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FACTORIZATION OF DIFFERENTIAL OPERATORS AND DECOMPOSITION OF SOLUTIONS OF HOMOGENEOUS EQUATIONS

To Professor Y. Akizuki on his 60-th birthday

By

SHIGETAKE MATSUURA

§ 1. Introduction

Let Ω be an open set in ν-dimensional Euclidean space $\boldsymbol{R}^{\nu}$ whose points are described by a fixed coordinate system $x=(x_1, \cdots, x_\nu)$. Let $L(X)$ be a polynomial of ν-variables $X=(X_1, \cdots, X_\nu)$ with complex coefficients. Replacing X by partial differentiations $D=(D_1, \cdots, D_\nu)$, $D_k=\frac{1}{i}\frac{\partial}{\partial x_k}$ $(i=\sqrt{-1})$, we get a partial differential operator with constant coefficients $L(D)$. Let $\mathcal{D}'(\Omega)$ be the space of distributions defined in Ω (See L. Schwartz [11]) and take a linear subspace E of $\mathcal{D}'(\Omega)$ which is stable under the operations of partial differentiations.

Let us consider a differential equation of the form

$$L(D)u = 0 \tag{1.1}$$

where u is an unknown element of E.

When we have a factorization of $L(X)$ into *mutually prime* factors[1)]

$$L(X) = P(X)Q(X), \tag{1.2}$$

it is very common in applied mathematics to seek for a general solution of (1.1) in the form of a sum

$$u = u_1+u_2, \qquad u_1, u_2 \in E, \tag{1.3}$$

where u_1 and u_2 are solutions of the equations corresponding to the factors, i.e.

$$\begin{aligned} P(D)u_1 &= 0, \\ Q(D)u_2 &= 0, \text{ in } \Omega. \end{aligned} \tag{1.4}$$

1) Factorizations are always considered in the polynomial ring $\boldsymbol{C}[X_1, \cdots, X_\nu]$ over the complex number field $\boldsymbol{C}$.

It is clear that an element of the form (1.3) with (1.4) is always a solution of (1.1). But a general solution of (1.1) cannot be decomposed in the form (1.3) with (1.4) unless the domain Ω and the factorization (1.2) are specified.[2]

Our subject in the present paper concerns the possibility of decompositions of solutions of the equation (1.1) into the form (1.3) with (1.4) in some special cases. It is said that this is also a problem proposed by Hadamard.

In §2 we shall treat a simplest case where Hilbert's Nullstellensatz gives all what we need. In §3 we give two lemmas which will simplify later proofs. In §4 we shall give an approximation theorem when Ω is convex and shall remark that L. Ehrenpreis' fundamental principle [3][3] will lead us to a precise result. In §5 we shall treat polynomial solutions in the case when the differential operator $L(D)$ is homogeneous. Some results in this paragraph will be exploited in the next. In §6 we shall treat (real) analytic solutions in the case when Ω is a simply connected domain in $\boldsymbol{R}^2$ and the differential operator $L(D)$ is homogeneous, and shall give a generalization of a classical theorem in function theory.

When Ω is the whole space $\boldsymbol{R}^\nu$, V. P. Palamodov [9] solved the problem for a various kind of spaces of ordinary and generalized functions by giving an explicit form of a general solution of the equation (1.1) by means of his detailed study of Fourier transformation of generalized functions.

I thank here my colleagues for helpful discussions and especially Prof. M. Yamaguchi for his critical reading of the manuscipt.

§2. A simplest case

A simplest situation is realized when the hypersurfaces in $\boldsymbol{C}^\nu$ defined by $P(X)=0$ and $Q(X)=0$ are disjoint[4]. In this case the following theorem holds.

Theorem 2.1. *If the hypersurfaces defined by $P(X)=0$ and $Q(X)=0$ are disjoint, then for any open set Ω in $\boldsymbol{R}^\nu$ and for any linear subspace E of $\mathcal{D}'(\Omega)$ which is stable under the operations of partial differentiations, every solution u in E of equation* (1.1) *is decomposed uniquely in the form* (1.3) *with* (1.4).

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2) It is the case even in a classical theorem in function theory. C.f. also Theorem 4.1 and §6.

3) Complete proof has not yet been published.

4) The well known result in the case of ordinary differential equations corresponds to this case.

Proof. In the polynomial ring $\boldsymbol{C}[X_1, \cdots, X_\nu]$, consider the ideal $\mathfrak{A}=(P, Q)$ generated by the polynomials $P(X)$ and $Q(X)$. Since the affine variety corresponding to this ideal is the empty set by assumption, $\mathfrak{A}$ should coincide with the whole ring $\boldsymbol{C}[X_1, \cdots, X_\nu]$ according to Hilbert's Nullstellensatz (See [12]). Thus there exist two polynomials $R(X)$ and $S(X)$ such that

$$1 = R(X)Q(X)+S(X)P(X)\,. \tag{2.1}$$

Substituting the differentiations D for X and applying on an element u of E, we get

$$u = R(D)Q(D)u+S(D)P(D)u\,. \tag{2.2}$$

If u is a solution of (1.1) then (2.2) gives clearly a decomposition of the form (1.3) with

$$\begin{aligned} u_1 &= R(D)Q(D)u\,, \\ u_2 &= S(D)P(D)u \end{aligned} \tag{2.3}$$

satisfying (1.4). Now, if we assume that u is decomposed in the form (1.3) with (1.4) in two ways:

$$\begin{aligned} u &= u_1+u_2\,, \\ u &= v_1+v_2\,. \end{aligned}$$

Then,

$$w = u_1-v_1 = -u_2+v_2$$

should satisfy the simultaneous equations

$$P(D)w = Q(D)w = 0\,. \tag{2.4}$$

Substituting this element w in the identity (2.2), we get $w=0$. That is, the decomposition of the form (1.3) is unique.

REMARK 1. Although u_1 and u_2 are uniquely determined, the polynomials $R(X)$ and $S(X)$ in (2.3) are not uniquely determined. But we cannot in general reduce $R(X)$ and $S(X)$ into constants. Therefore even when u is an ordinary function solution of (1.1), (1.3) might decompose u into a sum of distribution.

REMARK 2. If the space E contains the exponential functions, the decomposition of the form (1.3) is unique only when the surfaces $P(X)=0$ and $Q(X)=0$ are disjoint. Since, if $P(X)=0$ and $Q(X)=0$ admit a simultaneous solution $\zeta\in\boldsymbol{C}^\nu$,

$$w = e^{i\langle x,\zeta\rangle}, \langle x, \zeta\rangle = x_1\zeta_1+\cdots+x_\nu\zeta_\nu\,,$$

is a simultaneous solution of (2.4). Hence for any decomposition in the form (1.3) with (1.4): $u=u_1+u_2$, $u=(u_1+w)+(u_2-w)$ also gives such a decomposition.

§ 3. Lemmas.

