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AN EXTENSION OF A RESULT OF KOMATSU AND TAKASHIMA

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Introduction

It has been shown by Komatsu and Takashima (1984) [7] that in dimensions higher than one the Brownian path has Hausdorff dimension 2 for quasi-all paths. This result extends the result of Taylor (1953) [10] who proved it held a.s. with respect to Wiener measure. In this paper we prove:

Theorem.

Let A be any time set of dimension α, and let $\{B(t): t\geq 0\}$ be a d-dimensional Brownian motion. The image of A by $\{B(t): t\geq 0\}$ has Hausdorff dimension equal to the minimum of 2α and d quasi-everywhere.

This result extends the following result found in Kahane (1968) [5]:

The image of a time set of Hausdorff dimension α by d-dimensional Brownian motion has dimension equal to the minimum of 2α and d a.s..

Fukushima (1984) [2] proved that Levy's modulus of continuity held quasi-everywhere rather than just a.s.. Consequently it is easy to see that for a fixed set A of Hausdorff dimension α, the dimension of the image of A is bounded by 2α quasi-everywhere. Hence, the Theorem will be established once we have shown that for fixed A, the dimension of the image of A by B has dimension greater than the minimum of 2α and d quasi-everywhere. We shall in fact explicitly perform the calculations only for the case $d=1$ and $\alpha<d/2$, but the method we exhibit will also work for higher dimensions.

In Section 1 we give a general plan of the method; in the Section 2 we collect together the building blocks for the final proof which appears in Section 3.

1. Pre-requisites

For $\alpha>0$ one defines the function ϕ_α^δ on a set A by

$$\phi_\alpha^\delta(A) = \inf \sum_{i=1}^{\infty} |A_i|^\alpha ,$$

where the infimum is taken over all sequences of intervals A_i such that

$$A \subset \bigcup_{i=1}^{\infty} A_i \text{ and } |A_i| < \delta \quad (|A_i| \text{ denotes the length of the interval } A_i).$$

We define the set function ϕ_α by taking the (increasing) limit of ϕ_α^δ as δ tends to zero:

$$\phi_\alpha(A) = \operatorname*{limit}_{\delta \to 0} \phi_\alpha^\delta(A) .$$

The Hausdorff dimension of a set A is defined to be $\inf_\alpha$: $\phi_\alpha(A)=0$. With these definitions, it is easy to see that the Hausdorff dimension could also have been defined as $\sup_\alpha$: $\phi_\alpha(A)=\infty$.

The Hausdorff dimension of a set A is equal to its *capacitory dimension*, which is defined as the supremum over β such that

$$\exists \text{ a probability measure } \mu \text{ supported on A such that } \sup_y \int \frac{\mu(dx)}{|x-y|^\beta} < \infty .$$

This condition is equivalent to

$$\exists \text{ a probability measure } \mu \text{ supported on A such that } \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta} < \infty .$$

The equality of these two dimensions (Hausdorff and capacitory) implies that if there exists a measure μ supported on a set A such that for each n

$$\iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^{\alpha-1/n}} < \infty ,$$

then the Hausdorff dimension of A is greater than or equal to α. This is how Kahane proved his result: it was shown that for a positive measure μ supported on A (which can be assumed to be closed) and a standard d-dimensional Brownian motion $\{B(t): t \geq 0\}$, then the (random) measure ν defined by

$$\nu(E) = \mu\{t: B(t) \in E\}$$

is supported by the Brownian image of A and that (for $2\beta < d$)

$$\iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^\beta} < \infty$$

implies that a.s.

$$\iint \frac{\nu(dx)\,\nu(dy)}{|x-y|^{2\beta}} < \infty . \tag{1}$$

Our method of proving the Theorem is showing that a.s. the local time integral (1) is finite for all paths visited by the Ornstein-Uhlenbeck process. Let us fix the set A. Without loss of generality, we may assume that A is a closed subset of the interval [0, 1] and that $\exists$ a probability measure μ supported by A such that $\exists$ C such that $\forall\, y, r$, $\mu\{(y-r, y+r)\} \leq Cr^\alpha$. This last condi-

tion of course implies

$$\forall\ \beta<\alpha,\ \sup_y \int \frac{\mu(dx)}{|x-y|^\beta}<M_\beta<\infty\ .$$

The existence of such a measure μ does not follow merely from the fact that the Hausdorff dimension of A is equal to α (see Taylor (1961) [11] for details). Let $\{O_s(\cdot): s\geq 0\}$ be the Ornstein-Uhlenbeck process on Wiener space. Define the measures ν_s to be

$$\nu_s(E)=\mu\{t: O_s(t)\in E\}.$$

Our result will be established if we can show for each s and every n,

$$F_n(s)=\iint \frac{\nu_s(dx)\,\nu_s(dy)}{|x-y|^{2(\alpha-1/n)}}<\infty$$

for every s and every n a.s., which will follow provided we show $\{F_n(s): s\geq 0\}$ is a.s. continuous.

