Title	Doubly transitive groups of even degree whose one-point stabilizer has a normal subgroup isomorphic to PSL(3,2^n)
Author(s)	Hiramine, Yutaka
Citation	Osaka Journal of Mathematics. 1979, 16(3), p. 817-830
Version Type	VoR
URL	https://doi.org/10.18910/9948
rights	
Note	

DOUBLY TRANSITIVE GROUPS OF EVEN DEGREE WHOSE ONE POINT STABILIZER HAS A NORMAL SUBGROUP ISOMORPHIC TO $PSL(3,2^n)$

YUTAKA HIRAMINE

1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha \in \Omega$. By [4], the product of all minimal normal subgroups of G_α is the direct product $A \times N$, where A is an abelian group and N is 1 or a nonabelian simple group.

In this paper we consider the case $N \simeq PSL(3,q)$ with q even and prove the following:

Theorem. *Let G be a doubly transitive permutation group on Ω of even degree and let α, $\beta \in \Omega$ $(\alpha \neq \beta)$. If G_α has a normal subgroup N^α isomorphic to $PSL(3,q)$, $q=2^n$, then N^α is transitive on $\Omega - \{\alpha\}$ and one of the following holds:*

(i) *G has a regular normal subgroup E of order $q^3=2^{3n}$, where n is odd and G_α is isomorphic to a subgroup of $\Gamma L(3,q)$. Moreover there exists an element g in $Sym(\Omega)$ such that $\alpha^g=\alpha$, $(G_\alpha)^g$ normalizes E and $A\Gamma L(3,q)\geq (G_\alpha)^g E\geq ASL(3,q)$ in their natural doubly transitive permutation representation.*

(ii) *$|\Omega|=22$, $G^\Omega=M_{22}$ and $N^\alpha\simeq PSL(3,4)$.*

(iii) *$|\Omega|=22$, $G^\Omega=Aut(M_{22})$ and $N^\alpha\simeq PSL(3,4)$.*

We introduce some notations.

$V(n,q)$: a vector space of dimension n over $GF(q)$

$\Gamma L(n,q)$: the group of all semilinear automorphism of $V(n,q)$

$A\Gamma L(n,q)$: the semidirect product of $V(n,q)$ by $\Gamma L(n,q)$ in its natural action

$ASL(n,q)$: the semidirect product of $V(n,q)$ by $SL(n,q)$ in its natural action

$F(X)$: the set of fixed points of a nonempty subset X of G

$X(\Delta)$: the global stabilizer of a subset Δ $(\subseteq\Omega)$ in X

X_Δ : the pointwise stabilizer of Δ in X

X^Δ : the restriction of X on Δ

$Sym(\Delta)$: the symmetric group on Δ

X^H :   the set of H-conjugates of X

$|X|_p$:   the maximal power of a prime p dividing the order of X

$I(X)$:   the set of involutions contained in X

E_m :   an elementary abelian group of order m

Other notations are standard and taken from [1].

2. Preliminaries

Lemma 2.1 *Let G be a doubly transitive permutation group on Ω of even degree, $\alpha \in \Omega$ and N^α a normal subgroup of G_α isomorphic to $PSL(2,q)$, $Sz(q)$ or $PSU(3,q)$ with $q(>2)$ even.   Then $N^\alpha \simeq PSL(2,q)$, $N^\alpha \not\simeq Sz(q)$, $PSU(3,q)$, N^α is transitive on $\Omega - \{\alpha\}$ and one of the following holds:*

(i)   *G has a regular normal subgroup E of order q^2, $N_\beta^\alpha = N^\alpha \cap N^\beta \simeq E_q$ and G_α is isomorphic to a subgroup of $\Gamma L(2,q)$.   Moreover there exists an element g in $Sym(\Omega)$ such that $\alpha^g = \alpha$, $(G_\alpha)^g$ normalizes E and $A\Gamma L(2,q) \geq (G_\alpha)^g E \geq ASL(2,q)$ in their natural doubly transitive permutation representation.*

(ii)   *$|\Omega| = 6$ and $G^\Omega = A_6$ or S_6.*

Proof.   By Theorem 2 of [2], it suffices to consider the case that $N_\beta^\alpha = N^\alpha \cap N^\beta \simeq E_q$ and G has a regular normal subgroup of order q^2.   Since $|N^\alpha : N_\beta^\alpha| = q^2 - 1$, N^α is transitive on $\Omega - \{\alpha\}$.

Let E be the regular normal subgroup of G.   Then we may assume $\Omega = E$, $\alpha = 0 \in E$ and the semidirect product $GL(E)E$ is a subgroup of $Sym(\Omega)$. There is a subgroup H of $GL(E)$ such that $H \simeq \Gamma L(2,q)$ and $HE \simeq A\Gamma L(2,q)$. Let L be the normal subgroup of H isomorphic to $SL(2,q)$.   Then $L_\beta \simeq E_q$ for $\beta \in \Omega - \{\alpha\}$.   Hence $(N^\alpha)^{\Omega - \{\alpha\}} \simeq L^{\Omega - \{\alpha\}}$ and so there are an automorphism f from N^α to L and $g \in Sym(\Omega)$ satisfying $\alpha^g = \alpha$ and $(\beta^x)^g = (\beta^g)^{f(x)}$ for each $\beta \in \Omega - \{\alpha\}$ and $x \in N^\alpha$.   From this, $(\beta^g)^{g^{-1}xg} = (\beta^x)^g = (\beta^g)^{f(x)}$, so that $g^{-1}xg = f(x)$. Hence $g^{-1}N^\alpha g = L$.

Set $S = L_\beta$, $X = Sym(\Omega) \cap N(L)$, $D = C_X(L)$ and $Y = N_L(S)$.   By the properties of $A\Gamma L(2,q)$, L is transitive on $\Omega - \{\alpha\}$, $|F(S)| = q$ and $Y/S \simeq Z_{q-1}$.   Hence D is semi-regular on $\Omega - \{\alpha\}$ and $Y^{F(S)}$ is regular on $F(S) - \{\alpha\}$ and so $D \simeq D^{F(S)} \leq Y^{F(S)}$ because $[D, N^\alpha] = 1$.   Therefore $D \leq Z_{q-1}$.   Since X/DL is isomorphic to a subgroup of the outer automorphism group of $SL(2,q)$, we have $|X| \leq |\Gamma L(2,q)|$, while $\Gamma L(2,q) \simeq H \leq X$.   Hence $X = H$ and X normalizes E.   Therefore, as $(G_\alpha)^g \trianglerighteq (N^\alpha)^g = L$, we have $(G_\alpha)^g \leq H$.   Thus Lemma 2.1 is proved.

Lemma 2.2 *Let G be a doubly transitive permutation group on Ω of even degree and N^α a nonabelian simple normal subgroup of G_α, $\alpha \in \Omega$.   If $C_G(N^\alpha) \neq 1$, then $N_\beta^\alpha = N^\alpha \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega - \{\alpha\}$.   Moreover $C_G(N^\alpha) = 0(N^\alpha)$.*

Proof.   See Lemma 2.1 of [2].

Lemma 2.3 *Let G be a transitive permutation group on a finite set Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then*

$$|F(M)| = |N_G(M)| \times |\{M^g \mid M^g \subseteq H, g \in G\}| / |H| .$$

Proof.   See Lemma 2.2 of [2].

