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# LAWS OF THE ITERATED LOGARITHM FOR SOME SYMMETRIC DIFFUSION PROCESSES

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## 1. Introduction

In this paper we obtain three results concerning laws of the iterated logarithms (LILs) for certain functionals of some Markov processes.

The first is for symmetric diffusions whose transition densities satisfy upper and lower bounds similar to those of Aronson for uniformly elliptic divergence form operators in $\mathbf{R}^d$. We suppose the transition densities $p_t(x, y)$ are symmetric in $x$ and $y$ and satisfy an estimate of the form

$$\begin{aligned} & c_1 t^{-d_s/2} \exp\left(-c_2\left(\frac{d(x, y)^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ \leq\ & p_t(x, y) \\ \leq\ & c_3 t^{-d_s/2} \exp\left(-c_4\left(\frac{d(x, y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \end{aligned}$$

where $d(x, y)$ is the distance between $x$ and $y$ and $c_1, c_2, c_3, c_4, d_s$, and $d_w$ are constants. Examples of such processes include ones associated to uniformly elliptic operators in divergence form in $\mathbf{R}^d$, of course, but also Brownian motions whose state space is an affine nested fractal, such as the Sierpinski gasket, and Brownian motions on Sierpinski carpets. (See the Appendix of this paper and also [14], [6] and [3].)

For such processes we first prove a large deviations principle similar to that of Schilder for Brownian motion (cf. [24]). When the state space is a fractal, one cannot prove as much as in the case of Brownian motion; in fact, it can be shown (see [10]) that the direct analog for Schilder's theorem is not true. Nevertheless, the large deviations principle that we do prove is sufficient to obtain a functional law of the iterated logarithm similar to that of Strassen; see Theorem 2.11. This is the content of Section 2.

Next in Section 3 we consider arbitrary Markov processes, not necessarily continuous nor symmetric, and look at functionals of the path that are nondecreasing, continuous, subadditive, and satisfy a uniform scaling property. For these functionals we

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show that one has an upper bound for a LIL. We give several examples to illustrate the hypotheses of Theorem 3.1.

Finally, for our third result, in Section 4, we restrict attention to Brownian motion whose state space is an affine nested fractal or a Sierpinski carpet. Such processes have local times $\ell_t(x)$ and one can ask about limsup and liminf LILs for $L^*(t) = \sup_x \ell_t(x)$. Some results on the limsup LIL were obtained in [16]; we complete these and then obtain the corresponding liminf LIL. Such processes also have a range that has nonzero $\mu$-measure, where $\mu$ is the invariant measure for the state space; there is no analog of this fact for diffusions in $\mathbf{R}^d$ except for the uninteresting case when $d = 1$. It thus makes sense to talk about an LIL for the $\mu$-measure of the range, and this is also obtained in Section 4. We comment that neither $L^*(t)$ nor the $\mu$-measure of the range is a continuous functional of the path, so cannot be handled by the techniques of Section 2.

Section 5 is an appendix recalling a few facts about fractals and diffusions on fractals.

## 2. Large deviations and Strassen's LIL for diffusion processes with Aronson-type estimates

In this section, we consider diffusion processes on a complete metric space $E$ whose transition densities satisfy Aronson-type estimates. We show that a functional type LIL holds for these processes.

### 2.1. Diffusion processes with Aronson-type estimates and their properties

Let $(E, d)$ be a locally compact complete separable connected metric space which enjoys the *midpoint property*, i.e., for each $x, y \in E$ there exists $z \in E$ such that $d(x, z) = d(z, y) = (1/2)d(x, y)$. Let $\mu$ be a $\sigma$-finite Borel measure whose support is $E$ which satisfies

$$\mu(B(x, 2r)) \le M_1 \mu(B(x, r)) \quad \text{for all } x \in E, r > 0, \tag{2.1}$$

$$\mu(B(x, 1)) \le M_2 \quad \text{for all } x \in E, \tag{2.2}$$

for some constants $M_1, M_2 > 0$ where $B(x, r) = \{y \in E : d(x, y) < r\}$. (2.1) is often called a doubling condition of a measure.

Let $(\Omega, \mathcal{F}, \{P^x\}, \{X(t)\})$ a diffusion process on $E$ which is symmetric with respect to $\mu$. We assume the following for the process.

ASSUMPTION 2.1. There exists a jointly continuous symmetric transition density $p_t(x, y)$ for $X(t)$ with respect to $\mu$ on $E$ which satisfies the Chapman-Kolmogorov equations and the following:

$$c_{2.1} t^{-d_s/2} \exp(-c_{2.2} \Psi(d(x, y), t)) \le p_t(x, y) \le c_{2.3} t^{-d_s/2} \exp(-c_{2.4} \Psi(d(x, y), t))$$

for all $0 < t < \infty$, $x, y \in E$, where $\Psi(z, t) = (z^{d_w} t^{-1})^{1/(d_w - 1)}$, and $d_s \geq 1$, $d_w > 1$, and $c_{2.1}, c_{2.2}, c_{2.3}, c_{2.4}$ are positive constants.

The setup here is nearly the same as that of the diffusions on fractals studied in [2]. Note, though, under our setup we only have $\mu(B(x, r)) \leq M_3(r^\gamma \vee 1)$ for some $\gamma > 0$ and do not require the lower bound in general.

There are various examples of diffusion processes which have these estimates.
(a) $E = \mathbf{R}^d$, $\{X(t)\}$ is the diffusion whose generator $\mathcal{L}$ is the divergence form elliptc operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j},$$

where $\{a_{ij}(x)\}$ is bounded, symmetric, measurable and uniformly elliptic. Assumption 2.1 then holds with $d_s = d, d_w = 2$ ([1]).
(b) Brownian motion on a Riemannian manifold $M$. Suppose there is a single $C^\infty$ map from $\mathbf{R}^d$ onto $M$ such that with respect to these coordinates the coefficients of the Riemannian metric are bounded and uniformly elliptic. Then it may be shown that the infinitesimal generator of Brownian motion on $M$ will be a nondegenerate time change of an operator such as the one in (a) and that Assumption 2.1 holds.
(c) Diffusions on fractals. Another class of processes which satisfy these conditions are diffusion processes on fractals. Concrete examples are:

(1) Brownian motion on affine nested fractals ([14])
(2) Brownian motion on Sierpinski carpets ([5], [6])

See the Appendix for the definition of these fractals and of diffusions on them. Note that (1) contains Brownian motion on nested fractals whose heat kernel estimate was obtained in [19]; Brownian motion on the Sierpinski gasket ([7]) is a typical case. As we will discuss in the Appendix, Assumption 2.1 holds for these examples.

We now summarize some facts that may be deduced from our assumptions on $(E, d, \mu)$ and Assumption 2.1. For $A \subset E$, we set $\sigma_A = \inf\{t \geq 0 : X(t) \in A\}$.

**Proposition 2.2.** (1) *For each $x, y \in E$, there exists a geodesic path $\{\gamma(t) : 0 \leq t \leq 1\}$ such that $\gamma(0) = x, \gamma(1) = y$ and $d(\gamma(s), \gamma(t)) = |t - s| d(x, y)$ for all $0 \leq s \leq 1$.*
(2) *For each $x, y \in E$,*

$$c_{2.4} d(x, y)^{d_w/(d_w - 1)} \leq -\lim_{\epsilon \to 0} \epsilon^{1/(d_w - 1)} \log p_\epsilon(x, y) \leq c_{2.2} d(x, y)^{d_w/(d_w - 1)}.$$

(3) *There exist* $c_{2.5}, c_{2.6} > 0$ *which depend only on* $t_0 > 0$ *such that*

$$P^x(\sup_{s\le t} d(X_s, X_0) \ge \lambda) \le c_{2.5} \exp\Big( - c_{2.6}\Big(\frac{\lambda^{d_w}}{t}\Big)^{1/(d_w-1)}\Big),$$

*for all* $\lambda > 0, 0 < t \le t_0, x \in E$.

(4) *There exist* $c_{2.7}, c_{2.8} > 0$ *which depend only on* $t_0 > 0$ *such that*

$$P^x(\sigma_{B(x,r)} \le t) \le c_{2.7} \exp\Big( - c_{2.8}\Big(\frac{r^{d_w}}{t}\Big)^{1/(d_w-1)}\Big),$$

*for all* $r > 0, 0 < t \le t_0, x \in E$.

(5) *There exist* $c_{2.9}, \beta > 0$ *such that for each* $x, x', y \in E, t > 0$,

$$|p_t(x, y) - p_t(x', y)| \le c_{2.9}\{d(x, x')t^{-1/d_w}\}^{\beta} t^{-d_s/2}.$$

Proof. (1) and (2) are straightforward from our assumption. (3) and (4) are proved in the same way as Lemma 3.9($a$) and (3.11) of [2]. (5) follows by the argument in [13], Section 3. In fact, Assumption 2.1 also implies a parabolic Harnack inequality; cf. [13], Section 3 or [6], Section 7, although we do not need this fact. □

In this situation, we can also obtain a $0-1$ law analogous to the one which was obtained in [6] Theorem 8.4 for the case of Brownian motion on the Sierpinski carpet.

**Proposition 2.3.** *Suppose* $A$ *is a tail event*: $A \in \cap_t \sigma\{X_u : u \ge t\}$. *Then, either* $P^x(A)$ *is* 0 *for all* $x$ *or else it is* 1 *for all* $x$.

