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Osaka J. Math.
17 (1980), 325–327

NOTE ON A PROBLEM OF DIEPENBROCK

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(Received July 5, 1979)

In this note we shall discuss a problem of Diepenbrock. Let $(\mathcal{X}, \mathcal{A})$ be the n-dimensional Euclidean Borel space and let $\mathcal{P}$ be the totality of continuous probability measures on $(\mathcal{X}, \mathcal{A})$, $\mathcal{A}'$ the weak completion of $\mathcal{A}$ and $\mathcal{P}'$ be the extension of the $\mathcal{P}$ to $\mathcal{A}'$. Then there is no measure on $\mathcal{A}$ or $\mathcal{A}'$ w.r.t. which every element in $\mathcal{P}$ or $\mathcal{P}'$ has a density.

1. Introduction

Let $\mathcal{X}$ be the n-dimensional Euclidean space and $\mathcal{A}$ be the Borel field. A probability measure P on $(\mathcal{X}, \mathcal{A})$ is said to be continuous if $P(\{x\})=0$ holds for all points in $\mathcal{X}$. Let $\mathcal{P}$ be the totality of continuous probability measures on $(\mathcal{X}, \mathcal{A})$. We define $\mathcal{A}'=\{A\subset\mathcal{X}$; for all P in $\mathcal{P}$ there exists a set B_P in $\mathcal{A}$ and N_P in $\mathcal{A}$ such that $A\triangle B_P\subset N_P$ and $P(N_P)=0\}$. Extend each P in $\mathcal{P}$ to $\mathcal{A}'$ by defining $P'(A)=P(B_P)$.

A measure m on $(\mathcal{X}, \mathcal{A})$ is said to be a dominating measure for $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ if each P in $\mathcal{P}$ has a density w.r.t. m. $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is said to be weakly dominated if there exists a localizable dominating measure.

Diepenbrock ([1] Section 11) showed that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is not weakly dominated under the continuum hypothesis and raised the following problem: Is $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$, where $\mathcal{P}'=\{P'; P\in\mathcal{P}\}$, weakly dominated? The aim of the present note is to show, without any set theoretic assumption, neither of them is weakly dominated. In fact we shall show, more strongly, that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ or $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$ does not have any dominating measure.

2. The proof

In this section we shall prove the following

Theorem. *$(\mathcal{X}, \mathcal{A}, \mathcal{P})$ or $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$ does not have any dominating measure.*

Lemma 1 (Kuratowski [2] p. 451). *For any uncountable Borel subset B of $\mathcal{X}$ there exists a Borel isomorphism f from B to $\mathcal{X}$ (i.e., f is one-to-one, onto and bimeasurable). So we have $f(\mathcal{A}_B)=\mathcal{A}$, where $\mathcal{A}_B=\{A\in\mathcal{A}; A\subset B\}$.*

Lemma 2. *Let B be an uncountable Borel subset of $\mathcal{X}$. Then there exists a P in $\mathcal{P}$ such that $P(B)=1$.*

Proof. Take any A in $\mathcal{A}$ with $A\subset B$. Let n be a normal distribution on $(\mathcal{X}, \mathcal{A})$. Using f in Lemma 1, we define $Q(A)=n(f(A))$. For any A in $\mathcal{A}$ put $P(A)=Q(A\cap B)$. Then it follows that $P(B)=1$ and $P(\{x\})=0$ for all x in $\mathcal{X}$. So P belongs to $\mathcal{P}$.

REMARK 1. By Lemma 2, we have $\mathcal{A}''=\mathcal{A}$, where $\mathcal{A}''=\{A\subset\mathcal{X}$; there exists a set B in $\mathcal{A}$ and a set N in $\mathcal{A}$ such that $A\triangle B\subset N$ and $P(N)=0$ for all P in $\mathcal{P}\}$.

Lemma 3. *Let B be an uncountable Borel subset of $\mathcal{X}$. Then there exists a family $\{B_i;\, i\in I\}$ of Borel sets such that $B_i\subset B$, $B_i\neq\phi$, $B_i\cap B_j=\phi$ $(i\neq j)$, I is uncountable and B_i is uncountable for each i.*

Proof. By Lemma 1, there exists a Borel isomorphism f between B and $\mathcal{X}$.

