Nagai, H.
Osaka J. Math.
20 (1983), 863-879

IMPULSIVE CONTROL OF SYMMETRIC MARKOV PROCESSES AND QUASI-VARIATIONAL INEQUALITIES

HIDEO NAGAI

(Received February 17, 1982)

By introducing the notion of impulsive control of a diffusion process A. Bensoussan—J.L. Lions ([1]) showed that if the solutoin of a quasi-variational inequality has sufficient regularity (twice differentiability and continuity), it turns out to be a pay-off function. Furthermore they constructed the optimal strategy out of the solution. But the regularity problems remained open. On the other hand M. Robin ([7]) has set up an impulsive control problem of a general Markov process with a Feller transition semi-group and has constructed the optimal strategy out of the pay-off function which was characterized however in terms of the semi-group rather than the generator of the basic Markov process. As for the characterization by means of the quasi-variational inequality the regularity of the solution was still assumed in order to identify the solution with the pay-off function like that of Bensoussan-Lions. Regularity problems of elliptic or parabolic quasi-variational inequalities have been studied by L.A. Cafarelli—A. Friedman and others (cf. [2], [5]) under the condition that the diffusion and drift coefficients have sufficient regularity. Cafarelli-Friedmans' work, combined with Robin's, establishes completely the relationship between impulsive control problems and quasi-variational inequalities with respect to nice diffusion processes.

Our objective is to extend this relationship to general symmetric Markov processes associated with regular Dirichlet spaces. We prove that the pay-off function is characterized by the weak (maximum) solution of the quasi-variational inequality defined on the Dirichlet space (Theorem 2 in §2). Since we assume only that the Dirichlet space is regular, Theorem 2 establishes the relationship for a wide class of processes. It applies as well to symmetric diffusion process with measurable coefficients and symmetric Markov processes with non local generators (cf. [4]).

Our approach is more potential theoretic than others and accordingly the regularity questions can be dispensed with. Indeed we use the potential theory of Dirichlet spaces and Markov processes developed in [4]. The same method has been used in [6] to establish the relationship between variational inequalities and optimal stoppings and in [8] to include stopping games.

We would like to express our hearty thanks to Professors M. Fukushima and T. Sirao for valuable advice and also to Mr. S. Sato for useful discussions.

1. Quasi-variational inequalities on regular Dirichlet spaces

Let $m(dx)$ be a non-negative everywhere dense Radon measure on a locally compact Hausdorff space S with countable base. Suppose that $(\mathcal{F}, \mathcal{E})$ is a regular Dirichlet space relative to $L^2(dm)$:

i) $\mathcal{F}$ is a dense linear subspace of $L^2(dm)$,

ii) $\mathcal{E}$ is a symmetric bilinear form on $\mathcal{F}\times\mathcal{F}$,

iii) $\mathcal{F}$ is closed with respect to $\mathcal{E}_1$-norm, where $\mathcal{E}_1(u, v)=\mathcal{E}(u, v)+(u, v)$, (u, v) denoting inner product of $L^2(dm)$,

iv) unit contraction operates, that is, if $v=(0\vee u)\wedge 1$, $u\in\mathcal{F}$, then $v\in\mathcal{F}$ and $\mathcal{E}(v, v)\leqq\mathcal{E}(u, u)$,

v) $\mathcal{F}\cap C_0(S)$ is dense in $\mathcal{F}$ with $\mathcal{E}_1$-norm as well as in $C_0(S)$ with uniform norm, $C_0(S)$ denoting the space of all continuous functions on S with compact support.

DEFINITION 1.1. The capacity of a subset of S is defined as follows: for open set $A\subset S$

$$\operatorname{Cap}(A) = \begin{cases} \inf\ \{\mathcal{E}_1(u, u);\ u\in L_A\} & \text{if } L_A\neq\phi\,, \\ \infty & \text{otherwise,} \end{cases}$$

where $L_A=\{u\in\mathcal{F};\ u\geqq 1 \text{ m-a.e. on } A\}$ and for general set $B\subset S$

$$\operatorname{Cap}(B)=\inf\ \{\operatorname{Cap}(A):\ B\subset A,\ A \text{ is open}\}.$$

DEFINITION 1.2. A subset B of S with $\operatorname{Cap}(B)=0$ is called almost polar and "quasi-everywhere" or "q.e." will mean "except for an almost polar set".

Let $S_\Delta=S\cup\Delta$ be the one point compactification of S. When S is already compact, Δ is regarded as an isolated point. Any function on S is extended to a function on $S\cup\Delta$ by setting $f(\Delta)=0$.

DEFINITION 1.3. A function f defined q.e. on S is said to be quasi-continuous (resp. quasi-continuous in the restricted sense) provided that for each $\varepsilon>0$ there exists an open set $G\subset S$ such that $\operatorname{Cap}(G)<\varepsilon$ and $f|_{S-G}$ (resp. $f|_{S_\Delta-G}$) is continuous.

It is known that each $u\in\mathcal{F}$ admits a quasi-continuous modification $\tilde{u}$ in the restricted sense in the case that $(\mathcal{F}, \mathcal{E})$ is a regular Dirichlet space: $u=\tilde{u}$ m-a.e. and $\tilde{u}$ is quasi-continuous in the restricted sense (cf. [4]). Hereafter $\tilde{u}$ denotes a quasi-continuous modification of $u\in\mathcal{F}$. $\tilde{\mathcal{F}}$ denotes the subset of $\mathcal{F}$ consisting of all quasi-continuous functions in the restricted sense.

Let $\nu(dx)$ be a given non-negative Radon measure of finite energy integral, that is, there exists for each $\alpha>0$ a unique function $U_\alpha\nu\in\mathcal{F}$ such that

$$(1.1)\qquad \mathcal{E}_\alpha(U_\alpha\nu, v)=\int_S v(x)\nu(dx)\quad \text{for each } v\in\mathcal{F}\cap C_0(S)\,.$$

Suppose that M is an operator defined on $\tilde{\mathcal{F}}$ such that

$(M.1)$ Mu is a Borel function for any $u\in\tilde{\mathcal{F}}$,

$(M.2)$ $Mu_1(x)\leqq Mu_2(x)$ ${}^\forall x$ if $u_1(x)\leqq u_2(x)$ q.e.,

$(M.3)$ $Mu(x)\geqq 0$ ${}^\forall x$ if $u(x)\geqq 0$ q.e. and

$(M.4)$ $\lim_{n\to\infty} Mu_n(x)=Mu(x)$ ${}^\forall x$ if $u_n(x)\downarrow u(x)$ q.e.