Lemma 2.4 *Let H be a transitive permutation group on a finite set Δ and N a normal subgroup of H. Assume that a subgroup X of N satisfies $X^H = X^N$. Then*
  (i) $|F(X) \cap \beta^N| = |F(X) \cap \gamma^N|$ *for β, $\gamma \in \Delta$.*
  (ii) $|F(X)| = |F(X) \cap \beta^N| \times r$, *where r is the number of N-orbits on Δ.*

Proof.   By the same argument as in the proof of Lemma 2.4 of [3], we obtain Lemma 2.4.

2.5   Properties of $PSL(3,q)$, $q=2^n$.

Let $N_1 = SL(3,q)$, $S_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in GF(q) \right\}$, $A_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b \in \right.$

$\left. GF(q) \right\}$, $B_1 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| b, c \in GF(q) \right\}$ and $Z = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \middle| d \in GF(q), d^3 = 1 \right\}$.

Then $|Z| = (3, q-1)$ and $\bar{N}_1 = N_1/Z$ is isomorphic to $PSL(3,q)$. Set $N = \bar{N}_1$, $S = \bar{S}_1$, $A = \bar{A}_1$ and $B = \bar{B}_1$. Then the following hold.
  (i) N is a nonabelian simple group of order $q^3(q-1)^3(q+1)(q^2+q+1)/(3,q-1)$.

  (ii) $|S| = q^3$, $S' = \Phi(S) = Z(S) = \{x^2 \mid x \in S\} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| b \in GF(q) \right\} \simeq E_q$,

$S/S' \simeq E_{q^2}$ and S is a Sylow 2-subgroup of N.

  (iii) $S = \langle A, B \rangle$, $A \cap B = Z(S)$, $I(S) \subseteq A \cup B$ and each elementary abelian subgroup of S is contained in A or B. Let $z \in I(S) - Z(S)$. Then $C_S(z) = A$ or B.

  (iv)   Set $M_1 = A^N$, $M_2 = B^N$. Then $M_1 \neq M_2$ and $M_1 \cup M_2$ is the set of all subgroup of N isomorphic to E_{q^2}.

  (v)   Let z be an involution of N. Then $I(N) = z^N$ and $|C_N(z)| = (q-1)q^3/(3,q-1)$.

  (vi)   Let E denote A or B. Then $|N_N(E)| = (q-1)^2(q+1)q^3/(3,q-1)$, $N_N(E)/E \simeq Z_k \times PSL(2,q)$, where $k = (q-1)/(3,q-1)$ and $N_N(E)$ is a maximal subgroup of N.

  (vii)   Set $M = (N_N(E))'$. If $q > 2$, then $M = M'$, $M \trianglerighteq E$, $M/E \simeq PSL(2,q)$ and M acts irreducibly on E.

  (viii)   Set $\Delta = E^N$. Then $|\Delta| = q^2+q+1$ and by conjugation N is doubly transitive on Δ, which is an usual doubly transitive permuation representation

of N. If $C \in \{A,B\} - \{E\}$, $|F(C)| = q+1$, C is a Sylow 2-subgroup of $N_{F(C)}$ and C is semi-regular on $\Delta - F(C)$.

Lemma 2.6 ([6]).   *Let notations be as in (2.5) and set $G = Aut(N)$.   Then the following hold.*

(i)   *There exist in G a diagonal automorphism d, a field automorphism f and a graph automorphism g and satisfy the following:*

$$G = \langle g, f, d \rangle N \trianglerighteq H_1 = \langle f, d \rangle N \trianglerighteq H_2 = \langle d \rangle N, \quad H_1 = P\Gamma L(3,q), \quad H_2 = PGL(3,q)$$

$$H_2/N \simeq Z_r, \text{ where } r = (3, q-1), \quad G/H_1 \simeq Z_2, \quad H_1/H_2 \simeq Z_n \text{ and } G/H_2 \simeq Z_2 \times Z_n.$$

(ii) $M_1 = A^{H_1}$, $M_2 = B^{H_1}$ and $A^g = B$.

Lemma 2.7   *Let $N = PSL(3,q)$, where $q = 2^n$. Let R be a cyclic subgroup of N of order $q+1$ and Q a nontrivial subgroup of R. Then $N_N(Q) = N_N(R) \simeq Z_k \times D_{2(q+1)}$, where $k = (q-1)/(3, q-1)$ and $D_{2(q+1)}$ is a dihedral group of order $2(q+1)$.*

Proof.   We consider the group N as a doubly transitive permutation group on $\Delta = PG(2,q)$ with $q^2 + q + 1$ points.   By (2.5) (i), R is a cyclic Hall subgroup of N and so we may assume $R \leq N_\alpha$, where $\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in PG(2,q)$.   Since $|N_{\alpha\beta}| = (q-1)^2 q^2/(3, q+1)$ for $\alpha \neq \beta \in \Delta$ and $(q+1, (q-1)^2 q^2) = 1$, R is semiregular on $\Delta - \{\alpha\}$.   Hence $N_N(Q) \leq N_\alpha$.   Put $E = 0_2(N_\alpha)$.   Then $N_\alpha = N_N(E)$ by (2.5) (viii) and $N_N(Q)E/E \simeq Z_k \times D_{2(q+1)}$ by (2.5) (vi).   Since $N_N(Q) \cap E = C_E(Q) = 1$ by (2.5) (v).   Hence $N_N(Q) \simeq Z_k \times D_{2(q+1)}$.   As R is cyclic, $N_N(R) \leq N_N(Q)$.   Thus $N_N(Q) = N_N(R) \simeq Z_k \times D_{2(q+1)}$.

Lemma 2.8   *Let $N = PSL(3,q)$, $q = 2^n$ and let $H(\neq N)$ be a subgroup of N of odd index.   Then $H \leq N_N(E)$ for an elementary abelian subgroup E of N of order q^2.*

Proof.   Let S, A and B be as in (2.5) and let Δ be as in Lemma 2.7.   Since $|N:H|$ is odd, H contains a Sylow 2-subgroup of N and so we may assume $S \leq H$.   Set $\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.   Then $S \leq N_\alpha = N_N(A)$, $S_\beta = B$, $S_\gamma = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| a \in GF(q) \right\} \simeq E_q$ and hence $|\alpha^S| = 1$, $|\beta^S| = q$ and $|\gamma^S| = q^2$.

If $\alpha^H = \{\alpha\}$, $H \leq N_\alpha = N_N(A)$ and the lemma holds.   By (2.5) (i), $(q^2+1, |N|) = 1$.   Hence $\alpha^H \neq \{\alpha\} \cup \gamma^S$, so that we may assume either $\alpha^H = \{\alpha\} \cup \beta^H$ or $\alpha^H = \Delta$.

If $\alpha^H = \{\alpha\} \cup \beta^H$, $\alpha^H = F(B)$ and B is a unique Sylow 2-subgroup of $H_{F(B)}$ by (2.5) (viii).   Hence $H \trianglerighteq B \simeq E_{q^2}$ and the lemma holds.