To show $\{F_n(s): s\geq 0\}$ is a.s. continuous we use the following criterion of Kolmogorov:

A process $\{Z(t): t\geq 0\}$ has a continuous version if $\exists\ \beta>0$ and an integer k such that for all s and t,

$$E[|X(t)-X(s)|^k]<K|t-s|^{1+\beta} \tag{2}$$

(see e.g. Berman (1970) [1]).

If a process has a version with continuous paths, then any separable version must also have continuous paths (see Jain and Marcus (1978) [4]). We show conditon (2) holds for $F_n(\cdot)$ (Section 3) and that $\{F_n(s): s\geq 0\}$ is separable (Lemma 1 below), thereby establishing the Theorem.

Lemma 1. *For each n, the process $\{F_n(s): s\geq 0\}$ is separable.*

Proof. Define $F_n^M(s)$ as follows:

$$F_n^M(s)=\iint \frac{\nu_s(dx)\,\nu_s(dy)}{\max(|x-y|^{2(\alpha-1/n)},\ 1/M)}\ .$$

Then by the continuity of the Ornstein-Uhlenbeck process, $F_n^M(\cdot)$ is a.s. continuous. Letting M go to infinity, we see that the process $\{F_n(s): s\geq 0\}$ must be separable.

It only remains to show that condition (2) holds for each $F_n(\cdot)$. Meyer (1982) [8] noted that a realization of the Ornstein-Uhlenbeck process is

$$\{O_s(\cdot): s\geq 0\}=\{e^{-s/2}\,W(e^s,\ \cdot): s\geq 0\}\ ,$$

where W denotes the Brownian sheet. Therefore F_n has continuous sample paths if and only if

$$V_n(s) = \iint \frac{\mu(dx)\,\mu(dy)}{|W(s,x)-W(s,y)|^{2(\alpha-1/n)}}\,.$$

has continuous sample paths, where $s \geq 1$. Purely for greater notational simplicity we will work with $V_n(\cdot)$ rather than $F_n(\cdot)$.

2. Details

The following lemmas will be used in the proof of the Theorem given in Section 3. The proofs rest on tiresome calculations and are therefore omitted.

Lemma 2. *Let X and Y be Gaussian random variables with respective means m_x and m_y and variances τ_x^2 and τ_y^2. Then for $\alpha<1$ and some C,*

$$E\left[\left|\frac{1}{|X|^\alpha}-\frac{1}{|X-Y|^\alpha}\right|\right]<C(|m_y|^{1-\alpha}+|\tau_y|^{1-\alpha})/\tau_x\,.$$

Unfortunately, Lemma 2 gives a poor bound when either $|m_y|$ or $|\tau_y|$ is greater than $|\tau_x|$. For these cases we use the following:

Lemma 3. *Let X and Y be as in Lemma 2. Then for some finite C,*

$$E\left[\left|\frac{1}{|X|^\alpha}-\frac{1}{|X-Y|^\alpha}\right|\right]<\frac{C}{\tau_x^\alpha}\,.$$

Lemma 4. *Let $\{B(t): t\geq 0\}$ be a Brownian motion and $T=\{0, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}$ be a set of time points. Then given the values of B in T, the conditional distribution of $B(x)-B(y)$ is Gaussian with variance greater than*

$$\min\{(\max\{d(\{x\},T)/2, d(\{y\},T)/2\}), |x-y|/2\}\,.$$

Here $d(E,F)$ is defined to be equal to the infimum of the distance between a point in the set E and a point in the set F. The conditional mean of $B(x)-B(y)$ is a Gaussian variable with mean zero and variance bounded by $|x-y|$.

In the succeeding section we consider terms of the form

$$\prod_{i=1}^{n}\left|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2\alpha}}-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2\alpha}}\right|\,.$$

Let $\mathcal{F}_r$ denote the σ-field generated by $W(s,u_i)$, $W(s,v_i)$, $W(s+h,u_i)$, $W(s+h,v_i)$, $i=1,2,\cdots r$. Given $\mathcal{F}_r$, the conditional distribution of $(W(s,u_{r+1})-W(s,v_{r+1})$, $W(s+h,u_{r+1})-W(s+h,v_{r+1}))$ is that of (Y, Y+Z), where Y and Z are independently distributed as $N(c_Y, s\cdot k_r)$ and $N(c_Z, h\cdot k_r)$ respectively. The conditional means c_Y and c_Z are $\mathcal{F}_r$-measurable random variable with means zero and variances

bounded by $s\cdot(v_{r+1}-u_{r+1})$ and $h\cdot(v_{r+1}-u_{r+1})$ respectively. The term k_r is bounded below by

$$(3)\qquad d_{r+1}=\min\{\max\{d(u_{r+1},\{0,u_i,v_i\})/2,\, d(v_{r+1},\{0,u_i,v_i\})/2\},\ |u_{r+1}-v_{r+1}|/2\},\ i\leq r\,.$$

By applying Lemmas 2 and 3 and iterating Lemma 4, we obtain the following:

