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DONALDSON-FRIEDMAN CONSTRUCTION AND DEFORMATIONS OF A TRIPLE OF COMPACT COMPLEX SPACES

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1. Introduction and statements of the results

Let Z be a three-dimensional complex manifold and M a (real) oriented four-manifold. Z is said to be *a twistor space* of M if there exist a C^∞-map $\pi : Z \to M$ and a fixed point free anti-holomorphic involution $\sigma : Z \to Z$ such that the following conditions are fulfilled:
(1) π gives Z a C^∞ S^2-bundle structure over M. The fiber $L_p := \pi^{-1}(p)$ for any $p \in M$ is a complex submanifold of Z (which is biholomorphic to the complex projective line $\boldsymbol{P}^1$),
(2) σ preserves each L_p and the automorphism on M induced by σ is the identity,
(3) for any $p \in M$, $N_{L_p/Z}$ (= the holomorphic normal bundle of L_p in Z) is isomorphic to $\mathcal{O}(1)^{\oplus 2}$, where $\mathcal{O}(1)$ denotes the line bundle of degree one over L_p.
π is called *the twistor fibration*, σ *the real structure* and L_p *a twistor line*. A complex subspace X on Z is said to be *real* if $\sigma(X) = X$.

A fundamental theorem of Penrose's twistor theory is that there exists a natural one to one correspondence between twistor spaces Z of M and self-dual conformal structures $[g]$ on M [2].

Let $T_{Z/M}$ be the vertical C^∞ tangent bundle with respect to π. The Levi-Civita connection of corresponding self-dual metric g on M naturally induces a holomorphic structure on $T_{Z/M}$. When we regard $T_{Z/M}$ as a holomorphic line bundle with this complex structure, we denote it by $K_Z^{-\frac{1}{2}}$. This is called the *fundamental line bundle* and satisfies $(K_Z^{-\frac{1}{2}})^{\otimes 2} \simeq K_Z^{-1}$ (=the anticanonical bundle of Z) and $\overline{\sigma^* K_Z^{-\frac{1}{2}}} \simeq K_Z^{-\frac{1}{2}}$ (biholomorphically). The complete linear system $|K_Z^{-\frac{1}{2}}|$ is called the *fundamental system* and an element of the fundamental system is called *a fundamental divisor* [19].

Basic examples of compact self-dual manifolds are the Euclidean 4-sphere S^4 and the complex projective plane CP^2 with Fubini-Study metric. Their twistor spaces are the 3-dimensional projective space $\boldsymbol{P}^3$ and some flag manifold $\boldsymbol{F}$ respectively. Hitchin

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[8] showed that these are the only examples of compact Kählarian twistor spaces. Later, Poon [18] discovered twistor spaces of $2CP^2$ (= connected sum of two complex projective planes) whose algebraic dimensions are three, i.e. Moishezon. Then LeBrun [13] constructed such twistor spaces of nCP^2 (= connected sum of n complex projective planes) for any $n \geq 0$.

In the following, we let Z denote a twistor space of nCP^2 ($n \geq 0$) for some self-dual metric g on nCP^2, where $0CP^2$ means S^4 by convention. The fundamental system played important roles to study algebro-geometric structures of Z [8, 18, 13, 19, 14, 15, 12, 9]. In particular, Pedersen-Poon [14] proved that a real irreducible fundamental divisor S on Z is non-singular and can be blown-down to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ preserving the real structure. Moreover, they showed that the resulting real structure τ_0 on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is given by (anti-podal map)$\times$(complex conjugate). (The number of times of the blowing-ups is readily seen to be $2n$.) Then it is natural to ask whether the converse is also true: Let $\mu : S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be a rational surface obtained by $2n$-times blowing-ups preserving the real structure (including the case of infinitely near points). Then does there exist a twistor space Z of nCP^2 which has a real fundamental divisor biholomorphic to S? The purpose of this paper is to obtain partial answers to this problem by a detailed investigation of the construction by Donaldson-Friedman [4]. (See Remark 2.E.) Our main result is the following:

Theorem 1.1. *Let n be any positive integer. Let $C_0 \subseteq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be a real non-singular rational (resp. elliptic) curve of bidegree $(2,1)$ (resp. $(2,2)$). Then there exists a set of n points $\{p_1, \cdots, p_n\}$ on C_0 (which may be infinitely near) with the following property: Let $\overline{p}_i \in C_0$ $(i = 1, \cdots, n)$ be the τ_0 conjugate point of p_i and $\mu : S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ the blowing-up at $\{p_1, \overline{p}_1, \cdots, p_n, \overline{p}_n\}$. Then there exists a twistor space Z of nCP^2 which has a real fundamental divisor biholomorphic to S.*

REMARK 1.A. Strictly speaking, the blown-up points $p_i \in C_0$ cannot be on some closed subset A_0 on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. See Proposition 2.1.

REMARK 1.B. In the following proof of this theorem, we also show that the above (Z, S) satisfies $H^2(\Theta_Z(-S)) = 0$. Hence by the theorem of Horikawa [10], S is costable with respect to deformations of Z: namely for any real small deformation $\hat{S}$ of S, there exists a real small deformation $\hat{Z}$ of the twistor space Z which contains $\hat{S}$ as an element of $|K_{\hat{Z}}^{-\frac{1}{2}}|$

We recall that *an elementary divisor* is by definition a divisor on Z whose intersection number with a twistor line is equal to 1. An interesting property of these twistor spaces is the following:

Proposition 1.2. *Let $C_0 \subseteq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be a real non-singular curve of bidegree $(2,1)$ or $(2,2)$ and $\{p_1, \cdots, p_n\}$ $(n \geq 4)$ be any set of points on C_0 (which may be* *infinitely near). Let $\mu : S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the blowing-up at $\{p_1, \overline{p}_1, \cdots, p_n, \overline{p}_n\}$ and C the proper transform of C_0. Let Z be a twistor space of nCP^2 and assume that Z has S as a real irreducible fundamental divisor. Then Z has no elementary divisor. Further, in case of bidegree $(2,1)$, Z is a Moishezon 3-fold.*

Combining these two results, we obtain a negative answer to the question posed by Pedersen-Poon [15, p.687, Question], which was recently proved by Kreussler [12] using a completely different method:

Corollary 1.3. *For any $n \geq 4$, there exists a Moishezon twistor space of nCP^2 which has no elementary divisors.*

Theorem 1.1 is proved by induction on n, based on the construction of Donaldson-Friedman [4]. It is easy to see that Theorem 1.1 holds for the case $n = 1$. We assume that Theorem 1.1 holds for the case n, i.e. let Z_1 be a twistor space of nCP^2 which has a real irreducible fundamental divisor $S_1 = S$ as in Theorem 1.1 and let $C_1 \subseteq S_1$ be the proper transform of C_0. In Section 2, using (Z_1, S_1, C_1) we construct a triple of real normal crossing varieties $Z' \supseteq S' \supseteq C'$, where S' (resp. C') is a real Cartier divisor on Z' (resp. S'). Then we shall state four propositions (2.2–2.5) which are necessary to prove Theorem 1.1. Then we will prove Proposition 1.2. In Section 3, we study deformations of the pair (S', C') and prove Propositions 2.2 and 2.3. Next in Section 4, which is the main part of this paper, we prove Propositions 2.4 and 2.5: i.e. we study deformations of the triple (Z', S', C') and show that the triple (Z', S', C') can always be smoothed to give a twistor space of $(n+1)CP^2$ of the desired type. To this end, we need a deformation theory of a triple of compact complex spaces. In the final section, we shall develop this as a natural generalization of the theory of Ran [20] and obtain natural long exact sequences containing forgetting maps.

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2. The problem and the main construction

Let $S_0 := \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the product of two complex projective lines, $(z_0 : z_1)$ (resp. $(w_0 : w_1)$) homogeneous coordinates on the first (resp. second) factor, and $\pi_i (i = 1, 2)$ the projection to the i-th factor. We have $H^2(S_0, \boldsymbol{Z}) \simeq \mathrm{Pic} S_0 \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$. Let $\mathcal{O}(a,b) := \pi_1^* \mathcal{O}(a) \otimes \pi_2^* \mathcal{O}(b)$ denote a holomorphic line bundle over S_0, where $\mathcal{O}(k)$ denotes the holomorphic line bundle over $\boldsymbol{P}^1$ of degree k. Let τ_0 denote the real structure on S_0 which is defined by $((z_0 : z_1), (w_0 : w_1)) \mapsto ((-\overline{z_1} : \overline{z_0}), (\overline{w_0} : \overline{w_1}))$. τ_0 has no fixed points. Let $S^1 := \{(w_0 : w_1) \in \boldsymbol{P}^1 \mid w_0, w_1 \in \boldsymbol{R}\} \subseteq \boldsymbol{P}^1$ be the real circle of the second factor and set $A_0 := p_2^{-1}(S^1) \simeq \boldsymbol{P}^1 \times S^1$.

