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ON STRONGLY SEPARABLE ALGEBRAS

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Let A be a separable algebra over a commutative ring R [1] with center C. There exists an element $\sum u_i \otimes v_i^0 \in A \otimes A^0$ such that

i) $\sum xu_i \otimes v_i^0 = \sum u_i \otimes (v_i x)^0 \qquad (x \in A)$,
ii) $\sum u_i v_i = 1$

The mapping $x \to \sum u_i x v_i$ is a projection $A \to C$, and C is a C-direct summand of A. While, the mapping $\lambda: x \to \sum v_i x u_i$ yields a C-direct decomposition $A = \lambda(A) \oplus [A, A]$, where $[A, A]$ is the C-linear subspace spanned by all $[x, y] = xy - yx$ [2]. In many important cases $\lambda(A)$ coincides with C, and we have $A = C \oplus [A, A]$. Giving the name strongly separable algebras, T. Kanzaki studied such a class of algebras. He first studied strongly separable algebras over a field [3], and then the general case was studied in utilizing the former results [4]. The objective of this note is to re-establish his results more simply, and to generalize in some respects.

Theorem 1. *The following three statements are equivalent for an algebra A over R.*

1) *There exists an element* $s = \sum v_i \otimes u_i^0 \in A \otimes A^0$ *such that*
 i) $\sum v_i x \otimes u_i^0 = \sum v_i \otimes (xu_i)^0 \qquad (x \in A)$,
 ii) $\sum v_i u_i = 1$.
2) *A is separable and* $A = C \oplus [A, A]$
3) *A is separable and there exists a symmetric* C-linear map λ: $A \to C$ such that $\lambda(1) = 1$.

REMARK. A is called *strongly separable* if A satisfies these conditions. The original definition by Kanzaki was given by the condition 1), formulated in another way, and he proved the equivalence of 1) and 2) [4, Th. 1].

Proof. 1)$\Rightarrow$2). Put $\lambda(x) = \sum v_i x u_i$. λ is a C-linear map $A \to A$, is identity on C, and vanishes on $[A, A]$. ($\lambda(xy) = \sum (v_i x) y u_i = \sum v_i y (xu_i) = \lambda(yx)$.) It follows $C \cap [A, A] = 0$. For any $x \in A$, we have $x = \sum v_i u_i x$

$=\sum u_i x v_i+\sum [v_i, u_i x]\in C+[A, A]$. Hence we have $A=C\oplus[A, A]$. 1 is represented as

$$1=\sum v_i u_i=\sum u_i v_i+\sum [v_i, u_i]=1+0,$$

so that we have $\sum u_i v_i=1$. Since we have moreover $\sum x u_i\otimes v_i^0=\sum u_i\otimes(v_i x)^0$, A is separable.

2)$\Rightarrow$3) Take the projection $\lambda: A\to C$.

3)$\Rightarrow$1) As A is central separable over C, the mapping $\eta: A\underset{C}{\otimes}A^0\to \mathrm{Hom}_C(A, A)$ defined by $\eta(a\otimes b^0)(x)=axb$ is an isomorphism [1]. Let, in particular, $\lambda=\eta(\sum v_i\otimes u_i^0)$. Then we have

ii) $\sum v_i u_i=\lambda(1)=1$.

Further

$$\begin{aligned}\eta(\textstyle\sum v_i y\otimes u_i^0)(x)&=\textstyle\sum v_i y x u_i=\lambda(yx)\\&=\lambda(xy)=\textstyle\sum v_i x y u_i=\eta(\sum v_i\otimes(y u_i)^0).\end{aligned}$$

This means

i) $\sum v_i y\otimes u_i^0=\sum v_i\otimes(y u_i)^0$.

Hence $s_C=\sum v_i\underset{C}{\otimes}u_i^0$ is the desired element in $A\underset{C}{\otimes}A^0$.

Now, C is separable over R, and there exists $\sum p_j\otimes q_j\in C\otimes C$ such that $\sum c p_j\otimes q_j=\sum p_j\otimes q_j c$ $(c\in C)$ and $\sum p_j q_j=1$. The mapping $a\underset{C}{\otimes}b^0\to \sum a p_j\underset{R}{\otimes}(b q_j)^0$ is a well-defined ring homomorphism $\sigma: A\underset{C}{\otimes}A^0\to A\underset{R}{\otimes}A^0$ such that $\tau\sigma=$ identity, where τ donotes the natural epimorphism $A\underset{R}{\otimes}A^0\to A\underset{C}{\otimes}A^0$. Put $s=\sigma(s_C)=\sum v_i p_j\otimes(u_i q_j)^0$. We have

i) $\sum v_i p_j u_i q_j=\sum v_i u_i\sum p_j q_j=1$.