If $\alpha^H = \Delta$, by (2.5) (iv), $N_H(A)^{F(A)}$ is transitive and so $|H|$ is divisible by $q+1$. Since $(q^2+q+1, q+1)=1$, $|H_\alpha|$ is divisible by $q+1$. By (2.5) (vi) and by the structure of $PSL(2,q)$, $Z_m \times PSL(2,q) \simeq H_\alpha/A \leq N_N(A)/A$, where m is a divisor of $(n-1)/(3,n-1)$. Therefore $|N: H| \leq q-1$. We now consider the action of N on the coset $\Gamma = N/H$. As $|\Gamma| \neq 1$ and N is a simple group, N^Γ is faithful. But N has a cyclic subgroup of order $q+1$ and so $|\Gamma| > q+1$, which implies $|N: H| > q+1$, a contradiction.

Lemma 2.9 Let $N = PSL(3,q)$, where $q = 2^{2m}$ and t a field automorphism of N of order 2. Let S be a t-invariant Sylow 2-subgroup of N. Then the following hold.

(i) $Z(\langle t \rangle S) \simeq E_{\sqrt{q}}$.

(ii) If S_1 is a subgroup of $\langle t \rangle S$ isomorphic to S, then $S_1 = S$.

Proof. Since $C_S(t)$ is isomorphic to a Sylow 2-subgroup of $PSL(3, \sqrt{q})$, $Z(C_S(t)) \simeq E_{\sqrt{q}}$ and $Z(C_S(t)) \leq Z(S)$ by (2.5) (ii). Hence $Z(\langle t \rangle S) = Z(\langle t \rangle S) \cap \langle t \rangle C_S(t) \cap C(Z(S)) = Z(\langle t \rangle S) \cap C_S(t) = Z(C_S(t)) \simeq E_{\sqrt{q}}$. Thus we have (i).

Suppose $S_1 \neq S$. Then $\langle t \rangle S = S_1 S \trianglerighteq S_1$ and $[\langle t \rangle S: S] = [S_1: S_1 \cap S] = 2$. If $Z(S_1) \not\leq S$, we have $S_1 = \langle z \rangle \times (S_1 \cap S)$ for an involution z in $Z(S_1) - S$. By (2.5) (ii), $z \in \Phi(S_1)$ and so $S_1 = \langle z, S_1 \cap S \rangle = S_1 \cap S$, a contradiction. Hence $Z(S_1) \leq S$.

If $Z(S_1) = Z(S)$, $E_q \simeq Z(S) \leq Z(S_1 S) = Z(\langle t \rangle S) \simeq E_{\sqrt{q}}$ by (i), which is a contradiction. Hence $Z(S_1) \neq Z(S)$.

Let z be an involution in $Z(S_1) - Z(S)$. Then $C_S(z) \simeq E_{q^2}$ by (2.5) (iii). On the other hand, $S_1 \leq C_{\langle t \rangle S}(z)$ and $[C_{\langle t \rangle S}(z): C_S(z)] = 1$ or 2. From this S_1 has an elementary abelian subgroup of index 2. Hence $q = 2$, a contradiction. Thus we have (ii).

3. Proof of the theorem

Throughout the rest of the paper, G^Ω always denote a doubly transitive permutation group satisfying the hypotheses of the theorem.

Since $G_\alpha \trianglerighteq N^\alpha$, $|\beta^{N^\alpha}| = |\gamma^{N^\alpha}|$ for $\beta, \gamma \in \Omega - \{\alpha\}$ and so $|\Omega| = 1 + r |\beta^{N^\alpha}|$, where r is the number of N^α-orbits on $\Omega - \{\alpha\}$. Hence r is odd and N_β^α is a proper subgroup of N^α of odd index for $\alpha \neq \beta \in \Omega$. Therefore, by Lemma 2.8 $N_\beta^\alpha \trianglerighteq A$ for some elementary abelian subgroup A of order q^2. Let S be a Sylow 2-subgroup of N_β^α. Then, by (2.5) (iii) there exists a unique elementary abelian 2-subgroup B of S such that $A \simeq B \simeq E_{q^2}$ and $A \neq B$. Set $M_1 = A^{N^\alpha}$, $M_2 = B^{N^\alpha}$ and $K = G_\alpha(M_1) = G_\alpha(M_2)$. By (2.5) (iv), $M_1 \cup M_2$ is the set of all elementary abelian 2-subgroup of N^α of order q^2 and G_α acts on $\{M_1, M_2\}$, so that $G_\alpha/K \leq Z_2$. Hence K is transitive on $\Omega - \{\alpha\}$.

(3.1) Let $E = A$ or B. Then $N_{G_\alpha}(E)$ is transitive on $F(E) - \{\alpha\}$.

Proof. If $E^h \leq K_\beta$ for some $h \in K$, $E^h \leq N^\alpha \cap K_\beta = N_\beta^\alpha$. Since $E^{N^\alpha} = E^K$ and $A^K \neq B^K$, E^h is conjugate to E in N_β^α. By a Witt's theorem $N_K(E)$ is transitive on $F(E) - \{\alpha\}$. Thus $N_{G_\alpha}(E)$ is transitive on $F(E) - \{\alpha\}$.

(3.2)   *If $q = 2$, G^Ω is of type* (i) *of the theorem.*

Proof. Assume $q = 2$. We note that $PSL(3,2)$ is isomorphic to $PSL(2,7)$. It follows from [3] that G has a regular normal subgroup R.

Since K is transitive on $\Omega - \{\alpha\}$, by Lemmas 2.3 and 2.4

$$|F(A)| = 1 + \frac{|N^\alpha \cap N(A)|}{|N_\beta^\alpha|} r = \frac{24r}{|N_\beta^\alpha|} + 1 \quad \text{and}$$

$$|F(B)| = 1 + \frac{|N^\alpha \cap N(B)| \, |N_\beta^\alpha : N_\beta^\alpha \cap N(B)|}{|N_\beta^\alpha|} r = \frac{24r}{|N_\beta^\alpha \cap N(B)|} + 1 \, .$$

Let $E = A$ or B. As $N_R(E) \neq 1$, $N_G(E)^{F(E)}$ is doubly transitive by (3.1). Hence $E \leq N^\beta$ and $|F(A)| = 2^a$, $|F(B)| = 2^b$ for some integers a, b. From this $S = \langle A, B \rangle \leq N^\alpha \cap N^\beta$ and $|N_\beta^\alpha : N^\alpha \cap N^\beta|$ is odd. Hence, if $S^g \leq G_{\alpha\beta}$, $S^g \leq N_\beta^\gamma \cap N_\beta^\gamma$, where $\gamma = \alpha^g$ and so $S^g \leq N^\alpha \cap N^\beta$. Since S and S^g are Sylow 2-subgroups of $N^\alpha \cap N^\beta$, S^g is conjugate to S in $N^\alpha \cap N^\beta$. By a Witt's theorem $N_G(S)^{F(S)}$ is a doubly transitive permutation group with a regular normal subgroup $N_R(S)$. Hence $|F(S)| = 2^c$ for an integer c. By Lemmas 2.3 and 2.4,

$$|F(S)| = 1 + \frac{8 \times |N_\beta^\alpha : S|}{|N_\beta^\alpha|} r = r + 1 = 2^c \, .$$

Let z be an involution of $Z(S)$ and assume $z^g \in G_\alpha$ for some $g \in G$. Then $z^g \in N_\alpha^\gamma$, where $\gamma = \alpha^g$. Since $|N_\alpha^\gamma : N^\gamma \cap N^\alpha|$ is odd, z^g is contained in N^α. By (2.5) (v), z^g is conjugate to z in N^α. Hence $C_G(z)$ is transitive on $F(z)$ and by Lemmas 2.3 and 2.4,

$$|F(z)| = 1 + \frac{8 \times |I(N_\beta^\alpha)|}{|N_\beta^\alpha|} r \, .$$

Suppose $N_\beta^\alpha = S$. Then $|F(A)| = 3r + 1 = 2^a = 2^c + 2r$ and $|F(z)| = 5r + 1$. Hence $r = 1$. Since $N_R(A) = C_R(A) \leq C_G(z)$ and $N_R(A) \simeq E_4$, $|F(z)|$ is divisible by 4. But $|F(z)| = 5r + 1 = 6$. This is a contradiction.