Let Z be a twistor space of nCP^2, $\pi : Z \to nCP^2$ the twistor fibration and σ the real structure on Z. Let $K_Z^{-\frac{1}{2}}$ be the fundamental line bundle over Z. Then Pedersen and Poon [14] proved the following:

Proposition 2.1. *Let $S \in |K_Z^{-\frac{1}{2}}|$ be a real irreducible fundamental divisor. Then S is non-singular and can be blown-down to S_0 preserving the real structure. The resulting real structure on S_0 is τ_0. Let $\mu : S \to S_0$ be such a blowing-down map. Then the blown-up points on S_0 are never on A_0. Further, $A := \mu^{-1}(A_0)$ is the set of twistor lines on S, which are parameterized by S^1.*

We recall that since $c_1^2(S) = -\frac{1}{8}c_1^3(Z) = 8 - 2n$ [8], the number of times of blowing-ups is necessarily $2n$. We are interested in the quenstion as to whether the converse is also true:

Question 2.A. [1] For any given $2n$-times blowing-up $\mu : S \to S_0$ preserving the real structure, with blown-up points not lying on A_0, does there exist a twistor space Z of nCP^2 which has a real fundamental divisor biholomorphic to S? [2]

Though we could not give a complete answer to this question (cf. Remark 2.E below), we prove the existence of Z for special types of S. To state our result precisely, we introduce the following:

DEFINITION 2.B. We say that a non-singular rational surface S is *of type* (a, b) if S is obtained by blowing-up S_0 and if any blown-up points on S_0 are on one and the same irreducible curve $C_0 \in |\mathcal{O}(a, b)|$. (Some or all of the blown-up points are allowed to coincide, but in such cases the iterated blown-ups are required on the proper transforms of C_0.) We call such S *real* if the above blowing-ups preserve τ_0, C_0 is real on S_0, and if the blown-up points do not lie on A_0.

(Although this definition does not uniquely determine the type of a given rational surface, it is sufficient for our purpose.)

Next we will explain our main construction to prove Theorem 1.1. The notations given in the following construction will also be used throughout Sections 2,3 and 4.

Let Z_1 be a twistor space of nCP^2 and assume that there exists a real irreducible fundamental divisor $S_1 \in |K_{Z_1}^{-\frac{1}{2}}|$ whose type is $(2, 1)$ or $(2, 2)$. Let $\mu : S_1 \to S_0$ be a blowing-down map as in Definition B and $C_1 \subseteq S_1$ the proper transform of C_0 .

[1] From the differential geometric point of view, this is equivalent to the following: *Let $M^\circ = \boldsymbol{P}^1 \times H^2$ (H^2 denotes the upper half plane) and $p_1, \cdots, p_n$ a set of n points on M°, which may be infinitely near. Let $\tilde{M}^\circ$ be the complex surface obtained by blowing-up $\{p_1, \cdots, p_n\}$. Then does there exist a scalar-flat Kähler metric on $\tilde{M}^\circ$ which is asymptotically isometric to the standard metric on M°?*

[2] Of course, the biholomorphic map is required to preserve the real structure.

Let $p_1 \in C_1$ be any point not lying on $A_1 := \mu^{-1}(A_0)$. Then there exists a unique twistor line $L_1 \subseteq Z_1$ through p_1 and $\overline{p}_1$. Let $\sigma_1 : Z_1' \to Z_1$ be the blowing-up along L_1 and $Q_1 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ the exceptional divisor. Let S_1' and C_1' be the proper transforms of S_1 and C_1 respectively and set $l_1 := \sigma_1^{-1}(p_1)$ and $\overline{l}_1 := \sigma_1^{-1}(\overline{p}_1)$. Then since S_1 intersect L_1 transversally at p_1 and $\overline{p}_1$, $\sigma_1|_{S_1'} : S_1' \to S_1$ is the blowing-ups at $\{p_1, \overline{p}_1\}$ and $\{l_1, \overline{l}_1\}$ are the exceptional curves. Further, we set $p_1' := l_1 \cap C_1'$ and $\overline{p}_1' := \overline{l}_1 \cap C_1'$.

On the other hand, let $Z_2 := \boldsymbol{F} := \{(x, l) \in \boldsymbol{P}^2 \times \boldsymbol{P}^{2^*} \mid x \in l\}$ ($\boldsymbol{P}^{2^*}$ denotes the dual projective plane) be a flag manifold, which is the twistor space of $\boldsymbol{C}\boldsymbol{P}^2$ with Fubini-Study metric. We fix any twistor line $L_2 \subseteq Z_2$ and let $\{D_2, \overline{D}_2\}$ be the (unique) pair of elementary divisors on Z_2 such that $D_2 \cap \overline{D}_2 = L_2$ (transversal). It is easy to see $D_2 \simeq \Sigma_1 \simeq \overline{D}_2$, where $\Sigma_1 := \boldsymbol{P}(\mathcal{O}(1) \oplus \mathcal{O})$ is the non-minimal Hirzebruch surface. Let $\sigma_2 : Z_2' \to Z_2$ be the blowing-up along L_2, Q_2 the exceptional divisor and $D_2'(\simeq D_2), \overline{D}_2'(\simeq \overline{D}_2)$ the proper transforms of $D_2, \overline{D}_2$ respectively. D_2' and $\overline{D}_2'$ are disjoint and these define disjoint sections l_2 and $\overline{l}_2$ of $\sigma_2|_{Q_2} : Q_2 \to L_2$ respectively.

Next, let $\phi : Q_1 \to Q_2$ be a biholomorphic map preserving the real structures such that $\phi(l_1) = l_2$ and $\phi(\overline{l}_1) = \overline{l}_2$. (The existence of such an isomorphism is clear.)

Then following Donaldson-Friedman [4] and Kim-Pontecorvo [11], we set

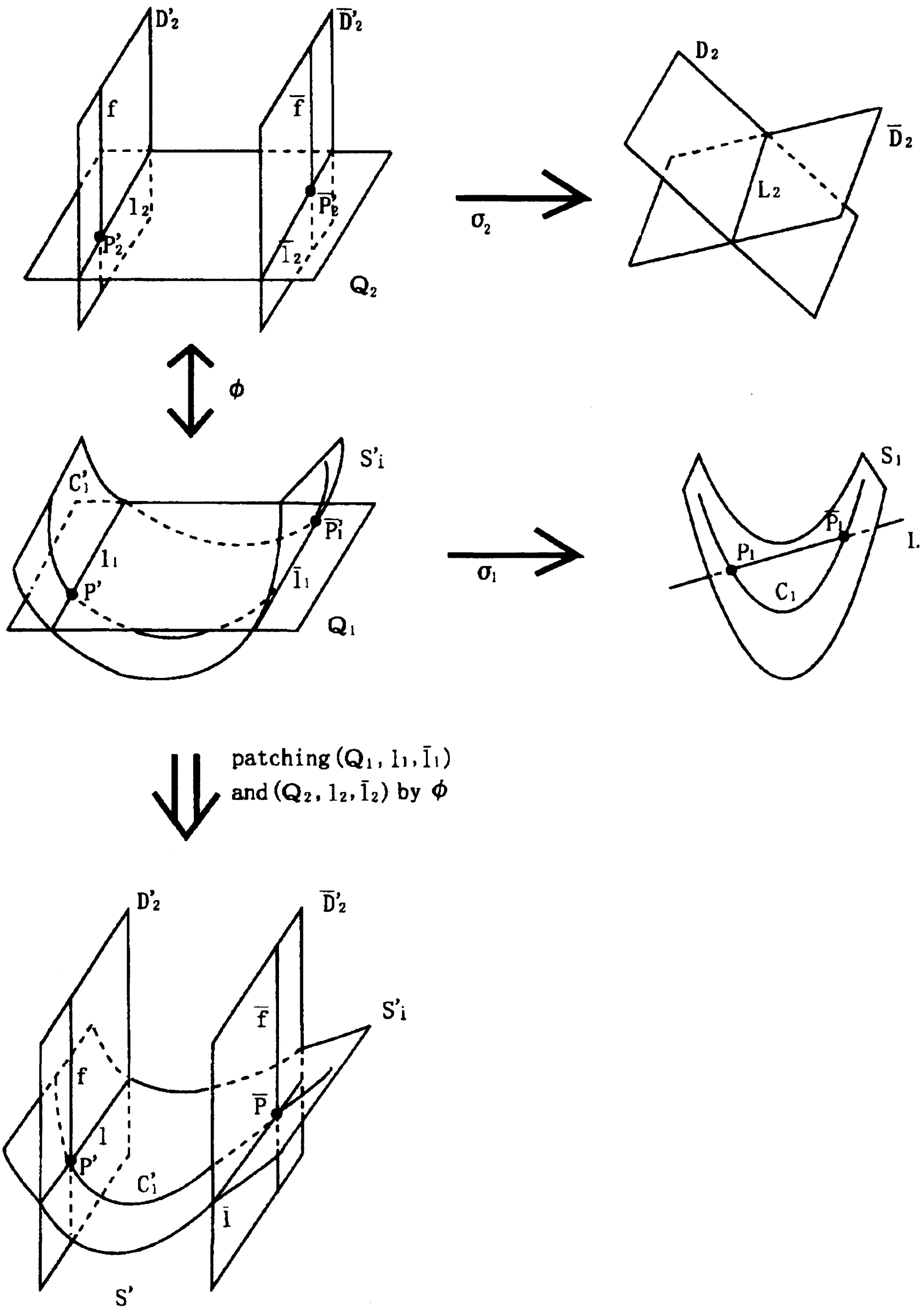
$$Z' := Z_1' \cup_Q Z_2',$$
$$S' := S_1' \cup_{l,\overline{l}} (D_2' \amalg \overline{D}_2') = D_2' \cup_l S_1' \cup_{\overline{l}} \overline{D}_2'.$$