We consider the following quasi-variational inequality:

$$(1.2)\qquad \begin{cases}\mathcal{E}_\alpha(u, v-u)\geqq\langle\nu, \tilde{v}-\tilde{u}\rangle\ {}^\forall\tilde{v}\leqq M\tilde{u}\ \text{q.e.}\\ \tilde{u}\leqq M\tilde{u}\ \text{q.e.}\end{cases}$$

Theorem 1. *The above quasi-variational inequality* (QVI) (1.2) *has the maximum solution.*

Put $u_0=U_\alpha\nu$ and $V_1=\{v\in\mathcal{F};\ \tilde{v}\leqq M\tilde{u}_0\ \text{q.e.}\}$, then we have the unique solution of the following variational inequality (VI) (1.3):

$$(1.3)\qquad \begin{cases}\mathcal{E}_\alpha(u, v-u)\geqq\langle\nu, \tilde{v}-\tilde{u}\rangle\ {}^\forall v\in V_1\\ u\in V_1\,,\end{cases}$$

because (1.3) is equivalent to

$$(1.4)\qquad \begin{cases}\mathcal{E}_\alpha(u-U_\alpha\nu, u-U_\alpha\nu)\leqq\mathcal{E}_\alpha(v-U_\alpha\nu, v-U_\alpha\nu)\ {}^\forall v\in V_1\\ u\in V_1\end{cases}$$

and V_1 is the closed convex subset of Hilbert space $(\mathcal{F}, \mathcal{E}_\alpha)$. Let us denote the solution by u_1. In the same way we can inductively take the solution u_n of the VI:

$$(1.5)\qquad \begin{cases}\mathcal{E}_\alpha(u, v-u)\geqq\langle\nu, \tilde{v}-\tilde{u}\rangle\ {}^\forall v\in V_n\\ u\in V_n\,,\end{cases}$$

$$V_n=\{v\in\mathcal{F};\ \tilde{v}\leq M\tilde{u}_{n-1}\ \text{q.e.}\}\quad \text{for each } n\,.$$

At first we note the properties of the solution u_n of the VI (1.5).

Lemma. *The above u_n has the following properties*
(i) *$U_\alpha\nu-u_n$ is an α-almost excessive function and the unique element which minimizes its α-energy integral in the closed convex subset $U_\alpha\nu-V_n$ of $(\mathcal{F}, \mathcal{E}_\alpha)$:*

$$(1.6) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_n,\ U_\alpha\nu-u_n)\leqq\mathcal{E}_\alpha(U_\alpha\nu-v,\ U_\alpha\nu-v)\ {}^\forall v\in V_n$$

for each n,
(ii) (1.7) $u_n\leqq u_{n-1}$ *m-a.e. for each n,*
(iii) (1.8) $u_n\geqq 0$ *m-a.e. for each n and*
(iv) $\{u_n\}$ *is a $\mathcal{E}_\alpha$-Cauchy sequence.*

Proof. (i) Since u_n is the solution of (1.5) it satisfies the following inequality:

$$(1.9) \quad \mathcal{E}_\alpha(u_n-U_\alpha\nu,\ v-u_n)\geqq 0\ {}^\forall v\in V_n\ .$$

If $w\geqq 0$, $\in\mathcal{F}$, then $u_n-w\in V_n$. Therefore it holds that

$$(1.10) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_n,\ w)\geqq 0\ {}^\forall w\geqq 0\ m\text{-a.e.},\ w\in\mathcal{F}\ .$$

that is $U_\alpha\nu-u_n$ is α-almost excessive because (1.10) is equivalent to

$$(1.11) \quad u_n\geqq 0,\ e^{-\alpha t}T_t u_n\leqq u_n\ m\text{-a.e.},\ {}^\forall t>0\ .$$

Here T_t is the L^2-Markov semigroup corresponding to Dirichlet form $\mathcal{E}$ (cf. [4]). The latter half of (i) follows directly if V_1 in (1.4) is replaced by V_n.

(ii) Inquality (1.7) with $n=1$ is obvious because $U_\alpha\nu-u_1$ is α-almost excessive and $U_\alpha\nu=u_0$. Assume that it holds for n, then $M\tilde{u}_n\leqq M\tilde{u}_{n-1}\ {}^\forall x$. Therefore $\tilde{u}_{n+1}\leqq M\tilde{u}_{n-1}$ q.e.. Since $\tilde{u}_n\leqq M\tilde{u}_{n-1}$ q.e. by definition we have $\tilde{u}_n\vee\tilde{u}_{n+1}\leqq M\tilde{u}_{n-1}$ q.e.. On the other hand $U_\alpha\nu-u_n\vee u_{n+1}=(U_\alpha\nu-u_n)\wedge(U_\alpha\nu-u_{n+1})$ is α-almost excessive because both $U_\alpha\nu-u_{n+1}$ and $U_\alpha\nu-u_n$ are α-almost excessive. So it follows that

$$(1.12) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_n\vee u_{n+1},\ U_\alpha\nu-u_n\vee u_{n+1})\leqq\mathcal{E}_\alpha(U_\alpha\nu-u_n,\ U_\alpha\nu-u_n)$$

from $U_\alpha\nu-u_n\geqq U_\alpha\nu-u_n\vee u_{n+1}$. By (i) of present Lemma we conclude that $u_n\vee u_{n+1}=u_n$, that is, $u_{n+1}\leqq u_n$ m-a.e..

(iii) Since $\widetilde{U_\alpha\nu}\geqq 0$ q.e. we have $M\tilde{u}_0\geqq 0\ {}^\forall x$. Furthermore $\tilde{u}_1\leqq M\tilde{u}_0$ q.e. by definition, so we have $\tilde{u}_1\vee 0\leqq M\tilde{u}_0$ q.e.. Both $U_\alpha\nu-u_1$ and $U_\alpha\nu$ being α-almost excessive, $U_\alpha\nu-u_1\vee 0=(U_\alpha\nu-u_1)\wedge U_\alpha\nu$ is α-almost excessive. Therefore it follows that

$$(1.13) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_1\vee 0,\ U_\alpha\nu-u_1\vee 0)\leqq\mathcal{E}_\alpha(U_\alpha\nu-u_1,\ U_\alpha\nu-u_1)$$

from $U_\alpha\nu-u_1\vee 0\leq U_\alpha\nu-u_1$ m-a.e.. It implies that $u_1\vee 0=u_1$, that is $u_1\geqq 0$ m-a.e.. We can inductively show $u_n\geqq 0$ m-a.e. by similar argument.

(iv) Since $U_\alpha\nu-u_n\leqq U_\alpha\nu-u_m$ m-a.e., $n\leqq m$, and $U_\alpha\nu-u_n\leqq U_\alpha\nu$ m-a.e. for each n by (ii) and (iii) it holds that

$$(1.14) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_n,\ U_\alpha\nu-u_n)\leqq\mathcal{E}_\alpha(U_\alpha\nu-u_m,\ U_\alpha\nu-u_m)\leqq\mathcal{E}_\alpha(U_\alpha\nu,\ U_\alpha\nu)$$

for each $n\leqq m$. Therefore $\mathcal{E}_\alpha(U_\alpha\nu-u_n,\ U_\alpha\nu-u_n)$ monotonously increases to a finite number. Since $w_n = U_\alpha\nu - u_n$ is α-almost excessive

$$\begin{aligned}0 &\leqq \mathcal{E}_\alpha(w_n - w_m, w_n - w_m) = \mathcal{E}_\alpha(w_n, w_n) - 2\mathcal{E}_\alpha(w_n, w_m) + \mathcal{E}_\alpha(w_m, w_m)\\ &\leqq \mathcal{E}_\alpha(w_m, w_m) - \mathcal{E}_\alpha(w_n, w_n), \quad n \leqq m\,.\end{aligned}$$

Hence w_n is a $\mathcal{E}_\alpha$-cauchy sequence, so u_n is also.