Case 1. $n>1$: For all $i\in I\equiv R$, put $B_i=\{f^{-1}(x)$; the last coordinate of x is equal to $i\}$.

Case 2. $n=1$: For all $i\in I\equiv(0, 1)$ let $0.\, i_1i_2i_3\cdots$ be the non terminating decimal expansion of i. Put $B_i=\{f^{-1}(x);\, x\in(0, 1), (x_2, x_4, x_6, \cdots)=(i_1, i_2, i_3, \cdots)\}$.

We proceed to prove that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ does not have any dominating measure. Assume that there exists a dominating measure m for $(\mathcal{X}, \mathcal{A}, \mathcal{P})$. For any uncountable Borel set B we have $m(B)>0$ by Lemma 2. Let $\{B_i;\, i\in I\}$ be an uncountable family given in Lemma 3. Each B_i is uncountable, so we have $m(B_i)>0$. $B_i\subset B$ and $B_i\cap B_j=\phi$ $(i\neq j)$ imply that $m(B)=\infty$. Therefore m is not sigma-finite on B.

If P in $\mathcal{P}$ has a density g w.r.t. m, then $[g>0]$ is a sigma-finite set w.r.t. m. Because of continuity of P, $[g>0]$ must be uncountable. So m is not sigma-finite on $[g>0]$ by the above discussion. But this is a contradiction.

Next we shall prove that $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$ does not have any dominating measure. Again assume that there exists a dominating measure m for $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$.

Step 1: Let B be any uncountable Borel subset of $\mathcal{X}$. For any A in $\mathcal{A}'$ such that $A\supset B$, we have $m(A)=\infty$ and in fact m is not sigma-finite on A.

Step 2: For any P' put $g=dP'/dm$. Then we can write $[g(x)>0]=\bigcup_{i=1}^{\infty}A_i$, $0<m(A_i)<\infty$, $A_i\in\mathcal{A}'$ because $[g(x)>0]$ is sigma-finite w.r.t. m. For each $i\geqq 1$ we define a finite non-zero measure m_i on $(\mathcal{X}, \mathcal{A}')$ by $m_i(A)=m(A_i\cap A)$. If there exists a point x in $\mathcal{X}$ such that $m_i(\{x\})>0$, x must belong to A_i. P' is an extension of P and P is continuous, so we have

$$0=P(\{x\})=\int_{\{x\}}gdm=g(x)m(\{x\})\,.$$

Therefore we have $g(x)=0$. But $x \in A_i \subset [g(x)>0]$. This is a contradiction. Hence m_i is continuous.

Since m_i is a continuous finite measure on $(\mathcal{X}, \mathcal{A}')$, we can easily show that the completion of $\mathcal{A}$ by $m_i | \mathcal{A}$ contains $\mathcal{A}'$. Therefore there exists a set $B \in \mathcal{A}$ contained in A_i and a set N in $\mathcal{A}$ such that $A_i - B \subset N$ and $m_i | \mathcal{A}(N)=0$. Hence $m_i(B)=m_i(A_i)$.

Step 3: A_i is uncountable, because $m_i(A_i)>0$ holds and m_i is continuous. B is also uncountable by the same reason. Since B is a Borel set, by step 1, we have $m(A_i)=\infty$. But this is a contradiction.

REMARK 2. To show that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ does not have a dominating measure it is not necessary to take the totality of all continuous probability measures as $\mathcal{P}$. It is sufficient to assume that $\mathcal{P}$ satisfies the following: For every uncountable $B \in \mathcal{A}$ there is $P_B \in \mathcal{P}$ with $P_B(B)=1$. For example $\mathcal{P}$ can be taken to be all probability measures which are continuous and singular relative to Lebesgue measure.

Acknowledgement. The authors thank Professor T. Onoyama of Keio University and Professor H. Morimoto of Osaka City University for making this collaboration possible.

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