These are normal crossing varieties obtained by identifying Q_1 and Q_2, $\{l_1, \overline{l}_1\}$ and $\{l_2, \overline{l}_2\}$ respectively by using ϕ and $\phi|_{l_1 \amalg \overline{l}_1}$. (Hence we denote $Q := Q_i, l := l_i$ and $\overline{l} := \overline{l}_i$ $(i = 1, 2)$.) By construction, Z' has a real structure and it preserves S'. S' is a Cartier divisor on Z'. We further proceed as follows: We put $\phi(p_1') =: p_2' \in l_2$ and $\phi(\overline{p}_1') =: \overline{p}_2' \in \overline{l}_2$ and let $f \ni p_2'$ (resp. $\overline{f} \ni \overline{p}_2'$) be the fibers of $D_2' \to l_2$ (resp. $\overline{D}_2' \to \overline{l}_2$) through p_2' (resp. $\overline{p}_2'$). Then we set

$$C' := C_1' \underset{p', \overline{p}'}{\cup} (f \amalg \overline{f}) = f \underset{p'}{\cup} C_1' \underset{\overline{p}'}{\cup} \overline{f}.$$

It is obvious that C' is a Cartier divisor on S' and is preserved by the real structure on S'. Thus we obtained a triple of normal crossing varieties $Z' \supseteq S' \supseteq C'$ which has a real structure.

As for deformations of the pair (S', C'), we have the following two propositions which will be proved in Section 3:


D'_2
$\bar{D}'_2$
f
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P'_2
$\bar{P}'_2$
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Q_2
σ_2
D_2
$\bar{D}_2$
L_2
ϕ
S'_1
C'_1
$\bar{P}'_1$
l_1
P'
$\bar{l}_1$
Q_1
σ_1
S_1
$\bar{P}_1$
P_1
L_1
C_1
patching $(Q_1, l_1, \bar{l}_1)$
and $(Q_2, l_2, \bar{l}_2)$ by ϕ
D'_2
$\bar{D}'_2$
S'_1
$\bar{f}$
f
$\bar{P}$
l
P'
C'_1
$\bar{l}$
S'

Proposition 2.2. *Let (S', C') be the pair of normal crossing varieties as above. Then the Kuranishi family of deformations of the pair (S', C') is unobstructed.*

Proposition 2.3. *Let $\{\mathcal{S} \to B', \mathcal{C} \to B'$ with $\mathcal{C} \hookrightarrow \mathcal{S}\}$ be the Kuranishi family of the pair (S', C'), where B' denotes a sufficiently small open ball in $\boldsymbol{T}^1_{S',C'}$ containing* 0.

If $t \in B'$ is away from some hypersurface in B' through 0, *then the following hold:*

(1) *the fibers S_t of the Kuranishi family are rational surfaces of type* $(2,1)$ *or* $(2,2)$ *according as S_1 is of type* $(2,1)$ *or* $(2,2)$ *respectively,*

(2) $c_1^2(S_t) = 8 - 2(n+1)$,

(3) *moreover if $t \in B'$ is real with respect to the real structure on $\boldsymbol{T}^1_{S',C'}$, then S_t is real in the sense of Definition B.*

(See Section 3 for the notation $\boldsymbol{T}^1_{S',C'}$ and real structure on it.)

The following condition is necessary for our induction proof of Theorem 1.1 to work.

Condition 2.C. $H^2(Z_1, \Theta_{Z_1}(-S_1)) = 0$.

As for deformations of the triple (Z', S', C'), we have the following two propositions which will be proved in Section 4:

Proposition 2.4. *Let (Z_1, S_1, C_1) be as above and assume that Condition 2.C is satisfied. Let (Z', S', C') be the triple of normal crossing varieties constructed as above. Then the Kuranishi family of deformations of the triple (Z', S', C') is unobstructed.*

Proposition 2.5. *Let $\{\mathcal{Z} \to B, \mathcal{S} \to B, \mathcal{C} \to B$ with $\mathcal{C} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{Z}\}$ be the Kuranishi family of the triple (Z', S', C'), where B is a sufficiently small open ball in $\boldsymbol{T}^1_{Z',S',C'}$ containing* 0. *If $t \in B$ is away from some hypersurface in B through* 0, *then* (1) *below holds. In addition, if such a t is real, then* (2) *and* (3) *below also hold.*

(1) *Z_t, S_t and C_t are non-singular complex manifolds.*

(2) *Z_t has a real structure σ_t and (Z_t, σ_t) has a structure of a twistor space of* $(n+1)\boldsymbol{CP}^2$.

(3) *S_t is a real fundamental divisor on Z_t and they satisfy* $H^2(Z_t, \Theta_{Z_t}(-S_t)) = 0$.

(See Section 5 for the notation $\boldsymbol{T}^1_{Z',S',C'}$ and also for the Kuranishi family and the obstruction of deformations of a triple of compact complex spaces.)

Theorem 1.1 are proved by combining these four propositions with the aid of some observations.

REMARK 2.D. Without considering the curve C_1, we can show the following:

Let Z_1 be a twistor space of nCP^2 with real irreducible fundamental divisor S_1. Let $p \in S_1$ be any points not lying on A_1 ($=$ the set of twistor lines on S_1). Let $\mu : S_1' \to S_1$ be the blowing-ups at p and $\overline{p}$. Then there exists a twistor space Z of $(n+1)CP^2$ with real irreducible fundamental divisor S which is a small deformation *of S_1'.*

The proof of this statement is lengthy but along the same line as that of [11]. So we omit it. But the author could not prove the stronger version of the above statement in the sense that the structure of S is *biholomorphic to* S_1'. (Of course this immediately gives the completely affirmative answer to Question 2.A.) This is the reason why we consider the curve C_1.

REMARK 2.E. (1) The complex structure of Z is not uniquely determined even if the complex structure of fundamental divisor is given, i.e. there is a continuous family of twistor spaces with biholomorphic fundamental divisors, whereas the rough geometric structure of Z is determined.
(2) Let Z be a twistor space of nCP^2 constructed by LeBrun [13]. Then Z has a real irreducible fundamental divisor S of type $(0,2)$ except that C_0 in Definition B is reducible. Conversely, if Z is a twistor space of nCP^2 which has a real irreducible fundamental divisor of type $(0,2)$, we can easily show that Z is necessarily a LeBrun twistor space. (In fact, one can use the argument of the proof of Proposition 1.2 below to reduce to a result of Poon [19, Theorem 3.1].)
(3) Let Z be a twistor space of nCP^2 constructed by Pedersen-Poon [16, Section 7]. Z has an effective action of the 2-dimensional torus T^2. We showed [9] that the T^2-equivariant part of the fundamental system on Z is a pencil and that generic elements of this pencil are non-singular toric surfaces obtained by T^2-equivariant blowing-up of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Hence Question 2.A is affirmative for some toric surfaces. We may prove this result also by the method developed in the present paper (i.e. consider the triple containing the cycle of rational curves), instead of using the T^2-action.
(4) The existence of Z over nCP^2 which has a real irreducible fundamental divisor S of type (2,1) was very recently proved by Kreussler [12] using very different method. But the author does not know whether or not twistor spaces constructed in this paper are biholomoprphic to those in [12].

REMARK 2.F. Let Z be a twistor space of nCP^2 which has an effective divisor. Then the scalar curvature of the self-dual metric on nCP^2 corresponding to Z is of positive type. In fact, if the scalar curvature is of negative type, there is no effective divisor on Z [5, Théorème 2]. Further, if the scalar curvature is of type 0, nCP^2 must be covered by a scalar-flat Kähler surface [17, Corollary 4.3]. But since nCP^2 is simply connected, nCP^2 with the complex orientation reversed must admit a scalar-flat Kähler metric, which is impossible since the intersection form of the 4-manifold is negative definite. Thus, the scalar curvatures of each self-dual metric of nCP^2 which we treat is of positive type. So we may use the vanishing theorem of Hitchin [7] for our twistor spaces.