Proof of Theorem 1. As the result of (ii) and (iv) of Lemma there exists u such that $\mathcal{E}_\alpha(u_n - u, u_n - u) \to 0$ and $\tilde{u}_n \downarrow \tilde{u}$ q.e.. We can now prove that this function u is a solution of the quasi-variational inequality (1.2). We at first note that it follows that

$$\mathcal{E}_\alpha(U_\alpha\nu - u_n, U_\alpha\nu - u_n) \leqq \mathcal{E}_\alpha(U_\alpha\nu - v, U_\alpha\nu - v)\ {}^\forall\tilde{v} \leqq M\tilde{u} = \lim_{n\to\infty} M\tilde{u}_n \text{ q.e.}$$

from (1.6) because $\tilde{u}_n \downarrow \tilde{u}$ q.e. implies $M\tilde{u}_n \downarrow M\tilde{u}$. Therefore it holds that

$$(1.15)\quad \mathcal{E}_\alpha(U_\alpha\nu - u, U_\alpha\nu - u) \leqq \mathcal{E}_\alpha(U_\alpha\nu - v, U_\alpha\nu - v),\ {}^\forall\tilde{v} \leqq M\tilde{u} \text{ q.e.}$$

since $\mathcal{E}_\alpha(u_n - u, u_n - u) \to 0$. On the other hand, since $\tilde{u} \leqq \tilde{u}_n \leqq M\tilde{u}_{n-1}$ q.e. for each n we have

$$(1.16)\quad \tilde{u} \leqq \lim_{n\to\infty} M\tilde{u}_n = M\tilde{u} \text{ q.e..}$$

(1.15) with (1.16) is equivalent to the QVI (1.2).

Now we are going to prove that the above solution u of QVI (1.2) is the maximum. Take another solution w of the QVI

$$\begin{cases}\mathcal{E}_\alpha(w, v - w) \geqq \langle \nu, \tilde{v} - \tilde{w} \rangle\ {}^\forall\tilde{v} \leqq M\tilde{w} \text{ q.e.}\\ \tilde{w} \leqq M\tilde{w} \text{ q.e.}\end{cases}$$

In the same way as Lemma we can see $U_\alpha\nu - w$ is α-excessive, so $U_\alpha\nu \geqq w$. Therefore $M\widetilde{U_\alpha\nu} \geqq \tilde{w}$ q.e.. That is $w \in V_1$. Since $U_\alpha\nu - u_1 \vee w = (U_\alpha\nu - u_1) \wedge (U_\alpha\nu - w)$ is α-almost excessive and $U_\alpha\nu - u_1 \vee w \leqq U_\alpha\nu - u_1$ it holds that

$$(1.17)\quad \mathcal{E}_\alpha(U_\alpha\nu - u_1 \vee w, U_\alpha\nu - u_1 \vee w) \leqq \mathcal{E}_\alpha(U_\alpha\nu - u_1, U_\alpha\nu - u_1)\,.$$

Hence we have $u_1 \geqq w$ by similar argument as (iii) of Lemma. In the same way we can inductively see $u_n \geqq w$ for each n, which implies $u \geqq w$.

2. Impulsive control of symmetric Markov processes

Let $X = \{\Omega, \mathcal{B}, \mathcal{B}_t, P_x, X_t, \theta_t\}$ be an m-symmetric standard Markov process of function space type with the state space S. We assume that its Dirichlet space $(\mathcal{F}, \mathcal{E})$ is regular. We are now going to repeat Robin's construction of controlled process (cf. [7]) with a little modification and set up an impulsive control problem.

Consider the infiinte product space $\Omega_\infty=\Omega\times\Omega\times\Omega\times\cdots$ and define its sub-σfields by

$$(2.1)\qquad \mathcal{B}_t^n = \Pi_n^{-1}(\mathcal{B}_t)^{\otimes n}$$

where Π_n is the projection from Ω_∞ to the n-th product $(\Omega)^n$. $\mathcal{B}^n$ is similarly defined. For $\underline{\underline{\omega}}=(\omega_1, \omega_2, \cdots)\in\Omega_\infty$, we let

$$(2.2)\qquad \begin{aligned}(\theta_{n,t}\underline{\underline{\omega}})(s) &= (\theta_t\omega_1(s), \cdots\cdots\cdots, \theta_t\omega_n(s)) \\ &= (\omega_1(t+s), \cdots\cdots\cdots, \omega_n(t+s))\,.\end{aligned}$$

We note that, if $\sigma(\underline{\underline{\omega}})$ is a $\mathcal{B}^n$-measurable function on Ω_∞, then $\sigma(\underline{\underline{\omega}})=\tilde{\sigma}(\omega_1, \omega_2, \cdots, \omega_n)$, $\tilde{\sigma}$ being a $(\mathcal{B})^{\otimes n}$-measurable function on $(\Omega)^n$. Such an identification of σ and $\tilde{\sigma}$ will be made below without mentioning explicitly. It is further noticed that P_x for each $x\in S$ can be regarded as a probability measure on $(\Omega_\infty, \mathcal{B}^1)$.

A family of subsets $\{\Gamma_x\}_{x\in S}$ of S is called *admissible* if the following condition (Γ) is satisfied:

(Γ) if $x_n\to x, x_n, x\in S$ and $y_n\in\Gamma_{x_n}$, then there exist $y\in\Gamma_x$ and $\{y_{n_k}\}\subset\{y_n\}$ such that $y_{n_k}\to y$.

A sequence $v=\{(\tau_i, \xi_i)_{i=1}^\infty\}$ of the pairs of random variables τ_i and ξ_i on Ω_∞ is called an *admissible control* if the following conditions (v.1)~(v.3) are satisfied for a given admissible $\{\Gamma_x\}$:

(v.1) τ_i is a $\mathcal{B}_t^i$-stopping time such that $\tau_i\leqq\tau_{i+1}$ for each i and $\lim_{i\to\infty}\tau_i=\infty$

(v.2) ξ_i is $\Gamma_{X_{\tau_i}(\omega_i)}$-valued $\mathcal{B}_{\tau_i}^i$-measurable random variable for each i

(v.3) for each N with $\mathrm{Cap}(N)=0$ there exists $\tilde{N}\supset N$ with $\mathrm{Cap}(\tilde{N})=0$ such that $P_x^i(\xi_i\in\tilde{N})=0$, $x\in S-\tilde{N}$ for each i, where P_x^i is a probability measure on $(\Omega_\infty, \mathcal{B}^i)$ specified below.

The set of all admissible controls are denoted by $\underline{\underline{V}}$. Let us define, for $y\in S$, an element $\delta_y\in\Omega$ by

$$(2.3)\qquad \delta_y(t) = y \quad {}^\forall t\geqq 0$$

and denote by ε_{δ_y} the probability measure on $(\Omega, \mathcal{B})$ which is concentrated on δ_y.