Suppose $N_\beta^\alpha \neq S$. Then $N_\beta^\alpha = N_{N^\alpha}(A)$ as $N_{N^\alpha}(A) \simeq S_4$. From this, $|F(B)| = 2^b = 2^c + 2r$ and so $r = 1$. Hence $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 8$. Thus $|R| = 8$ and $G_\alpha \simeq GL(3,2)$, hence $G \simeq AL(2,3)$.

By (3.2), it suffices to consider the case $q > 2$ to prove the theorem. From now on we assume the following.

Hypothesis (*): $q = 2^n \geq 4$

(3.3)   *The following hold.*

    (i) $|N_\beta^\alpha/N^\alpha \cap N^\beta|$ *is odd.*

    (ii) *Let* $\gamma \in \Omega$ *and* S_0 *a 2-subgroup of* N^γ.   *Then* $F(S_0) = \{\delta \in \Omega \,|\, S_0 \leq N^\delta\}$.

    Proof. Suppose false and let T be a Sylow 2-subgroup of $N_\alpha^\beta N_\beta^\alpha$ such that $T \geq S$. Then $T \neq S$. Set $S_1 = T \cap N_\beta^\alpha$ and $S_2 = T \cap N^\alpha \cap N^\beta$. Then S_1 is a Syow 2-subgroup of N_α^β, $S_1 \neq S$ and S_1, S_2 and S are normal subgroups of T. By Lemma 2.2, $S_1 N^\alpha/N^\alpha$ is isomorphic to a subgroup of the outer automorphism group of N^α. It follows from Lemma 2.6 that $S_1 N^\alpha/N^\alpha$ is abelian of 2-rank at most 2. Since $S_1 N^\alpha/N^\alpha \simeq S_1/S_2$ and $S_1 \simeq S$, we have $S_1/S_2 \leq E_4$ by (2.5) (ii).

    Let A_1, B_1 be the subgroups of S_1 such that $A_1 \simeq B_1 \simeq E_{q^2}$ and $A_1 \cap S_2 \leq A$, $B_1 \cap S_2 \leq B$. Since $A_1/A_1 \cap S_2 \simeq A_1 S_2/S_2 \leq S_1/S_1 \leq E_4$ and by the hypothesis (*), $q \geq 4$, we have $|A_1 \cap S_2| \geq q^2/4$. Therefore, if $A_1 \cap S_2 \leq Z(S)$, then $q = 4$, $A_1 \cap S_2 = Z(S)$ and $T = A_1 S$ and so $Z(S) \leq Z(T)$, contrary to Lemma 2.9. Hence $A_1 \cap S_2 \not\leq Z(S)$. Similarly $B_1 \cap S_2 \not\leq Z(S)$.

    Let $x \in A_1 \cap S_2 - Z(S)$. Then $x \in A^y \leq S$ for each $y \in A_1$ and so A_1 normalizes A. Hence A_1 normalizes B. Similarly B_1 normalizes A and B. From this $T = \langle A_1, B_1 \rangle S \geq A$, B and so $S_1 N^\alpha \leq K$. Hence $S_1 N^\alpha/N^\alpha \simeq S_1/S_2 \simeq Z_2$, so that there exists a field automorphism t of order 2 such that $T = \langle t \rangle S \rhd S$. Since $S_1 \leq T$ and $S_1 \simeq S$, we have $S_1 = S$ by Lemma 2.9, a contradiction. Thus (i) holds.

    Let $\delta \in F(S_0) - \{\gamma\}$. Then $S_0 \leq N_\delta^\gamma$. Since $N_\delta^\gamma \geq N^\gamma \cap N^\delta$ and $|N_\delta^\gamma/N^\gamma \cap N^\delta|$ is odd by (i), $S_0 \leq N^\gamma \cap N^\delta \leq N^\delta$. Hence $F(S_0) \subseteq \{\delta \in \Omega \,|\, S_0 \leq N^\delta\}$. The converse implication is clear. Thus (ii) holds.

    (3.4) *The following hold.*

    (i) $N_G(B)^{F(B)}$ *is doubly transitive.*

    (ii) *If* $F(A) \neq \{\alpha, \beta\}$, $N_G(A)^{F(A)}$ *is doubly transitive.*

    Proof. Let $E = A$ or B. By (3.3) (i), S is a Sylow 2-subgroup of N_β^α. Therefore, by a similar argument as in (3.1), $N_{G_\beta}(E)$ is transitive on $F(E) - \{\beta\}$. Suppose $N_G(E)^{F(E)}$ is not doubly transitive. Then, $F(E) = \{\alpha, \beta\}$ by (3.1) and (3.3). Since $N_{N^\alpha}(E)$ acts on $F(E)$ and fixes $\{\alpha\}$, we have $N_{N^\alpha}(E) \leq N_\beta^\alpha$. On the other hand $N_{N^\alpha}(E)$ is a maximal subgroup of N^α by (2.5) (vi). Hence $N_{N^\alpha}(E) = N_\beta^\alpha$. If $E = B$, then $N_\beta^\alpha \not\rhd A$, a contradiction. Thus $E = A$ and (3.4) follows.

    (3.5) *The following hold.*

    (i) *Put* $M = (N_{N^\alpha}(A))'$. *Then* $F(M) = F(A)$.

    (ii) $N_\beta^\alpha = N_\gamma^\alpha$ *for each* $\gamma \in F(A) - \{\alpha\}$.

    Proof. Suppose $F(M) \neq F(A)$. Then $M \not\leq N_G(A)_{F(A)}$. It follows from (3.4) that $F(A) \neq \{\alpha, \beta\}$ and $N_G(A)^{F(A)}$ is doubly transitive. Moreover by (2.5) (vii) $N_{G_\alpha}(A)^{F(A)} \geq M^{F(A)} \simeq PSL(2, q)$ as $q > 2$. By Lemma 2.1, $r = 1$ and either (1) $q =$

4 and $N_G(A)^{F(A)} = A_6$ or S_6 or (2) $|F(A)| = q^2$.