Proof of Proposition 1.2. Let Z, $\mu : S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $C_0 \subseteq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and C $(\subseteq S)$ be as in Proposition 1.2. Following [14], we decompose $S = (\boldsymbol{P}^1 \times H^+)\tilde{}\amalg A \amalg (\boldsymbol{P}^1 \times H^-)\tilde{}$. (Here, $(\boldsymbol{P}^1 \times H^\pm)\tilde{}$ denote n points blowing-ups of $\boldsymbol{P}^1 \times H^\pm$, where $H^\pm$ denote the upper and lower half planes. When $n = 0$, this decomposition is given as follows: $S_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1 = \boldsymbol{P}^1 \times (H^+ \amalg S^1 \amalg H^-) = (\boldsymbol{P}^1 \times H^+) \amalg (\boldsymbol{P}^1 \times S^1) \amalg (\boldsymbol{P}^1 \times H^-)$. The case $n \geq 1$ is similar.)

Let $\{E_1, \cdots, E_n, \overline{E}_1, \cdots, \overline{E}_n\}$ be the exceptional curves of μ with $E_i \cdot E_j = -\delta_{ij}$, $1 \leq i, j \leq n$, where we may assume $E_i \subseteq (\boldsymbol{P}^1 \times H^+)\tilde{}$ and $\overline{E}_i \subseteq (\boldsymbol{P}^1 \times H^-)\tilde{}$. Further, by using the twistor fibration, we regard $(\boldsymbol{P}^1 \times H^\pm)\tilde{}$ (with the complex orientation reversed) as a subset in nCP^2 (whose complement is S^1) and set $\alpha_i := [E_i] \in H^2(nCP^2, \boldsymbol{Z})$ for $1 \leq i \leq n$. Then $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal basis of $H^2(nCP^2, \boldsymbol{Z})$.

Let $\mathcal{D}$ be a holomorphic line bundle on Z whose restriction to a twistor line has degree 1. If $\mathcal{D}$ has a non-zero section, the first Chern class of $\mathcal{D}$ must satisfy [19, Lemma 1.9]

$$c_1(\mathcal{D}) = \frac{1}{4}c_1(Z) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i \alpha_i, \quad \sigma_i = \pm 1.$$

Then considering the structure of $\pi|_S : S \to nCP^2$ [14] (π denotes the twistor fibraiton), we have

$$\begin{aligned}
\mathcal{D}|_S &= \frac{1}{4}c_1(Z)|_S + \frac{1}{2}\sum_{i=1}^{n} \sigma_i \alpha_i|_S \\
&= -\frac{1}{2}K_S + \frac{1}{2}\sum_{i=1}^{n} \sigma_i (E_i - \overline{E}_i) \\
&= -\frac{1}{2}(\mu^*\mathcal{O}(-2,-2) + \sum_{i=1}^{n} E_i + \sum_{i=1}^{n} \overline{E}_i) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i (E_i - \overline{E}_i) \\
&= \mu^*\mathcal{O}(1,1) - \frac{1}{2}\sum_{i=1}^{n}(1 - \sigma_i)E_i - \frac{1}{2}\sum_{i=1}^{n}(1 + \sigma_i)\overline{E}_i,
\end{aligned} \tag{1}$$

where $\mu : S \to S_0$ denotes the prescribed blowing-down map.

Now we claim that $H^i(\mathcal{D} \otimes K_Z^{\frac{1}{2}}) = 0$ for any $i \geq 0$. First, since $\mathcal{D} \otimes K_Z^{\frac{1}{2}}|_L \simeq \mathcal{D}|_L \otimes K_Z^{\frac{1}{2}}|_L \simeq \mathcal{O}_L(1) \otimes \mathcal{O}_L(-2) \simeq \mathcal{O}_L(-1)$, we have $H^0(\mathcal{D} \otimes K_Z^{\frac{1}{2}}) = 0$. Next, by Serre duality, we have $H^i(\mathcal{D} \otimes K_Z^{\frac{1}{2}}) \simeq H^{3-i}(\mathcal{D}^{-1} \otimes K_Z^{\frac{1}{2}})$ and the right-hand-side vanishes for $3-i=1$ (by the Hitchin vanishing) and $3-i=0$. Finally, by Riemann-Roch [8], we have

$$\chi(\mathcal{D}\otimes K_Z^{\frac{1}{2}})=0,$$

and hence we also have $H^1(\mathcal{D}\otimes K_Z^{\frac{1}{2}})=0$.

Therefore by the cohomology exact sequence of

$$0\to\mathcal{D}\otimes K_Z^{\frac{1}{2}}\to\mathcal{D}\to\mathcal{D}|_S\to 0,$$

we have

$$H^i(\mathcal{D})\simeq H^i(\mathcal{D}|_S)\quad\text{for any}\quad i\geq 0. \tag{2}$$

Then in our situation, since we obtain S by blowing-up $\boldsymbol{P}^1\times\boldsymbol{P}^1$ on the non-singular curve C_0 of type (2,1) or (2,2), $H^0(\mathcal{D}|_S)=0$ by the formula (1) and the assumption that $n\geq 4$. Thus by (2) for $i=0$, we conclude that there exists no elemetary divisor.

For the Moishezon part, we refer to [12]. $\square$

3. Deformations of the pair

The purpose of this section is to prove Propositions 2.2 and 2.3. Although these propositions contain statements for the two types of rational surfaces (of type (2,1) and (2,2)), no independent treatment will be needed except the proofs of Lemma 3.3 and Proposition 2.3. We use the following notations:

For a compact complex space X,

Ω_X : the sheaf of Kähler differentials on X,
$\Theta_X^p := \mathcal{E}xt^p_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$: the p-th local Ext-sheaf,
$\Theta_X := \Theta_X^0 = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$: the tangent sheaf,
$\boldsymbol{T}_X^p := \operatorname{Ext}^p_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$: the p-th global Ext-group,

and for a Cartier divisor Y on X,

$\Omega_{X,Y} := \{\alpha\in\Omega_X\otimes\mathcal{O}_X(Y)\mid f\cdot d\alpha\in\Omega_X$ (f is a local defining equation of Y in X)$\}$, i.e. $\Omega_{X,Y}$ is the sheaf of Kähler differentials on X which have at worst logarithemic poles along Y,
$\Theta_{X,Y}^p := \mathcal{E}xt^p_{\mathcal{O}_X}(\Omega_{X,Y},\mathcal{O}_X)$,
$\Theta_{X,Y} := \Theta_{X,Y}^0$,
$\boldsymbol{T}_{X,Y}^p := \operatorname{Ext}^p_{\mathcal{O}_X}(\Omega_{X,Y},\mathcal{O}_X)$.

As for deformations, $\boldsymbol{T}_X^1$ (resp. $\boldsymbol{T}_{X,Y}^1$) is the Zariski tangent space of the Kuranishi family of deformations of X (resp. deformations of the pair (X,Y)) and $\boldsymbol{T}_X^2$

(resp. $\boldsymbol{T}^2_{X,Y}$) is the obstruction space. In particular, if $\boldsymbol{T}^2_X$ (resp. $\boldsymbol{T}^2_{X,Y}$) vanishes, the parameter space of the Kuranishi family is non-singular and the tangent space at the reference point is naturally identified with $\boldsymbol{T}^1_X$ (resp. $\boldsymbol{T}^1_{X,Y}$). Moreover, we have the local to global spectral sequences:

$$E_2^{p,q} := H^p(X, \Theta^q_X) \Rightarrow \boldsymbol{T}^{p+q}_X,$$
$$E_2^{p,q} := H^p(X, \Theta^q_{X,Y}) \Rightarrow \boldsymbol{T}^{p+q}_{X,Y}.$$

Finally, if $\mathcal{F}$ is any $\mathcal{O}_X$-module, we set $\mathcal{F}(\pm Y) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\pm Y)$ and $\mathcal{F}|_Y := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, where $\mathcal{O}_X(\pm Y)$ denote the invertible sheaves defined by $\pm Y$.

Througout this section, (S', C') denotes the pair of normal crossing varieties as in Proposition 2.2.

