Restoring the naturalness of deduction

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Bibliographical introduction

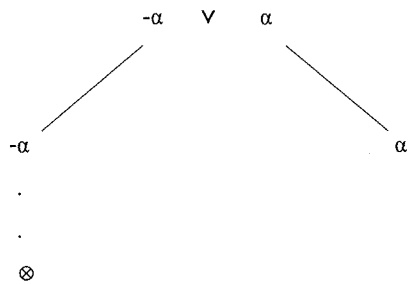
I have been developing various informal logical methods, mainly for linguists, over a good many years. The methods have been cobbled together from a variety of sources, so it is easiest to mention them here, rather than sprinkle the text with endless references.

The core parts of my versions of Natural Deduction and Reductio ad Absurdum come from Quine (1974). Reductio ad Absurdum is called the "main method" by Quine in that work. The ideas on branching which I have incorporated into my versions were mostly influenced by Hodges (1977). Various other points were influenced by Carnap (1958). My ways of dealing with modal logics are mostly due to Hughes and Cresswell (1968).

Although there are later editions of the works of Quine and Hughes and Cresswell, they have not led to any changes in my own methods. Previous efforts of mine include Stirk (1985, 1994, 1995 and 2004)

Preliminaries

In a previous paper (Stirk 2004), I presented a way of tackling proofs in modal propositional calculus (MPC) using a form of Natural Deduction (ND). Although that method undoubtedly works, it is uncomfortably close to Reductio ad Absurdum (RaA). In RaA, one shows that a proposition α is logically true by proving that -α is inconsistent. In ND one shows that α is logically true by deducing it from a tautology. However it is possible to disguise what is essentially a proof by RaA as one by ND :



This begins with an obvious tautology in the form of an alternation, so that branching is possible. In the left hand branch, -α is shown to be inconsistent, leaving only the branch containing α. Of course this is technically ND, but all the hard work takes place in the left hand branch, and is done by RaA. The ND proofs in my previous paper were not so blatantly artificial as that, but they could not be called genuine all the same.

In this paper I want to develop a real ND method for tackling proofs in modal logic. I deal mainly with the MPC system S5, as that is the one most commonly seen in models of the semantics of natural language.

RaA and S5

It is easiest to begin by showing how RaA may be adapted to proofs in S5.

Clearly there is an analogy between the possibility operator M and the existential quantifier, and the necessity operator L and the universal quantifier. The following relations hold between the quantifiers :

$$-(\exists x)\text{-}Fx \equiv (x)Fx \qquad \text{and} \qquad -(x)\text{-}Fx \equiv (\exists x)Fx$$

A little thought shows that

$$-M\text{-}p \equiv Lp \qquad \text{and} \qquad -L\text{-}p \equiv Mp$$

for if it is not possible that not-p, then p is necessarily true, and if it is not necessarily true that not-p, then p is possible. Thus the modal operators will behave in the same way with regard to negation : -MLp, for instance, will be equivalent to LM-p.

The analogy is not complete, though, as the last two expressions show. The modal operators make a proposition out of another proposition, and that is why they can be iterated. If p is a proposition, so are Mp, LMp, LLMp and an infinite number of others. On the other hand, quantifiers must be related to variables in the expressions to which they are prefixed. Thus Rxy is not a proposition, and nor is (y)Rxy. ($\exists$x)(y)Rxy is a proposition, but ($\exists$z)($\exists$x)(y)Rxy is not, and so on, at least if we do not allow vacuous quantifiers.