Now let

$$(3.1)\qquad \begin{aligned} P(X) &= P_1(X)^{\mu_1}\cdots P_m(X)^{\mu_m}, \\ Q(X) &= Q_1(X)^{\nu_1}\cdots Q_n(X)^{\nu_n} \end{aligned}$$

be the factorizations into irreducible polynomials. Then, in the factorization (1.2) of $L(X)$, that $P(X)$ and $Q(X)$ are mutually prime means that in (3.1) there is no element common to $\{P_1,\cdots,P_m\}$ and $\{Q_1,\cdots,Q_n\}$. Thus, when the surjectivity of differential operators with constant coefficients is known[5] for E (See Introduction), the following elementary Lemmas 3.1 and 3.2 will reduce the problem to the case when $P(X)$ and $Q(X)$ are themselves irreducible and distinct.

Lemma 3.1. *Let E be an abelian group (written additively) and let $p_1,\cdots,p_m$; $q_1,\cdots,q_n$ be a commuting family of not necessarily distinct surjective endomorphisms. If we have that*

$$\mathrm{Ker}\,(p_iq_j) = \mathrm{Ker}\,(p_i)+\mathrm{Ker}\,(q_j)^{6)}$$

for any pair of i, j ($i=1,\cdots,m$; $j=1,\cdots,n$), then, we have that

$$\mathrm{Ker}\,(p_1\cdots p_mq_1\cdots q_n) = \mathrm{Ker}\,(p_1\cdots p_m)+\mathrm{Ker}\,(q_1\cdots q_n).$$

A topological version of the above lemma is the following

Lemma 3.2. *Let E be a topological abelian group (written additively) and let $p_1,\cdots,p_m$; $q_1,\cdots,q_n$ be a commuting family of topological homomorphisms[7] of E onto itself. If we have that*

$$\mathrm{Ker}\,(p_iq_j) = \overline{\mathrm{Ker}\,(p_i)+\mathrm{Ker}\,(q_j)}$$

for any pair of i, j ($i=1,\cdots,m$; $j=1,2,\cdots,n$), then we have that

$$\mathrm{Ker}\,(p_1\cdots p_mq_1\cdots q_n) = \overline{\mathrm{Ker}\,(p_1\cdots p_m)+\mathrm{Ker}(q_1\cdots q_n)}$$

("$\overline{}$" *means the closure operation*).

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5) C.f. [3], [5], [7], [8].

6) $\mathrm{Ker}\,(p_i)=\{x\,;p_i(x)=0\}$; for two subgroups E_1, E_2, $E_1+E_2=\{x+y\,;x\in E_1, u\in E_2\}$.

7) A topological homomorphism means an algebraic homomorphism which is continuous and open.

For the convenience of proving the above lemmas, we add two more

Lemma 3.1.' *Let E be an abelian group and let p and q be two commuting surjective endomorphisms. Then the following two conditions are equivalent.*

(1°) $\qquad \mathrm{Ker}\,(pq) = \mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)\,;$

(2°) $\qquad p\,\mathrm{Ker}\,(q) = \mathrm{Ker}\,(q)\,.$

Lemma 3.2.' *Let E be a topological abelian group and let p and q be two commuting surjective topological homomorphisms. Then the following two conditions are equivalent.*

(1°) $\qquad \mathrm{Ker}\,(pq) = \overline{\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)}\,;$

(2°) $\qquad \overline{p\,\mathrm{Ker}\,(q)} = \mathrm{Ker}\,(q)\,.$

Proof of Lemma 3.1'. (1°) *implies* (2°). It is clear that $p\,\mathrm{Ker}\,(q) \subseteq \mathrm{Ker}\,(q)$ since p and q are commuting. Now x be any element of $\mathrm{Ker}\,(q)$. Since p is surjective, there exists an element y of E such that $x=p(y)$. Thus we have $pq(y)=q(x)=0$, i.e. $y\in\mathrm{Ker}\,(pq)$. Thus, by (1°) there exist $y_1\in\mathrm{Ker}\,(p)$, $y_2\in\mathrm{Ker}\,(q)$ such that $y=y_1+y_2$. Hence $x=p(y)=p(y_1)+p(y_2)=p(y_2)$. Thus we get $x=p(y_2)\in p\,\mathrm{Ker}\,(q)$. This means that $\mathrm{Ker}\,(q)\subseteq p\,\mathrm{Ker}\,(q)$.

(2°) *implies* (1°). That $\mathrm{Ker}\,(pq)\supseteq\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)$ is clear, since p and q are commuting. Let x be any element of $\mathrm{Ker}\,(pq)$, i.e. $pq(x)=0$. Put $y=p(x)$. Then $y\in\mathrm{Ker}\,(q)$. Therefore, by (2°), we have an element $z\in\mathrm{Ker}\,(q)$ such that $y=p(z)$. Hence $u=x-z\in\mathrm{Ker}\,(p)$, since $p(u)=p(x)-p(z)=0$. Thus we get that $x=u+z\in\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)$. This means that $\mathrm{Ker}\,(pq)\subseteq\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)$.

Proof of Lemma 3.1. By assumption we get that $p_i\,\mathrm{Ker}\,(q_j)=\mathrm{Ker}\,(q_j)$, $(i=1, 2, \cdots, m)$ according to Lemma 3.1'. These relations express that $p_1, \cdots, p_m$ can be considered as surjective endomorphisms of $\mathrm{Ker}\,(q_j)$. Thus we get that $p_1\cdots p_m\ \mathrm{Ker}\,(q_j)=\mathrm{Ker}\,(q_j)$, since any composition of surjective mappings is surjective. Since condition (1°) in Lemma 3.1' is symmetric for p and q, we get that $q_j\ \mathrm{Ker}\,(p_1\cdots p_m)=\mathrm{Ker}\,(p_1\cdots p_m)$. By the same argument again we have that $q_1\cdots q_n\ \mathrm{Ker}\,(p_1\cdots p_m)=\mathrm{Ker}\,(p_1\cdots p_m)$. This means that $\mathrm{Ker}\,(p_1\cdots p_m\ q_1\cdots q_n)=\mathrm{Ker}\,(p_1\cdots p_m)+\mathrm{Ker}\,(q_1\cdots q_n)$ according to Lemma 3.1'.

Proof of Lemma 3.2.' (1°) *implies* (2°). It is clear that $\overline{p\cdot\mathrm{Ker}\,(q)}\subseteq\mathrm{Ker}\,(q)$, since $p\,\mathrm{Ker}\,(q)\subseteq\mathrm{Ker}\,(q)$ and $\mathrm{Ker}\,(q)$ is closed by the continuity of q. Now le x be an arbitrary element of $\mathrm{Ker}\,(q)$ and V be a neighbourhood of 0. Since p is surjective, there exists an element y of E such

that $x=p(y)$. Since $pq(y)=q(x)=0$, we have that $y\in \mathrm{Ker}\,(pq)$. Since p is continuous we can take a neighbourhood W of 0 such that $p(W)\subseteq V$. Now according to condition (1°), we can take two elements y_1 and y_2 such that $y_1\in \mathrm{Ker}\,(p)$, $y_2\in \mathrm{Ker}\,(q)$ and $y_1+y_2-y\in W$. From this it follows that $p(y_2)-x\in p(W)\subseteq V$ or $p\,\mathrm{Ker}\,(q)\cap(x+V)\neq\phi$. Since V is arbitrary, this implies that $x\in\overline{p\,\mathrm{Ker}\,(q)}$.