Proposition 3.1. *We have $H^2(\Theta_{S',C'}) = 0$ and $\boldsymbol{T}^2_{S',C'} = 0$. In particular, the Kuranishi family of deformations of the pair (S', C') is unobstructed.*

To prove this, we need the following two lemmas:

Lemma 3.2. $\Theta^i_{S',C'} \simeq \begin{cases} \mathcal{O}_l \oplus \mathcal{O}_{\bar{l}} & \text{for } i = 1, \\ 0 & \text{for } i \geq 2. \end{cases}$

Lemma 3.3. $H^2(\Theta_{S'_1,C'_1}) = 0.$

Lemma 3.2 can be proved almost in the same way as in Lemma 5.3 in [9]. So we omit it. The proof of Lemma 3.3 will be given after that of Proposition 3.1.

Proof of Proposition 3.1. For the first one, we consider the exact sequence

$$\begin{aligned} 0 \to \Theta_{S',C'} \to \nu_*(\Theta_{S'_1,C'_1+l+\bar{l}} \oplus \Theta_{D'_2,f+l} \oplus \Theta_{\overline{D}'_2,\bar{f}+\bar{l}}) \\ \to \Theta_l(-p') \oplus \Theta_{\bar{l}}(-\bar{p}') \to 0, \end{aligned} \tag{3}$$

where ν denotes the normalization of S'. It is easy to check that the cohomology of the middle term is isomorphic to the direct sum of each cohomology group. Further, since $D'_2 \simeq \Sigma_1$ and l (resp. f) is a (+1)-section (resp. a fiber) of $\Sigma_1 \to \boldsymbol{P}^1$, it is easy to see that $H^2(\Theta_{D'_2,f+l}) = 0 = H^2(\Theta_{\overline{D}'_2,\bar{f}+\bar{l}})$. Then by the cohomology exact sequence of (3) and the fact that $H^i(\Theta_l(-p')) \simeq H^i(\mathcal{O}_l(1)) = 0$, we have

$$H^2(\Theta_{S',C'}) \simeq H^2(\Theta_{S'_1,C'_1+l+\bar{l}}). \tag{4}$$

But we have $H^2(\Theta_{S'_1,C'_1+l+\bar{l}}) \simeq H^2(\Theta_{S'_1,C'_1})$, as is easily shown by the exact sequence

$$0 \to \Theta_{S'_1,C'_1+l+\bar{l}} \to \Theta_{S'_1,C'_1} \to \mathcal{O}_l(-1) \oplus \mathcal{O}_{\bar{l}}(-1) \to 0. \tag{5}$$

(We recall that l and $\bar{l}$ are (-1)-curves on S'_1 and they intersect C'_1 transversally.) Hence, using Lemma 3.3, we get

$$H^2(\Theta_{S',C'}) = 0. \tag{6}$$

Next we show that $\boldsymbol{T}^2_{S',C'} = 0$. Since $\Theta^i_{S',C'} = 0$ for $i \geq 2$ (Lemma 3.2), the local to global spectral sequence associated to $\boldsymbol{T}^i_{S',C'}$ induces a long exact sequence

$$\begin{aligned} 0 \to H^1(\Theta_{S',C'}) \to \boldsymbol{T}^1_{S',C'} \to H^0(\Theta^1_{S',C'}) \to H^2(\Theta_{S',C'}) \\ \to \boldsymbol{T}^2_{S',C'} \to H^1(\Theta^1_{S',C'}) \to \cdots . \end{aligned} \tag{7}$$

Now that we have $\Theta^1_{S',C'} \simeq \mathcal{O}_l \oplus \mathcal{O}_{\bar{l}}$ (Lemma 3.2), $H^1(\Theta^1_{S',C'}) = 0$. Hence by (6) and (7), we get $\boldsymbol{T}^2_{S',C'}$=0, as claimed. □

Proof of Lemma 3.3. We recall that S'_1 is a rational surface of type (2,1) (resp.(2,2)). Then it is easy to see that $C'_1 + L$ (resp. C'_1) is an anticanonical element on S'_1, where L is any irreducible curve of bidegree (0,1). Hence we have

$$\begin{aligned} H^2(\Theta_{S'_1}(-C'_1)) &\simeq H^0(\Omega_{S'_1} \otimes K_{S'_1} \otimes \mathcal{O}_{S'_1}(C'_1))^* \\ &\simeq H^0(\Omega_{S'_1}(-L))^* \quad (\text{resp. } H^0(\Omega_{S'_1})^*) \\ &= 0 \end{aligned}$$

by Serre duality and the rationality of S'_1. (Here, for a vector space V, V^* denotes the dual vector space of V.) Then the claim immediately follows from the cohomology sequence of

$$0 \to \Theta_{S'_1}(-C'_1) \to \Theta_{S'_1,C'_1} \to \Theta_{C'_1} \to 0.$$ □

Let $\{\mathcal{S} \to B', \mathcal{C} \to B'$ with $\mathcal{C} \hookrightarrow \mathcal{S}\}$ be the Kuranishi family of the pair (S', C'). By Proposition 3.1, B' can be regarded as a small open ball in $\boldsymbol{T}^1_{S',C'}$ containing 0. By (7) and Lemma 3.2, we have the following exact sequence

$$0 \to H^1(\Theta_{S',C'}) \to \boldsymbol{T}^1_{S',C'} \stackrel{(e,\bar{e})}{\to} H^0(\mathcal{O}_l \oplus \mathcal{O}_{\bar{l}}) \to 0. \tag{8}$$

We note that the real structure on $H^0(\mathcal{O}_l \oplus \mathcal{O}_{\bar{l}}) \simeq \boldsymbol{C} \oplus \boldsymbol{C}$ is given by $(z,w) \mapsto (\overline{w}, \overline{z})$ for appropriate coordinates.

Proof of Proposition 2.3. Let $\eta \in B' \subseteq \boldsymbol{T}^1_{S',C'}$ be any element such that $e(\eta) \neq 0$ and $e(\overline{\eta}) \neq 0$. Let Δ' be any non-singular holomorphic curve in $\boldsymbol{T}^1_{S',C'}$ through 0 whose tangent vector at 0 is η and let $\{\tilde{\mathcal{S}}|_{\Delta'} \to \Delta', \tilde{\mathcal{C}}|_{\Delta'} \to \Delta'\}$ be the restriction on Δ' of the Kuranishi family of the pair (S', C'). What we have to prove is that if we choose Δ' sufficiently small, any fiber (S_t, C_t), $t \in \Delta' - 0$ of this family is a rational surface of type (2,1) (resp. (2,2)) whose Chern number satisfies $c_1^2(S_t) = 8 - 2(n+1)$. First, the smoothness of S_t is obvious by the choice of η. Next, we consider the commutative diagram

$$\begin{array}{ccc} \boldsymbol{T}^1_{S',C'} & \to & \boldsymbol{T}^1_{C'} \\ \downarrow & & \downarrow \\ H^0(\Theta^1_{S',C'}) & \stackrel{\alpha}{\to} & H^0(\Theta^1_{C'}) \end{array}$$

where the horizontal arrows are natural forgetting maps and the vertical arrows are the maps induced by the local to global spectral sequence. It is easy to see $\Theta^1_{C'} \simeq \boldsymbol{C}_p \oplus \boldsymbol{C}_{\overline{p}}$, where $\boldsymbol{C}_p$ (resp. $\boldsymbol{C}_{\overline{p}}$) denotes the sheaf whose support is p (resp. $\overline{p}$) and the stalk at p (resp. $\overline{p}$) is $\boldsymbol{C}$. It is clear that α is isomorphic. Hence, C' is automatically smoothed if S' is, and C_t $(t \in \Delta' - 0)$ is a non-singular rational (resp. elliptic) curve since both f and $\overline{f}$ are non-singular rational curves.

By the assumption on S_1, we have a birational morphism $\mu_1 : S_1 \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$. We set $\mu := \mu_1 \cdot \sigma_1|_{S_1'} : S_1' \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and let m_0 and m_0' be any curves of bidegree (1,0) and (0,1) respectively which do not pass through the blown-up points of μ. Let m, m' be the proper transforms of m_0, m_0' respectively. Then it is clear that m and m' are stable under any small deformations of S' (in the sense of Kodaira), and that the self-intersection number of each curve are unchanged. Let $m_t, m_t' \subseteq S_t$ $(t \neq 0)$ be such preserved curves. Then it is obvious that the map $\mu_t := \Phi_{|m_t|} \times \Phi_{|m_t'|} : S_t \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is a birational morphism and that $\mu_t(C_t)$ is an element of $|\mathcal{O}(2,1)|$ (resp. $|\mathcal{O}(2,2)|$).