Nevertheless, the analogy is helpful in working out how to adapt RaA to modal logic. Of course it is best to start with really obvious examples. The propositions

"Lp" and "-p" should certainly be inconsistent, while "Mp" and "-p" should be consistent, for p might be true in a world other than the one where -p is true. A good trick is to represent possible worlds as rectangles : propositions taken to be true in that possible world are written inside the rectangle. For instance :

1.	Lp
2.	-p

Here we are imagining a world in which both Lp and -p are true. These are our premises. Line 1 states that p is true in every world, including this particular world. So we can continue :

1.	Lp	
2.	-p	
3.	p	1
4.	⊗	2,3

Deducing Line 3 from Line 1 is, of course, very similar to instantiating a universal quantifier. If something is true in every world, it must be true in this one. So an inconsistency is reached, expressed in Line 4. Now let us try :

1.	Mp
2.	-p

This time we continue :

1. Mp	3. p 1
2. -p	

Line 3 appears in a different world, so there is no inconsistency. This is akin to instantiating an existential quantifier with a new individual name. The operator M is instantiated with a new world.

Tautologies are necessarily true, not to say necessarily necessarily true :

1. MM(p. -p)	2. M(p. -p) 1	3. p. -p 2
		4. $\otimes$ 3

The example above shows this for the tautology p $\vee$ -p. Clearly the same kind of proof would work for any other tautology, preceded by any number of L's. In the RaA proof, each M is removed by instantiation with a new world, until finally a world containing the negation of the tautology is reached.

So far so good : we go on to try more substantial examples.

In predicate calculus we can prove that although $(\exists x)(y)Rxy \supset (y)(\exists x)Rxy$ is logically true, $(y)(\exists x)Rxy \supset (\exists x)(y)Rxy$ is not. The analogues of these expressions in modal prepositional calculus would be MLp $\supset$ LMp and LMp $\supset$ MLp. We test these with RaA :

1.	MLp	
2.	ML-p	

3.	Lp	1

4.	L-p	2
5.	-p	4
6.	p	3
7.	⊗	5,6

That shows that MLp ⊃ LMp is logically true, like its predicate calculus counterpart. Then how about LMp ⊃ MLp?

1.	LMp	
2.	LM-p	
3.	Mp	1
4.	M-p	2

5.	p	3
7.	M-p	2

6.	-p	4
8.	Mp	1

Clearly there is no point continuing with this : we cannot reach a world with both p and -p in it. LMp ⊃ MLp is consistent. The analogy with the predicate calculus formulae holds.

The formula L(p ⊃ q) ⊃ . Lp ⊃ Lq is easily shown to be logically true :

1.	L(p⊃q)	
2.	Lp	
3.	M-q	

4.	-q	3
5.	p	2
6.	p⊃q	1
7.	q	5,6
8.	⊗	4,7

The formula $Mp\supset Lq. \supset L(p \supset q)$ illustrates a further point about RaA :

1.	$Mp\supset Lq$	
2.	$M(p.-q)$	
4.	Mp	3
5	Lp	1,4

3.	$p.-q$	2
6.	q	5
7.	$\otimes$	3,6

Although in general an RaA proof proceeds by removing L's and M's and looking for an inconsistency among the propositional variables, it is sometimes convenient to add an M or L, as above in going from line 3 to line 4. We need to be completely certain about the legitimacy of doing such a thing, however. As adding operators is a vital part of ND, let us turn to that right away.

ND

In predicate calculus, an existential quantifier may be removed in the process of existential instantiation (EI), and a universal quantifier may be removed by universal instantiation (UI). Existential quantifiers can be added by existential generalisation (EG), and universal ones by universal generalisation (UG). Let us try to think out what the equivalents would be in modal propositional calculus.

EI : If $M\alpha$ is true in some world, then there is at least one other world where α is true. That sounds like the situation with RaA, but there is a complication here:

1.	$M\alpha$

2.	α

The world where α is true may not be any world : α may not be true in other worlds. We need to have something analogous to *flagging*, to prevent any mistaken UG taking place. A simple way to do this is to put a small $\exists$ sign at the top left of the rectangle, to show that this is a "special" world :

1. $M\alpha$	$\exists$ 2. α

A rectangle without a small $\exists$ represents any world : in the diagram above, the world where $M\alpha$ is true is any world. Of course the following instance of EI should also be correct :

$\exists$ 1. $M\alpha$	$\exists$ 2. α

where the world containing $M\alpha$ is also a flagged world.