(2°) *implies* (1°). It is clear that $\mathrm{Ker}\,(pq)\supseteq\overline{\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)}$. Now let x be an element of $\mathrm{Ker}\,(pq)$. Put $y=p(x)$. Then $p\in \mathrm{Ker}\,(q)$. Let V be a neighbourhood of 0. Since p is a surjective open mapping, $p(V)$ is also a neighbourhood of 0. According to condition (2°), $y\in\overline{p\,\mathrm{Ker}\,(q)}$. Hence $p\,\mathrm{Ker}\,(q)\cap(y+p(V))\neq\phi$, i.e. there exists an element $x_2\in \mathrm{Ker}\,(q)$ such that $p(x_2)\in y+p(V)$. Thus for some $v\in V$, $p(x_2)-y=p(v)$, that is, $p(x+v-x_2)=0$. Putting $x_1=x+v-x_2$, we get $x_1\in \mathrm{Ker}\,(p)$ and $x+v=x_1+x_2$. Thus we get that $(x+V)\cap(\mathrm{Ker}\,(p)+\mathrm{Ker}\,(q))\neq\phi$. Since V is arbitrary, this means that $x\in \mathrm{Ker}\,(p)+\mathrm{Ker}\,(q)$.

Proof of Lemma 3.2. We can proceed in an analoguous way to in the proof of Lemma 3.1, using Lemma 3.2′ instead of Lemma 3.1′, and expressing the continuity of mappings through the closure operation.

REMARKS. The assumption that mappings be topological homomorphisms is satisfied when E is a Fréchet space and mappings are continuous surjective linear mappings, according to the *homomorphism theorem* (See Bourbaki [1].)

§4. Indefinitely differentiable solutions

In this paragraph we shall give an approximation theorem for C^∞ (indefinitely continuously differentiable) solutions, i.e. the fact that any C^∞-solution can be approximated by decomposable C^∞-solutions when the domain Ω is convex. And then we shall show how a result announced by Ehrenpreis [3] leads to the exact decomposition. We shall mainly exploit results and methods developped in Malgrange [7].

Let $\mathcal{E}(\Omega)$ be the space of indefinitely continuously differentiable functions defined in Ω with the standard topology (See [11]). It is a Fréchet space. Its dual $\mathcal{E}'(\Omega)$ is the space of distributions with compact supports in Ω. The Fourier transforms of the elements of $\mathcal{E}'(\Omega)$ are completely characterized by Paley-Wiener's theorem (See [11]).

Theorem 4.1. *Let Ω be a convex open set in $\boldsymbol{R}^\nu$. Then a necessary and sufficient condition for any solution $u\in\mathcal{E}(\Omega)$ of the equation*

$$P(D)Q(D)u = 0 \tag{4.1}$$

to be approximated by the solutions of the form u_1+u_2 *with*

$$P(D)u_1 = 0,\ Q(D)u_2 = 0,\ u_2 \in \mathcal{E}(\Omega) \tag{4.2}$$

in the topology of $\mathcal{E}(\Omega)$ *is that the polynomials* $P(X)$ *and* $Q(X)$ *are mutually prime.*

Proof. *Necessity of the condition*: Assume the contrary and let $G(X)$ be the greatest common divisor and $H(X)$ be the least common multiple of $P(X)$ and $Q(X)$. By assumption, $G(X)$ is not a constant and we have

$$P(X)Q(X) = H(X)G(X) \tag{4.3}$$

Since an element of the form u_1+u_2 with (4.2) is always a solution of the equation

$$H(D)u = 0 \tag{4.4}$$

and since the space of all solutions of this equation is a closed subspace of $\mathcal{E}(\Omega)$, all the solutions in $\mathcal{E}(\Omega)$ of the equation (4.1) should be solutions of (4.4), for we assumed that every solutions of (4.1) could be approximated by decomposable solutions. According to the formula (4.3), this means that $G(D)H(D)u=0$, $u\in\mathcal{E}(\Omega)$ implies $H(D)u=0$. But since $H(D)$ is a surjective mapping of $\mathcal{E}(\Omega)$ to $\mathcal{E}(\Omega)$ (See [7]), any element of $\mathcal{E}(\Omega)$ is of the form $H(D)u$. Therefore the above argument shows that the equation $G(D)u=0$, $u\in\mathcal{E}(\Omega)$ implies $u=0$. But since $G(X)$ is not a constant, there exist a point $\zeta\in\boldsymbol{C}^\nu$ with $G(\zeta)=0$. We have thus a non-zero solution in $\mathcal{E}(\Omega)$, $u(x)=e^{i<x,\zeta>}$, of the equation $G(D)u=0$. This contradiction proves the necessity of the condition.

Sufficiency of the condition: According to Lemma 3.2 and Remark in the previous paragraph, we have to prove the sufficiency only in the case when $P(X)$ and $Q(X)$ are distinct irreducible polynomials. Let $\mathcal{U}$ be the totality of solutions u in $\mathcal{E}(\Omega)$ of the equation (4.1), $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) be the totality of solutions u_1 (resp. u_2) in $\mathcal{E}(\Omega)$ of the equation $P(D)u_1=0$ (resp. $Q(D)u_2=0$). We have to show that $\mathcal{U}_1+\mathcal{U}_2$ is dense in $\mathcal{U}$. To this end, according to Hahn-Banach's theorem (See [1]), it is enough to show that any element in $\mathcal{E}'(\Omega)$ which is orthogonal to $\mathcal{U}_1$ and $\mathcal{U}_2$ is also orthogonal to $\mathcal{U}$. Now, let T be an element of $\mathcal{E}'(\Omega)$ orthogonal to $\mathcal{U}_1$ and $\mathcal{U}_2$. According to [7], there exist two elements S_1 and S_2 in $\mathcal{E}'(\Omega)$ such that

$$T = P(-D)S_1 = Q(-D)S_2\,.$$

Taking their Fourier transforms, we get that

$$\hat{T}(\zeta) = P(-\zeta)\hat{S}_1(\zeta) = Q(-\zeta)S_2(\zeta)\,,$$

$\hat{T}(\zeta)$, $\hat{S}_1(\zeta)$ and $\hat{S}_2(\zeta)$ being entire analytic fuctions of $\zeta \in \boldsymbol{C}^\nu$ of Paley-Wiener type, i.e. of exponential type and slowly increasing on $\boldsymbol{R}^\nu$. Now consider the following function