Proof. We follow the proof of [6]. Let $\epsilon > 0$ and fix $x_0 \in E$. By the martingale convergence theorem, $E^{x_0}[1_A|\mathcal{F}_t] \to 1_A$ almost surely as $t \to \infty$. Choose $t_0$ large enough so that

$$E^{x_0}|E^{x_0}[1_A|\mathcal{F}_{t_0}] - 1_A| < \epsilon. \tag{2.3}$$

Write $Y$ for $E^{x_0}[1_A|\mathcal{F}_{t_0}]$. Using Proposition 2.2(3), choose $M$ large so that

$$P^{x_0}(\sup_{s\le t_0} d(X_s, x_0) > M t_0^{1/d_w}) < \epsilon.$$

For each $t$, by Proposition 2.2(5) we have the continuity of $P_t f(x)$ in $x$ with a modulus depending only on $t$ and $\|f\|_\infty$. We choose $t_1$ large so that

$$|P_{t_1} f(x) - P_{t_1} f(x_0)| < \epsilon \|f\|_\infty, \quad d(x, x_0) \le M t_0^{1/d_w}. \tag{2.4}$$

We note

$$|P^{x_0}(A) - E^{x_0}(Y; A)| = |E^{x_0}(1_A; A) - E^{x_0}(Y; A)| < \epsilon. \tag{2.5}$$

Since $A$ is a tail event, there exists $C$ such that $A = C \circ \theta_{t_0+t_1}$. Let $f(z) = P^z(C)$. By the Markov property at time $t_1$,

$$E^w(1_C \circ \theta_{t_1}) = E^w E^{X(t_1)} 1_C = E^w f(X_{t_1}) = P_{t_1} f(w). \tag{2.6}$$

By the Markov property at time $t_0$ and (2.6),

$$E^{x_0}(Y; A) = E^{x_0}[Y E^{X(t_0)}(1_C \circ \theta_{t_1})] = E^{x_0}[Y P_{t_1} f(X_{t_0})], \tag{2.7}$$

while

$$P^{x_0}(A) = E^{x_0} 1_A = E^{x_0} E^{X(t_0)}(1_C \circ \theta_{t_1}) = E^{x_0}[P_{t_1} f(X_{t_0})]. \tag{2.8}$$

If $d(X_{t_0}, x_0) \leq M t_0^{1/d_w}$, then $|P_{t_1} f(X_{t_0}) - P_{t_1} f(x_0)| < \epsilon$ by (2.4). Since

$$\begin{aligned} E^{x_0}[Y P_{t_1} f(X_{t_0})] = {} & E^{x_0}[Y P_{t_1} f(X_{t_0}); A_{t_0}] \\ & + E^{x_0}[Y P_{t_1} f(X_{t_0}); d(X_{t_0}, x_0) > M t_0^{1/d_w}], \end{aligned}$$

then

$$|E^{x_0}[Y P_{t_1} f(X_{t_0}); A_{t_0}] - P_{t_1} f(x_0) E^{x_0}[Y; A_{t_0}]| \leq \epsilon,$$

where $A_{t_0} = \{d(X_{t_0}, x_0) \leq M t_0^{1/d_w}\}$. Also

$$E^{x_0}[Y; A_{t_0}] = E^{x_0} Y - E^{x_0}[Y; d(X_{t_0}, x_0) > M t_0^{1/d_w}].$$

Hence

$$|E^{x_0}[Y P_{t_1} f(X_{t_0})] - P_{t_1} f(x_0) E^{x_0} Y| \leq 3\epsilon. \tag{2.9}$$

Similarly

$$|E^{x_0} P_{t_1} f(X_{t_0}) - P_{t_1} f(x_0)| \leq 3\epsilon. \tag{2.10}$$

Combining (2.5), (2.7), (2.8), (2.9), and (2.10),

$$|P^{x_0}(A) - P^{x_0}(A) \mathbf{E}^{x_0} Y| \leq 7\epsilon.$$

Using this and (2.3),

$$|P^{x_0}(A) - P^{x_0}(A) P^{x_0}(A)| \leq 8\epsilon.$$

Since $\epsilon$ is arbitrary, we deduce $P^{x_0}(A) = [P^{x_0}(A)]^2$, or $P^{x_0}(A)$ is 0 or 1. Since $P^x(A) = \mathbf{E}^x P_{t_1} f(X_{t_0}) = P_{t_0}(P_{t_1} f)(x)$ is continuous in $x$ (by Proposition 2.2(5)) and $E$ is connected, then $P^x(A)$ is either identically 0 or identically 1. □

Note that invariant events (i.e., an event $B$ which satisfies $B \circ \theta_t = B$ for all $t \geq 0$) are tail events, hence by this proposition they are trivial. It follows that there are no nonconstant bounded harmonic functions on $E$.

**2.2. Schilder-type large deviations** We will now prepare notation and lemmas for the results on large deviations of the process. For fixed $T > 0$, let $\Omega_x \equiv C_x([0, T] \to E) = \{\phi \in C([0, T] \to E) : \phi(0) = x\}$, furnished with the uniformly continuous topology. For $\phi \in \Omega_x$, define an $I$-functional by

$$I_x(\phi) = \limsup_{|\Delta| \to 0} \sum_{\substack{0=t_0<t_1<\cdots t_N=T, \\ \Delta=\{t_0,t_1,\ldots,t_N\}}} \left(\frac{d(\phi(t_i), \phi(t_{i-1}))^{d_w}}{(t_i - t_{i-1})}\right)^{1/(d_w-1)}$$

where $I_x(\phi) = \infty$ if the right hand side is $\infty$. Here we set $|\Delta| \equiv \max_{1 \leq i \leq N}(t_i - t_{i-1})$. When $\phi \in C([0, T] \to E)$ (no restriction on $\phi(0)$), we denote the corresponding $I$-functional as $I(\phi)$. Note that if $\Delta\phi(t) \equiv \lim_{s \to t+0} d(\phi(s), \phi(t))/(s - t)$ exists for all $0 \leq t < T$ and is continuous, then the $I$-functional can be expressed as

$$I_x(\phi) = \int_0^T (\Delta\phi(t))^{d_w/(d_w-1)} dt.$$

For $\Delta : 0 = t_0 < t_1 < t_2 \cdots < t_m = T$ and $\phi \in \Omega_x$, we set $\Pi_\Delta \phi = \{\phi(t_1), \ldots, \phi(t_m)\}$. Also, define $\phi_\Delta \in \Omega_x$ by taking points $\{\phi(t_j)\}$ and joining the successive ones by geodesic paths. If there is more than one geodesic path between two such points, it is immaterial which one is chosen. Thus, $\phi_\Delta$ is a piecewise geodesic path and $\phi_\Delta(t_j) = \phi(t_j)$ $(0 \leq j \leq m)$. We then have the following.

**Lemma 2.4.** (a) *On $C([0, T] \to E)$ we have*

$$\inf_{\substack{\phi(\alpha)=a, \\ \phi(\beta)=b}} I(\phi) = \left(\frac{d(a, b)^{d_w}}{\beta - \alpha}\right)^{1/(d_w-1)},$$

*where the infimum is attained by the geodesic path on $E$.*
(b) *On $C_x([0, T] \to E)$ we have*

$$\inf_{\substack{\phi(t_i)=x_i, \\ i=1,\ldots,m}} I_x(\phi) = I_x(\phi_\Delta) = \sum_{i=1}^m \left(\frac{d(x_i, x_{i-1})^{d_w}}{t_i - t_{i-1}}\right)^{1/(d_w-1)}$$

*where $\Delta : 0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq T$, $x_0 = x, x_1, \ldots, x_m \in E$ and $\phi_\Delta$ is a piecewise geodesic path with $\phi_\Delta(t_j) = x_j$ $(0 \leq j \leq m)$.*

Proof. Note that (b) is an obvious extension of (a). For (a), it is enough to consider the case $\alpha = 0, \beta = T$, as otherwise the infimum is attained by the path which does not move in the intervals $[0, \alpha]$ and $[\beta, T]$. Now let $\phi(t_i) = y_i$, $f(i) = d(y_i, y_{i-1})/(t_i - t_{i-1})$ for $1 \le i \le N$ and define $L(\phi) = \sum_i d(y_i, y_{i-1})$. Then,

$$\begin{aligned}
&\sum_{\substack{0=t_0<t_1<\cdots t_N=T,\\ \Delta=[t_0,t_1,\ldots,t_N]}} \left(\frac{d(\phi(t_i), \phi(t_{i-1}))^{d_w}}{(t_i - t_{i-1})}\right)^{1/(d_w-1)} \\
&\quad = \sum_i f(i)^{1/(d_w-1)} d(y_i, y_{i-1}) \\
&\quad = L(\phi) \sum_i \left(\frac{1}{f(i)}\right)^{-1/(d_w-1)} \frac{d(y_i, y_{i-1})}{L(\phi)} \\
&\quad \ge L(\phi) \Big(\sum_i \frac{t_i - t_{i-1}}{L(\phi)}\Big)^{-1/(d_w-1)} \\
&\quad = L(\phi)^{d_w/(d_w-1)} T^{-1/(d_w-1)} \\
&\quad \ge \left(\frac{d(a,b)^{d_w}}{T}\right)^{1/(d_w-1)}.
\end{aligned}$$

Here we use Jensen's inequality for the first inequality and the second inequality holds because $L(\phi) \ge d(a, b)$. As $x^{-1/(d_w-1)}$ is strictly convex, the equalities hold if and only if $f(i)$ is constant and $L(\phi) = d(a, b)$. In this case we have the geodesic with the natural parameterization. We thus obtain the result. □

Using the results, we see for $\phi \in \Omega_x$ and $0 \le \alpha \le \beta \le T$

$$\begin{aligned}
I_x(\phi) &\ge \left(\frac{d(\phi(\alpha), x)^{d_w}}{\alpha}\right)^{1/(d_w-1)} + \left(\frac{d(\phi(\beta), \phi(\alpha))^{d_w}}{\beta - \alpha}\right)^{1/(d_w-1)} \\
&\ge \left(\frac{d(\phi(\beta), \phi(\alpha))^{d_w}}{\beta - \alpha}\right)^{1/(d_w-1)}.
\end{aligned}$$

Thus,

$$d(\phi(\alpha), \phi(\beta)) \le I_x(\phi)^{(d_w-1)/d_w} (\beta - \alpha)^{1/d_w}. \tag{2.11}$$

For $\psi, \phi \in \Omega_x$, define $\|\psi - \phi\| = \sup_{0 \le t \le T} d(\psi(t), \phi(t))$.