UI : If $L\alpha$ is true in a world, then α is true in any world. That means that all of the following examples of UI should be correct :

(i)

1. $L\alpha$	2. α

(ii)

∃
1. $L\alpha$

2. α

(iii)

1. $L\alpha$
2. α

(iv)

∃
1. $L\alpha$
2. α

Of course, (iv) is a weak case, since we are just stating that α is true in a flagged world rather than any world.

EG : If α is true in a world, flagged or not, then $M\alpha$ should be true in any world. That suggests the following cases :

(i)

1. α	2. Mα

(ii)

∃ 1. α	2. Mα

(iii)

1. α 2. Mα

(iv)

∃ 1. α 2. Mα

Notice that there could be weaker forms of (i) and (ii), where the world in which Mα is written is a flagged world, rather than any world.

UG : If α is true in *any* world, then Lα is true in any world :

(i)

1. α

2. Lα

(ii)

1. α
2. Lα

There will also be a weakened form of (i), where Lα appears in a flagged world. Notice also that EG case (iii) is itself a weak form of UG case (ii).

We should try those ideas out first on something simple : $p \supset Mp$ seems suitable. To begin with, we imagine a world in which p is true :

∃
1. p

Notice the little ∃. This cannot be any world, because there may be worlds where p is false. But even so, if p is true in this world, Mp must be also, since there is at least one world where p is true - namely, this one! This is EG case (iv) :

∃
1. p
2. Mp

It tells us that in worlds where p is true, Mp is also true. In other worlds, p is false. Thus, in any world at all, $p \supset Mp$ must be true. There can be no world where p is true but Mp false.

Clearly in using ND we must be careful in general to flag the "starting" world. Otherwise we might be misled into mistakes like this :

1. p
2. Lp

and imagining that $p \supset Lp$ was logically true.

$Lp \supset p$ is of course similarly straightforward. Tautologies are even more so, since in this case we can start by assuming that they are true in any world :

1.	$p \vee -p$	
2.	$L(p \vee -p)$	1
3.	$LL(p \vee -p)$	2

Since we are in any world, we can apply UG over and over again, as we like. There may be any number of L's in front of a tautology - the result is still logically true. Unusually, we have something briefer to demonstrate with ND than RaA!

Now let us try MLp ⊃ LMp :

∃	∃	
1. MLp	2. Lp 1	3. p 2
5. LMp 4		4 Mp 3

That works according to the rules we have worked out. It is clearly a weak result, for simply by omitting the step from line 3 to line 4 we could show that MLp ⊃ Lp :

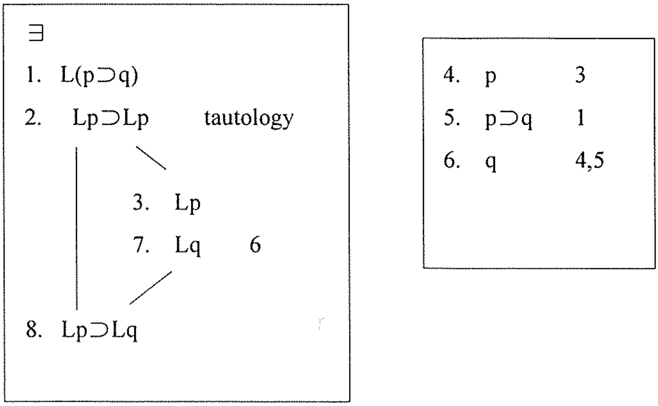
∃	∃	
1. MLp	2. Lp 1	3. p 2
4. Lp 3		

It may also be worthwhile to try what happens with LMp⊃ MLp in ND :