Now we claim that $C_t^2 = 4 - 2(n+1)$ (resp. $8 - 2(n+1)$). To show this, we consider the exact sequence

$$0 \to \mathcal{O}_{S'}(C') \to \nu_*(\mathcal{O}_{S_1'}(C_1') \oplus \mathcal{O}_{D_2'}(f) \oplus \mathcal{O}_{\overline{D}_2'}(\overline{f})) \to \mathcal{O}_l(p') \oplus \mathcal{O}_{\overline{l}}(\overline{p}') \to 0,$$

from which it immediately follows that $H^i(\mathcal{O}_{S'}(C')) \simeq H^i(\mathcal{O}_{S_1'}(C_1'))$ for any $i \geq 0$. Further, it is easy to check from Riemann-Roch that $\chi(\mathcal{O}_{S_1'}(C_1')) = 4 - 2n$ (resp. $7 - 2n$). Hence we have $\chi(\mathcal{O}_{S'}(C')) = 4 - 2n$ (resp. $7 - 2n$). Then by the invariance of the Euler characteristic under flat deformations, we have $\chi(\mathcal{O}_{S_t}(C_t)) = 4 - 2n$ (resp. $7 - 2n$). Therefore, by the exact sequence

$$0 \to \mathcal{O}_{S_t} \to \mathcal{O}_{S_t}(C_t) \to \mathcal{O}_{C_t}(C_t) \to 0,$$

we have

$$\begin{aligned}\chi(\mathcal{O}_{C_t}(C_t)) &= \chi(\mathcal{O}_{S_t}(C_t)) - \chi(\mathcal{O}_{S_t}) \\ &= (4-2n)-1 \quad (\text{resp. } (7-2n)-1) \\ &= 3-2n \quad (\text{resp. } 6-2n)\end{aligned}$$

Then by Riemann-Roch on C_t, we have

$$\begin{aligned}C_t^2 &= \chi(\mathcal{O}_{C_t}(C_t)) - 1 + g(C_t) \\ &= 2-2n \quad (\text{resp. } 6-2n),\end{aligned} \tag{9}$$

where $g(C_t)$ denotes the genus of C_t. Hence the claim follows.

Next we claim that $c_1^2(S_t) = 8-2(n+1)$. To see this, following [11], let $\beta : \mathcal{S} \to S_1' \times \boldsymbol{C}$ be the blowing-up with center $(l_1 \times 0) \amalg (\bar{l}_1 \times 0)$ and we regard $\mathcal{S}$ as a family over $\boldsymbol{C}$ in a natural manner. The fiber over 0 is biholomorphic to S' and all the other fibers are biholomorphic to S_1'. Thus, S' can be smoothed to S_1'. It is clear that the Chern number of the above S_t $(t \in \Delta' - 0)$ is equal to that of S_1'. But it is obvious that $c_1^2(S_1') = 8-2(n+1)$ and hence we have proved the last claim.

Therefore with the aid of (9) and the fact that $\mu_t(C_t) \subseteq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is bidegree (2,1) (resp. (2,2)), we have shown that S_t $(t \in \Delta' - 0)$ is a rational surface of type (2,1) (resp. (2,2)).

Finally, the claim for the real structure can be proved by using the same technique of Donaldson-Friedman [4, pp. 225–226]. □

4. Deformations of the triple

The purpose of this section is to study deformations of the triple (Z', S', C') which is constructed in Section 2, depending on the results of Section 5. We shall freely use the notations in Section 2.

First we will prove the following proposition which will be needed to prove the unobstructedness of deformations of the triple (i.e. Proposition 2.4).

Proposition 4.1. *Let Z_1 be a twistor space of nCP^2 and S_1 a real irreducible fundamental divisor. Let $Z_2 = \boldsymbol{F}$ be the flag twistor space, and (Z', S') the pair of normal crossing varieties, where $Z' = Z_1' \cup Z_2'$, $S' = S_1' \cup (D_2' \amalg \overline{D}_2')$ as in Section 2. Then we have*

$$H^2(\Theta_{Z'}(-S')) \simeq H^2(\Theta_{Z_1}(-S_1)),$$

$$H^3(\Theta_{Z'}(-S')) = 0.$$

Here we need not assume that S_1 is of type (2,1) or (2,2). For the proof, we need the following three lemmas which will be proved after the proof of the proposition itself.

Lemma 4.2. *The restriction map*

$$r : H^1(\Theta_{Z_2',Q_2}(-D_2' - \overline{D}_2')) \to H^1(\Theta_{Q_2}(-l_2 - \bar{l}_2))$$

is surjective.

Lemma 4.3. $H^i(\Theta_{Z_2',Q_2}(-D_2' - \overline{D}_2')) = 0$ *for* $i \geq 2$.

Lemma 4.4. *We have a natural isomorphism:*

$$H^i(\Theta_{Z_1',Q_1}(-S_1')) \simeq H^i(\Theta_{Z_1}(-S_1)) \quad \text{for any } i \geq 0.$$

Proof of Proposition 4.1. We have the following exact sequence of sheaves on Z':

$$\begin{aligned} 0 \to \Theta_{Z'}(-S') \to \nu_*(\Theta_{Z_1',Q_1}(-S_1') \oplus \Theta_{Z_2',Q_2}(-D_2' - \overline{D}_2')) \\ \to \Theta_Q(-l - \bar{l}) \to 0, \end{aligned} \tag{10}$$

where $\nu : Z_1' \amalg Z_2' \to Z'$ denotes the normalization. Since ν is a finite morphism, it is easy to show

$$\begin{aligned} H^i(\nu_*(\Theta_{Z_1',Q_1}(-S_1') \oplus \Theta_{Z_2',Q_2}(-D_2' - \overline{D}_2'))) &\simeq \\ H^i(\Theta_{Z_1',Q_1}(-S_1')) \oplus H^i(\Theta_{Z_2',Q_2}(-D_2' - \overline{D}_2')) \end{aligned}$$

for any $i \geq 0$. On the other hand, it is easy to check that $H^i(\Theta_Q(-l - \bar{l})) = 0$ for $i \geq 2$. Then by Lemmas 4.2, 4.3, 4.4 and the long exact sequence of cohomology groups induced by (10), we have

$$\begin{aligned} H^i(\Theta_{Z'}(-S')) &\simeq H^i(Z_1', \Theta_{Z_1',Q_1}(-S_1')) \\ &\simeq H^i(Z_1, \Theta_{Z_1}(-S_1)) \end{aligned}$$

for $i = 2, 3$. Further, by Serre duality, we always have $H^3(\Theta_{Z'}(-S')) = 0$, completing the proof. □

In the following proofs of Lemmas 4.2 and 4.3 we write $\boldsymbol{F}$ and $\boldsymbol{F}'$ for Z_2 and Z_2' respectively and omit the subscript 2 for simplicity.

Proof of Lemma 4.2. We have the following exact sequence of sheaves on $\boldsymbol{F}'$:

$$0 \to \Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}') \to \Theta_{\boldsymbol{F}',Q}(-D' - \overline{D}') \to \Theta_Q(-l - \bar{l}) \to 0. \tag{11}$$

We note that r is induced by the third arrow of this sequence. Hence we have only to show that $H^2(\Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}')) = 0$.

We consider the exact sequence

$$0 \to \Theta_{\boldsymbol{F}'} \xrightarrow{\sigma_*} \sigma^*\Theta_{\boldsymbol{F}} \to \mathcal{O}_Q(1,1) \to 0 \tag{12}$$

(Here, the cokernel of σ_* is naturally isomorphic to $\Theta_{Q/L} \otimes \mathcal{O}_Q(-1)$, where $\Theta_{Q/L}$ denotes the relative tangent sheaf associated to $Q \to L$ and $\mathcal{O}_Q(-1)$ denotes the tautological line bundle over Q, which is isomorphic to the normal bundle of Q in $\boldsymbol{F}'$. Further, it is easy to see that $\Theta_{Q/L} \simeq \mathcal{O}_Q(2,0)$ and $\mathcal{O}_Q(-1) \simeq \mathcal{O}_Q(-1,1)$, where $\mathcal{O}_Q(0,1)$ denotes the pullback bundle of the hyperplane bundle over L. Hence we have $\Theta_{Q/L} \otimes \mathcal{O}_Q(-1) \simeq \mathcal{O}_Q(1,1)$.)