For a given $v=\{(\tau_i, \xi_i)_{i=1}^\infty\}\in\underline{\underline{V}}$, we are interested in the process $X_t(\omega_1)$ governed by P_x up to time $\tau_1(\omega_1)$. $X_t(\omega_1)$ is stopped at time τ_1 and then our interest is switched to the process $X_{\tau_1(\omega_1)+t}(\omega_2)$, $t\geqq 0$, governed by $P_{\xi_1(\omega_1)}$ up to time $\tau_2(\omega_1, \omega_2)$ and so forth. To formulate such a process, we construct probability measures P_x^n on $(\Omega_\infty, \mathcal{B}^n)$, $n=1, 2, \cdots$, as follows:

First let

$$P_x^1 = P_x \quad \text{on} \quad (\Omega_\infty, \mathcal{B}^1)$$

We can construct a probability measure P_x^2 on $(\Omega_\infty, \mathcal{B}^2)$ such that

$$(2.4)\quad \begin{cases} P_x^2 = P_x^1 \quad \text{on} \quad \mathcal{B}_{\tau_1}^1(\subset \mathcal{B}^1) \\ P_x^2(\theta_{2,\tau_1}^{-1}B \mid \mathcal{B}_{\tau_1}^1) = \varepsilon_{\delta X_{\tau_1}} \otimes P_{\xi_1}(B) \quad P_x^1\text{-a.s.} \quad \text{on} \quad \{\tau_1 < +\infty\} \end{cases}$$

for each $B \in \mathcal{B}^2$. Then the process $X_{\tau_1+t}(\omega_2)$, $t \geqq 0$, is Markovian with respect to $(\mathcal{B}_{\tau_1+t}^2, P_x^2)$ under the condition $\mathcal{B}_{\tau_1}^1$. We define the probability measure P_x^{n+1} on $(\Omega_\infty, \mathcal{B}^{n+1})$ inductively by

$$(2.5)\quad \begin{cases} P_x^{n+1} = P_x^n \quad \text{on} \quad \mathcal{B}_{\tau_n}^n(\subset \mathcal{B}^n) \\ P_x^{n+1}(\theta_{n+1,\tau_n}^{-1}B \mid \mathcal{B}_{\tau_n}^n) = \varepsilon_{\delta X_{\tau_1}(\omega_1)} \otimes \cdots \otimes \varepsilon_{\delta X_{\tau_n}(\omega_n)} \otimes P_{\xi_n}(B) \\ \qquad\qquad P_x^n\text{-a.s.} \quad \text{on} \quad \{\tau_n < +\infty\} \end{cases}$$

where $B \in \mathcal{B}^{n+1}$.

We are now in a position to formulate our main theorem. Consider the Dirichlet space $(\mathcal{F}, \mathcal{E})$ associated with the process X. We suppose that a non-negative Radon measure $\nu(dx)$ of finite energy integral and non-negative continuous function $k(x, \xi)$, $x, \xi \in S$, are given which are to define a pay-off function. It is known that a non-negative continuous additive functional $A_t(\omega)$ on X corresponds to $\nu(dx)$:

$$(2.6)\quad E_x\Big[\int_0^\infty e^{-\alpha s} dA_s\Big] = U_\alpha \nu \quad \text{q.e.}$$

(cf. [4]). Let for $\underline{\underline{\omega}} = (\omega_1, \omega_2, \cdots) \in \Omega_\infty$

$$(2.7)\quad \underline{\underline{A}}_t = \begin{cases} A_t(\omega_1), \quad 0 \leqq t \leqq \tau_1 \\ \underline{\underline{A}}_{\tau_1} + A_{t-\tau_1}(\theta_{\tau_1}\omega_2), \ \tau_1 < t \leqq \tau_2 \\ \underline{\underline{A}}_{\tau_{n-1}} + A_{t-\tau_{n-1}}(\theta_{\tau_{n-1}}\omega_n), \ \tau_{n-1} < t \leqq \tau_n \end{cases}$$

and

$$(2.8)\quad y_t(\underline{\underline{\omega}}) = X_t(\omega_{k+1}) \quad \text{if} \quad t \in [\tau_k, \tau_{k+1}).$$

We can now define the pay-off function $u^*(x)$ by

$$(2.9)\quad u^*(x) = \inf_{v \in \underline{\underline{V}}} J_x(v)$$

$$(2.10)\quad J_x(v) = \lim_{n \to \infty} J_x^n(v)$$

$$(2.11)\quad J_x^n(v) = E_x^n\Big[\int_0^{\tau_n} e^{-\alpha t} d\underline{\underline{A}}_t + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i)\Big].$$

We then introduce the operator M by

$$(2.12)\quad M\phi(x) = q\text{-}\operatorname*{essinf}_{y \in \Gamma_x} \{\phi(y) + k(x, y)\}$$
$$\equiv \sup\{c\colon \mathrm{Cap}\{y \in \Gamma_x;\ \phi(y) + k(x, y) < c\} = 0\}$$

for $\phi \in \mathcal{F}$. The fact that this operator M satisfies $(M.1) \sim (M.4)$ will be shown later (§3). Recall that Theorem 1 then guarantees the existence of the maximum solution of the QVI (1.2) associated with the present data $(\mathcal{F}, \mathcal{E})$, ν and M.

Theorem 2. *The pay-off function $u^*(x)$ defined by (2.9) is a quasi-continuous modification of the maximum solution u of the QVI (1.2) corresponding to the above $(\mathcal{F}, \mathcal{E})$, ν and M.*

REMARK. We note that if $\nu(dx)=f(x)dm$ with a Borel function f in $L^2(dm)$, $J_x^n(v)$ is written as

$$(2.13)\quad J_x^n(v) = E_x^n\left[\int_0^{\tau_n} e^{-\alpha t} f(y_t)dt + \sum_{i=1}^{n} e^{-\alpha\tau_i} k(X_{\tau_n}(\omega_n), \xi_n)\right]$$

In the next section we study the operator M defined by (2.12). All assumptions and notations in section 2 are assumed thruogh the following sections.

3. Operator M

DEFINITION 3.1. A sequence $\{F_k\}$ of closed sets such that $F_k \uparrow$ and Cap $(S\text{-}F_k) \downarrow 0$, $k \to \infty$ is called a nest on S. A nest $\{F_k\}$ is said to be (m)-regular if for each k $m(U(x) \cap F_k) \neq 0$ for any $x \in F_k$ and any open neighborhood $U(x)$ of x.

Let Q be a countable family of quasi-continuous function in the restricted sense on S. Then it is known that there exists a regular nest $\{F_k\}$ on S such that $u|_{F_k \cup \Delta}$ is continuous for each k for any function $u \in Q$.

Lemma 3.1. *For any function $\phi \in \tilde{\mathcal{F}}$ $M\phi$ is a Borel function and has the following representation:*

$$(3.1)\qquad M\phi(x) = \lim_{n\to\infty} \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x, y)\}$$

where $\{F_n\}$ is a regular nest and Γ_x^n is a subset of S which satisfies (Γ).