Lemma 5. *Given numbers $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, $u_i<v_i$ for each i, and numbers d_i, $i=1,\cdots,n$, defined as in* (3). *Then for $s\geq 1$,*

$$E\left[\prod_{i=1}^{n}\left|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}\right|\right]$$
$$\leq\prod_{i=1}^{n}\left\{\frac{C}{d_{i+1}^{\alpha-1/n}},\frac{C_n h^{1/2-\alpha+1/n}}{d_{i+1}^{1/2}}\right\}$$
$$\leq\prod_{i=1}^{n}\left\{\min\left\{\frac{C}{d_{i+1}^{\alpha-1/n}},\frac{C_n h^{1/2n}}{d_{i+1}^{\alpha-1/2n}}\right\}\right\}.$$

Once it is noted that the variance of c_Z is bounded by h, the proof of Lemma 5 is easy.

The following Lemma is well known, but I have been unable to find a reference.

Lemma 6. *Let μ be a measure on* $[0,1]$ *such that $\forall\, y, r$, $\mu\{(y-r,y+r)\}\leq Cr^{\alpha}$ for some constant C not depending on y or r. Then for some K not depending on h or y,*

$$\forall\, y, h,\qquad \int_{|x-z|<h}\frac{\mu(dx)}{|x-y|^{\alpha-\varepsilon}}<Kh^{\varepsilon}\,.$$

This lemma gives the following corollary which will be used in Section 3.

Corollary. *Given any fixed points $\{u_1, v_1, u_2, v_2, \cdots, u_r, v_r\}$ and a measure μ as in Lemma* 6, *the integral*

$$\int_{u_{r+1}<v_{r+1}}\mu(du_{r+1})\,\mu(dv_{r+1})\min\left\{\frac{C}{d_{r+1}^{\alpha-1/n}},\frac{C_n h^{1/2n}}{d_{r+1}^{\alpha-1/2n}}\right\}$$

is majorized by $K_{r,n}\,h^{1/2n}$, where $K_{r,n}$ depends on r and n only and is otherwise independent of the particular $\{u_i\}$ and $\{v_i\}$.

We are now prepared to assemble the proof of the Theorem.

3. Proof of the Theorem

As was noted in Section 1, the final result will be established if we can show $\forall\, n$ and $s\geq 1$

$$V_n(s) = \int \frac{\mu(dx)\,\mu(dy)}{|W(s,x)-W(s,y)|^{2(\alpha-1/n)}}$$

is continuous. Let us now consider n to be fixed. By Lemma 1 and Kolmogorov's criterion, it is sufficient to find a k such that

$$E[|V_n(s+h)-V_n(s)|^{2k}] \leq Ch^{1+\beta}$$

for some fixed C and $\beta>0$ and all h. Now $E[(V_n(s+h)-V_n(s))^{2k}] =$

$$E\Big[\sum_{A\subset\{1,2,\cdots,2k\}}(-1)^{|A|}\int\cdots\int\prod_{i\in A}\frac{\mu(du_i)\,\mu(dv_i)}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}$$
$$\int\cdots\int\prod_{i\in A^c}\frac{\mu(du_i)\,\mu(dv_i)}{|W(s+h,v_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}$$
$$\leq\int\cdots\int\prod_{i=1}^{2k}\mu(du_i)\,\mu(dv_i)\,E\Big[\prod_{i=1}^{2k}\Big|\frac{1}{|W(s,u_i)-W(s,v_i)|^{2(\alpha-1/n)}}$$
$$-\frac{1}{|W(s+h,u_i)-W(s+h,v_i)|^{2(\alpha-1/n)}}\Big|\Big].$$

By Lemma 5 this last quantity is majorized by

$$\int\cdots\int\prod_{i=1}^{2k}\mu(du_i)\,\mu(dv_i)\,\min\left\{\frac{C}{d_{r+1}^{\alpha-1/n}},\ \frac{C_n h^{1/2n}}{d_{r+1}^{\alpha-1/2n}}\right\},$$

which by Lemma 6 is less than or equal to

$$\prod_{i=1}^{2k}K_{i,n}h^{1/2n}\leq Q_{2k,n}h^{k/n}$$

for some constant $Q_{2k,n}$ not depending on h. Taking k greater than n, we complete the proof.

4. Related Questions

Given this Theorem, it is of interest to further investigate the case $2\alpha\geq d$. We know that if the Hausdorff dimension of a set A is greater than 1/2, then the image of A by one-dimensional Brownian motion a.s. has interior points (see Kaufmann (1975) [6] and Pitt (1978) [9]). By creating an appropriate continuous local time for the Ornstein-Uhlenbeck process, we can easily extend this result from a.s. to q.e.. Kahane (1968) [5] also showed that if a set has positive capacity w.r.t. $t^{-1/2}$, then a.s. its image by Brownian motion has positive Lebesgue measure. A simple martingale argument show s that this result also holds q.e.. Unfortunately we cannot find an extension of the converse (see Hawkes (1977) [3]) from a.s. to q.e..

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