On the other hand, we have the following isomorphisms of sheaves:

$$\Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}') \simeq \sigma^*\Theta_{\boldsymbol{F}}(-D - \overline{D}) \otimes \mathcal{O}_{\boldsymbol{F}'}(Q),$$

$$\mathcal{O}_{\boldsymbol{F}}'(-D' - \overline{D}')|_Q \simeq \mathcal{O}_Q(-l - \bar{l}) = \mathcal{O}_Q(-2,0),$$

$$\mathcal{O}_{\boldsymbol{F}}'(-Q)|_Q \simeq N_{Q/\boldsymbol{F}'}^{-1} = \mathcal{O}_Q(1,-1).$$

Then by tensoring $\mathcal{O}_{\boldsymbol{F}'}(-Q - D' - \overline{D}')$ with (12) and using the above isomorphisms, we get the exact sequence

$$0 \to \Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}') \to \sigma^*\Theta_{\boldsymbol{F}}(-D - \overline{D}) \otimes \mathcal{O}_{\boldsymbol{F}'}(Q) \to \mathcal{O}_Q \to 0.$$

Taking cohomology of this exact sequence, we get

$$H^2(\Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}')) \simeq H^2(\sigma^*\Theta_{\boldsymbol{F}}(-D - \overline{D}) \otimes \mathcal{O}_{\boldsymbol{F}'}(Q)). \tag{13}$$

On the other hand, by the Leray spectral sequence for σ, it is easy to show that

$$H^i(\boldsymbol{F}', \sigma^*\Theta_{\boldsymbol{F}}(-D - \overline{D}) \otimes \mathcal{O}_{\boldsymbol{F}'}(Q)) \simeq H^i(\boldsymbol{F}, \Theta_{\boldsymbol{F}}(-D - \overline{D})) \tag{14}$$

for any $i \geq 0$. Now since $D + \overline{D} \in |K_{\boldsymbol{F}}^{-\frac{1}{2}}|$,

$$\begin{aligned} H^2(\Theta_{\boldsymbol{F}}(-D - \overline{D})) &\simeq H^1(\Omega_{\boldsymbol{F}} \otimes K_{\boldsymbol{F}} \otimes \mathcal{O}_{\boldsymbol{F}}(D + \overline{D}))^* \\ &\simeq H^1(\Omega_{\boldsymbol{F}} \otimes K_{\boldsymbol{F}}^{\frac{1}{2}})^* \end{aligned}$$

by Serre duality. But since $K_{\boldsymbol{F}}^{-\frac{1}{2}}$ is ample, $H^1(\Omega_{\boldsymbol{F}} \otimes K_{\boldsymbol{F}}^{\frac{1}{2}}) = 0$ by the Akizuki-Nakano vanishing and hence we have $H^2(\Theta_{\boldsymbol{F}}(-D - \overline{D})) = 0$. Therefore, by (13) and (14) we obtain $H^2(\Theta_{\boldsymbol{F}'}(-Q - D' - \overline{D}')) = 0$. □

Proof of Lemma 4.3. By considering the exact sequence

$$0 \to \Theta_{\boldsymbol{F}',Q} \xrightarrow{\sigma_*} \sigma^*\Theta_{\boldsymbol{F}} \to \sigma^* N_{L/\boldsymbol{F}} \to 0$$

and checking that $H^i(\sigma^* N_{L/\boldsymbol{F}}) \simeq H^i(N_{L/\boldsymbol{F}}) = 0$ for any $i \geq 1$, we have $H^j(\Theta_{\boldsymbol{F}',Q}) \simeq H^j(\sigma^*\Theta_{\boldsymbol{F}})$ for $j \geq 2$. But by the Leray spectral sequence, we have $H^j(\sigma^*\Theta_{\boldsymbol{F}}) \simeq H^j(\Theta_{\boldsymbol{F}})$ for any $j \geq 0$ and the latter vanishs for $j \geq 2$. Hence $H^j(\Theta_{\boldsymbol{F}',Q}) = 0$ for $j \geq 2$. Then considering the exact sequence

$$0 \to \Theta_{\boldsymbol{F}',Q+D'+\overline{D}'} \to \Theta_{\boldsymbol{F}',Q} \to N_{D'/\boldsymbol{F}'} \oplus N_{\overline{D}'/\boldsymbol{F}'} \to 0,$$

(which is valid since Q, D' and $\overline{D}'$ intersect transversally) and checking that $H^i(N_{D'/\boldsymbol{F}'}) = 0 = H^i(N_{\overline{D}'/\boldsymbol{F}'})$ for any $1 \leq i \leq 3$, we get

$$H^j(\Theta_{\boldsymbol{F}',Q+D'+\overline{D}'}) \simeq H^j(\Theta_{\boldsymbol{F}',Q}) = 0 \quad \text{for } j \geq 2. \tag{15}$$

Finally, checking that $H^i(\Theta_{D',l}) = 0 = H^i(\Theta_{\overline{D}',\overline{l}})$ for any $i \geq 1$ and using (15) and the cohomology exact sequence of

$$0 \to \Theta_{\boldsymbol{F}',Q}(-D' - \overline{D}') \to \Theta_{\boldsymbol{F}',Q+D'+\overline{D}'} \to \Theta_{D',l} \oplus \Theta_{\overline{D}',\overline{l}} \to 0,$$

we have $H^i(\Theta_{\boldsymbol{F}',Q}(-D' - \overline{D}')) = 0$ for $i \geq 2$. □

Proof of Lemma 4.4. Tensoring $\mathcal{O}_{Z_1'}(-S_1') \simeq \sigma^*\mathcal{O}_{Z_1}(-S_1)$ with the exact sequence $0 \to \Theta_{Z_1'} \to \sigma^*\Theta_{Z_1} \to \mathcal{O}_{Q_1}(1,1) \to 0$ (cf. the proof of Lemma 4.2), we get an exact sequence of sheaves

$$0 \to \Theta_{Z_1'}(-S_1') \to \sigma^*(\Theta_{Z_1}(-S_1)) \to \mathcal{O}_{Q_1}(1,-1) \to 0.$$

(Here, we use the fact that $\mathcal{O}_{Z_1'}(-S_1')|_{Q_1} \simeq \mathcal{O}_{Q_1}(-l - \overline{l}) = \mathcal{O}_{Q_1}(0,-2)$.) Considering the cohomology exact sequence of this sequence and the Leray spectral sequence, we have

$$H^i(\Theta_{Z_1'}(-S_1')) \simeq H^i(\Theta_{Z_1}(-S_1)) \tag{16}$$

for any $i \geq 0$. On the other hand, we have an exact sequence

$$0 \to \Theta_{Z_1',Q_1}(-S_1') \to \Theta_{Z_1'}(-S_1') \to \mathcal{O}_{Q_1}(-1,-1) \to 0.$$

Then the associated cohomology sequence of this and (16) imply the desired isomorphisms. □

Next we show the unobstructedness of deformations of the triple (Z', S', C').

Proposition 4.5. *Let* (Z_1, S_1) *and* (Z', S') *be as in Proposition 4.1. Assume further that* S_1 *is a (real) rational surface of type* $(2,1)$ *or* $(2,2)$. *Then we have* $H^2(\Theta_{Z',S',C'}) = 0$ *and* $\boldsymbol{T}^2_{Z',S',C'} = 0$. (See Section 5 for the notations.) *In particular, the Kuranishi family of deformations of the triple* (Z', S', C') *is unobstructed.*

Proof. First we show that

$$\Theta^i_{Z',S',C'} \simeq \begin{cases} \mathcal{O}_Q & i = 1, \\ 0 & i \geq 2. \end{cases}$$

It is obvious that the assumptions of Proposition 5.6 are satisfied and hence we immediately get the first one (i.e. the case $i = 1$). For $i \geq 2$, by Proposition 5.5, we have a long exact sequence

$$\cdots \to \Theta^i_{Z'}(-S') \to \Theta^i_{Z',S',C'} \to \Theta^i_{S',C'} \to \Theta^{i+1}_{Z'}(-S') \to \cdots . \tag{17}$$

We recall ([4] and Lemma 3.2) that

$$\Theta^i_{Z'} \simeq \begin{cases} \mathcal{O}_Q & i = 1, \\ 0 & i \geq 2, \end{cases} \qquad \Theta^i_{S',C'} \simeq \begin{cases} \mathcal{O}_l \oplus \mathcal{O}_{\bar{l}} & i = 1, \\ 0 & i \geq 2. \end{cases}$$

Hence by (17), we have $\Theta^i_{Z',S',C'} = 0$ for any $i \geq 2$.

Next we show that $H^2(\Theta_{Z',S',C'}) = 0$. It is easy to see that the restriction map $\Theta_{Z',S',C'} \to \Theta_{S',C'}$ is surjective. Hence by (17), we have a short exact sequence

$$0 \to \Theta_{Z'}(-S') \to \Theta_{Z',S',C'} \to \Theta_{S',C'} \to 0. \tag{18}$$

Then by Proposition 3.1, it suffices to show that $H^2(\Theta_{Z'}(-S')) = 0$. But this is an immediate consequence of Proposition 4.1 and Condition 2.C. Hence we have $H^2(\Theta_{Z',S',C'}) = 0$.

Finally, let us consider the local to global spectral sequence for $\boldsymbol{T}^i_{Z',S',C'}$. Now since we have $\Theta^i_{Z',S',C'} = 0$ for $i \geq 2$, we get a long exact sequence

$$\begin{aligned} 0 \to H^1(\Theta_{Z',S',C'}) \to \boldsymbol{T}^1_{Z',S',C'} \to H^0(\Theta^1_{Z',S',C'}) \to H^2(\Theta_{Z',S',C'}) \to \\ \boldsymbol{T}^2_{Z',S',C'} \to H^1(\Theta^1_{Z',S',C'}) \to \cdots . \end{aligned} \tag{19}$$

Hence, since $\Theta^1_{Z',S',C'} \simeq \mathcal{O}_Q$ and $H^2(\Theta_{Z',S',C'}) = 0$, we have $\boldsymbol{T}^2_{Z',S',C'} = 0$. □

The following diagram is a key to prove Proposition 2.5 and Theorem 1.1.