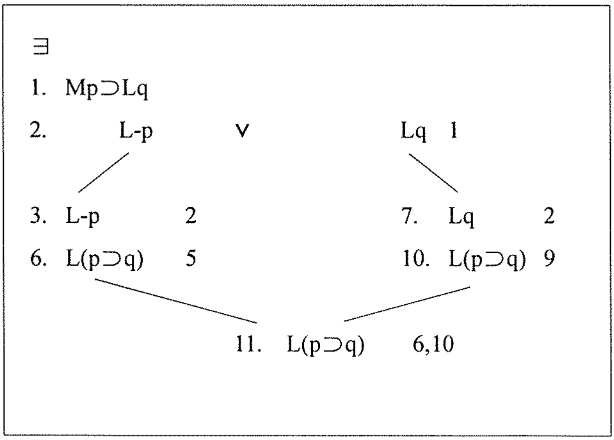
∃		∃
1. LMp	2. Mp 1	3. p 2

Clearly there is no way to get Lp in any world : the best we can establish is LMp ⊃ Mp, which is trivial.

As a more substantial example, let us try L (p⊃ q) ⊃. Lp ⊃ Lq



This shows that other features of ND, like branching, can be employed. A proof that Mp ⊃ Lq . ⊃ L (p⊃ q) provides a more complex example of this. It is simplest to begin by rewriting Mp ⊃ Lq as L-p∨ Lq :



4.	-p	3
5.	p⊃q	4

8.	q	7
9.	p⊃q	8

It is worth noting that the line numbers are sufficient to identify a world as "belonging" to a particular branch in a proof. There is no need for any more graphic devices than the rectangles themselves.

The unsatisfactory version of ND which I included in my (2004) now becomes unnecessary. I was led to it in the first place by a difficulty in proving Lp ⊃ LLp. This formula presents no problem to the current version of ND :

∃		
1.	Lp	
4.	LLp	3

2.	p	1
3.	Lp	2

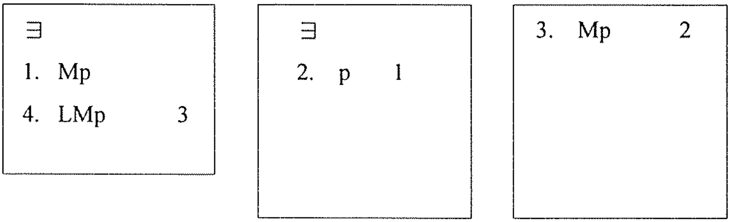
The problem disappears due to the more sophisticated version of UG used here.

To illustrate one last point, let us go on to demonstrate the logical truth of Mp⊃ LMp. This is of course equivalent to ML-p ⊃ L-p, or, replacing -p by p, MLp ⊃ Lp, proved above. But tackling it directly leads first to :

∃	
1.	Mp

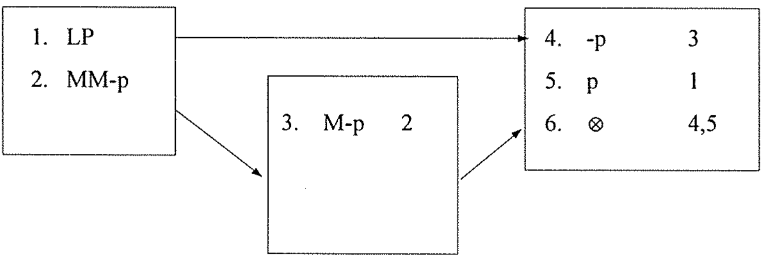
∃		
2.	p	1

That seems to leave us at an impasse, for with only two worlds, neither of which is any world, where can we get L from? The answer is just to imagine any world : since p is true in at least one world, we can be certain that the new world contains Mp, and the proof can be completed :

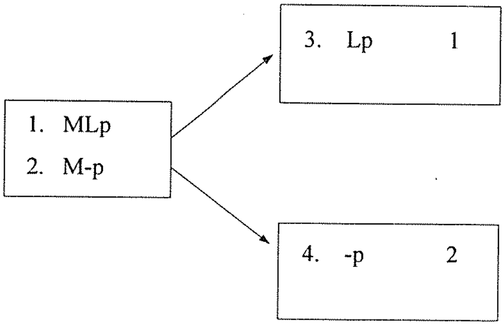


ND in systems other than S5

Fond though I am of ND, there is no doubt that RaA is much easier to use, especially when it comes to systems other than S5, when more attention must be paid to the accessibility relation. With RaA, arrows are sufficient to indicate accessibility in every case. For example, we can easily show that in S4, where the accessibility relation is transitive but not symmetric, Lp ⊃ LLp is logically true :