$$F(\zeta) = \hat{S}_1(\zeta)/Q(-\zeta) = \hat{S}_2(\zeta)/P(-\zeta).$$

$F(\zeta)$ is analytic everywhere except at those points where $P(-\zeta)$ and $Q(-\zeta)$ vanish simultaneously. The set of these exceptional points is an algebraic set of complex dimension at most $\nu-2$, since $P(-X)$ and $Q(-X)$ are distinct irreducible polynomials. Now, arguing as in Hörmander [4, Lemma 2], we can claim that $F(\zeta)$ is really an entire function. Thus we get that $\hat{T}(\zeta)$ should be of the form

$$\hat{T}(\zeta) = P(-\zeta)Q(-\zeta)F(\zeta).$$

From this formula, according to [7], $F(\zeta)$ should be of Paley-Wiener type, and there exists a distribution S of compact support such that $F(\zeta)=\hat{S}(\zeta)$. Thus, taking the inverse Fourier transforms, we get that

$$T=P(-D)Q(-D)S. \tag{4.5}$$

Now, according to Lions' theorem of supports [6], S should be in $\mathcal{E}'(\Omega)$ since Ω is convex. (4.5) shows that T is orthogonal to $\mathcal{U}$. This completes the proof.

REMARK 1. The above theorem holds also for $\mathcal{D}'(\Omega)$ replacing $\mathcal{E}(\Omega)$. The arguments will be analoguous as in the above. The necessity of the condition follows from the surjectivity of differential operators in $\mathcal{D}'(\Omega)$ with convex Ω (see [8], [5]). The sufficiency can be proved from the duality between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ by using the characterization of the Fourier transforms of the elements of $\mathcal{D}(\Omega)$.

REMARK 2. Let us remark here that the above theorem can be sharpened if we admit a result announced in L. Ehrenpreis [3]*). His fundamental principle (a) (See [3, pp. 162–163]), here in our special case, takes the following form.**)

For a fixed pair of polynomials $P(X)$ and $Q(X)$, if Ω is a convex open set in $\boldsymbol{R}^\nu$, then the totality of elements of the form $P(-D)S-Q(-D)T$, S, $T\in\mathcal{E}'(\Omega)$ constitutes a *closed* subspace of $\mathcal{E}'(\Omega)$.

From this, we can deduce the following precise

*) C.f. footnote 3).

**) Ehrenpreis states his fundamental principle for a wide class of spaces which he calls analytically uniform and localizable and claims that $\mathcal{E}(\Omega)$ with convex Ω is such a space.

Theorem 4.2. Let Ω be a convex open set in $\boldsymbol{R}^{\nu}$, and $P(X)$ and $Q(X)$ be two mutually prime polynomials. Then every solution $u \in \mathcal{E}(\Omega)$ of (4.1) can be decomposed in a sum $u = u_1 + u_2$ with (4.2).

Proof. Using notations in the proof of the previous theorem, we are to show that the following continuous linear mapping

$$\Phi : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{U}$$

defined by $\Phi(u_1, u_2) = u_1 + u_2$ is surjective. (The topology of $\mathcal{U}_1 \times \mathcal{U}_2$ be the usual product topology.) Since $\mathcal{U}_1 \times \mathcal{U}_2$ and $\mathcal{U}$ are Fréchet spaces, for the proof of surjectivity of Φ, it is enough to show that the transposed mapping

$$\Phi' : \mathcal{U}' \to \mathcal{U}_1' \times \mathcal{U}_2'$$

is one-to-one and that the image $\Phi'(\mathcal{U}')$ is weakly closed in $\mathcal{U}_1' \times \mathcal{U}_2'$ (See [1]). According to Theorem 4.1, the image $\Phi(\mathcal{U}_1 \times \mathcal{U}_2)$ is dense in $\mathcal{U}$. Hence Φ' is one-to-one. To prove that $\Phi'(\mathcal{U}')$ is closed, it is enough to show that $\Phi'(\mathcal{U}') \cap (\boldsymbol{U}^\circ \times \boldsymbol{V}^\circ)$ is weakly closed for every pair of sufficiently small neighbourhoods $\boldsymbol{U}$ and $\boldsymbol{V}$ of zeros of $\mathcal{U}_1$ and $\mathcal{U}_2$ (See [1]). ($\boldsymbol{U}^\circ$, $\boldsymbol{V}^\circ$ denote the polars of $\boldsymbol{U}$ and $\boldsymbol{V}$ (See [1])). Since the topologies of $\mathcal{U}_1$ and $\mathcal{U}_2$ are induced by that of $\mathcal{E}(\Omega)$, we may assume that $\boldsymbol{U}^\circ$ and $\boldsymbol{V}^\circ$ be of the forms

$$\boldsymbol{U}^\circ = \{\tilde{U}_1 ;\ \tilde{U}_1 \in \mathcal{U}_1',\ |\langle \tilde{U}_1, u_1 \rangle| \leq p(u_1) \text{ for all } u_1 \in \mathcal{U}_1\} ,$$
$$\boldsymbol{V}^\circ = \{\tilde{U}_2 ;\ \tilde{U}_2 \in \mathcal{U}_2',\ |\langle \tilde{U}_2, u_2 \rangle| \leq p(u_2) \text{ for all } u_2 \in \mathcal{U}_2\} ,$$

where p denotes a continuous seminorm of $\mathcal{E}(\Omega)$. Now let us notice here that Φ' is the restriction mapping, i.e. $\Phi'(\tilde{U}) = (\tilde{U}_1, \tilde{U}_2)$ means that $\tilde{U}_i =$ the restriction of $\tilde{U}$ on $\mathcal{U}_i$ $(i=1, 2)$. This is clear from the duality formula

$$\langle \tilde{U}, u_1 + u_2 \rangle = \langle \tilde{U}_1, u_1 \rangle + \langle \tilde{U}_2, u_2 \rangle,\ u_1 \in \mathcal{U}_1,\ u_2 \in \mathcal{U}_2 .$$

Now, let $\{(\tilde{U}_1^{(\iota)}, \tilde{U}_2^{(\iota)}\}$ be a filter in $\Phi'((\mathcal{U}') \cap (\boldsymbol{U}^\circ \times \boldsymbol{V}^\circ)$ which converges weakly to an element $(\tilde{U}_1, \tilde{U}_2)$ in $\mathcal{U}_1' \times \mathcal{U}_2'$. We are to show that $(\tilde{U}_1, \tilde{U}_2)$ is in $\Phi'(\mathcal{U}')$. Let $\tilde{U}_i^{(\iota)}$ be the restriction of $\tilde{U}^{(\iota)} \in \mathcal{U}'$ on $\mathcal{U}_i$ $(i=1, 2)$ and $U^{(\iota)} \in \mathcal{E}'(\Omega)$ be an extention of $\tilde{U}^{(\iota)}$, this being possible by Hahn-Banach's theorem (See [1]). Again by Hahn-Banach's theorem, we can extend $\tilde{U}_i^{(\iota)}$ $(i=1, 2)$ to elements of $\mathcal{E}'(\Omega)$ preserving the seminorm inequalities that defined $\boldsymbol{U}^\circ$ and $\boldsymbol{V}^\circ$. Hence there exist $S_1^{(\iota)}, S_2^{(\iota)} \in \mathcal{E}'(\Omega)$ such that

$$|\langle U^{(\iota)}+P(-D)S_1^{(\iota)}, f\rangle| \leq p(f),\ f\in \mathcal{E}(\Omega),$$
$$|\langle U^{(\iota)}+Q(-D)S_2^{(\iota)}, f\rangle| \leq p(f),\ f\in \mathcal{E}(\Omega).$$ [9]