**Lemma 2.5.** (1) *The function $I_x(\phi)$ is lower semi-continuous. Further, for every $N > 0$, $\{\phi : I_x(\phi) \le N\}$ is compact.*
(2) *If $C \subset \Omega_x$ is closed in $\Omega_x$, then*

$$\lim_{\delta \to 0} \inf_{\phi \in C_\delta} I_x(\phi) = \inf_{\phi \in C} I_x(\phi),$$

*where* $C_\delta = \{\phi \in \Omega_x : \|\phi - \psi\| < \delta \quad \textit{for some } \psi \in C\}$.

Proof. For the lower semi-continuity, it is enough to show that if $I_x(\phi_n) \le N$ and $\|\phi_n - \phi\| \to 0$, then $I_x(\phi) \le N$. But this can be easily proved. Next, (2.11) shows that the elements of $F \equiv \{\phi : I_x(\phi) \le N\}$ are equicontinuous and $\{\phi(t) : \phi \in F\}$ is relative compact for each $t \in [0, T]$ (note that $E$ is locally compact). As $F$ is closed by the lower semi-continuity of $I_x$, (1) follows from Ascoli's theorem. Using (2.11) and (1), (2) can be proved in the same way as in [28], p. 159. □

Let $P_\epsilon^x$ be the law for $X^x(\epsilon t)$ where $X^x$ is the process starting at $x$. We now state our large deviation theorem.

**Theorem 2.6.** *There exist* $c_{2.10}, c_{2.11} > 0$ *such that for each* $A \subset C_x([0, T] \to E)$,

$$-c_{2.10} \inf_{\phi \in \mathrm{Int}A} I_x(\phi) \le \liminf_{\epsilon \to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x(A)$$
$$\le \limsup_{\epsilon \to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x(A) \le -c_{2.11} \inf_{\phi \in \mathrm{Cl}A} I_x(\phi).$$

REMARK. There are cases where one can not choose $c_{2.10} = c_{2.11}$. In these cases Schilder's large deviation theorem does not hold with its original form. Indeed, the following holds for Brownian motion on the Sierpinski gasket ([10]):
For each $z \in [2/5, 1)$, $A \subset C_x([0, T] \to E)$,

$$-\inf_{\phi \in \mathrm{Int}A} I_x^z(\phi) \le \liminf_{n \to \infty} \left(\left(\frac{2}{5}\right)^n z\right)^{1/(d_w-1)} \log P^x_{(2/5)^n z}(A)$$
$$\le \limsup_{n \to \infty} \left(\left(\frac{2}{5}\right)^n z\right)^{1/(d_w-1)} \log P^x_{(2/5)^n z}(A) \le -\inf_{\phi \in \mathrm{Cl}A} I_x^z(\phi),$$

where $\{I_x^z\}_{z \in [2/5,1)}$ is a family of (different) $I$-functions whose ratios are bounded from above and below by some positive constants.

Theorem 2.6 can now be proved following the argument of the corresponding proof of [27] (see also [10]). Although the strategy is the same, we state the key lemmas of the proof for the reader's convenience.

**Lemma 2.7.** *Let* $C \subset \Omega_x$ *be a closed set of the form* $\Pi_\Delta^{-1} A$, *where* $A \in E^m$ *is closed. Then*

$$\limsup_{\epsilon \to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x(C) \le -c_{2.11} \inf_{\phi \in C} I_x^z(\phi).$$

Proof. Using Proposition 2.2(2) and Lemma 2.4, this can be proved in the same way as Lemma 3.1 of [27]. □

For $m \in \mathbf{N}$, let $\Delta_m : 0 = t_0 < t_1 < t_2 \cdots < t_m = T$ be an equally spaced partition, i.e., $t_j = jT/m$ $(0 \le j \le m)$.

**Lemma 2.8.** *For every $\delta > 0$,*

$$\limsup_{m\to\infty} \limsup_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x(\|\phi - \phi_{\Delta_m}\| \ge \delta)) = -\infty.$$

Proof. Using Proposition 2.2(4), the proof is the same as Lemma 3.2 of [27]. □

From these lemmas, we can prove the third inequality of Theorem 2.6. Indeed, it is enough to prove the inequality when $A$ is closed. Let $I_x^\delta(\omega) = \inf_{\omega':\|\omega'-\omega\|<\delta} I_x(\omega')$ and define $T_\delta = \inf_{\omega\in C_\delta} I_x(\omega)$. If $\omega \in C$ then $I^\delta(\omega) \ge T_\delta$ and therefore

$$P_\epsilon^x[C] \le P_\epsilon^x[I_x^\delta(\omega) \ge T_\delta] \le P_\epsilon^x[\|\omega - \omega_{\Delta_m}\| \ge \delta] + P_\epsilon^x[I_x(\omega_{\Delta_m}) \ge T_\delta].$$

From Lemma 2.8,

$$\limsup_{m\to\infty} \limsup_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x[\|\omega - \omega_{\Delta_m}\| \ge \delta] = -\infty.$$

As the set $\{I_x(\omega_{\Delta_m}) \ge T_\delta\}$ is equal to

$$\left\{\omega : \sum_{i=1}^m \left(\frac{d(\omega(t_i), \omega(t_{i-1}))^{d_w}}{t_i - t_{i-1}}\right)^{1/(d_w-1)} \ge T_\delta\right\},$$

we see from Lemma 2.7 that

$$\limsup_{m\to\infty} \limsup_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x[I_x(\omega_{\Delta_m}) \ge T_\delta] \le -c_{2.11} T_\delta.$$

Combining these facts with $\lim_{\delta\to 0} T_\delta = \inf_{\phi\in A} I_x(\phi)$, which comes from Lemma 2.5(2), we obtain the third inequality of Theorem 2.6.

Next comes the lemma for the lower bound.

**Lemma 2.9.** *Let $f \in \Omega_x$, $V = \{\phi \in \Omega_x : \|\phi - f\| < \delta\}$ where $\delta > 0$. Then*

$$\liminf_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P_\epsilon^x(V) \ge -c_{2.10} I_x(f).$$

Proof. Using Proposition 2.2(2), Lemma 2.4 and the third inequality of Theorem 2.6, this can be proved in the same way as Lemma 3.4 of [27]. □

Using this lemma, we can prove the first inequality of Theorem 2.6. Indeed, it is enough to prove the inequality when $A$ is open. For $f \in A$, take a sphere $V$ around $f$ contained in $A$. Then, by the above lemma,

$$\liminf_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P^x_\epsilon(A) \geq \liminf_{\epsilon\to 0} \epsilon^{1/(d_w-1)} \log P^x_\epsilon(V) \geq -c_{2.10} I_x(f).$$

As this is true for all $f \in A$, we have the result. This concludes the proof of Theorem 2.6.

**2.3. Strassen's law** We now study the Strassen-type law of the iterated logarithm. From now on, we assume the following additional condition on $E$:

$$\begin{aligned} &\text{There exists a continuous map } F : E \to E \text{ such that} \\ &d(F(x), F(y)) = \eta^{-1} d(x, y) \text{ for all } x, y \in E \text{ where } 1 < \eta. \end{aligned} \tag{2.12}$$

Clearly, the examples in Subsection 2.1 satisfy this condition. Denote by 0 a fixed point of $F$ in $E$. For the process starting at $0 \in E$, set

$$\xi_n(t, \omega) = F^n(X(\eta^{nd_w}(\log n)^{1-d_w} t, \omega)),$$

where $F^n = \underbrace{F \circ \cdots \circ F}_{n \text{ times}}$. Then, we can prove the following proposition using Theorem 2.6 by a simple modification of the proof of Theorem 1.17 in [25].

**Proposition 2.10.** *For $P^0$-almost all $\omega$, the sequence $\{\xi_n(\cdot, \omega)\}_2^\infty$ has the following properties*:

(1) *$\{\xi_n(\cdot, \omega)\}_2^\infty$ is precompact in $C_0([0, T] \to E)$.*

(2) *If $\{\xi_{n'}(\cdot, \omega)\}_2^\infty$ is a convergent subsequence of $\{\xi_n(\cdot, \omega)\}_2^\infty$ and $\psi$ is its limit, then $c_{2.11} I_0(\psi) \leq 1$.*

(3) *If $\psi \in C_0([0, T] \to E)$ with $c_{2.10} I_0(\psi) \leq 1$, then there is a subsequence of $\{\xi_n(\cdot, \omega)\}_2^\infty$ which converges to $\psi$.*

*In particular, if $\Phi : C_0([0, T) \to E) \to \mathbf{R}$ is a continuous functional, then*

$$P^0\left(\frac{1}{c_{2.10}} \sup_{\psi\in K} \Phi(\psi) \leq \limsup_{n\to\infty} \Phi(\xi_n(\cdot)) \leq \frac{1}{c_{2.11}} \sup_{\psi\in K} \Phi(\psi)\right) = 1,$$

*where $K = \{\phi \in C_0([0, T] \to E) : I_0(\phi) \leq 1\}$.*

With the help of Proposition 2.3, we can obtain the following functional-type law of the iterated logarithm.