Proof. It holds that by definition

$$\text{Cap } \{y \in \Gamma_x;\ \phi(y) + k(x, y) < M\phi(x) - \varepsilon\} = 0$$

for any $\varepsilon > 0$. Take a regular nest $\{F_n\}$ such that $\phi|_{F_n \cup \Delta}$ is continuous for each n. Put

$$N_x^n = \{y \in \Gamma_x;\ \text{there exists a open neighborhood } U_y \text{ such that}$$
$$\text{Cap}(U_y \cap F_n \cap \Gamma_x) = 0\}$$

and define

$$\Gamma_x^n = \Gamma_x \cap (\bigcup_{y \in N_x^n} U_y)^c$$

by above U_y. Then it is obvious that Γ_x^n satisfies (Γ) because $(\bigcup_{y \in N_x^n} U_y)^c$ is closed. Since $\phi(\cdot)$ and $k(x, \cdot)$ are continuous on $\Gamma_x^n \cap F_n$ it follows that

$$\phi(y)+k(x,y)\geqq M\phi(x)-\varepsilon \ {}^{\forall}y\in\Gamma_x^n\cap F_n$$

from

$$\mathrm{Cap}\{y\in\Gamma_x^n\cap F_n;\ \phi(y)+k(x,y)<M\phi(x)-\varepsilon\}=0$$

Therefore

$$\lim_{n\to\infty}\inf_{y\in\Gamma_x^n\cap F_n}\{\phi(y)+k(x,y)\}\geqq M\phi(x)\,.$$

In order to get converse inequality put

$$c=\lim_{n\to\infty}\inf_{y\in\Gamma_x^n\cap F_n}\{\phi(y)+k(x,y)\}\ ,$$

then

$$\begin{aligned}&\mathrm{Cap}\{y\in\Gamma_x;\ \phi(y)+k(x,y)<c\}=\mathrm{Cap}\{\Gamma_x\cap(\bigcup_n F_n);\ \phi(y)+k(x,y)<c\}\\ &\leqq\sum_{n=1}^{\infty}\mathrm{Cap}\{y\in\Gamma_x\cap F_n;\ \phi(y)+k(x,y)<c\}\\ &=\sum_{n=1}^{\infty}\mathrm{Cap}\{y\in\Gamma_x^n\cap F_n;\ \phi(y)+k(x,y)<c\}\ .\end{aligned}$$

Hence $c\leqq M\phi(x)$. Now (3.1) has been proved. On the other hand, since $\inf_{y\in\Gamma_x^n\cap F_n}\{\phi(y)+k(x,y)\}$ is a lower semi-continuous function according to the following lemma, we have the conclusion that $M\phi(x)$ is a Borel function.

Lemma 3.2. *For any* $\phi\in\tilde{\mathcal{F}}$

$$M_n\phi(x)=\inf_{y\in\Gamma_x^n\cap F_n}\{\phi(y)+s(x,y)\}$$

is a lower semi-continuous function and has a measurable selection for each n.

This lemma is a trivial modification of Theorem A in §5, Chap. 2 of [3]. Because $\Gamma_x^n\cap F_n$ also satisfies (Γ) and $\phi(\cdot)$ and $k(x,\cdot)$ are continuous on F_n.

Lemma 3.3. *The operator* M *defined by* (2.12) *satisfies* $(M.1)\sim(M.4)$.

Proof. $(M.1)$ has been proved in Lemma 3.1. $(M.2)$ and $(M.3)$ are obvious. As to $(M.4)$ it is easily seen that

$$\lim_{n\to\infty}Mu_n(x)\geqq Mu(x)\,.$$

On the other hand

$$Mu_n(x)\leqq u_n(y)+k(x,y)\ {}^{\forall}y\in\Gamma_x^m\cap F_m\,,$$

so we have

$$\lim_{n\to\infty}Mu_n(x)\leqq u(y)+k(x,y)\ {}^{\forall}y\in\Gamma_x^m\cap F_m$$

for each m. Then it holds that

$$\lim_{n\to\infty} Mu_n(x) \leqq \lim_{m\to\infty} \inf_{y\in\Gamma_x^m \cup F_m} \{u(y)+k(x, y)\}$$
$$= Mu(x) .$$

4. Optimal stoppings of Markov processes

We prepare for the proof of Theorem 2 some lemmas on optimal stoppings of Markov processes with which regular Diiichlet spaces are assoicated.

Let ψ_n be a given Borel function and s_n be the unique solution of the following variational inequality:

$$\text{(4.1)} \quad \begin{cases} \mathcal{E}_\alpha(s_n, v-s_n) \geqq \langle \nu, \tilde{v}-\tilde{s}_n \rangle \; {}^\forall v \in \mathcal{F}, \; \tilde{v} \leqq \psi_n \text{ q.e.} \\ s_n \in \mathcal{F}, \; \tilde{s}_n \leqq \psi_n \text{ q.e.} \end{cases}$$

for each n.

Lemma 4.1. *Suppose that* $\psi_n(x) \downarrow \psi(x) \geqq 0 \; {}^\forall x$, *then* $\mathcal{E}_\alpha(s_n-s, s_n-s) \to 0$ *where* s *is the unique solution of*

$$\text{(4.2)} \quad \begin{cases} \mathcal{E}_\alpha(s, v-s) \geqq \langle \nu, \tilde{v}-\tilde{s} \rangle \; v \in \mathcal{F}, \; \tilde{v} \leqq \psi \text{ q.e.} \\ s \in \mathcal{F}, \; \tilde{s} \leqq \psi \text{ q.e.} \end{cases}$$

Proof. In a similar way as the proof of Theorem 1 we can easily show that $s_n \geqq s_{n+1}$, $s_n \geqq 0$ and $U_\alpha \nu - s_n$ is an α-almost excessive function for each n (cf. Lemma in §1) Therefore we have

$$\mathcal{E}_\alpha(U_\alpha \nu - s_n, U_\alpha \nu - s_n) \leqq \mathcal{E}_\alpha(U_\alpha \nu - s_m, U_\alpha \nu - s_m) \leqq \mathcal{E}_\alpha(U_\alpha \nu, U_\alpha \nu) \quad n \leqq m .$$

So there exists $s_0 \in \mathcal{F}$ such that $\mathcal{E}_\alpha(s_n - s_0, s_n - s_0) \to 0$. Furthermore s_0 satisfies

$$\text{(4.3)} \quad \begin{cases} \mathcal{E}_\alpha(U_\alpha \nu - s_0, U_\alpha \nu - s_0) \leqq \mathcal{E}_\alpha(U_\alpha \nu - v, U_\alpha \nu - v) \; {}^\forall \tilde{v} \leqq \lim_{n\to\infty} \psi_n = \psi \text{ q.e.} \\ \tilde{s}_0 \leqq \psi \text{ q.e.} \end{cases}$$

which is equivalent to (4.2). Hence we conclude $s_0 = s$ because of uniqueness of the solution of (4.2).