Since $s_C(x\underset{C}{\otimes}1^0-1\underset{C}{\otimes}x^0)=0$, we have, applying σ,

ii) $s(x\underset{R}{\otimes}1^0-1\underset{R}{\otimes}x^0)=0$.

Hence s has the desired properties. q.e.d.

Corollary [4, Lemma 1]. *A is strongly separable if and only if A is strongly separable over C and C is separable.*

We now study a strongly separable algebra A over its center C. Let A^* be the dual space of $A: A^*=\mathrm{Hom}_C(A, C)$, considered as a two-sided A-module by $a\xi(x)=\xi(xa)$, $\xi a(x)=\xi(ax)$. Then the mapping $x\otimes\xi\to x\xi$ yields an isomorphism $A\underset{C}{\otimes}(A^*)^A=A^*$ [1]. But $(A^*)^A=\{\xi\in A^*|$

$a\xi=\xi a,\ a\in A\}$ consists of symmetric linear forms. Since $A=C\oplus[A, A]$, this set consists of scalar multiples of λ. It follows that the mapping $\tilde{\lambda}: x\to x\lambda$ gives an isomorphism of two-sided A-modules $A\cong A^*$ (i.e. A is a symmetric algebra).

In [2], we defined the rank element of a projective module. Since A is C-finite projective, we can speak of the rank element $r_C(A)$ which is an element of C. Then, as a natural generalization of [3, Theorem], we have

Theorem 2. *A central separable algebra A over C is strongly separable if and only if $r_C(A)$ is a unit of C.*

Proof. Assume that A is strongly separable. Since A is C-finite projective, we have an isomorphism $\theta: A^*\otimes_C A^0\cong \mathrm{Hom}_C(A, A)$ by $\theta(\xi\otimes a^0)(x)=\xi(x)a$. Composing this with $\tilde{\lambda}\otimes 1: A\otimes A^0\cong A^*\otimes A^0$, we have an isomorphism $f: A\otimes_C A^0\cong\mathrm{Hom}_C(A, A)$, which is defined by $f(a\otimes b^0)(x)=\lambda(ax)b$. Let $\theta(\sum \xi_i\otimes u_i)=$ identity of A (u_i as above), and $\xi_i=w_i\lambda$ ($w_i\in A$). Then we have $x=\sum\lambda(xw_i)u_i$. Since

$$\begin{aligned} f(\textstyle\sum xw_i\otimes u_i^0)(y) &= \textstyle\sum (xw_iy)u_i = yx \\ &= \textstyle\sum (yw_i)u_ix = f(\sum w_i\otimes(u_ix))^0), \end{aligned}$$

we have

$$\sum xw_i\otimes u_i^0 = \sum w_i\otimes(u_ix)^0 \qquad (x\in A).$$

It follows that $\sum w_iu_i$ is in the center C, and we have

$$r_C(A) = \sum \xi_i(u_i) = \lambda(\textstyle\sum w_iu_i) = \sum w_iu_i .$$

But

$$\begin{aligned} 1 = \textstyle\sum v_iu_i &= \textstyle\sum \lambda(u_jw_i)v_ju_i = \sum\lambda(u_j)(v_jw_i)u_i \\ &= (\textstyle\sum\lambda(u_j)v_j)(\sum w_iu_i) . \end{aligned}$$

This shows that $r_C(A)=\sum w_iu_i$ is a unit of C.

Conversely assume that $r_C(A)=c$ is a unit of C. Consider the trace: $\mathrm{Tr}\,(x)=\sum\xi_i(xu_i)$, where $\theta(\sum\xi_i\otimes u_i)=1$ as above. We see that this is a symmetric linear form $A\to C$ and $r_C(A)=\mathrm{Tr}\,(1)$ [2]. Put $\lambda=c^{-1}\mathrm{Tr}$. Then λ is also a symmetric linear form $A\to C$, and satisfies $\lambda(1)=c^{-1}\mathrm{Tr}(1)=1$. Hence A is strongly separable. q.e.d.

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