This means that filters $\{U^{(\iota)}+P(-D)S_1^{(\iota)}\}$ and $\{U^{(\iota)}+Q(-D)S_2^{(\iota)}\}$ are contained in an equicontinuous set in $\mathcal{E}'(\Omega)$. Since an equicontinuous set is weakly compact, taking finer filters and keeping the same notations, we can make these filters converge weakly. Let V_1 and V_2 be their limits, i.e.

$$\begin{aligned} &\lim_{\iota}(U^{(\iota)}+P(-D)S_1^{(\iota)}) = V_1 \\ &\lim_{\iota}(U^{(\iota)}+Q(-D)S_2^{(\iota)}) = V_2. \end{aligned} \tag{4.6}$$

Then it is clear that V_1, V_2 are extensions of $\tilde{U}_1$, $\tilde{U}_2$. Moreover, from (4.6) we see that

$$V_1-V_2 = \lim_{\iota}(P(-D)S_1^{(\iota)}-Q(-D)S_2^{(\iota)}).$$

Hence, by Ehrenpreis' fundamental principle stated above, we can find $S_1, S_2\in \mathcal{E}'(\Omega)$ such that

$$V_1-V_2 = P(-D)S_1-Q(-D)S_2.$$

Thus, $U=V_1-P(-D)S_1=V_2-Q(-D)S_2$ is a simultaneous extension of $\tilde{U}_1$ and $\tilde{U}_2$. Hence $(\tilde{U}_1, \tilde{U}_2)=\Phi'(\tilde{U})$, where $\tilde{U}$ is the restriction of U on $\mathcal{U}$. This completes the proof.

§ 5. Polynomial solutions

A main result in this paragraph is Theorem 5.2, Corollary 5.3 will be used in the next section. Let $\mathcal{P}$ be the totality of complex valued polynomial functions of ν real variables, $\mathcal{P}_n$ be the totality of polynomials of degree at most n and $\mathcal{P}^{(n)}$ be the totality of homogeneous polynomials of degree n. $\mathcal{P}_n$ and $\mathcal{P}^{(n)}$ are finite dimensional vector space. We denote their dimensions by δ_n and $d^{(n)}$ respectively.

$$d^{(n)} = \frac{(n+\nu-1)!}{n!(\nu-1)!},\quad d_n = d^{(0)}+\cdots+d^{(n)}. \tag{5.1}$$

Theorem 5.1. *Let $L(X)$ be non-vanishing polynomial. Then $L(D)$ is a surjective endomorphism of $\mathcal{P}$, i.e.*

$$L(D)\mathcal{P} = \mathcal{P}. \tag{5.2}$$

9) Since $U^{(\iota)}$ is an extension of $\tilde{U}_1^{(\iota)}$ (resp. $\tilde{U}_2^{(\iota)}$), other extensions of $\tilde{U}_1^{(\iota)}$ (resp. $\tilde{U}_2^{(\iota)}$) should be of the form $U^{(\iota)}+P(-D)S_1$, $S_1\in\mathcal{E}'(\Omega)$ (resp. $U^{(\iota)}+Q(-D)S_2$, $S_2\in\mathcal{E}'(\Omega)$. For, any element which is orthogonal to $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is of the form $P(-D)S_1$ (resp. $Q(-D)S_2$).

More precisely, if the lowest degree of non-vanishing term of $L(X)$ is l, then

$$L(D)\mathcal{P}_{n+l} = \mathcal{P}_n \quad (n = 0, 1, 2, \cdots). \tag{5.3}$$

When $L(X)$ is homogeneous, then

$$L(D)\mathcal{P}^{(n+l)} = \mathcal{P}^{(n)}. \tag{5.4}$$

Proof. We proceed by induction on the number of variables ν. When $\nu=1$, $L(D)=L(D_1)$ should be of the form $L(D_1)=R(D_1)D_1^l$ with $R(0)\neq 0$. It is clear that $R(D_1)\mathcal{P}_{n+l}\subseteq\mathcal{P}_{n+l}$. That is, $R(D_1)$ may be regarded as an endomorphism of the finite dimensional vector space $\mathcal{P}_{n+l}$. Moreover $R(D_1)f\neq 0$ if f is a non-zero polynomial. To see this it is enough to compare the degrees of 1st and 2nd terms of the right hand side of the following formula

$$R(D_1)f = R(0)f+(F(D_1)-R(0))f,\ R(0)\neq 0\,.$$

Since an endomorphism of finite dimensional vector space whose kernel is zero should be surjective, we should have that $R(D_1)\mathcal{P}_{n+l}=\mathcal{P}_{n+l}$. Hence $L(D)\mathcal{P}_{n+l}=D_1^l\mathcal{P}_{n+l}=\mathcal{P}_n$.

Now assume that the theorem holds when the number of variables is $\nu-1$ and let us prove it in the case of ν variables ($\nu\geq 2$). By a suitable linear transformation of variables (the coefficient of transformation matrix being real), we may suppose that $L(X)$ contains the term $c\cdot X_1^l(c\neq 0)$. Now, we proceed by induction on n. Since $L(D)x_1^l$ is clearly a non vanishing constant, we get that $L(D)\mathcal{P}_l=\mathcal{P}_0$. Assuming that $L(D)\mathcal{P}_{n+l}=\mathcal{P}_n$, we are to show that $L(D)\mathcal{P}_{n+1+l}=\mathcal{P}_{n+1}$. Let f be an element of $\mathcal{P}_{n+1}$. Then $\partial f/\partial x_\nu\in\mathcal{P}_n$. Thus we can find an element $u\in\mathcal{P}_{n+l}$ such that $L(D)u=\partial f/\partial x_\nu$. Take an element v of $\mathcal{P}_{n+1+l}$ such that $\partial v/\partial x_\nu=u$ (a primitive of v with respect to the last variable x_ν). Thus we get that

$$\frac{\partial}{\partial x_\nu}\{L(D)v-f\} = 0\,.$$

This means that $g=L(D)v-f$ is a polynomial of degree at most $n+1$ and depending only on $x_1,\cdots,x_{\nu-1}$. Thus, by the induction hypothesis, we can find a polynomial w of degree at most $n+1+l$ and depending only on $x_1,\cdots,x_{\nu-1}$ such that $L(D)w=L(D_1,\cdots,D_{\nu-1},0)w=g$, since $L(D)$ contains the term $c\cdot D_1^l$. Thus $L(D)(v-w)=f$ and $v-w\in\mathcal{P}_{n+1+l}$. This proves (5.3) and (5.2). (5.4) follows from (5.2), since $L(D)\mathcal{P}^{(n+l)}\subseteq\mathcal{P}^{(n)}$.