Lemma 4.2. *Put*

$$t_n(x) = \inf_\tau E_x\Big[\int_0^\tau e^{-\alpha s} dA_s + e^{-\alpha\tau} M_n \phi(X_\tau)\Big] \text{ q.e.}$$

where $\phi \in \tilde{\mathcal{F}}$, *then* t_n *is a quasi-continuous modification of the solution* s_n *of the variational inequality* (4.1) *for each* n *in which* ψ_n *is considered* $M_n\phi$. *Furthermore there exists an optimal stopping time.*

Proof. Since $U_\alpha \nu - s_n$ is α-almost excessive by similar argument as Lemma 1.1 there corresponds a non-negative Radon measure μ_n of finite energy integral such that

$$\mathcal{E}_\alpha(U_\alpha\nu - s_n, v) = \int \mu_n(dx)\tilde{v}(x) \quad {}^\forall v \in \mathcal{F}.$$

Therefore it follows that

$$(4.4)\qquad \begin{cases} \int \mu_n(dx)\,(\tilde{s}_n(x) - \tilde{v}(x)) \geqq 0 \ {}^\forall \tilde{v} \leqq M_n\phi \text{ q.e., } v \in \mathcal{F} \\ \tilde{s}_n \leqq M_n\phi \text{ q.e., } s_n \in \mathcal{F} \end{cases}$$

from (4.1) with $\psi_n = M_n\phi$. Put

$$(4.5)\qquad L_n = \{x \in \bigcup_{k=1}^{\infty} F_k;\ \tilde{s}_n(x) < M_n\phi(x)\},$$

where $\{F_k\}$ is a regular nest corresponding to the family of quasi-continuous functions $\{\tilde{s}_n\}$. Take an arbitrary point $x_0 \in L_n$, then $x_0 \in F_{k_0}$ for some k_0. On the other hand, Since $M_n\phi(x)$ is a lower semi-continuous function there exists a sequence of continuous functions $c_j^n(x)$ such that $c_j^n(x) \uparrow M_n\phi(x), j \to \infty, {}^\forall x$. Therefore $c_{j_0}^n(x_0)s > s_n(x_0)$ for sufficiently large j_0, which implies that there exists a neighborhood $U(x_0)$ of x_0 such that

$$s_n(x) < c_{j_0}^n(x) \quad {}^\forall x \in F_{k_0} \cap U(x_0).$$

Accordingly there exists a neighborhood $V(x_0)$ and $v_n \in \mathcal{F} \cap C_0(S)$ such that

$$\bar{V}(x_0) \subset U(x_0),$$

$$\text{Supp } v_n \subset U(x_0),\ v_n(x) > 0 \text{ on } V(x_0)$$

and

$$\tilde{s}_n(x) + v_n(x) \leqq M_n\phi(x)$$

because the Dilichlet space $(\mathcal{F}, \mathcal{E})$ is regular. Therefore

$$-\int \mu_n(dx) v_n(x) \geqq 0$$

which implies $\mu_n(V(x_0)) = 0$. Since $x_0 \in L_n$ is arbitrary we conclude that

$$(4.6)\qquad \mu_n(L_n) = 0.$$

Next, we have

$$(4.7)\qquad \tilde{s}_n(x) \leqq M_n\phi(x) \text{ q.e.}$$

On the other hand let $S-N$ be a defining set of the additive functional A_t (cf. [1]) and put $\tau_n = \inf\{t;\ X_t \in L_n^c \cap \{S-N\}\}$, then

$$(4.8)\qquad P_x(X_{\tau_n} \in L_n^c \cap (S-N)) = 1 \quad x \in (\bigcup_{n=1}^{\infty} F_n) \cap (S-N)$$

for the benifit of lower semi-continuity of $M_n\phi$ and quasi-continuity of s_n.

From (4.6), (4.7) and (4.8) in addition to the fact that there corresponds a non-negative additive functional A_t^n to the α-almost excessive function $U_\alpha\nu-s_n$ such that

$$\widetilde{U_\alpha\nu-s_n}=E_x[\int_0^\infty e^{-\alpha s}dA_s^n] \text{ q.e.}$$

our present lemma follows in the same way as Theorem in [6].

Lemma 4.3. *Put*

$$(4.9)\qquad t(x)=\inf_\tau E_x[\int_0^\tau e^{-\alpha s}dA_s+e^{-\alpha\tau}M\phi(X_\tau)]\,,$$

then $t(x)$ *is a quasi-continuous modification of the solution of the variational inequality* (4.2) *in which* ψ *is considered as* $M\phi(x)$.

Proof. Let $s(x)$ be the solution of (4.2) with $\psi=M\phi$, then $U_\alpha\nu-s$ is α-almost excessive and there corresponds a non-negative continuous additive functional A_t^0 such that

$$(4.10)\qquad U_\alpha\nu(x)-s(x)=E_x[\int_0^\infty e^{-\alpha t}dA_t^0] \text{ q.e..}$$

On the other hand we have

$$(4.11)\qquad \tilde{s}(x)\leqq M\phi(x) \text{ q.e..}$$

From (4.10) and (4.11) it follows that

$$\begin{aligned}\tilde{s}(x)&=E_x[\int_0^\infty e^{-\alpha t}dA_t]-E_x[\int_0^\infty e^{-\alpha t}dA_t^0]\\&=E_x[\int_0^\tau e^{-\alpha t}dA_t]-E_x[\int_0^\tau e^{-\alpha t}dA_t^0]+E_x[e^{-\alpha\tau}\tilde{s}(X_\tau)]\\&\leqq E_x[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}\tilde{s}(X_\tau)]\\&\leqq E_x[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}M\phi(X_\tau)] \text{ q.e.,}\end{aligned}$$

for any stopping time τ. Therefore it holds that

$$(4.12)\qquad \tilde{s}(x)\leqq t(x) \text{ q.e..}$$

Now it is clear that

$$t(x)\leqq\inf_\tau E_x[\int_0^\tau e^{-\alpha t}dA_t+e^{-\alpha\tau}M_n\phi(X_\tau)]=\tilde{s}_n(x) \text{ q.e..}$$

Since $\mathcal{E}_\alpha(s_n-s,\, s_k-s)\to 0$ by lemma 4.1 we obtain $\tilde{s}_n(x)\downarrow\tilde{s}(x)$ q.e.. Hence

$$(4.13)\qquad t(x)\leqq\tilde{s}(x) \text{ q.e.}$$

(4.12) and (4.13) give our conclusion.

5. Proof of Theorem 2

Now we are going to prove Theorem 2. Let us introduce the set $\underline{\underline{V}}_n$ of admissible controls which have n jump times at most:

$$(5.1) \qquad \underline{\underline{V}}_n = \{v \in \underline{\underline{V}};\ \tau_{n+1}(\underline{\underline{\omega}}) = \infty\}$$

for each n. Put

$$(5.2) \qquad u_n^*(x) = \inf_{v \in \underline{\underline{V}}_n} E_x^{n+1}\Big[\int_0^{\tau_{n+1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^n e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i),\ \xi_i)\Big]$$

and

$$(5.3) \qquad w_n(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha\tau} M w_{n-1}(X_\tau)\Big]$$

for each n where $w_0(x) = \widetilde{U_\alpha \nu}(x)$.

Theorem 2 is a consequence of the following two propositions.

Proposition 5.1. *$w_n(x)$ is a quasi-continuous modification of the solution u_n of the variational inequality* (1.5).

Proposition 5.2. *It holds that*

$$(5.4) \qquad w_n(x) = u_n^*(x) \text{ q.e..}$$

Proposition 5.1 is a direct consequence of Lemma 4.3. For the proof of Proposition 5.2 we prepare the following two lemmas.