If (1) holds, $|F(A)| = 1 + |N_{N^\alpha}(A): N_\beta^\alpha| = 1 + 2^6 \cdot 3 \cdot 5/|N_\beta^\alpha| = 6$ and so $|N_\beta^\alpha| = 2^6 3$. Hence $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 1 + 2^6 \cdot 3^2 \cdot 5 \cdot 7/2^6 \cdot 3 = 2 \cdot 53$. Let z be an involution of $N^\alpha \cap N^\beta$. Then, by (2.5) (v) and (3.3), $z^G \cap G_\alpha = z^{G_\alpha}$, so that $C_G(z)^{F(z)}$ is transitive by a Witt's theorem. On the other hand $|F(z)| = 1 + \dfrac{|C_{N^\alpha}(z)| \times |I(N_\beta^\alpha)|}{|N_\beta^\alpha|} = 1 + 2^6 \cdot 3^3/2^6 \cdot 3 = 10$. In particular $|C_G(z)|$ is divisible by 5. Let R be a Sylow 5-subgroup of $C_G(z)$. Then $|\Omega|$, $|G_\alpha: N^\alpha|$ and $|N_\beta^\alpha|$ are not divisible by 5 and so $F(R) = \{\gamma\}$ and $R \leq N^\gamma$ for some $\gamma \in \Omega$. Therefore $\langle z \rangle \times R \leq N^\gamma$ by (3.3) (ii). But $|C_{N^\gamma}(z)| = 2^6$ by (2.5) (v). This is a contradiction.

If (2) holds, $q^2 = |F(A)| = 1 + |N_{N^\alpha}(A): N_\beta^\alpha|$, hence $|N_\beta^\alpha| = (q-1)q^3/(3, q-1)$. From this $|\Omega| = 1 + |N^\alpha: N_\beta^\alpha| = 1 + (q-1)(q+1)(q^2+q+1) = q(q^3+q^2-1)$. Hence $|G|_2 = |\Omega|_2 \times |G_\alpha|_2 = q \times |G_\alpha: K| \times |K|_2$. On the other hand $|N_G(A)|_2 = |F(A)| \times |N_{G_\alpha}(A)|_2 = q^2 |K|_2$ because $K = N_{G_\alpha}(A)N^\alpha$. Therefore $q^2 |K|_2 = |N_G(A)|_2 \leq |G|_2 = q \times |G_\alpha: K| \times |K|_2 \leq 2q|K|_2$ and we obtain $q = 2$, contrary to the hypothesis (*). Thus we have (i).

Let $\gamma \in F(A) - \{\alpha\}$. By (i) and (3.4) (ii), $N_\gamma^\alpha \trianglerighteq A$ and $M \leq N_\gamma^\alpha$. Since $N_{N^\alpha}(A)/M \simeq Z_k$, where $k = (q-1)/(3, q-1)$ and $|N_\beta^\alpha/M| = |N_\gamma^\alpha/M|$, we have $N_\beta^\alpha = N_\gamma^\alpha$. Thus (ii) holds.

(3.6) $B \notin A^G$ and $G_\alpha = K$.

Proof   If $B \in A^G$, by (3.4) (i), there is an element $g \in G_{\alpha\beta}$ such that $B = A^g$. Hence $N_\beta^\alpha = g^{-1}N_\beta^\alpha g \trianglerighteq g^{-1}Ag = B$ and so M normalizes $\langle A, B \rangle = S$, a contradiction.

(3.7)   Set $L = (N_{N^\alpha}(B))'$. Then $r = 1$, $L_{F(B)} = B$, $L^{F(B)} = L/B \simeq PSL(2, q)$, $L_\beta = S$ and one of the following holds.
   (i) $C_G(N^\alpha) = 1$, $|F(B)| = 6$, $q = 4$ and $N_G(B)^{F(B)} = A_6$ or S_6.
   (ii) $C_G(N^\alpha) \leq Z_{q-1}$, $|F(B)| = q^2$ and $N_G(B)^{F(B)}$ has a regular normal subgroup.

Proof.   By (3.4) (i), $N_G(B)^{F(B)}$ is doubly transitive. If $L \leq G_{\alpha\beta}$, then $L \leq N_\beta^\alpha$ and so $B \trianglelefteq L = L' \leq (N_\beta^\alpha)' = M$. Therefore $L = M$ and $M \trianglerighteq \langle A, B \rangle = S$, a contradiction. Hence $L \nleq G_{\alpha\beta}$. From this $N_{G_\alpha}(B)^{F(B)} \trianglerighteq L^{F(B)} \simeq PSL(2, q)$ and (3.7) follows from Lemmas 2.1 and 2.2.

(3.8)   If (i) of (3.7) occurs, then we have (ii) or (iii) of the theorem.

Proof.   Since $|F(B)| = 1 + |N_{N^\alpha}(B): N_{N_\beta^\alpha}(B)| = 6$ and $|N_\beta^\alpha: N_{N_\beta^\alpha}(B)| = |N_\beta^\alpha: N_{N_\beta^\alpha}(S)| = 5$, we have $|N_\beta^\alpha| = 2^6 \cdot 3 \cdot 5$. Hence $N_\beta^\alpha = N_{N^\alpha}(A)$ and so $|\Omega - \{\alpha\}| = |N^\alpha: N_\beta^\alpha| = 21$. By (3.6), $PSL(3, 4) \leq (G_\alpha)^{\Omega - \{\alpha\}} \leq P\Gamma L(3, 4)$ in their natural doubly transitive permutation representation and hence (3.8) follows from Satz 7 of [7].

In the rest of this paper, we consider the case (ii) of (3.7). From now on

we assume the following.

Hypothesis (∗∗): $r=1$, $q=2^n>2$, $|F(B)|=q^2$ and $N_G(B)^{F(B)}$ is a doubly transitive permutation group with a regular normal subgroup.

(3.9) *The following hold.*

(i) $N_\beta^\alpha=N^\alpha\cap N^\beta=M$ and $|N_\beta^\alpha|=(q-1)(q+1)q^3$.

(ii) *n is odd.*

(iii) $|F(A)|=q$.

Proof. Since $q^2=|F(B)|=1+|N_{N^\alpha}(B):N_{N_\beta^\alpha}(B)|$ by (3.7), we have $|N_{N_\beta^\alpha}(B)|=|N_{N^\alpha}(B)|/(q^2-1)=(q-1)q^3/(3,q-1)$. As $N_\beta^\alpha\trianglerighteq A$, $N_{N_\beta^\alpha}(B)=N_{N_\beta^\alpha}(\langle A,B\rangle)=N_{N_\beta^\alpha}(S)$. On the other hand, from (2.5) (vi) $|N_{N_\beta^\alpha}(S)|=|N_\beta^\alpha: M|\times|N_M(S)|=|N_\beta^\alpha: M|\times(q-1)q^3$. Therefore $(3,q-1)=1$ and $|N_\beta^\alpha: M|=1$. Hence $N_\beta^\alpha=M$ and n is odd. By (3.3) (i) and (2.5) (vii), $N_\beta^\alpha=N^\alpha\cap N^\beta$. Hence $F(A)=1+|N_{N^\alpha}(A)|/|N^\alpha|=q$. Thus we have (3.9).

(3.10) *Put $m=|G_\alpha: N^\alpha|$. Then the following hold.*

(i) *m is odd and S is a Sylow 2-subgroup of G_α.*

(ii) $|\Omega|=q^3$ and $|G|=q^6(q-1)^2(q+1)(q^2+q+1)m$.