**Theorem 2.11.** *Let* $\Phi : C_x([0,T) \to E) \to \mathbf{R}$ *be a continuous functional such that* $\limsup_{n\to\infty} \Phi(\xi_n(\cdot))$ *is measurable with respect to the the tail* $\sigma$*-field. Then there exists a constant* $C(\Phi) \in [-\infty,\infty]$ *with* $(1/c_{2.10})\sup_{\psi\in K}\Phi(\psi) \le C(\Phi) \le (1/c_{2.11})\sup_{\psi\in K}\Phi(\psi)$ *which depends only on* $\Phi$ *such that*

$$P^x(\limsup_{n\to\infty} \Phi(\xi_n(\cdot)) = C(\Phi)) = 1 \qquad \forall x \in E. \tag{2.13}$$

Proof. By the assumption on $\Phi$, $\{\limsup_{n\to\infty}\Phi(\xi_n(\cdot)) = C(\Phi)\}$ is a tail event. Thus, by Proposition 2.3, the probability of the event is either 0 for all $x \in E$ or else 1 for all $x \in E$. By Proposition 2.10, the latter occurs for some $C(\Psi)$. □

Taking $\Phi(\xi) = \sup_{0\le t\le 1} d(\xi(t),0)$ in the above theorem and noting that $d(\xi_n(s,\omega),0) = \eta^{-n} d(X(\eta^{nd_w}(\log n)^{1-d_w}s),0)$ and $\eta^n \sim t^{1/d_w}(\log\log t)^{1-1/d_w}$ if and only if $t \sim \eta^{nd_w}(\log n)^{1-d_w}$, we have the following classical law of iterated logarithm. Note that by (2.11) it is easy to see $0 < \sup_{\psi\in K}\Phi(\psi) < \infty$.

**Corollary 2.12.** *There exists* $c_{2.12} > 0$ *such that*

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{1-1/d_w}} = c_{2.12} \qquad P^x\text{-a.s.},\ x \in E. \tag{2.14}$$

We remark that (2.14) may be proved directly via hitting time estimates.

## 3. Upper bounds for LILs

In this section we prove that the limsup result for an LIL for a functional $F$ will follow if $F$ is subadditive and has a uniform scaling property. Here we do not require that our Markov process have continuous paths or satisfy Aronson-type estimates. See [8] for other properties of functionals of this type.

**Theorem 3.1.** *Let* $\{X(t)\}_{t\ge 0}$ *be any strong Markov process on a topological space* $E$. *Suppose* $\{F_t\}_{t\ge 0}$ *is a continuous adapted non-decreasing functional of* $\{X(t)\}_{t\ge 0}$ *satisfying the following*:

(1) (*Uniform scaling near* $\infty$) *There exists a constant* $\beta > 0$ *such that*

$$\sup_{x,\lambda} P^x(F_{\lambda^\beta} > b\lambda) \to 0 \quad \text{as } b \to \infty.$$

(2) (*Subadditivity*) $F_t - F_s \le F_{t-s}\circ\theta_s$ *for all* $0 \le s \le t$.

*Then, there exists a constant* $0 < K < \infty$ *such that,*

$$\limsup_{t\to\infty} \frac{F_t}{t^{1/\beta}(\log\log t)^{1-1/\beta}} \le K \quad P^x\text{-a.s. } \omega,\ \forall x \in E.$$

Proof. We use $\ell\ell$ to abbreviate log log. For $1 \le i \le [\ell\ell t]$, set $A_i = F_{it/[\ell\ell t]} - F_{(i-1)t/[\ell\ell t]}$ and define $B_i = A_i/(t/[\ell\ell t])^{1/\beta}$. We first prove that there exists $a > 0$ such that

$$M \equiv \sup_{x,t} E^x \exp(aB_1) < \infty. \tag{3.1}$$

In fact, this can be obtained by the following routine argument using subadditivity (2) (cf. [9]). For $0 \le s \le 1$, set $B_s = F_{ts/[\ell\ell t]}/(ts/[\ell\ell t])^{1/\beta}$. We will show that there exists $b > 0$ so that

$$P^x(B_1 > bn) \le \frac{1}{2^n} \qquad \forall n \in \mathbf{N}, t > 0, x \in E. \tag{3.2}$$

$n = 1$ is easily obtained by our uniform scaling assumption (1). Now, set $T_n = \inf\{s \ge 0 : B_s \ge bn\}$. Then

$$\begin{aligned} P^x(B_1 > b(n+1)) &= P^x(B_1 > b(n+1), T_n < 1) = P^x(B_1 - B_{T_n} > b, T_n < 1) \\ &\le E^x[E^{X_{T_n}}(B_{1-T_n} > b); T_n < 1] \le \frac{1}{2} \sup_y P^y(B_1 > bn), \end{aligned}$$

for all $x \in E$, where we use the continuity of $F_t$ for the second equality. We thus obtain (3.2) by induction. From this, we obtain $P^x(B_1 > \lambda) \le c_1 \exp(-c_2\lambda)$ for all $x \in E, t > 0$, which is sufficient for (3.1).

Now define $\phi(t) = (t/[\ell\ell t])^{1/\beta}[\ell\ell t]$. Noting that $F_t/\phi(t) \le 1/[\ell\ell t] \sum_{i=1}^{[\ell\ell t]} B_i$ and using Chebyshev's inequality, we have

$$P^x(F_t \ge \phi(t)\lambda) \le P^x\left(\sum_{i=1}^{[\ell\ell t]} B_i \ge \lambda[\ell\ell t]\right) \le \exp(-a\lambda[\ell\ell t]) E^x \exp\left(a \sum_{i=1}^{[\ell\ell t]} B_i\right), \tag{3.3}$$

for all $\lambda > 0$. As

$$E^x[\exp(aB_i)|\mathcal{F}_{(i-1)t/[\ell\ell t]}] \le E^x[E^{X_{(i-1)t/[\ell\ell t]}}[\exp(aB_1)]] \le \sup_y E^y[\exp(aB_1)],$$

by iterating the conditioning, we have $E^x \exp(a \sum_{i=1}^{[\ell\ell t]} B_i) \le M^{[\ell\ell t]}$. Thus, taking $\lambda$ large so that $-a\lambda + \log M \le -p$ for some $p > 1$, we have by (3.3),

$$P^x(F_t \ge \phi(t)\lambda) \le \exp(-p[\ell\ell t]) \le c(\log t)^{-p}.$$

Taking $t = e^k$ and using Borel-Cantelli lemma, we have $P^x$-a.s., $\forall x \in E$,

$$\limsup_{k\to\infty} \frac{F_{e^k}}{\phi(e^k)} \le \lambda.$$

For $e^k \le t < e^{k+1}$, we have

$$\frac{F_t}{\phi(t)} \le \frac{F_{e^{k+1}}}{\phi(e^{k+1})}\frac{\phi(e^{k+1})}{\phi(e^k)} = \frac{F_{e^{k+1}}}{\phi(e^{k+1})}e^{1/\beta}\left(\frac{[\log(k+1)]}{[\log k]}\right)^{1-1/\beta},$$

which completes the proof. □

Here are some examples of upper bounds that can be obtained by means of this theorem. In each case, of course, the bounds are already in the literature.

(1) Let $X_t$ be a diffusion on $\mathbf{R}^d$ associated to a uniformly elliptic operator in either nondivergence or divergence form and let $F_t = \sup_{s\le t}|X_s - X_0|$. It is easy to check that the conditions of Theorem 3.1 are met, so the upper bound for a LIL holds.

(2) Let $X_t$ be a symmetric stable process on the line with index $\alpha \in (1, 2]$. Let $\ell_t(x)$ be the local times for $X_t$ and let $L^*(t) = \sup_x \ell_t(x)$. Then $F(t) = L^*(t)$ satisfies the hypotheses, and so the upper bound for a LIL for $L^*(t)$ follows.

(3) Let $X_t$ be a symmetric stable process of index $\alpha \in (1, 2]$ and let $F_t$ be the Lebesgue measure of the range of $X_t$. Again the hypotheses in Theorem 3.1 hold, and consequently an upper bound for the LIL for the range.

## 4. LIL for local times and the range for Brownian motion on fractals

In this section, we will prove laws of the iterated logarithm for the local time and the range of Brownian motion on fractals. The base space $E$ and diffusion process $\{X(t)\}$ (or the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$) we will treat in this section is either of the following (see the Appendix for the definitions):

(1) Brownian motion on affine nested fractals

(2) Brownian motion on Sierpinski carpets with $d_s < 2$

As mentioned, the process enjoys Assumption 2.1.

Here we list some other properties of the processes. See [15], [21], [19], [14], [16], [4], [5], [22], [6] for the proof.

**Proposition 4.1.** *$(E, d, \mu)$ and $\{X(t)\}, (\mathcal{E}, \mathcal{F})$ have the following properties.*

(a) *There exist constants $d_f \ge 1$ and $c_{4.1}, c_{4.2} > 0$ such that*

$$c_{4.1}r^{d_f} \le \mu(B(x, r)) \le c_{4.2}r^{d_f}, \quad \forall r \ge 0. \tag{4.1}$$

(b) *$\mathcal{F} \subset C(E, \mathbf{R})$ and for all $u \in \mathcal{F}$, $x, y \in E$,*

$$|u(x) - u(y)|^2 \le R(x, y)\mathcal{E}(u, u), \tag{4.2}$$

*where $R(\cdot, \cdot)$ is the resistance metric defined by*

$$R(p, q)^{-1} = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}, f(p) = 1, f(q) = 0\} \quad \forall p \ne q \in E$$

*and* $R(p, p) \equiv 0$. *Further, this metric is comparable to* $d$, *i.e.,*

$$c_{4.3}d(x, y)^{d_c} \leq R(x, y) \leq c_{4.4}d(x, y)^{d_c} \quad \forall x, y \in E, \tag{4.3}$$

*for some positive constants* $c_{4.3}, c_{4.4}$ *and* $d_c$.