Lemma 5.3. *It holds that*

$$(5.5) \qquad w_n(x) = \lim_{k_1 \uparrow \infty} \cdots \operatorname{im}_{k_n \uparrow \infty} w^n_{k_n \cdots k_1}(x) \text{ q.e.}$$

where

$$(5.6) \qquad w^n_{k_n \cdots k_1}(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha\tau} M_{k_n} w^{n-1}_{k_{n-1} \cdots k_1}(X_\tau)\Big] \quad n = 2, 3, \cdots$$

and

$$(5.7) \qquad w^1_{k_1}(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha\tau} M_{k_1} \widetilde{U_\alpha \nu}(X_\tau)\Big].$$

Proof. Because of Lemma 4.1 it follows that

$$(5.8) \qquad w^1_{k_1}(x) \downarrow w_1(x) \text{ q.e.}, \ k_1 \uparrow \infty$$

from $M_{k_1} \widetilde{U_\alpha \nu}(x) \downarrow M \widetilde{U_\alpha \nu}(x)\ {}^\forall x,\ k_1 \uparrow \infty$, in the same way as the the proof of Lemma 4.3. Let us assume that

$$(5.9) \qquad w_{n-1}(x) = \lim_{k_1 \uparrow \infty} \cdots \lim_{k_n \uparrow \infty} w^{n-1}_{k_{n-1}\cdots k_1}(x) \text{ q.e.}.$$

Then it follows that

$$(5.10) \quad \lim_{k_n \uparrow \infty} w^n_{k_n \cdots k_1}(x) = \inf_\tau E_x[\int_0^\tau e^{-\alpha s} dA_s + e^{-\alpha\tau} M w^{n-1}_{k_{n-1}\cdots k_1}(X_\tau)]$$

from $M_{k_n} w^{n-1}_{k_{n-1}\cdots k_1}(x) \downarrow M w^{n-1}_{k_{n-1}\cdots k_1}(x)$ ${}^\forall x$, $k_n \uparrow \infty$, in the same way as above. On the other hand it holds that

$$(5.11) \quad M w_{n-1}(x) = \lim_{k_1 \uparrow \infty} \cdots \lim_{k_{n-1} \uparrow \infty} M w^{n-1}_{k_{n-1}\cdots k_1}(x) \ {}^\forall x$$

by our assumption and the property of M. Making use of Lemma 4.1 we obtain our present lemma from (5.10) and (5.11).

Lemma 5.4. *Let $\bar{X}=(\bar{\Omega}, \bar{\mathcal{B}}, \bar{\mathcal{B}}_t, \bar{P}_x, \bar{X}_t)$ be a m-symmetric Markov process associated with a regular Dirichlet space $(\mathcal{F}, \mathcal{E})$ and*

$$H(M\phi; x) = \inf_\tau \bar{E}_x[\int_0^\tau e^{-\alpha s} d\bar{A}_s + e^{-\alpha\tau} M\phi(\bar{X}_\tau)] \quad \phi \in \tilde{\mathcal{F}},$$

then it holds that

$$\bar{E}_x[\int_\sigma^\tau e^{-\alpha s} d\bar{A}_s + e^{-\alpha\tau} M\phi(\bar{X}_\tau) \,|\, \bar{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma} H(M\phi; \bar{X}_\sigma)$$

for any stopping time σ, τ such that $\sigma \leqq \tau$. Here $\bar{A}_t$ is an additive functional of $\bar{X}$ corresponding to the Radon measure $\nu(dx)$ of finite energy integral.

Proof. At first we note that $H(M\phi; x) \in \tilde{\mathcal{F}}$, $U_\alpha \nu(x) - H(M\phi; x)$ is α-almost excessive and $H(M\phi; x) \leqq M\phi(x)$ q.e. by Lemma 4.3. Therefore $e^{-\alpha t}\{U_\alpha \nu(\bar{X}_t) - H(M\phi; \bar{X}_t)\}$ is a $(\bar{P}_x, \bar{\mathcal{B}}_t)$ supermartingale for q.e. x. Hence

$$\begin{aligned} &\bar{E}_x[e^{-\alpha\tau}\{U_\alpha \nu(\bar{X}_\tau) - H(M\phi; \bar{X}_\tau)\} \,|\, \bar{\mathcal{B}}_\sigma] \\ &\leqq e^{-\alpha\sigma}\{U_\alpha \nu(\bar{X}_\sigma) - H(M\phi; \bar{X}_\sigma)\} \quad \bar{P}_x\text{—a.s.} \quad \text{q.e. } x. \end{aligned}$$

So we have

$$\bar{E}_x[\int_\sigma^\tau e^{-\alpha s} d\bar{A}_s + e^{-\alpha\tau} H(M\phi; \bar{X}_\tau) \,|\, \bar{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma} H(M\phi; \bar{X}_\sigma).$$

Accordingly it follows that

$$\bar{E}_x[\int_\sigma^\tau e^{-\alpha s} d\bar{A}_s + e^{-\alpha\tau} M\phi(\bar{X}_\tau) \,|\, \bar{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma} H(M\phi; \bar{X}_\sigma)$$

from $H(M\phi; x) \leqq M\phi(x)$ q.e.

Proof of Proposition 5.2. Let $v \in \underline{\underline{V}}_n$, $v = \{(\tau_k, \xi_k)_{k=1,2,\cdots n}, \tau_{n+1} = \infty\}$, then

$$J^n_x(v) = E^{n+1}_x[\int_0^\infty e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^n e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i), \xi_i)]$$

$$= E_x^{n+1}[\int_0^{\tau_n} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n} e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha\tau_n}\int_0^{\infty} e^{-\alpha s} dA_s(\theta_{\tau_n}\omega_{n+1})]$$

$$= E_x^{n+1}[\int_0^{\tau_n} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n} e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha\tau_n} E_{\xi_n}[\int_0^{\infty} e^{-\alpha s} dA_s]]$$

$$\geqq E_n^x[\int_0^{\tau_n} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i), \xi_i) + e^{-\alpha\tau_n} M\widetilde{U_\alpha \nu}(X_{\tau_n}(\omega_n))]$$

$$= E_x^n[\int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i} k(X_{\tau_i}(\omega_i), \xi_i)$$

$$+ E_x^n[\int_{\tau_{n-1}}^{\tau_n} e^{-\alpha s} dA_{s-\tau_{n-1}}(\theta_{\tau_{n-1}}\omega_n) + e^{-\alpha\tau_n} M\widetilde{U_\alpha \nu}(X_{\tau_n}(\omega_n)) \mid \mathcal{B}_{\tau_{n-1}}^{n-1}]]$$

$$\geqq E_x^n[\int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i} k(X_{\tau_i}, \xi_i) + e^{-\alpha\tau_{n-1}} H(M\widetilde{U_\alpha \nu};\, X_{\tau_{n-1}})]$$

$$= E_x^{n-1}[\int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i} k(X_{\tau_i}, \xi_i) + e^{-\alpha\tau_{n-1}} w_1(X_{\tau_{n-1}})]$$

$$\geqq E_x^1[\int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha\tau_1} k(X_{\tau_1}, \xi_1) + e^{-\alpha\tau_1} w_{n-1}(X_{\tau_1})]$$

$$\geqq E_x^1[\int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha\tau_1} M w_{n-1}(X_{\tau_1})] \geqq w_n(x) \quad \text{q.e.}$$