Proof. Set $C^\alpha=C_G(N^\alpha)$. By (3.6), (3.9) (ii) and Lemma 2.6, $|G_\alpha/C^\alpha N^\alpha|$ is odd. Since $C^\alpha\cap N^\alpha=1$, $m=|G_\alpha/C^\alpha N^\alpha|\cdot|C^\alpha|$ and so m is odd by Lemma 2.2. Therefore S is a Sylow 2-subgroup of G_α and so (i) holds.

Since $|\Omega|=1+|N^\alpha: N_\beta^\alpha|$, $|\Omega|=q^3$ by (3.9). From this $|G|=|\Omega|\times|G_\alpha|=q^3 m|N^\alpha|=q^6(q-1)^2(q+1)(q^2+q+1)m$. Thus (ii) holds.

(3.11) *Let z be an involution of G_α. Then $|F(z)|=q^2$. In particular B is semi-regular on $\Omega-F(B)$.*

Proof. By (3.10) (ii), z is contained in N^α. By (2.5) (vii) and (3.9) (ii), $|I(N_\beta^\alpha)|=|N_\beta^\alpha: N_{N_\beta^\alpha}(S)|\times(q^2-q)+q^2-1=(q+1)(q^2-q)+q^2-1=(q-1)(q+1)^2$, hence $|F(z)|=1+q^3(q-1)\times(q-1)(q+1)^2/q^3(q-1)(q+1)=q^2$ by Lemma 2.3. As $|F(B)|=q^2$, B is semi-regular on $\Omega-F(B)$.

(3.12) *Set $\Delta=F(B)$. Then the following hold.*

(i) $G_\Delta\trianglerighteq B$ and B is a Sylow 2-subgroup of G_Δ.

(ii) $G(\Delta)=N_G(B)$ and $|N_G(B)|=q^5(q-1)^2(q+1)m$.

Proof. Since $N_{N^\alpha}(B)\leq N^\alpha(\Delta)\neq N^\alpha$ and $N_{N^\alpha}(B)$ is a maximal subgroup of N^α, we have $N_{N^\alpha}(B)=N^\alpha(\Delta)$. By (3.7), B is a normal Sylow 2-subgroup of $(N^\alpha)_\Delta$ and (i) follows immediately from (3.10) (i).

Since $G(\Delta)\trianglerighteq G_\Delta$ and B is a characteristic subgroup of G_Δ by (i), we have $G(\Delta)\leq N_G(B)$. The converse implication is clear. Thus $G(\Delta)=N_G(B)$. By (3.6), $G_\alpha=N_{G_\alpha}(B)N^\alpha$ and so $|N_{G_\alpha}(B): N_{N^\alpha}(B)|=|G_\alpha: N^\alpha|=m$. Hence $|N_G(B)|$

$$= |F(B)| \times |N_{G_\alpha}(B)| = q^2 m \times |N_{N^\alpha}(B)| = q^5(q-1)^2(q+1)m. \quad \text{Thus we have (ii)}.$$

(3.13)   *Let T_1 be a Sylow 2-subgroup of $N_G(B)$ and T_2 a Sylow 2-subgroup of $N_G(T_1)$. Then $T_1 \neq T_2$. Let x be an element of $T_2 - T_1$ and set $U = BB^x$. Then $U \simeq E_{q^4}$ and for each $\gamma \in \Omega$, $U_\gamma \simeq E_{q^2}$, $U_\gamma \in B^G$, $\gamma^U = F(U_\gamma)$ and $|\gamma^U| = q^2$. Moreover $U_\gamma = U_\delta$ for all $\delta \in \gamma^U$.*

Proof.   If $B \cap B^x \neq 1$, by (3.11) and (3.12)(i), we have $B = B^x$ and so $x \in T_1$, contrary to the choice of x. Hence $B \cap B^x = 1$. As $T_1 \trianglerighteq B$ and $T_1 = T_1^x \trianglerighteq B^x$, $U = B \times B^x$ and $U \simeq E_{q^4}$.

Let $\gamma \in \Omega$ and put $D = U_\gamma$. Then $F(D) \supseteq \gamma^U$ as U is abelian. Therefore $|U:D| = |\gamma^U| \leq q^2$ by (3.11), while $|D| \leq q^2$ because D is an elementary abelian subgroup of N^γ. Hence $D \simeq E_{q^2}$ and $|F(D)| = |\gamma^U| = q^2$. By (3.6) and (3.9)(iii), $D \in B^G$. Since $U_\gamma \leq U_\delta \simeq E_{q^2}$ for each $\delta \in \gamma^U$, we have $U_\gamma = U_\delta$.

(3.14)   *Let U be as in (3.13). Let $\Gamma = \{X_i \mid 1 \leq i \leq s\}$ be the set of U-orbits on Ω and set $B_i = U_\gamma$ for $\gamma \in X_i$ with $1 \leq i \leq s$. Then the following hold.*

(i) $s = q, \Omega = \bigcup_{i=1}^{q} X_i$ and $|X_i| = q^2$.

(ii)   *B_i is semi-regular on $\Omega - X_i$ and $B_i \cap B_j = 1$ for each i, j with $i \neq j$.*

Proof.   By (3.10)(ii) and (3.13), $|X_i| = q^2$ and $|\Omega| = q^3$, hence $s = q$. Cleary $\Omega = \bigcup_{i=1}^{q} X_i$. Thus we have (i).

By (3.13)(ii), B_i is conjugate to B for each i. Hence B_i is semi-regular on $\Omega - X_i$ by (3.11). Therefore, if $B_i \cap B_j \neq 1$, then $X_i = F(B_i) = F(B_j) = X_j$, so that $i = j$. Thus we have (ii).

(3.15)   *Set $Y = \{B_i \mid 1 \leq i \leq q\}$ and let $D \in Y$. Then $N_G(D) \leq N_G(U)$ and U is a unique Sylow 2-subgroup of $C_G(D)$.*

Proof.   Suppose $N_G(D) \not\leq N_G(U)$. Since $[N_G(D), U] \not\leq U$, there exist $g \in N_G(D)$ and $B_i \in Y - \{D\}$ such that $(B_i)^g \not\leq U$. Set $D_1 = (B_i)^g$. Since $[D_1, D] = [B_i, D]^g = 1$, it follows from (3.10)(i) that $F(D_1) \cap F(D) = \phi$ and so D is regular on $F(D_1)$ by (3.11). Hence $F(D_1) = \gamma^D = \gamma^U$ for $\gamma \in F(D_1)$. By (3.14), $F(D_1) = F(B_j)$ for some $B_j \in Y$. By (3.12)(i), $D_1 = B_j$, so that $D_1 \leq U$, a contradiction. Thus we have $N_G(D) \leq N_G(U)$. Hence $U \leq 0_2(C_G(D))$. Since $U \leq C_G(B)$, $C_G(B)$ is transitive on $F(B)$. Hence $|C_G(B)|_2 = |F(B)| \times |C_{G_\alpha}(B)|_2 = q^4$ by (3.10)(i). Therefore $|C_G(D)|_2 = q^4$ as $D \in B^G$ and so U is a unique Sylow 2-subgroup of $C_G(D)$.