(c) $\{X(t)\}$ *is point recurrent, and each-one-point set of* $E$ *has positive capacity.*

(d) *For all* $n \in \mathbf{N}, t > 0$ *and* $x \in E$,

$$X(\tau_1^n t) \text{ under } P_x \text{ is equal in law to } \alpha_1^n X(t) \text{ under } P_{\alpha_1^{-n}x}. \tag{4.4}$$

(e) $\{X(t)\}$ *admits a local time* $\ell_t(\omega, y)$ *which is jointly continuous in* $t, y$ *and satisfies*

$$\int_B \ell_t(\omega, y)\mu(dy) = \int_0^t 1_B(X_s(\omega))ds \quad \forall B \subset \mathcal{B}(E). \tag{4.5}$$

(f) *There exist* $\rho, \theta, c_{4.5}, c_{4.6} > 0$ *such that for all* $a > 0, 0 < \delta < 1$ *and* $x \in E$,

$$P^x(\sup_{d(y_1,y_2)\leq\delta} |\ell_t(\omega, y_1) - \ell_t(\omega, y_2)| \geq a) \leq c_{4.5}t^{d_s}\delta^{-2d_f} \exp(-c_{4.6}t^{1-\rho}a^{\rho}\delta^{-\rho\theta/2}).$$

REMARK. 1. It is proved that the $d_f$ in (a) is the Hausdorff dimension of $(E, d, \mu)$. It is expressed as $d_f = \log(1/\mu_1)/\log\eta$.
2. (b) is usually mentioned only for the compact fractal $\hat{E}$ ([14], [16], [18]), but by an easy argument, one can also show it for the unbounded fractal $E$.
3. The process constructed in [5], [6] might not have (4.4) because of the lack of uniqueness. But using the averaging method in [22] one can construct a process which satisfies (4.4). Alternately, we may in place of (4.4) use the fact that $\alpha_1^{-n}X(\tau_1^n t)$ is again a process on the Sierpinski carpet satisfying all the same estimates that $X(t)$ does; this is all that will be needed.

**4.1. Results of Fukushima-Shima-Takeda ([16])** Under the above framework, Donsker-Varadhan's large deviation theory for Markov processes can be applied and the results of [16] hold exactly in the same way. In this subsection, we list the main theorems we use.

Define the occupation time distribution $L_t$ for $\{X(t)\}$ as

$$L_t(\omega, B) = \frac{1}{t}\int_0^t 1_B(X_s(\omega))ds, \quad \forall B \subset \mathcal{B}(E). \tag{4.6}$$

For each $\omega$, $L_t(\omega, \cdot)$ is an element of the space $\mathcal{M}$ of subprobability measures on $E$.

Note that

$$L_t(\omega, B) = \frac{1}{t}\int_B \ell_t(\omega, y)\mu(dy).$$

By definition and by Proposition 4.1(d), $L_t$ and $\ell_t$ enjoys the following self-similarity property for all $n \in \mathbf{N}$, $t > 0$ and $x, y \in E$:

$$L_t(\omega, \alpha_1^n \cdot) \text{ under } P_x \sim L_{\tau_1^{-n}t}(\omega, \cdot) \text{ under } P_{\alpha_1^{-n}x}, \tag{4.7}$$

$$\ell_t(\omega, y) \text{ under } P_x \sim (\tau_1\mu_1)^n \ell_{\tau_1^{-n}t}(\omega, \alpha_1^{-n}y) \text{ under } P_{\alpha_1^{-n}x}, \tag{4.8}$$

where '$\sim$' means 'is equal in law to.' Define the $I$-functional on the space $\mathcal{M}$ of sub-probability measures on $E$ in terms of the Dirichlet form by

$$I_{\mathcal{E}}(\beta) = \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}) & \beta \prec \mu,\ \sqrt{f} \in \mathcal{F} \text{ for } f = \dfrac{d\beta}{d\mu} \\ 0 & \text{otherwise} \end{cases} \qquad \beta \in \mathcal{M}. \tag{4.9}$$

Denote the distribution of the occupation time distribution with respect to $P_x$ by $Q_{t,x}$:

$$Q_{t,x}(A) = P_x(L_t(\omega, \cdot) \in A), \quad A \subset \mathcal{B}(\mathcal{M}).$$

$\mathcal{M}$ is endowed with the vague topology. We also consider the space $\mathcal{M}_1$ of all probability measures on $E$ endowed with the weak topology. Then the following large deviation principle holds:

**Theorem 4.2.** (i) *For any closed subset $K$ of $\mathcal{M}$,*

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in E} \log Q_{t,x}(K) \le -\inf_{\beta\in K} I_{\mathcal{E}}(\beta). \tag{4.10}$$

(ii) *Let $\beta$ be a probability measure on $E$ with $\beta(G) = 1$ for a bounded connected open set $G \subset E$. Let $O$ be a neighborhood of $\beta$ in $\mathcal{M}_1$ and $G'$ be a bounded connected open set with $G' \supset \overline{G}$. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in G} P_x\,(L_t(\omega, \cdot) \in O, t < \sigma_{G'}) \ge -I_{\mathcal{E}}(\beta). \tag{4.11}$$

Now, define a sequence $\{t_m,\ m = 1, 2, \ldots\}$ of times by

$$\frac{t_m}{\log\log t_m} = \tau_1^m, \tag{4.12}$$

and set

$$\hat{L}_{t_m}(\omega, B) = L_{t_m}(\omega, \alpha_1^m B), \quad B \in \mathcal{B}(E). \tag{4.13}$$

Then, the following holds:

**Proposition 4.3.** (1) *For $P_x$-a.e. $\omega$, $x \in E$,*

$$\bigcap_N \overline{\bigcup_{m \geq N} \{\hat{L}_{t_m}(\omega, \cdot)\}} = \{\beta \in \mathcal{M} : I_{\mathcal{E}}(\beta) \leq 1\} (\equiv C). \tag{4.14}$$

(2) *If $\Phi$ is a continuous functional on $\mathcal{M}$ in the vague topology, then, for $P_x$-a.e. $\omega$, $x \in E$,*

$$\limsup_{m \to \infty} \Phi(\hat{L}_{t_m}(\omega, \cdot)) = \sup_{\beta \in C} \Phi(\beta). \tag{4.15}$$

From this, the authors in [16] deduced Chung's law of the iterated logarithm with the help of Proposition 2.3. Using the distance $d$, the result can be expressed as follows.

**Proposition 4.4.** *There exist $c_{4.7} > 0$ such that*

$$\liminf_{t \to \infty} \left( \frac{\log \log t}{t} \right)^{1/d_w} \sup_{0 \leq s \leq t} d(X_s, X_0) = c_{4.7} \quad P_x \text{ -a.e. } \omega, \ x \in E. \tag{4.16}$$

Next, set

$$\hat{\ell}_{t_m}(\omega, y) = \frac{1}{t_m \mu_1^m} \ell_{t_m}(\omega, \alpha_1^m y).$$

Let $\mathcal{A}$ be the totality of non-negative uniformly continuous functions $f$ on $E$ with $\int_E f d\mu \leq 1$. The space $\mathcal{A}$ is equipped with the topology of uniform convergence on compact subsets of $E$. For $f \in \mathcal{A}$, we denote $I_{\mathcal{E}}(f d\mu)$ by $I_{\mathcal{E}}(f)$.

**Proposition 4.5.** (1) *For $P_x$-a.e., $x \in E$,*

$$\bigcap_N \overline{\bigcup_{m \geq N} \{\hat{\ell}_{t_m}(\omega, \cdot)\}} = \{ f \in \mathcal{A} : I_{\mathcal{E}}(f) \leq 1\}.$$

(2) *For a continuous functional $\Phi$ on $\mathcal{A}$,*

$$\limsup_{m \to \infty} \Phi(\hat{\ell}_{t_m}(\omega, \cdot)) = \sup_{f \in \{f \in \mathcal{A} : I_{\mathcal{E}}(f) \leq 1\}} \Phi(f), \qquad P_x\text{-a.e. } \omega, \ x \in E.$$

From this the authors in [16] derived the result that

$$\limsup_{m \to \infty} \left( \frac{t_m}{\log \log t_m} \right)^{d_s/2} \frac{1}{t_m} \ell_{t_m}(\omega, 0) = b_0 \quad P_x \text{ -a.e. } \omega, \ x \in E, \tag{4.17}$$

where $b_0 = \sup\{f(0) : f \in \mathcal{A}, \sqrt{f} \in \mathcal{F}, \mathcal{E}(\sqrt{f}, \sqrt{f}) \leq 1\}$, which was shown to be positive and bounded. By using Proposition 2.3 again, they derived

**Proposition 4.6.** *There exist* $c_{4.8} > 0$ *such that*

$$\limsup_{t\to\infty} \left(\frac{t}{\log\log t}\right)^{d_s/2} \frac{1}{t}\ell_t(\omega, 0) = c_{4.8} \quad P_x \text{ -a.e. } \omega, \ x \in E. \tag{4.18}$$

**4.2. LIL for the supremum of local times** Set $L^*(t) = \sup_{x\in E} \ell(t, x)$. Note that by (4.8) we have

$$L^*(t) \text{ under } P_x \sim (\tau_1\mu_1)^n L^*(t/\tau_1^n) \text{ under } P_{\alpha_1^{-n}x}. \tag{4.19}$$

Our first assertion in this subsection is a kind of extension of Proposition 4.6.