In order to get the converse inequality take a sequence $\{\tilde{\tau}_j\}_{j=1,2,\cdots n}$ of stopping times such that τ_j minimizes

$$E_x[\int_0^{\tau} e^{-\alpha s} dA_s + M_{k_j} w_{k_{j-1}\cdots k_1}^{j-1}(X_\tau)]\,.$$

Furthermore take a sequence of functions $y_{k_j}(x)$, $j=1, 2, \cdots, n$ such that

$$M_{k_j} w_{k_{j-1}\cdots k_1}^{j-1}(x) = w_{k_{j-1}\cdots k_1}^{j-1}(y_{k_j}(x)) + k(x, y_{k_j}(x))$$

Put

$$\hat{\tau}_1 = \hat{\tau}_{i-1} + \tilde{\tau}_{n+1-i}(\theta_{\tau_{i-1}}\omega_i), \quad \hat{\tau}_1 = \tilde{\tau}_n(\omega_1)$$

and

$$\hat{\xi}_1 = y_{k_{n+1-i}}(X_{\hat{\tau}_i}(\omega_i))\,.$$

Then $\hat{v} = \{(\tilde{\tau}_i, \hat{\xi}_i)_{i=1,2,\cdots n}, \hat{\tau}_{i+1} = \infty\} \in \underline{\underline{V}}_n$

and

$$w_{k_n\cdots k_1}^n(x) = E_x[\int_0^{\tilde{\tau}_n} e^{-\alpha s} dA_s + e^{-\alpha\tilde{\tau}_n} M_{k_n} w_{k_{n-1}\cdots k_1}^{n-1}(X_{\tau_n})]$$

$$= E_x[\int_0^{\tilde{\tau}_n} e^{-\alpha s} dA_s + e^{-\alpha\tilde{\tau}_n}\{w_{k_{n-1}\cdots k_1}^{n-1}(y_{k_n}(X_{\tilde{\tau}_n})) + k(X_{\tilde{\tau}_n}, y_{k_n}(X_{\tilde{\tau}_n})\}]$$

$$= E_x^1[\int_0^{\hat{\tau}_1} e^{-\alpha s} dA_s + e^{-\alpha\hat{\tau}_1} k(X_{\hat{\tau}_1}, y_{k_n}(X_{\hat{\tau}_1}))$$

$$+e^{-\alpha\hat{\tau}_1}E_{\hat{\xi}_1}^{\hat{}}[\int_0^{\hat{\tau}_{n-1}} e^{-\alpha s}dA_s+e^{-\alpha\tilde{\tau}_{n-1}}M_{k_{n-1}}w_{k_{n-2}\cdots k_1}^{n-2}(X_{\tilde{\tau}_{n-1}})]$$

$$= E_x^{n+1}[\int_0^\infty e^{-\alpha s}d\underline{\underline{A}}_s+\sum_{i=1}^n e^{-\alpha\hat{\tau}_i}k(X_{\hat{\tau}_i}, \hat{\xi}_i)]$$

$$\geqq u_n^*(x) \quad \text{q.e..}$$

Therefore making use of Lemma 5.3 we conclude that

$$w_n(x) \geqq u_n^*(x) \quad \text{q.e..}$$

Proof of Theorem 2. By Propositions 5.1 and 5.2 u_n^* is a quasi-continuous modification of the solution u_n of the variational inequality (1.5) for each n. Since u_n converges to the maximum solution u of the QVI (1.2) in $\mathcal{E}_\alpha$-norm: $\mathcal{E}_\alpha(u_n-u, u_n-u)\to 0$, $n\to\infty$ we have

$$u_n^*(x) \to u(x) \quad \text{q.e., } n \to \infty$$

taking a sub-sequence if necessary. On the other hand it holds that

$$u_n^*(x) \downarrow u^*(x) \quad \text{q.e., } n \to \infty$$

by the next lemma. This completes the proof of Theorem 2.

Lemma 5.5. *It holds that $u_N^*(x) \downarrow u^*(x)$ q.e., $N\to\infty$.*

Proof. For each $\varepsilon>0$ there exists $v=v(x)=\{(\tau_i, \xi_i)_{i=1}^\infty\}\in\underline{\underline{V}}$ such that

$$u^*(x)\geqq\lim_{n\to\infty} E_x^n[\int_0^{\tau_n} e^{-\alpha s}d\underline{\underline{A}}_s+\sum_{i=1}^n e^{-\alpha\tau_i}k(X_{\tau_i}, \xi_i)]-\varepsilon$$

So for any N it holds that

$$u^*(x)\geqq E_x^N[\int_0^{\tau_N} e^{-\alpha s}d\underline{\underline{A}}_s+\sum_{i=1}^N e^{-\alpha\tau_i}k(X_{\tau_i}, \xi_i)]-\varepsilon$$

Put $v^N=\{(\tau_i, \xi_i)_{i=1,2,\cdots N}, \tau_{N+1}=\infty\}$, then $v^N\in\underline{\underline{V}}_N$. Therefore from

$$E_x^{N+1}[\int_0^{\tau_{N+1}} e^{-\alpha s}d\underline{\underline{A}}_s+\sum_{i=1}^{N+1} e^{-\alpha\tau_i}k(X_{\tau_i}, \xi_i)]\geqq u_N^*(x)$$

and $u_N^*(x)\geqq u^*(x)$ it follows that

$$|u_N^*(x)-u^*(x)|\leqq E_x^{N+1}[\int_{\tau_N}^\infty e^{-\alpha s}d\underline{\underline{A}}_s]+2\varepsilon\,.$$

Since $\tau_N\to\infty$ as $N\to\infty$ we obtain

$$\lim_{N\to\infty} u_N^*(x) = u^*(x) \quad \text{q.e..}$$

$u_N^*(x)\geqq u_{N+1}^*(x)$ q.e. is obvious.

References

[1] A. Bensoussan and J.L. Lions: *Nouvelles méthodes en contrôle impulsionnel,* Appl. Math. Optim. **1** (1975), 289–312.

[2] L.A. Caffarelli and A. Friedman: *Regularity of the solution of the quasi-variational inequality for the impulse control problem,* I, II, Comm. Partial Differential Equations **3** (1978), 745–753; **4** (1979), 279–291.

[3] E.B. Dynkin and A.A. Yushkevich: Controlled Markov processes, Springer, 1979.

[4] M. Fukushima: Dirichlet forms and Markov processes, North-Holland, Kodansha, 1980.

[5] M. Matzeu and A. Vivaldi: *On the regular solutions of a non-linear parabolic quasi-variational inequality related to a stochastic control,* Comm. Partial Differential Equations **4** (1979), 1123–1147.

[6] H. Nagai: *On an optimal stopping problem and a variational inequality,* J. Math. Soc. Japan **30** (1978), 303–312.

[7] M. Robin: *Contrôle impulsionnel des processus de Markov,* Thèse de Doctrat L'université Paris IX.

[8] J. Zabczyk: *Stopping games for symmetric Markov processes,* to appear.

Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Fukazawa-cho, Setagaya-ku
Tokyo, Japan