(3.16) $|N_G(U)| = q^6(q-1)^2(q+1)m.$

Proof.   Let S_1 be a Sylow 2-subgroup of $N_G(U)$ and S_2 be a Sylow 2-subgroup of $N_G(S_1)$. Suppose $S_1 \neq S_2$ and let w be an element of $S_2 - S_1$.

Set $\gamma = \alpha^{w^{-1}}$. Then $(U_\gamma)^w \in B^G$ by (3.13) and $(U_\gamma)^w \leq (G_\gamma)^w = G_\alpha$. Since U and U^w are normal subgroups of S_1, $\langle B, (U_\gamma)^w \rangle$ is 2-subgroup of $G_\alpha \cap S_1 = S$. Hence $B = (U_\gamma)^w$ by (2.5) (iii) and (3.6). Therefore $U, U^w \leq C_G(B)$, so that $U = U^w$ by (3.15) and $w \in S_2 \cap N_G(U) = S_1$, contrary to the choice of w. Hence $S_1 = S_2$ and S_1 is a Sylow 2-subgroup of G. It follows from (3.10) that $|S_1| = q^6$.

We now consider the action of $N_G(U)$ on $\Gamma = \{X_i \mid 1 \leq i \leq q\}$. Set $\Delta = F(B)$. By (3.12), $S_1(\Delta) \leq G(\Delta) = N_G(B)$ and $|N_G(B)|_2 = q^5$ and so $|S_1 : S_1(\Delta)|$ is divisible by q. Hence S_1 is transitive on Γ and so $N_G(U)$ is transitive on Γ. Therefore $|N_G(U)| = q \times |N_G(U) \cap N_G(B)| = q \times |N_G(B)| = q^6(q-1)^2(q+1)m$ by (3.12) (ii) and (3.15).

(3.17) *Let R be a cyclic subgroup of N_β^α of order $q+1$. Then $|F(R)| = q$ and R is semi-regular on $\Omega - F(R)$.*

Proof. Since $N_\beta^\alpha / A \simeq PSL(2, q)$, there exists a cyclic subgroup R of N_β^α of order $q+1$. Let $Q \neq 1$ be a subgroup of R. Then, by Lemma 2.7 $|F(Q)|$
$$= 1 + \frac{|N_{N^\alpha}(Q)| \times |N_\beta^\alpha : N_{N_\beta^\alpha}(Q)|}{|N_\beta^\alpha|} = 1 + \frac{2(q-1)(q+1)}{2(q+1)} = q.$$ Thus (3.17) holds.

(3.18) *Let $V \in U^G$. If $V \neq U$, then $|F(U_\gamma) \cap F(V_\gamma)| = 1$ or q for $\gamma \in \Omega$.*

Proof. Suppose $\gamma \neq \delta \in F(U_\gamma) \cap F(V_\gamma)$. By (3.13), $U_\gamma, V_\gamma \in B^G$ and so by (3.3) (ii), $U_\gamma, V_\gamma \leq N^\gamma \cap N^\delta$. Set $H = O_2(N_\delta^\gamma)$. Then, by (3.6) and (3.9) (i), $U_\gamma H$ and $V_\gamma H$ are Sylow 2-subgroups of N_δ^γ. If $U_\gamma H = V_\gamma H$, then $U_\gamma = V_\gamma$ and $U, V \leq C_G(U_\gamma)$. By (3.15) we have $U = V$, a contradiction. Therefore $U_\gamma H \neq V_\gamma H$. Set $X = \langle U_\gamma, V_\gamma \rangle$. Then $XH = N_\delta^\gamma$ because $N_\delta^\gamma / H \simeq PSL(2, q)$, $q = 2^n$, and $PSL(2, q)$ is generated by its two distinct Sylow 2-subgroups. Hence $N_\delta^\gamma \trianglerighteq X \cap H$. By (2.5) (iii), $E_q \simeq U_\gamma \cap H \leq X \cap H$. Since N_δ^γ acts irreducibly on H by (2.5) (vii), $X \cap H = H$ and hence $H \leq X$. From this $X = N_\delta^\gamma$. Thus, by (3.5)(i) and (3.9), $|F(U_\gamma) \cap F(V_\gamma)| = |F(X)| = |F(N_\delta^\gamma)| = q$.

(3.19) *Let Q be a cyclic subgroup of $N_{N^\alpha}(B)$ of order $q+1$, $V \in U^G$ and set $P = N_Q(V)$. Then the following hold.*
  (i) *Q is semi-regular on $\Omega - F(Q)$ and $|F(Q)| = q$.*
  (ii) *If $P \neq 1$ and $V \geq D \in B^G$, then P normalizes D and $|F(P) \cap F(D)| = 1$.*

Proof. Since $N_{N^\alpha}(B)/B \simeq PSL(2, q)$, there exists a cyclic subgroup Q of $N_{N^\alpha}(B)$ of order $q+1$. Clearly Q is a cyclic Hall subgroup of N^α, hence Q is conjugate to R defined in (3.17). By (3.17), Q is semi-regular on $\Omega - F(Q)$ and $|F(Q)| = q$. Thus (i) holds.

Suppose $P \neq 1$ and let $\gamma \in F(P)$. Then, by (3.9) (i), $P \leq N^\gamma$ and hence P normalizes $N^\gamma \cap V$. By (3.10) (i) and (3.13), $N^\gamma \cap V = V_\gamma$ and $V_\gamma \in B^G$ and so $P \leq N_{N^\gamma}(V_\gamma)$ and $N_G(V_\gamma)^{F(V_\gamma)} \simeq N_G(B)^{F(B)}$. Hence we have $F(P) \cap F(V_\gamma) = \{\gamma\}$ by (3.7). As $|F(P)| = q$ by (i), (ii) holds.

(3.20) *Let $V \in U^G - \{U\}$ and let Q be a cyclic subgroup of $N_{N^\alpha}(B)$ of order $q+1$. Then $N_Q(V) = 1$.*

Proof. Set $P = N_Q(V)$ and assume $P \neq 1$. Let $\gamma \in \Omega - F(Q)$ and set $B_1 = U_\gamma$, $B_2 = V_\gamma$. By (3.15), Q normalizes U and so by (3.19) Q normalizes B_1. Similarly P normalizes B_2. Therefore $F(B_1) \cap F(B_2) \ni \gamma^P \neq \{\gamma\}$ as $P \neq 1$ and P is semi-regular on $\Omega - F(Q)$. By (3.18), we have $|F(B_1) \cap F(B_2)| = q$. Since P acts on $F(B_1) \cap F(B_2)$ and $|P|$ divides $q+1$, P fixes at least two points of $F(B_1) \cap F(B_2)$, which contradicts to (3.19).