Corollary 5.1. *Let H_n be the space of those polynomials u of degree $\leq n$ which satisfy* (1.1) *and $H^{(n)}$ be the space of those homogeneous polynomials of degree n which satisfy* (1.1). *Then we have*:

(i) $$\dim H_n = \begin{cases} d_n - d_{n-l}, & \text{if } n \geq l \\ d_n, & \text{if } n < l \end{cases}$$

(ii) *If $L(X)$ is homogeneous,*

$$\dim H^{(n)} = \begin{cases} d^{(n)} - d^{(n-l)}, & \text{if } n \geq l \\ d^{(n)}, & \text{if } n < l. \end{cases}$$

Proof. By the above theorem we have that $\mathcal{P}_{n+l}/H_{n+l}$ is isomorphic to $\mathcal{P}_n$, and if $L(X)$ is homogeneous we have that $\mathcal{P}^{(n+l)}/H^{(n+l)}$ is isomorphic to $\mathcal{P}^{(n)}$. Equating the dimensions respectively we get the conclusion stated above.

Corollary 5.2. *Assume that $\nu \geq 2$ and $L(X_1, \cdots, X_{\nu-1}, 0) \neq 0$, then we have: for any $n \geq 1$*

(i) $$\frac{\partial}{\partial x_\nu} H_n = H_{n-1};$$

(ii) $$\frac{\partial}{\partial x_\nu} H^{(n)} = H^{(n-1)}, \text{ when } L(X) \text{ is homogeneous.}$$

Proof. As in the proof of Theorem 5.1, we may assume that $L(X)$ contains the term $c \cdot X_1^l$ $(c \neq 0)$. Let u be an element of H_{n-1}. Then it is clear that there exists an element $v \in \mathcal{P}_n$ such that $\frac{\partial v}{\partial x_\nu} = u$. Then we have that $\frac{\partial}{\partial x_\nu} L(D)v = 0$. Hence $g = L(D)v$ is an element of $\mathcal{P}_{n-l}$ depending only on $x_1, \cdots, x_{\nu-1}$ (or $g=0$, if $n < l$). Then, by Theorem 5.1, there exists an element $w \in \mathcal{P}_n$ which depends only on $x_1, \cdots, x_{\nu-1}$ such that $L(D)w = g$. Thus we have that $L(D)(v-w) = 0$, i.e. $v - w \in H_n$. Since $\frac{\partial w}{\partial x_\nu} = 0$, we have $\frac{\partial (v-w)}{\partial x_\nu} = u$. This proves (i). (ii) follows from (i) since, $L(X)$ being homogeneous, each homogeneous part of a solution polynomial should also be a solution.

Theorem 5.2. *Let $L(X)$ be a homogeneous polynomial and $L(X) = P(X)Q(X)$ be a factorization. Let $H_1^{(n)}$ (resp. $H_1^{(n)}$) be the space of those homogeneous polynomials u_1 (resp. u_2) of degree n such that $P(D)u_1 = 0$ (resp. $Q(D)u_2 = 0$). If $P(X)$ and $Q(X)$ are mutually prime, then we have that for any n*

$$H^{(n)} = H_1^{(n)} + H_2^{(n)}.$$

(Since $P(X)$ and $Q(X)$ should also be homogeneous, this is the same to say that polynomial solutions of (1.1) decompose in the form (1.3) with (1.4).)

Proof. Since the statement has no sense for $\nu=1$, we proceed by induction on ν beginning with the case $\nu=2$.[10] When $\nu=2$, homogeneous polynomial splitting into linear factors, by a suitable real linear transformation of variables we may assume that

$$P(X) = (X_1-\alpha_1 X_2)^{\mu_1}\cdots(X_1-\alpha_m X_2)^{\mu_m}$$
$$Q(X) = (X_1-\beta_1 X_2)^{\nu_1}\cdots(X_1-\beta_n X_2)^{\nu_n}.$$

Here, $\{\alpha_1,\cdots,\alpha_m\}$ and $\{\beta_1,\cdots,\beta_n\}$ are disjoint, since $P(X)$, $Q(X)$ are mutually prime. Since differential operators are surjective for the space of polynomials, according to Lemma 3.1, we have only to treat the case

$$P(X) = X_1-\alpha X_2,$$
$$Q(X) = X_1-\beta X_2$$

where α and β are distinct complex numbers. Now we proceed by induction on n. For $n=0$, $H^{(0)}=H_1^{(0)}+H_2^{(0)}$ is clear. Assuming the statement for n, let us prove it for $n+1$. Take an element $u\in H^{(n+1)}$. Then $\frac{\partial u}{\partial x_2}\in H^{(n)}$. Hence, by the hypothesis of induction, there exist $v_1\in H_1^{(n)}$ $v_2\in H_2^{(n)}$ such that $\frac{\partial u}{\partial x_2}=v_1+v_2$. Hence, by Corollary 5.2, there exist $u_1\in H_1^{(n+1)}$, $u_2\in H_2^{(n+1)}$ such that $\frac{\partial u_1}{\partial x_2}=v_1$, $\frac{\partial u_2}{\partial x_2}=v_2$. Hence $\frac{\partial}{\partial x_2}(u-u_1-u_2)=0$. Thus $f=u-u_1-u_2$ is a homogeneous polynomial of degree $(n+1)$ depending only on x_1. Since f clearly satisfies

$$(D_1-\alpha D_2)(D_1-\beta D_2)f = 0,$$

f should be a constant or of the form cx_1. Therefore it only remains to prove that $x_1\in H_1^{(1)}+H_2^{(1)}$. But since $\tilde{u}_1=\alpha x_1+x_2\in H_1^{(1)}$ and $\tilde{u}_2=\beta x_1+x_2\in H_2^{(1)}$, we have a desired decomposition:

$$x_1 = \frac{1}{\alpha-\beta}\tilde{u}_1+\frac{-1}{\alpha-\beta}\tilde{u}_2.$$

Now assume that the statement is true for ν and let us prove it for $\nu+1$ ($\nu\geq 2$). We may again assume that $P(X)$ and $Q(X)$ are irreducible and distinct homogeneous polynomials. Therefore $P(X)=0$ and $Q(X)=0$ respectively define two distinct irreducible hypersurfaces V_P and V_Q in the ν-dimensional complex projective space $\boldsymbol{P}_\nu(\boldsymbol{C})$ that is realized as the hyperplane at infinity of $\boldsymbol{C}^{\nu+1}$. Therefore the sections of V_P and V_Q by a general hyperplane F (isomorphic to $\boldsymbol{P}_{\nu-1}(\boldsymbol{C})$) are again two

