND proof will do the trick here too :

1. Mp

10. LMp 9

2.	p	1	*
7.	**-p**	**5**	
8.	**⊗**	**2,7**	

3. –Mp∨Mp tautology

4. L-p ∨ Mp 3

5. L-p 6. Mp

9. Mp 6,8

The world to the left is flagged with an asterisk, but not the world to the right, since tautologies have to be true in all worlds. If the world on the left is accessible to the one on the right, then the lines in bold type follow, culminating in the "universal generalisation" of line 10.

The concept of Natural Deduction proofs in which certain branches are nullified by an inconsistency is somehow intriguing. I have a hunch that they could be useful in more substantial examples than the ones above. Watch this space!

In their classic work "An Introduction to Modal Logic", Hughes and Cresswell (1968) prove that formulae of modal prepositional calculus are logically true by demonstrating that they cannot be false. This of course is different from RaA, which proves that some conjunction of formulae cannot be true. However, Hughes and Cresswell's method would be difficult to apply to predicate calculi.

They also show how a certain system of natural deduction may be adapted to deal with modal prepositional calculus (Hughes and Cresswell, 1968, Appendix I). However this system is nothing like so elegant as Quine's.

Bibliography

Hughes, G.E. and Cresswell, M.J. (1968) "An Introduction to Modal Logic" (Methuen)

Quine, W.V. (1974) "Methods of Logic" (Routledge and Kegan Paul, 3rd edition)

Stirk, I.C. (1985) "Intensional Logic without Tears" (大阪外大英米研究)
(1994) "Proofs in Intensional Logic" (大阪外大英米研究)
(1995) "The Joys of Natural Deduction" (大阪外大英米研究)

Ian C. Stirk

November 2003