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Osaka J. Math.
6 (1969), 143–146

A KÜNNETH FORMULA FOR EQUIVARIANT K-THEORY

Dedicated to Professor Atuo Komatu for his 60th birthday

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(Received December 9, 1968)

1. In this note we prove the following theorem for equivariant K-theory which is a generalization of Atiyah's Künneth formula for K-theory [1].

Theorem. *Let X and Y be compact Hausdorff spaces on which operate compact Lie groups G and H respectively. If the orbit spaces X/G and Y/H are of finite covering dimension and X (or Y) is locally G—(or H—) contractible, there holds an exact sequence*

$$0 \to \sum_{i+j=k} K_G^i(X) \otimes K_H^j(Y) \to K_{G\times H}^k(X\times Y) \to \sum_{i+j=k+1} \operatorname{Tor}(K_G^i(X), K_H^j(Y)) \to 0$$

where indices i, j and k are regarded as elements of Z_2.

The author wishes to express his thanks to Professor S. Araki for his many valuable suggestions and discussions.

2. Let G and H be compact Lie groups, (X, A) be a compact G-pair and (Y, B) a compact H-pair. Put

$$h_{1,(Y,B)}^*(X, A) = K_G^*(X, A) \otimes K_H^*(Y, B)$$
$$h_{2,(Y,B)}^*(X, A) = K_{G\times H}^*((X, A)\times(Y, B)) .$$

When $K_H^*(Y, B)$ is a free abelian group, $h_{1,(Y,B)}^*$ and $h_{2,(Y,B)}^*$ define Z_2-graded cohomology theories on the category whose objects are compact G-pairs.

If E is a G-vector bundle on X and F an H-vector bundle on Y, then $E\hat{\otimes}F$ is a $G\times H$-vector bundle on $X\times Y$. This defines a natural pairing

$$\mu' \colon K_G(X)\otimes K_H(Y) \to K_{G\times H}(X\times Y) .$$

And then we can extend this pairing to a homomorphism

$$\mu'' \colon K_G^{-m}(X, A)\otimes K_H^{-n}(Y, B) \to K_{G\times H}^{-m-n}((X, A)\times(Y, B))$$

making use of the canonical decomposition

$$\tilde{K}^{-1}_{G\times H}(X'\times Y')\cong\tilde{K}^{-1}_{G\times H}(X'\wedge Y')\oplus\tilde{K}^{-1}_{G\times H}(X')\oplus\tilde{K}^{-1}_{G\times H}(Y')$$

where X' and Y' are G-space and H-space with basepoints respectively. Clearly μ'' commutes with the Bott isomorphism and coboundary homomorphisms with respect to (X, A). Thus μ'' defines a cohomology operation

$$\mu:\ h^*_{1,(Y,B)}\to h^*_{2,(Y,B)}\,.$$

3. Let Z be a compact Hausdorff space with an action of a compact Lie group G on Z, Z' be a compact Hausdorff trivial G-space of finite covering dimension and $\pi:Z\to Z'$ a G-map. When $K^*_H(Y, B)$ is a free abelian group, we denote by $\mathfrak{S}_i$ the sheaves corresponding to the presheaves defined by

$$(h^q_{i,(Y,B)}\pi)(U)=h^q_{i,(Y,B)}(\pi^{-1}(\bar{U}))$$

for any open set U of Z' for $i=1$, 2. Then we get the following results by parallel discussions to [2], Lecture 3.

Proposition 1. *There are strongly convergent spectral sequences* $\{E_{r,(i,(Y,B))}\}$ *such that*

$$E_{2,(i,(Y,B))}=H^*(Z, \mathfrak{S}_i)$$

and $E_{\infty,(i,(Y,B))}$ *are the graded groups associated with filtrations of* $h^*_{i,(Y,B)}(Z)$ *respectively, and* μ *induces a morphism of these spectral sequences*

$$\{\mu_r\}:\ \{E_{r,(1,(Y,B))}\}\to\{E_{r,(2,(Y,B))}\}\,.$$

Next we show

Proposition 2. *Let G and H be compact Lie groups, and X'' and Y'' be compact Hausdorff G-space and H-space respectively. If the orbit spaces X''/G and Y''/H are of finite covering dimension, then we obtain isomorphisms*