(3.21) *Let T be a Sylow 2-subgroup of $N_G(U)$. Then, for each $V \in U^G - \{U\}$, $|T : N_T(V)|$ is divisible by q.*

Proof. Suppose $|T : N_T(V)| < q$ and set $T_1 = N_T(V)$. Then $|T_1| > q^5$ as $|T| = q^6$ by (3.16). Hence $q > |T_1 V : T_1| = |V : V \cap T_1|$ and so $|V \cap T_1| > q^3$. Therefore, for each $B_1 \in B^G$ such that $B_1 \leq V$, $q > |B_1(V \cap T_1) : V \cap T_1| = |B_1 : B_1 \cap T_1| = |B_1 : B_1 \cap T|$. Hence $|B_1 \cap T| > q$. Let $\gamma \in F(B_1 \cap T)$ and set $B_2 = U_\gamma$. Then $\langle B_1 \cap T, B_2 \rangle \leq N^\gamma \cap T$. As $|B_1 \cap T| > q$ by (2.5) (iii), $B_1 \cap T \cap B_2 \neq 1$. By (3.11), $\langle B_1 \cap T, B_2 \rangle \leq G_{F(B_2)}$. By (3.12) (i), we have $B_1 \cap T \leq B_2$, so that $F(B_1) = F(B_1 \cap T) = F(B_2)$. Again, by (3.12) (i), $B_1 = B_2$ and so $U, V \leq C_G(B_2)$. Therefore $U = V$ by (3.15), a contradiction.

(3.22) *Put $W = U^G$. Then $|W| = q^2 + q + 1$ and G^W is doubly transitive.*

Proof. Set $H = N_G(U)$. By (3.10) (ii) and (3.16), $|W| = |G : H| = q^2 + q + 1$. Let $V \in W - \{U\}$ and let Q be as defined in (3.20). By (3.15), $Q \leq H$ and by (3.20), Q acts semi-regularly on $W - \{U\}$. Hence $|V^H|$ is divisible by $q+1$. On the other hand, by (3.21), $|V^H|$ is divisible by q and so we have $|V^H| = q(q+1)$. Thus (3.22) holds.

(3.23) $G_W \cap U \neq 1$.

Proof. Suppose $G_W \cap U = 1$. Since $G \trianglerighteq G_W$ and $H \trianglerighteq U$, $[G_W, U] \leq G_W \cap U = 1$. Hence $G_W \leq C_G(U)$. By (3.15), U is a unique Sylow 2-subgroup of $C_G(U)$ and so $G_W \leq 0(G)$. On the other hand, as $|\Omega|$ is even and G is doubly transitive on Ω, we have $0(G) = 1$. Therefore $G_W = 1$ and hence G acts faithfully on W. Since U is not semi-regular on $W - \{U\}$, by [4], $PSL(n_1, q_1) \leq G \leq P\Gamma L(n_1 q_1)$ for some $n_1 \geq 3$ and q_1 with q_1 even. As $|W| = q^2 + q + 1 = q_1^{n_1 - 1} + \cdots + q_1 + 1$, $q(q+1) = q_1(q_1^{n_1 - 2} + \cdots + 1)$ and so $q = q_1$ and $n_1 = 3$. Therefore $PSL(3, q) \leq G \leq P\Gamma L(3, q)$. But $|P\Gamma L(3, q)|_2 = q^3$ by (3.9) (ii) and Lemma 2.6. Hence $q^3 = q^6$ by (3.10) (ii). This is a contradiction. Thus $G_W \cap U \neq 1$.

(3.24) G^Ω *has a regular normal subgroup.*

Proof. Since $G_W \leq N_G(U)$, $G_W \cap U$ is a normal subgroup of G_W. As $G_W \cap$

$U \leq O_2(G_W)$ and $G \trianglerighteq G_W$, $O_2(G_W)$ is a normal subgroup of G. Let E be a minimal normal subgroup of G contained in $O_2(G_W)$. Then E is an elementary abelian 2-subgroup of G and acts regularly on Ω.

(3.25)   *If* (ii) *of* (3.7) *occurs, we have* (i) *of the theorem.*

Proof.   By (3.9), (3.10) and (3.24), G has a regular normal subgroup E of order q^3, where $q = 2^n$ and $n \equiv 1$ (mod 2) and N^α is transitive on $\Omega - \{\alpha\}$. Moreover $G = G_\alpha E$ and G_α is isomorphic to a subgroup of $GL(E)$. As in the proof of Lemma 2.1, we may assume $\Omega = E$, $\alpha = 0 \in E$ and $GL(E)E \leq \mathrm{Sym}(\Omega)$. There exists a subgroup H of $GL(E)$ such that $H \simeq \Gamma L(3,q)$ and $HE \simeq A\Gamma L(3,q)$. Let L be a normal subgroup of H isomorphic to $SL(3,q)$. Since $q = 2^n$ and $n \equiv 1$ (mod 2), L is isomorphic to $PSL(3,q)$.

By (3.9) (i) and by the structure of $A\Gamma L(3,q)$, there exist an automorphism f from N^α to L and $g \in \mathrm{Sym}(\Omega)$ such that $\alpha^g = \alpha$ and $(\beta^x)^g = (\beta^g)^{f(x)}$ for each $\beta \in \Omega - \{\alpha\}$ and $x \in N^\alpha$. From this $(\beta^g)^{g^{-1}xg} = (\beta^x) = (\beta^g)^{f(x)}$ for each $\beta \in \Omega - \{\alpha\}$ and so $g^{-1}xg = f(x)$. Hence $g^{-1}N^\alpha g = L$.

Set $X = N(L) \cap \mathrm{Sym}(\Omega)$ and $D = C_X(L)$. Then D is semi-regular on $\Omega - \{\alpha\}$ as L is transitive on $\Omega - \{\alpha\}$. Put $T = f(A)$. Then $N_L(T)^{F(T)} \simeq Z_{q-1}$ and it is semi-regular on $F(T) - \{\alpha\}$ by (3.5) (i) and (3.9) (i), (iii). It follows that $D \leq Z_{q-1}$. Since X/DL is isomorphic to a subgroup of the outer automorphism group of $PSL(3,q)$ and $f(A)$ and $f(B)$ are not conjugate in $\mathrm{Sym}(\Omega)$ by the hypothesis (**) and (3.9) (ii), it follows from Lemma 2.6 (i) that $|X/DL| \leq n$. Hence $|X| \leq n(q-1)|L| = |\Gamma L(3,q)|$. On the other hand $\Gamma L(3,q) \simeq H < X$ and so $X = H$. Therefore $g^{-1}G_\alpha g \trianglerighteq g^{-1}N^\alpha g = L$ and $g^{-1}G_\alpha \leq X = H$. Thus we have (3.25).

The conclusion of the theorem now follows immediately from steps (3.2), (3.7), (3.8) and (3.25).

OSAKA KYOIKU UNIVERSITY

References

[1]   D. Gorenstein: Finite groups, Harper and Row, New York, 1968.

[2]   Y. Hiramine: *On doubly transitive permutation groups,* Osaka J. Math. **15** (1978), 613–631.

[3]   Y. Hiramine: *On some doubly transitive permutation group in which socle(G_α) is nonsolvable,* Osaka J. Math. **16** (1979), 797–816.

[4]   M. O'Nan: *A characterization of $L_n(q)$ as a permutation group,* Math. Z. **127** (1972), 301–314.

[5]   M. O'Nan: *Normal structure of one-point stabilizer of a doubly transitive permutation group* II, Trans. Amer. Math. Soc. **214** (1975), 43–74.

[6]   R. Steinberg: *Automorphism of finite linear groups,* Canad. J. Math. **12** (1960),

606–615.

[7]   H. Zassenhaus: *Über transitive Erweiterungen gewisser Gruppen aus Automorphismen endlicher mehrdimensionaler Geometrien,* Math. Ann. **111** (1935), 748–759.