**Proposition 4.7.** *There exists* $c_{4.9} > 0$ *such that*

$$\limsup_{t\to\infty} \frac{L^*(t)}{t^{1-d_s/2}(\log\log t)^{d_s/2}} = c_{4.9} \quad P^x\text{-a.e. } \omega, \ \forall x \in E.$$

Proof. Take $\Psi(f) = \sup_{x\in E} f(x)$ in Proposition 4.5(2), which is obviously continuous. Then one has the corresponding result to (4.17) for $L^*(t)$, so the proof is completed in the same way as above once we prove

$$b_* \equiv \sup\{\sup_{y\in E} f(y) : f \in \mathcal{A}, \sqrt{f} \in \mathcal{F}, \mathcal{E}(\sqrt{f}, \sqrt{f}) \leq 1\} < \infty.$$

Suppose $b_* = \infty$. Then there is a sequence $f_n$, $x_n$ such that $f_n(x_n) \geq n$. As $\sqrt{f_n} \in \mathcal{F}$ and $\mathcal{E}(\sqrt{f_n}, \sqrt{f_n}) \leq 1$, using (4.2) and (4.3) we have

$$|\sqrt{f_n(x)} - \sqrt{f_n(y)}|^2 \leq R(x, y)\mathcal{E}(\sqrt{f_n}, \sqrt{f_n}) \leq c_{4.4}d(x, y)^{d_c}$$

so that $|f_n(y)| \geq (\sqrt{n} - \sqrt{c_{4.4}})^2$ for all $y \in B(x_n, 1)$. Taking $n$ large, this contradicts the fact $f_n \in \mathcal{A}$, i.e., $\int_E f_n d\mu \leq 1$. □

We next prove a liminf estimate of $L^*(t)$. The proof is based on the proof of the corresponding results for symmetric stable processes due to P.S. Griffin ([17]). As before, we sometimes abbreviate $\ell\ell$ for log log.

**Proposition 4.8.** *There exists* $c_{4.10} > 0$ *such that*

$$\liminf_{t\to\infty} \frac{L^*(t)}{t^{1-d_s/2}(\log\log t)^{-1+d_s/2}} = c_{4.10} \quad P^x\text{-a.e. } \omega, \ \forall x \in E.$$

Proof. With the help of Proposition 2.3, it is enough to prove that there exists $c_1, c_2 > 0$ such that the following holds $P^x$-a.e. $\omega$, $\forall x \in E$.

$$c_1 \le \liminf_{t\to\infty} \frac{L^*(t)}{t^{1-d_s/2}(\log\log t)^{-1+d_s/2}} \le c_2.$$

We will first prove the upper estimate. Note that it is enough to show that there exist constants $\xi, c > 0$, $1/e < \beta < 1$ (the choice $1/e < \beta$ is for (4.22)) such that

$$P^x\Big(L^*(t) \le \xi\Big(\frac{t}{[\ell\ell t]}\Big)^{1-d_s/2}\Big) \ge c\beta^{[\ell\ell t]} \quad \forall x \in E. \tag{4.20}$$

Indeed, we can then apply the Borel-Cantelli lemma. Let $p \in (1, (\log\beta^{-1})^{-1})$ and set $t_k = \exp k^p$. Define

$$C_k = \Big\{\sup_x(\ell_{t_{k+1}}(\omega, x) - \ell_{t_k}(\omega, x)) \le \eta\Big(\frac{t_{k+1}}{[\ell\ell t_{k+1}]}\Big)^{1-d_s/2}\Big\}.$$

Then, the $C_k$ are independent events and using (4.20),

$$\sum P^x(C_k) \ge \sum P^x\Big(L^*(t_{k+1}) \le \eta\Big(\frac{t_{k+1}}{[\ell\ell t_{k+1}]}\Big)^{1-d_s/2}\Big) \ge c\sum k^{-p\log\beta^{-1}} = \infty.$$

Thus,

$$P^x(C_k \text{ i.o. }) = 1. \tag{4.21}$$

On the other hand, by Proposition 4.7 and by the choice of $\{t_k\}$,

$$\limsup_{k\to\infty} \frac{L^*(t_k)}{(t_{k+1}/[\ell\ell t_{k+1}])^{1-d_s/2}} = 0. \tag{4.22}$$

Since $L^*(t_{k+1}) \le L^*(t_k) + \sup_x(\ell_{t_{k+1}}(\omega, x) - \ell_{t_k}(\omega, x))$, by (4.21) and (4.22), we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{L^*(t)}{(t/\ell\ell t)^{1-d_s/2}} \le{}& \limsup_{k\to\infty} \frac{L^*(t_k)}{(t_{k+1}/[\ell\ell t_{k+1}])^{1-d_s/2}} \\ &+ \liminf_{k\to\infty} \sup_x \frac{(\ell_{t_{k+1}}(\omega, x) - \ell_{t_k}(\omega, x))}{(t_{k+1}/[\ell\ell t_{k+1}])^{1-d_s/2}} < \infty,\end{aligned}$$

which prove the upper bound. Thus, we will prove (4.20).
For this, we first choose $\beta', \epsilon > 0$ and $K \in \mathbf{N}$ so that the following is satisfied.

$$\begin{aligned}&P^x(X(s) \in \{A_1^k, \dots, A_1^{k+K}\}) \ge \beta' + 2\epsilon \\ &\qquad \forall s \in [\tau_1^{-1}, 1], \forall x \in \{A_1^{k-1}, \dots, A_1^{K+k-1}\}, \forall k \in \mathbf{N}.\end{aligned} \tag{4.23}$$

Here $\{A_1^k\}_{k=0}^\infty$ is a sequence of 1-complexes which satisfies $0 \in A_1^0$, $A_1^i \cap A_1^{i+1} \ne \emptyset$ $(\forall i)$ and there exists $L \in \mathbf{N}$ such that $\{x : d(A_1^i, x) \le \rho\} \cap A_1^j = \emptyset$, $\forall j \ge i + L, \forall i$.

See the Appendix for the definitions. An example of $\{A_i^k\}$ is a sequence of connected 1-complexes along a shortest path from 0 to infinity. In the case of the standard Sierpinski gasket (carpet), they are the intersections of the Sierpinski gasket (carpet) with the unit triangles (squares) bordering the $x$ axis. Then, (4.23) are easily verified using the lower estimate of Assumption 2.1.

Now, take $m \in \mathbf{N}$ such that $(\beta')^{1/M} > 1/e$ where $M \equiv \tau_1^m$. We set $\beta = (\beta')^{1/M}$. Then, by (4.4), we have, by defining $A_n^k = \alpha_1^n A_1^k$ for each $n \in \mathbf{N}, k \geq 0$,

$$P^x(X(Ms) \in \{A_m^k, \ldots, A_m^{k+K}\}) \geq \beta' + 2\epsilon \quad \forall s \in [\tau_1^{-1}, 1], \forall x \in \{A_m^{k-1}, \ldots, A_m^{K+k-1}\}, \forall k \in \mathbf{N}. \tag{4.24}$$

Further, we can take $\lambda, \rho, \epsilon > 0$ so that the following are satisfied.

$$P^x(L^*(M) \leq (\tau_1\mu_1)^{-1}\lambda) \geq 1 - \epsilon \qquad \forall x \in E, \tag{4.25}$$

$$P^x((X(0), X(M))^* > \eta^{-1}\rho) \leq \epsilon \qquad \forall x \in E. \tag{4.26}$$

Here we define $(X(s), X(s'))^* = \sup_{s \leq t \leq s'} d(X(s), X(t))$ for each $0 \leq s \leq s'$. Indeed, (4.26) is easily verified using Proposition 2.2(3). To show (4.25), assume this does not hold. Then for each $\epsilon > 0$, there exist $\{x_m\} \subset E$ such that $P^{x_m}(L^*(M) > m) \geq \epsilon$. Let $y_m = y_m(\omega) \in E$ be such that $l_M(\omega, y_m) \geq L^*(M) - \epsilon$ and choose $a > 0$ large in Proposition 4.1(f) so that

$$P^x\left(\sup_{d(y_1,y_2)\leq 1/2} |\ell_M(\omega, y_1) - \ell_M(\omega, y_2)| \geq a\right) \leq \epsilon/2 \quad \forall x \in E.$$

Then, by (4.1) and (4.5),

$$\begin{aligned}
\epsilon/2 &\leq P^{x_m}(L^*(M) \geq m, \sup_{d(y_1,y_2)\leq 1/2} |\ell_M(\omega, y_1) - \ell_M(\omega, y_2)| \leq a) \\
&\leq P^{x_m}(\ell_M(\omega, x) \geq m - a - \epsilon, \quad \forall x \in B(y_m, 1/2)) \\
&\leq P^{x_m}\Big(\int_0^M 1_{B(y_m,1/2)}(X_s)ds \geq (m - a - \epsilon)c_{4.1}(1/2)^{d_f}\Big),
\end{aligned}$$

which contradicts $\int_0^M 1_{B(y_m,1/2)}(X_s)ds \leq M$ when $m$ large. Thus (4.25) is verified.