10) The case $\nu=2$ cannot be reduced to the case $\nu=1$.

distinct irreducible hypersurfaces in $\boldsymbol{P}_{\nu-1}(\boldsymbol{C})$ if $\nu\geq 3$ or two disjoint finite point sets if $\nu=2$. Such a hyperplane F can be obtained by a linear transformation from a given coordinate hyperplane and moreover we may assume that the transformation matrix has real coefficients[11]. Thus we may assume, after a suitable (real) linear transformation of variables, that $X_\nu=0$ defines such a hyperplane F. Now we proceed by induction on n. For $n=0$ the statement is clear. Assuming it for n, let us prove it for $n+1$. Let u be an element of $H^{(n+1)}$. Then $\frac{\partial u}{\partial x_\nu}\in H^{(n)}$. Hence there exist $v_1\in H_1^{(n)}$, $v_2\in H_2^{(n)}$ such that $\frac{\partial u}{\partial x_\nu}=v_1+v_2$. By Corollary 5.2, we can find $u_1\in H_1^{(n+1)}$, $u_2\in H_2^{(n+1)}$ such that $\frac{\partial u_1}{\partial x_\nu}=v_1$, $\frac{\partial u_2}{\partial x_\nu}=v_2$. Thus $\frac{\partial}{\partial x_\nu}(u-u_1-u_2)=0$. Thus $f=u-u_1-u_2$ is in $H^{(n+1)}$ but is independent of x_ν. According to the above assumption, $P(X_1,\cdots,X_{\nu-1},0)$ and $Q(X_1,\cdots,X_{\nu-1},0)$ are mutually prime and, therefore by the induction hypothesis, there exist $f_1\in H_1^{(n+1)}$, $f_2\in H_2^{(n+1)}$ depending only on $x_1,\cdots,x_{\nu-1}$ such that $f=f_1+f_2$. Thus we get the desired decomposition

$$u=(u_1+f_1)+(u_2+f_2)\,.$$

This completes the proof.

Corollary 5.3. *Let $P(X)$ and $Q(X)$ be mutually prime homogeneous polynomials of two varaibles, then the space of those polynomials u which are simultaneous solutions of the equations*

$$P(D)u=Q(D)u=0 \tag{5.5}$$

constitutes a finite dimensional vector space. More precisely, if the orders of $P(D)$ and $Q(D)$ are l_1, and l_2, then polynomial solutions of (5.5) *are of degree at most l_1+l_2-1.*

Proof. We are to show that $\dim H_1^{(n)}\cap H_2^{(n)}=0$, if $n\geq l_1+l_2$. That is the same to say that $H^{(n)}=H_1^{(n)}+H_2^{(n)}$ is a direct sum or that

$$\dim H^{(n)}=\dim H_1^{(n)}+\dim H_2^{(n)}\,. \tag{5.6}$$

According to Corollary 5.1 and (5.1),

$$\begin{aligned}
\dim H^{(n)} &= d^{(n)}-d^{(n-l_1-l_2)} &&= l_1+l_2 \quad \text{if } n\geq l_1+l_2\\
\dim H_1^{(n)} &= d^{(n)}-d^{(n-l_1)} &&= l_1 \quad \text{if } n\geq l_1\\
\dim H_2^{(n)} &= d^{(n)}-d^{(n-l_2)} &&= l_2 \quad \text{if } n\geq l_2\,.
\end{aligned}$$

11) As for these elementary facts from algebraic geometry, see, for instance, [13].

Thus we see that (5.6) holds if $n \geq l_1 + l_2$.

REMARK. If $L(X)$ is not homogeneous, we can easily see, by simple examples, $H_n = H_{1,n} + H_{2,n}$ does not hold. That is, to decompose a polynomial solution u of $L(D)u = 0$, we should use u_1 and u_2 and of higher degree than that of u.

§ 6. Analytic solutions in a simply connected domain in $\boldsymbol{R}^2$.

It is a classical theorem in function theory that a real harmonic function defined in a simply connected domain Ω in $\boldsymbol{R}^2$ can be represented as the real part of a holomorphic function in Ω. This is equivalent to say that a complex valued continuous function u which satisfies

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u = 0$$

can be decomposed into a sum

$$u = u_1 + u_2$$

with

$$\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) u_1 = 0\,, \quad \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) u_2 = 0\,.$$

The condition that Ω is simply connected is essential as is easily seen by simple examples.

Since harmonic functions are necessarily (real) analytic, Theorem 6.1 below gives a generalization of the classical fact above.

In this paragraph, for a domain Ω in $\boldsymbol{R}^2$, $\mathcal{A}(\Omega)$ shall denote the totality of (complex-valued) real analytic functions defined in Ω.

Lemma 6.1. *Let Ω be a bounded convex domain in $\boldsymbol{R}^2$ and $L(X)$ be a non-zero homogeneous polynomial of two variables. Then $L(D)$ is a surjective mapping of $\mathcal{A}(\Omega)$ onto itself.*

Proof. Since $L(X)$ splits into linear factors we have only to treat the case where $L(X)$ is of the form

$$L(X) = X_1 - \alpha X_2\,. \tag{6.1}$$

First let us assume that α is not real. Then, $L(D)$ is an elliptic operator. Let f be an element of $\mathcal{A}(\Omega)$. Since f is in $\mathcal{E}(\Omega)$ and Ω is convex, according to [7], there exists an element $u \in \mathcal{E}(\Omega)$ such that $L(D)u = f$. But since $L(D)$ is elliptic, u should be in $\mathcal{A}(\Omega)$ (See [10]). This proves the surjectivity of $L(D)$. Now if α is real, by a suitable linear transformation, we may assume that

$$L(D) = D_1 .$$

Thus, in this case, the surjectivity of $L(D)$ is nothing but that every $f \in \mathcal{A}(\Omega)$ has a primitive $u \in \mathcal{A}(\Omega)$ with respect to x_1. Since Ω is bounded and convex, such a primitive can be obtained in the form

$$u(x_1, x_2) = \int_{\lambda(x_2)}^{x_1} f(t, x_2)dt$$

where $\lambda(x_2) = ax_2 + b$, (a, b: real constants) is so chosen that the intersecting points $(x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}, x_2^{(2)})$ of the line $x_1 - \lambda(x_2) = 0$ and the boundary of Ω attain the two extremal values of the second coordinate, i.e.

$$x_2^{(1)} = \inf_{\Omega} x_2 . \quad x_2^{(2)} = \sup_{\Omega} x_2 .$$

Such choice is possible, since Ω is bounded aud convex.