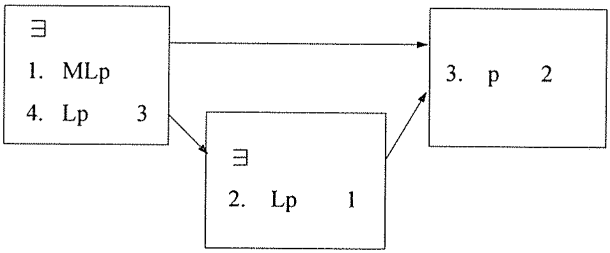


However, MLp ⊃ Lp is not logically true in S4 :

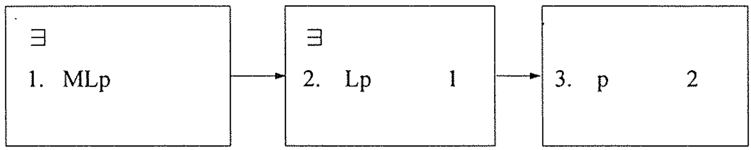


There is no inconsistency, as the world containing -p is not accessible to the one containing Lp.

That same formula, MLp ⊃ Lp, was found above to be logically true in S5, using ND. I repeat the proof below, this time adding S4 type arrows :

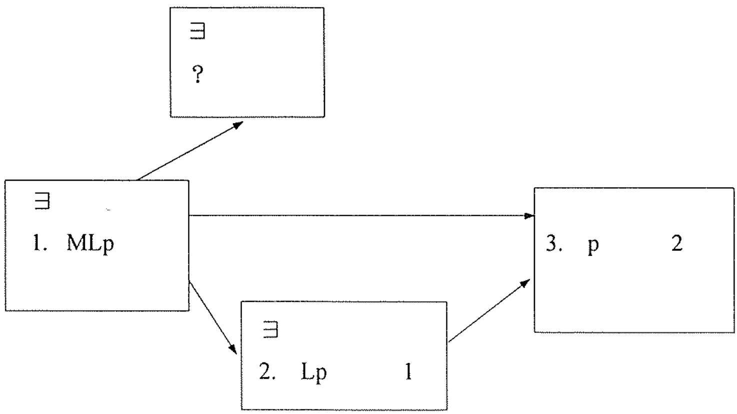


Unfortunately that makes MLp ⊃ Lp seem logically true in S4. The problem lies in the world on the right, containing p. It is not any world at all, but any world accessible from the middle one containing Lp. It may not be any world accessible from the one containing MLp. Of course with a bit of thought we could avoid this trap. The easiest way might be to forget about the arrow joining the left hand and the right hand worlds, and just to remember that the accessibility relation is transitive here :



The rule would be, roughly speaking, that if a chain of arrows passed through any ∃ world on its way, then subsequent worlds in the chain could not be any world. Thus above, we would not be entitled to deduce Lp in the world containing MLp. Different versions of ND could no doubt be devised to suit T and other systems, though they would be rather *ad hoc*.

Lovers of ND might still hope that a generally applicable version could be found. In predicate calculus it is sometimes useful to remember the connection between the existential quantifier and alternation. (∃x)Fx may be thought of as Fa ∨ Fb∨ Fc ... Similarly in modal logic we might think of Mα as meaning that α is true either in this world, or that one, or... The previous diagram might be revised in this way :



The world at the top is some world accessible from the one with MLp in it. The possible world containing Lp is specified, and in this alternative world we do not know what is the case : maybe Lp is true there also, or maybe not. This is indicated by the question mark. The presence of this world shows that the world containing p is not any world from the point of view of the one containing MLp, so we could not go on to conclude Lp there.

I am not sure if this idea could be developed very far. At least it shows that pondering on even the most elementary parts of logic may open up new vistas. Logic is a magic toy box in which something new can always be found.

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