Let $\gamma(t) = (t/[\ell\ell t])^{d_s/2}$, take $n = n(t) \in \mathbf{N}$ such that $\tau_1^n \leq t/[\ell\ell t] < \tau_1^{n+1}$. For $k = 1, 2, \ldots, [[\ell\ell t]/M] + 1$, set

$$\begin{aligned}
E_k = \Bigg\{ &\sup_x\{\ell_{ktM/[\ell\ell t]}(\omega, x) - \ell_{(k-1)tM/[\ell\ell t]}(\omega, x)\} \leq \lambda\left(\frac{t}{[\ell\ell t]}\right)^{1-d_s/2}, \\
&\left(X\left(\frac{ktM}{[\ell\ell t]}\right), X\left(\frac{(k-1)tM}{[\ell\ell t]}\right)\right)^* \leq \rho\gamma(t)^{1/d_f}, \\
&X\left(\frac{ktM}{[\ell\ell t]}\right) \in \{A_{(n+1)m}^k, \ldots, A_{(n+1)m}^{k+K}\}\Bigg\}.
\end{aligned}$$

We will show

$$P\Big(E_{k+1} \,\Big|\, X\Big(\frac{ktM}{[\ell\ell t]}\Big) = x\Big) \ge \beta' \quad \forall x \in \{A^k_{(n+1)m}, \dots, A^{k+K}_{(n+1)m}\}. \tag{4.27}$$

First, observe that by (4.19) and (4.25),

$$\begin{aligned} & P\Big(\sup_x\{\ell_{(k+1)tM/[\ell\ell t]}(\omega, x) - \ell_{ktM/[\ell\ell t]}(\omega, x)\} \le \lambda\Big(\frac{t}{[\ell\ell t]}\Big)^{1-d_s/2} \,\Big|\, X\Big(\frac{ktM}{[\ell\ell t]}\Big) = x\Big) \\ = \; & P^x\Big(L^*\Big(\frac{tM}{[\ell\ell t]}\Big) \le \lambda\Big(\frac{t}{[\ell\ell t]}\Big)^{1-d_s/2}\Big) \\ \ge \; & P^{\alpha_1^{-n-1}x}(L^*(M) \le (\tau_1\mu_1)^{-1}\lambda) \ge 1-\epsilon. \end{aligned}$$

Next, by (4.4) and (4.26),

$$\begin{aligned} & P\Big(\Big(X\Big(\frac{(k+1)tM}{[\ell\ell t]}\Big), X\Big(\frac{ktM}{[\ell\ell t]}\Big)\Big)^* \le \rho\gamma(t)^{1/d_f} \,\Big|\, X\Big(\frac{ktM}{[\ell\ell t]}\Big) = x\Big) \\ = \; & P^x\Big(\sup_{0\le s\le tM/[\ell\ell t]} d(X(s), X(0)) \le \rho\Big(\frac{t}{[\ell\ell t]}\Big)^{1/d_w}\Big) \\ \ge \; & P^{\alpha_1^{-n-1}x}(\sup_{0\le s\le M} d(X(s), X(0)) \le \eta^{-1}\rho) \ge 1-\epsilon. \end{aligned}$$

Finally, by (4.4) and (4.24),

$$\begin{aligned} & P\Big(\Big(X\Big(\frac{(k+1)tM}{[\ell\ell t]}\Big) \in \{A^{k+1}_{(n+1)m}, \dots, A^{k+1+K}_{(n+1)m}\} \,\Big|\, X\Big(\frac{ktM}{[\ell\ell t]}\Big) = x\Big) \\ = \; & P^x\Big(X\Big(\frac{tM}{[\ell\ell t]}\Big) \in \{A^{k+1}_{(n+1)m}, \dots, A^{k+1+K}_{(n+1)m}\}\Big) \\ = \; & P^{\alpha_1^{-n-1}x}(X(Ms) \in \{A^{k+1}_m, \dots, A^{k+1+K}_m\}) \quad [\text{ for some } s \in [\tau_1^{-1}, 1]] \\ \ge \; & \beta' + 2\epsilon. \end{aligned}$$

We thus obtain (4.27). Set $F_l = \cap_{k=1}^l E_k$ and denote $\mathcal{F}_t = \sigma\{X(s) : s \le t\}$. Noting that $X(tM/[\ell\ell t]\cdot[[\ell\ell t]/M]) \in \{A^{[[\ell\ell t]/M]}_{(n+1)m}, \dots, A^{[[\ell\ell t]/M]+K}_{(n+1)m}\}$ on $E_{[[\ell\ell t]/M]}$, we have by (4.27),

$$\begin{aligned} P^x(F_{[[\ell\ell t]/M]+1}) &= E^x\Big[\prod_{k=1}^{[[\ell\ell t]/M]} 1_{E_k} E\Big[1_{E_{[\ell\ell t]}} \Big| X\Big(\frac{tM}{[\ell\ell t]}\Big[\frac{[\ell\ell t]}{M}\Big]\Big)\Big]\Big] \\ &\ge \beta' P^x\left(F_{[[\ell\ell t]/M]}\right). \end{aligned}$$

Iterating this, we obtain

$$P^x(F_{[[\ell\ell t]/M]+1}) \ge (\beta')^{[[\ell\ell t]/M]+1} \ge c\beta^{[\ell\ell t]}. \tag{4.28}$$

Observing that

$$F_{[[\ell\ell t]/M]+1} \subset \left\{ L^*(t) \le \sup_x \sum_{k=1}^{[[\ell\ell t]/M]+1} \{ l_{ktM/[\ell\ell t]}(\omega, x) - l_{(k-1)tM/[\ell\ell t]}(\omega, x) \} \le \xi \left( \frac{t}{[\ell\ell t]} \right)^{1-d_s/2} \right\},$$

where $\xi \equiv (K + 2L)\lambda$, we obtain (4.20).

We next prove the lower estimate. Fix $p > 1$ and choose $\lambda$ small enough so that

$$P^x(L^*(1) \le \tau_1 \mu_1 \lambda) \le e^{-p} \qquad \forall x \in E. \tag{4.29}$$

To prove this, assume (4.29) does not hold. Then, there exist $\{x_n\} \subset E$ so that $P^{x_n}(L^*(1) \le 1/n) \ge e^{-p}$. Then, by (4.1) and (4.5),

$$P^{x_n}\left( \sigma_{B(x_n,1)} \le c_{4.2}/n \right) \ge P^{x_n}\left( \int_0^1 1_{B(x_n,1)}(X_s) ds \le c_{4.2}/n \right) \ge e^{-p}.$$

This contradicts Proposition 2.2(4) when $n$ is large so that (4.29) is verified.

Now, for $k = 1, 2, \ldots, [\ell\ell t]$, let

$$D_k = \left\{ \sup_x (l_{kt/[\ell\ell t]}(\omega, x) - l_{(k-1)t/[\ell\ell t]}(\omega, x)) \le \lambda \left( \frac{t}{[\ell\ell t]} \right)^{1-d_s/2} \right\}.$$

As the $D_k$ are independent events, we have by (4.19) and (4.29)

$$\begin{aligned} P^x(D_k) &= E^x\left( P^{X_{(k-1)t/[\ell\ell t]}}\left( L^*\left( \frac{t}{[\ell\ell t]} \right) \le \lambda \left( \frac{t}{[\ell\ell t]} \right)^{1-d_s/2} \right) \right) \\ &\le P^x(L^*(1) \le \lambda \tau_1 \mu_1) \le e^{-p}. \end{aligned}$$

Thus

$$P^x\left( L^*(t) \le \lambda \left( \frac{t}{[\ell\ell t]} \right)^{1-d_s/2} \right) \le P\left( \bigcap_{k=1}^{[\ell\ell t]} D_k \right) \le e^{-p[\ell\ell t]}.$$

Taking $t_k = 2^k$,

$$\begin{aligned} &P^x\left( L^*(t) \le 2^{-(1-d_s/2)} \lambda \left( \frac{t}{[\ell\ell t]} \right)^{1-d_s/2} \quad \text{for some } t \in [t_k, t_{k+1}) \right) \\ \le\ &P^x\left( L^*(t_k) \le \lambda \left( \frac{t_k}{[\ell\ell t_k]} \right)^{1-d_s/2} \right) \le e^{-p[\ell\ell 2^k]} \le ck^{-p}, \end{aligned}$$

whose sum converges since $p > 1$. Thus, by the Borel-Cantelli lemma, we obtain the result. □

**4.3. LIL for range** Define $R(t) = \mu(\{x : X(s) = x, \text{ for some } s \le t\})$. In this subsection, we will show a LIL for $R(t)$. First, note that from (4.4), $R(t)$ enjoys the

following self-similarity property for all $n \in \mathbf{N}$, $t > 0$ and $x \in E$:

$$R(\tau_1^n t) \text{ under } P_x \sim \mu_1^{-n} R(t) \text{ under } P_{\alpha_1^{-n} x}. \tag{4.30}$$

The limsup result of $R(t)$ can be deduced from Theorem 3.1.

**Proposition 4.9.** *There exists $c_{4.11} > 0$ such that the following holds,*

$$\limsup_{t\to\infty} \frac{R(t)}{t^{d_s/2}(\log\log t)^{1-d_s/2}} = c_{4.11} \quad P^x\text{-a.e. } \omega, \ \forall x \in E.$$

Proof. Using Proposition 2.2(3),

$$P^x(R(1) > b) \le P^x(\sup_{s\le 1} d(X_s, x_0) > b^{1/d_f}) \le c_{2.5} \exp(-c_{2.6} b^{d_w/(d_f(d_w-1))}).$$

Combining this with (4.30), $R(t)$ satisfies Theorem 3.1(1) with $\beta = 2/d_s$. It is clear that $R(t)$ satisfies Theorem 3.1(2) so that the upper estimate is obtained by Theorem 3.1.