Theorem 6.1. *Let $L(X)$ be a homogeneous polynomial and $L(X) = P(X)Q(X)$ be a factorization into mutually prime factors. Let Ω be a simply connected domain in $\boldsymbol{R}^2$. Then every solution $u \in \mathcal{A}(\Omega)$ of* (1.1) *can be decomposed into the form*

$$u = u_1 + u_2, \; u_1, u_2 \in \mathcal{A}(\Omega) \tag{6.2}$$

with (1.4).

Further, if the degree of $L(X)$ is l, then the decomposition (6.2) *is unique up to a certain polynomial of degree at most $l-1$.*

Proof. We proceed as follows. We first prove the decomposability locally and then extend it in the large by analytic continuation along polygons in Ω. Since the surjectivity of differential operators is not known for a general simply connected domain, localization shall be twofold.

1) *The case when Ω is convex.* According to Lemma 3.1 and Lemma 6.1, we may assume that

$$P(D) = D_1 - \alpha D_2$$
$$Q(D) = D_1 - \beta D_2$$

where α and β are distinct complex numbers. Let $a = (a_1, a_2)$ be a point in Ω. By a translation of coordinates we may assume that $a_1 = a_2 = 0$. Let $u \in \mathcal{A}(\Omega)$ be a solution of $P(D)Q(D)u = 0$. Put

$$f_0(x_2) = u(0, x_2) ,$$
$$f_1(x_2) = (D_1 u)(0, x_2) .$$

Now chose two analytic functions $g_0(x_2)$ and $h_0(x_2)$ solutions of the following ordinary differential equations in x_2

$$(6.3)\qquad \begin{aligned}(\alpha-\beta)D_2 g_0 &= f_1-\beta D_2 f_0\,,\\ (\alpha-\beta)D_2 h_0 &= \alpha D_2 f_0 - f_1\end{aligned}$$

with initial conditions

$$(6.4)\qquad \begin{aligned}g_0(0) &= f_0(0)\,,\\ h_0(0) &= 0\,.\end{aligned}$$

g_0, h_0 are analytic functions in x_2 defined near the origin. Now consider the Cauchy problems:

$$\begin{aligned}&(D_1-\alpha D_2)u_1 = 0 \text{ with } u_1(0, x_2) = g_0(x_2)\,, \text{ and}\\ &(D_1-\beta D_2)u_2 = 0 \text{ with } u_2(0, x_2) = h_0(x_2)\,.\end{aligned}$$

According to Cauchy-Kowalevski's theorem, analytic solutions u_1 and u_2 exist in a neighbourhood of the origin. Then

$$(6.5)\qquad v = u-u_1-u_2$$

is a solution of the Cauchy problem:

$$(6.6)\qquad \begin{aligned}&P(D)Q(D)v = 0\,, \text{ with}\\ &v(0, x_2) = (D_1 v)(0, x_1) = 0\,.\end{aligned}$$

In fact, conditions (6.3) and (6.4) were so chosen that (6.4) should satisfy (6.6). Hence, according to the uniqueness of solution to the Cauchy problem, v should vanish in a neighbourhood V of the origin. To summarize, we have shown that for each point $a\in\Omega$ and for every solution u of the equation

$$P(D)Q(D)u = 0, \quad u\in\mathcal{A}(\Omega)\,,$$

there exists a circular neighbourhood V_a of a in which u has a decomposition such that

$$u = u_1+u_2, \quad u_1, u_2 \in \mathcal{A}(V_a)$$

with

$$P(D)u_1 = 0, \quad Q(D)u_2 = 0 \text{ in } V_a\,.$$

Consider a covering $\{V_i\}_{i\in I}$ of Ω consisting of such circular neighbourhoods (I being an index set). Now we proceed to the global decomposability. Let $u\in\mathcal{A}(\Omega)$ be a solution of $P(D)Q(D)u=0$. For a pair of $i, j\in I$, consider the decompositions:

$$(6.7)\qquad \begin{aligned} u &= u_1^{(i)}+u_2^{(i)}, \qquad u_1^{(i)}, u_2^{(i)} \in \mathcal{A}(V_i), \\ u &= u_1^{(j)}+u_2^{(j)}, \qquad u_1^{(j)}, u_2^{(j)} \in \mathcal{A}(V_j), \end{aligned}$$

with

$$(6.8)\qquad \begin{aligned} P(D)u_1^{(i)} &= 0,\ Q(D)u_2^{(i)} = 0 \text{ in } V_i, \\ P(D)u_1^{(j)} &= 0,\ Q(D)u_2^{(j)} = 0 \text{ in } V_j. \end{aligned}$$

If the intersection $V_i \cap V_j$ is not empty,

$$(6.9)\qquad w = u_1^{(i)}-u_1^{(j)} = -u_2^{(i)}+u_2^{(j)}$$

should satisfy

$$(6.10)\qquad P(D)w = Q(D)w = 0 \text{ in } V_i \cap V_j.$$

Since $\alpha \neq \beta$, this equation implies that w is a constant on $V_i \cap V_j$. Therefore, adjusting by a constant, we can get a decomposition of u in the union $V_i \cup V_j$. Continuing this process, we can extend a given decomposition in a V_i along any polygon starting at a point in V_i. The resulting decomposition should be univalent since Ω is simply connected.

2) *The general case.* Let $L(X)=P(X)Q(X)$ be the given factorization into mutually prime factors. We denote by l_1 and l_2 the degrees of $P(X)$ and $Q(X)$ respectively[12]. Consider a covering $\{V_i\}_{i\in I}$ of Ω consisting of convex subdomains of Ω. And let $u \in \mathcal{A}(\Omega)$ be a solution of $L(D)u=0$. For each pair of V_i, V_j, according to case 1), we have decompositions of the form (6.7) with (6.8). If $V_i \cap V_j \neq \phi$, then the corresponding $w \in \mathcal{A}(V_i \cap V_j)$ defined by (6.9) should satisfy (6.10). By a translation of coordinates we may assume that $V_i \cap V_j$ contains the origin (0, 0). Let

$$(6.11)\qquad w = \sum_{n=0}^{\infty} w_n$$

be the Taylor expansion of w around the origin. w_n denotes the homogeneous part of degree n. Since $P(X)$ and $Q(X)$ are homogeneous polynomials, each term w_n should satisfy (6.10). Thus, according to Corollary 5.3,

$$(6.12)\qquad w_n = 0 \quad \text{if} \quad n \geq l_1+l_2 = l.$$

This shows that w should be equal to a polynomial of degree $\leq l-1$ around the origin and hence everywhere in $V_i \cap V_j$ because of its analyticity. Thus, adjusting by a polynomial of degree $\leq l-1$ we can get a decomposition of u in the union $V_i \cup V_j$. Thus, as in the case 1), we

13) Since we know the surjectivity of differential operators only for convex Ω, we cannot assume here that $P(X)$ and $Q(X)$ be linear factors.

can get a global docomposition because of simply-connectedness of Ω. The last statement in the theorem can be proved by the same argument as in the above, using the Taylor expansion (6.11) and (6.12). This completes the proof.

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