(i) $$K^*_G(X'')\otimes K^*_H(H/H_0)\cong K^*_{G\times H}(X''\times H/H_0)$$

for any closed subgroup H_0 of H, and
(ii) *when $K^*_H(Y'')$ is a free abelian group,*

$$K^*_G(X'')\otimes K^*_H(Y'')\cong K^*_{G\times H}(X''\times Y'')\,.$$

Proof. Since $K^*_H(H/H_0)\cong R(H_0)$ [2] and $R(H_0)$ is a free abelian group, we can apply Proposition 1. If we put $Z=X''$, $Z'=X''/G$, $(Y, B)=(H/H_0, \phi)$ and π: $X''\to X''/G$, the projection, then μ_2: $E_{2,(1,H/H_0)}\to E_{2,(2,H/H_0)}$ is an isomorphism. Because, when we write $\pi(x)=[x]$ for any element x in X'' and denote the isotropy subgroup of G at x by G_x,

$$\pi^{-1}[x]=G/G_x$$

and

$$\begin{aligned} h^*_{1,H/H_0}(\pi^{-1}[x]) &= K^*_G(\pi^{-1}[x])\otimes K^*_H(H/H_0) \\ &\cong K^*_G(G/G_x)\otimes K^*_H(H/H_0) \\ &\cong R(G_x)\otimes R(H_0) \\ &\cong R(G_x\times H_0) \qquad\qquad \text{by [3], Lemma 3.2} \\ &\cong K^*_{G\times H}(G\times H/G_x\times H_0) \\ &\cong K^*_{G\times H}(\pi^{-1}[x]\times H/H_0) \\ &= h^*_{2,H/H_0}(\pi^{-1}[x])\,. \end{aligned}$$

Hence μ induces an isomorphism of sheaves $\mathfrak{S}_1\cong\mathfrak{S}_2$. And so μ_2 induces an isomorphism of the spectral sequences

$$\{\mu_r\}:\ \{E_{r,(1,H/H_0)}\}\cong\{E_{r,(2,H/H_0)}\} \qquad r\geqq 2\,,$$

Since the both spectral sequences are strongly convergent by Proposition 1, this completes the proof of (i). We can prove (ii) by a parallel argument making use of (i).

Proof of Theorem. Suppose that X is locally G-contractible. Under this hypothesis and the condition that $\dim X/G<\infty$, L. Hodgkin [4] proved that there exist a compact differentiable manifold N on which operates G and G-map $f\colon X\to N$ such that $f^*\colon K^*_G(N)\to K^*_G(X)$ is an epimorphism and $K^*_G(N)$ is a free abelian group.

Then we get a short exact sequence

$$0\to \tilde{K}^*_G(M_f/X)\to K^*_G(M_f)\to K^*_G(X)\to 0$$

where M_f is the mapping cylinder. Since $K^*_G(M_f)\cong K^*_G(N)$, $K^*_G(M_f)$ and $\tilde{K}^*_G(M_f/X)$ are free abelian groups. Further, $\dim M_f/G\leqq \mathrm{Max}\,(\dim N,\ \dim X/G+1)$ [5] and so is $\dim(M_f/X)/G$. Therefore we can deduce

$$\begin{aligned} K^*_G(M_f)\otimes K^*_H(Y) &\cong K^*_{G\times H}(M_f\times Y)\,, \\ \tilde{K}^*_G(M_f/X)\otimes K^*_H(Y) &\cong \tilde{K}^*_{G\times H}(M_f\times Y/X\times Y) \end{aligned}$$

from Proposition 2, (ii).

Next consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0\to K^*_G(X)*K^*_H(Y) & \to & \tilde{K}^*_G(M_f/X)\otimes K^*_H(Y) & \to & K^*_G(M_f)\otimes K^*_H(Y) & \to & K^*_G(X)\otimes K^*_H(Y) & \to 0 \\ J\uparrow & & \cong\downarrow\mu & & \cong\downarrow\mu & & \mu\downarrow & \\ \to K^*_{G\times H}(X\times Y) & \to & \tilde{K}^*_{G\times H}(M_f\times Y/X\times Y) & \to & K^*_{G\times H}(M_f\times Y) & \longrightarrow & K^*_{G\times H}(X\times Y) & \to \end{array}$$

We see that there exists a homomorphism J: $K^*_{G\times H}(X\times Y)\to K^*_G(X)*K^*_H(Y)$ determined uniquely by the above diagram and so that the sequence

$$0\to K^*_G(X)\otimes K^*_H(Y)\xrightarrow{\mu} K^*_{G\times H}(X\times Y)\xrightarrow{J} K^*_G(X)*K^*_H(Y)\to 0$$

is exact.

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