Now, note that defining $Z(t) = \{y \in E : X(s) = y \text{ for some } s \le t\}$, we have

$$t = \int_{\{x\in Z(t)\}} l(t, x)\mu(dx) \le L^*(t)R(t). \tag{4.31}$$

Combining this with Proposition 4.8 we have the lower estimate. □

The liminf estimate is rather simple.

**Proposition 4.10.** *There exists $c_{4.12} > 0$ such that the following holds.*

$$\liminf_{t\to\infty} \frac{R(t)}{t^{d_s/2}(\log\log t)^{-d_s/2}} = c_{4.12} \quad P^x\text{-a.e. } \omega, \ \forall x \in E.$$

Proof. By Proposition 4.4, $P^x$-a.e. there exists $t_n \to \infty$ so that

$$\sup_{0\le s\le t_n} d(X_s, X_0) \le C_1' \left(\frac{t_n}{\log\log t_n}\right)^{1/d_w} \quad \text{i.o.}$$

As $R(t) \le (\sup_{s\le t} d(X_s, X_0))^{d_f}$, we have $P^x$-a.e.

$$R(t_n) \le C_1' \left(\frac{t_n}{\log\log t_n}\right)^{d_f/d_w} \quad \text{i.o.}$$

Noting $d_f/d_w = d_s/2$, we obtain the upper estimate.

Combining (4.31) with Proposition 4.7, we have the lower estimate. □

Remark. As we have seen in (2.14) and (4.16), the following holds $P^x$-a.e. $\omega$, $\forall x \in E$.

$$\limsup_{t\to\infty} \frac{\sup_{s\le t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{1-1/d_w}} = c_{2.12}, \quad \liminf_{t\to\infty} \frac{\sup_{s\le t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{-1/d_w}} = c_{4.7}.$$

If

$$R(t) = (\sup_{s\le t} d(X_s, X_0))^{d_f} \tag{4.32}$$

held, then the loglog order of the limsup of $R(t)$ would be $d_f - d_s/2$ instead of $1 - d_s/2$. On the other hand, the order of loglog for the liminf of $R(t)$ is $-d_s/2$, which is what one would expect if (4.32) held. This suggests that the trajectory of the process is essentially 1-dimensional at times when the limsup of $R(t)$ is attained whereas it is more like a uniform covering at times when the liminf of $R(t)$ is attained.

## 5. Appendix

In this appendix, we will briefly explain about affine nested fractals, Sierpinski carpets and diffusion processes (or Dirichlet forms) on them.

Let $\{\Psi_i\}_{i=1}^N$ be *similitude maps* on $\mathbf{R}^d$, i.e., $\Psi_i \mathbf{x} = \alpha_i^{-1} U_i \mathbf{x} + \beta_i$, $\mathbf{x} \in \mathbf{R}^d$ for some unitary maps $U_i$, $\alpha_i > 1$, $\beta_i \in \mathbf{R}^d$. We also assume the *open set condition* for $\{\Psi_i\}_{i=1}^N$, i.e., there is a non-empty, bounded open set $V$ such that $\{\Psi_i(V)\}$ are disjoint and $\cup_{i=1}^N \Psi_i(V) \subset V$. As $\{\Psi_i\}_{i=1}^N$ is a family of contraction maps, there exists a unique non-void compact set $\hat{E}$ such that $\hat{E} = \cup_{i=1}^N \Psi_i(\hat{E})$. Assume further that $\hat{E}$ is connected. We now give the definition of affine nested fractals and Sierpinski carpets following [14] and [6].

1. Affine nested fractals

Let $F$ be the set of fixed points of the $\Psi_i$'s, $1 \le i \le N$. A point $x \in F$ is called an essential fixed point if there exist $i, j \in \{1, \dots, N\}$, $i \ne j$ and $y \in F$ such that $\Psi_i(x) = \Psi_j(y)$. We write $F^{(0)}$ for the set of essential fixed points. Denote $\Psi_{i_1,\dots,i_n} = \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}$. We will call the set $\Psi_{i_1,\dots,i_n}(F^{(0)})$ an $n$-cell and $\Psi_{i_1,\dots,i_n}(\hat{E})$ an $n$-complex. Set $F^{(n)} = \cup_{i_1,\dots,i_n=1}^N \Psi_{i_1,\dots,i_n}(F^{(0)})$. Then, $\hat{E}$ is called a (compact) affine nested fractal if the following holds in addition to the above conditions:

(AN1) (symmetry) If $x, y \in F^{(0)}$, then reflection in the hyperplane $H_{xy} = \{z : |z - x| = |z - y|\}$ maps $F^{(n)}$ to itself.

(AN2) (nesting) If $\{i_1, \dots, i_n\}, \{j_1, \dots, j_n\}$ are distinct sequences, then

$$\Psi_{i_1,\dots,i_n}(\hat{E}) \bigcap \Psi_{j_1,\dots,j_n}(\hat{E}) = \Psi_{i_1,\dots,i_n}(F^{(0)}) \bigcap \Psi_{j_1,\dots,j_n}(F^{(0)}).$$

We say $\Psi_i(\hat{E})$ and $\Psi_j(\hat{E})$ are the same size if they can be mapped to each other by the composition of the reflection maps which appear in (AN1). In that case, the contraction rates of two maps are the same. When all the contraction rates are the same,

$\hat{E}$ is called a nested fractal ([23]).

2. Sierpinski carpets

Let $d \geq 2$, $F_0 = [0,1]^d$, and let $l \in \mathbf{N}$, $l \geq 3$ be fixed. Set $\mathcal{S} = \{\prod_{i=1}^d [(k_i - 1)/l, k_i/l] : 1 \leq k_i \leq l \ (1 \leq i \leq d)\}$. We assume that each $\Psi_i$ maps $F_0$ onto some element of $\mathcal{S}$. Set $F_1 = \cup_{i=1}^N \Psi_i(F_0)$. Then, $\hat{E}$ is called a (compact) Sierpinski carpet if the following holds in addition to the conditions mentioned above:

(SC1) (Symmetry) $F_1$ is preserved by all the isometries of the unit cube $F_0$.

(SC2) (Non-diagonality) Let $B$ be a cube in $F_0$ which is the union of $2^d$ distinct elements of $\mathcal{S}$. (So $B$ has side length $2l^{-1}$.) Then if $\mathrm{Int}(F_1 \cap B)$ is non-empty, it is connected.

(SC3) (Borders included) $F_1$ contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_d = 0\}$.

The assumptions (SC2) and (SC3) are included for technical reasons which are not essential. We will denote $l = \alpha_1$ to unify the notation, although each 1-complex is the same size in this framework.

Note that the biggest difference between the two examples is whether the fractal is finitely ramified or not, i.e., whether it can be disconnected by removing a certain finite number of points or not (affine nested is finitely ramified due to (AF2)). We also note that both of the examples have strong symmetry with respect to reflections.

Let $\hat{\mu}$ be a Bernoulli probability measure on $\hat{E}$ such that $\hat{\mu}(\Psi_i(\hat{E})) = \mu_i > 0$, where $\sum_i \mu_i = 1$ and $\mu_i = \mu_j$ if $\Psi_i(\hat{E})$ and $\Psi_j(\hat{E})$ are the same size. (Thus, for the case of nested fractals and for the case of Sierpinski carpets, $\mu_i \equiv 1/N \ \forall i$.) Then we can construct a local regular Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on $\mathbf{L}^2(\hat{E}, \hat{\mu})$ such that

$$\hat{\mathcal{E}}(f,g) = \sum_{i=1}^N \mu_i^{-1/S} \hat{\mathcal{E}}(f \circ \Psi_i, g \circ \Psi_i) \quad \forall f, g \in \hat{\mathcal{F}},$$

where $S \in [-\infty, -1) \cup (0, \infty]$ is determined during the procedure of the construction (here we use the convention $1/\pm\infty = 0$). Set $\tau_i = \mu_i^{-1-1/S}$. $\tau_i^{-1}$ is the time scaling factor for $\Psi_i(\hat{E})$. Denoting by $\{\hat{X}(t)\}$ the corresponding diffusion process, the law of $\{\hat{X}(t)\}$ is invariant under reflections with respect to $\hat{E}$.

Now assume without loss of generality that $\Psi_1(\mathbf{x}) = \alpha_1^{-1}\mathbf{x}$. Then, the affine nested fractal or the Sierpinski carpet $E$ is constructed as $E = \cup_{n=1}^\infty \alpha_1^n \hat{E}$. The local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $E$, whose restriction to $\hat{E}$ is $\hat{\mathcal{E}}$, can be constructed on $\mathbf{L}^2(E, \mu)$ (where $\mu$ is a Bernoulli measure on $E$ so that $\mu|_{\hat{E}} = \hat{\mu}$) and has the following scaling property:

$$\mathcal{E}(f,g) = \mu_1^{-1/S} \mathcal{E}(f \circ \Psi_i, g \circ \Psi_i) \quad \forall f, g \in \mathcal{F}.$$

There is also a metric (called shortest path metric) on $E$ which has the following scaling: $d(\alpha_1 x, \alpha_1 y) = \eta d(x,y)$ for all $x, y \in E$ by some constant $\eta > 1$. For the case of the Sierpinski carpet, $\eta = \alpha_1 = l$ and this metric is equivalent to the Euclidean metric,

but for the case of affine nested fractals, there is in general no relation (although they induce the same topology) between the two metrics. For these examples, Assumption 2.1 holds with $d_s = 2\log(1/\mu_1)/\log\tau_1 = 2S/(S+1)$, $d_w = \log\tau_1/\log\eta$. For the case of affine nested fractals, $0 < S < \infty$ so that $d_s < 2$, but for the Sierpinski carpets, $S$ could be less than $-1$, in which case $d_s$ could be greater than 2, when $d$ is large.

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