Proposition 4.6. *We have the following commutative and exact diagram:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(\Theta_{Z'}(-S')) & \to & H^1(\Theta_{Z',S',C'}) & \to & H^1(\Theta_{S',C'}) & \to & H^2(\Theta_{Z'}(-S')) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ext}^1(\Omega_{Z'}(S'),\mathcal{O}_{Z'}) & \to & \boldsymbol{T}^1_{Z',S',C'} & \xrightarrow{g} & \boldsymbol{T}^1_{S',C'} & \to & \mathrm{Ext}^2(\Omega_{Z'}(S'),\mathcal{O}_{Z'}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{O}_Q) & \to & H^0(\mathcal{O}_l \oplus \mathcal{O}_{\bar{l}}) & \to & H^1(\mathcal{O}_Q(-l-\bar{l})) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. The first (resp. third) row is obtained by the cohomology sequence of (18) (resp. (17) for $i = 1$) and Proposition 4.5. The second row is obtained by using Propositions 5.5 and 4.5. Next, since S' is a Cartier divisor on Z', we have

$$\begin{aligned}
\mathcal{E}xt^i(\Omega_{Z'}(S'),\mathcal{O}_{Z'}) &\simeq \mathcal{E}xt^i(\Omega_{Z'},\mathcal{O}_{Z'}) \otimes \mathcal{O}_{Z'}(-S') \\
&\simeq \begin{cases} \Theta^i_{Z'}(-S') & i = 0,1, \\ 0 & i \geq 2. \end{cases}
\end{aligned}$$

Hence we have $H^0(\mathcal{E}xt^1(\Omega_{Z'}(S'),\mathcal{O}_{Z'})) \simeq H^0(\mathcal{O}_Q(-l-\bar{l})) = 0$. Then the local to global spectral sequence associated to $\mathrm{Ext}^i(\Omega_{Z'}(S'),\mathcal{O}_{Z'})$ and Proposition 4.1 induces the first and forth columns.

Finally, the second (resp. third) column is induced from the local to global spectral sequence associated to $\boldsymbol{T}^i_{Z',S',C'}$ (resp. $\boldsymbol{T}^i_{S',C'}$) and Proposition 4.5 (resp. 3.1). The commutativity of the diagram follows from the naturality of the construction.

(As was shown in the proof of Proposition 2.5, we have $H^2(\Theta_{Z'}(-S')) = 0$ under Condition 2.C. So in such a case, we have $\mathrm{Ext}^2(\Omega_{Z'}(S'),\mathcal{O}_{Z'}) \simeq H^1(\mathcal{O}_Q(-l-\bar{l})) \simeq \boldsymbol{C}$.) □

Proof of Proposition 2.5. Let $\xi \in \boldsymbol{T}^1_{Z',S',C'}$ be any element whose image in $H^0(\mathcal{O}_Q)$ is non-zero in the above diagram. Let $\Delta \subseteq \boldsymbol{T}^1_{Z',S',C'}$ be any holomorphic curve through 0 whose tangent vector at 0 is ξ and let $\{\mathcal{Z}|_\Delta \to \Delta, \mathcal{S}|_\Delta \to \Delta, \mathcal{C}|_\Delta \to \Delta\}$ be the 1-dimensional family obtained by restricting the Kuranishi family of the triple (Z',S',C'). Then by the choice of ξ, the fiber Z_t, $t \in \Delta - 0$ is a non-singular 3-fold if we take Δ sufficiently small. Next, put $\eta := g(\xi) \in \boldsymbol{T}^1_{S',C'}$. Then by the commutativity of the diagram in Proposition 4.6, η is mapped to an element of

$H^0(\mathcal{O}_l \oplus \mathcal{O}_{\bar{l}})$ whose components are both non-zero. Hence, since the Kodaira-Spencer class of $\mathcal{S}' \to \Delta$ is η (up to a non-zero constant), S_t of the fiber is non-singular if $t \in \Delta - 0$. Similar argument using the commutative diagram in the proof of Proposition 2.3 shows that the fiber C_t of $\mathcal{C}' \to \Delta$ is non-singular if $t \in \Delta - 0$. Thus (Z_t, S_t, C_t) is a triple of complex manifolds.

Next, assume that the above ξ is a real element and Δ is a real curve in addition to the above condition. Then the family $\{\mathcal{Z}|_\Delta \to \Delta, \mathcal{S}|_\Delta \to \Delta, \mathcal{C}|_\Delta \to \Delta\}$ has a real structure and Z_t has a real structure of twistor space of $(n+1)\boldsymbol{CP}^2$ by Donaldson-Friedman, so long as $t \in \Delta - 0$ is real and Δ is sufficiently small. In this situation, S_t becomes a fundamental divisor on Z_t as remarked in [9].

Finally, by Proposition 4.1 and Condition 2.C, we have $H^2(\Theta_{Z'}(-S')) = 0$. Then by upper semi-continuity, if we take Δ sufficiently small, we have $H^2(\Theta_{Z_t}(-S_t)) = 0$. Thus we have completed the proof of the proposition. □

Finally, we prove our main theorem, the statement of which can be restated as follows by using the type of rational surfaces (Definition 2.B):

Theorem 4.7. *For any positive integer n, there exists a twistor space Z of $n\boldsymbol{CP}^2$ which has a real fundamental divisor S of type $(2,1)$. The same assertion also holds for a rational surface of type $(2,2)$.*

In the following proof of the theorem, we further prove that the above (Z, S) satisfies Condition 2.C (i.e. $H^2(\Theta_Z(-S)) = 0$) in addition to the above claims. (This is necessary for our induction proof to work.)

Proof. We prove that the theorem and Condition 2.C hold by induction on n. First we consider the case $n = 1$. Let S be a real fundamental divisor on $\boldsymbol{F}$, which is the twistor space of $\boldsymbol{CP}^2$. Since S is a two points blowing-up of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, it is clear that S is of type (2,1) and also of type (2,2). Further we have $H^2(\Theta_{\boldsymbol{F}}(-S)) = 0$ as in the last part of the proof of Lemma 4.2. Thus we have shown the claim for the case $n = 1$.

Next we assume that the statements for the case n: i.e. let Z_1 be a twistor space of $n\boldsymbol{CP}^2$ and assume that Z_1 has a real irreducible fundamental divisor S_1 of type (2,1) (resp. (2,2)) and that Condition 2.C. is satisfied. Then by Proposition 2.5, there exists a twistor space Z_t of $(n+1)\boldsymbol{CP}^2$ which has a real irreducible fundamental divisor S_t and they satisfy $H^2(Z_t, \Theta_{Z_t} \otimes (-S_t)) = 0$. Further by the proof of Proposition 2.3 and by virtue of the commutative diagram of Proposition 4.6, $g(t) \in \boldsymbol{T}^1_{S',C'}$ is not lying on the hypersurfaces of Proposition 2.3. Hence by Propositon 2.3, S_t is a (real) rational surface of type (2,1) (resp. (2,2)). Hence the case $n+1$ is proved. Therefore the claims of the theorem follow for any $n \geq 1$. □

5. Deformation theory of a triple of compact complex spaces

In this section, we shall discuss a deformation theory of holomorphic maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, where X, Y and Z are reduced compact complex spaces. The following construction is similar to that of Ran [20], where he considers a deformation theory of a holomorphic map $f : X \to Y$. First we will work on a homologically-algebraic category.

Let R, S and T be commutative rings and $\phi : R \to S, \psi : S \to T$ ring homomorphisms. We define a non-commutative ring $\mathcal{R}$ by

$$\mathcal{R} := \left\{ \begin{pmatrix} r & 0 & 0 \\ s_1 & s_2 & 0 \\ t_1 & t_2 & t_3 \end{pmatrix} \middle| \begin{array}{c} r \in R \\ s_1, s_2 \in S \\ t_1, t_2, t_3 \in T \end{array} \right\}$$

where the multiplicative structure is given by

$$\begin{pmatrix} r & 0 & 0 \\ s_1 & s_2 & 0 \\ t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} r' & 0 & 0 \\ s'_1 & s'_2 & 0 \\ t'_1 & t'_2 & t'_3 \end{pmatrix}$$
$$:= \begin{pmatrix} rr' & 0 & 0 \\ s_1\phi(r') + s_2 s'_1 & s_2 s'_2 & 0 \\ t_1\psi(\phi(r')) + t_2\psi(s'_1) + t_3 t'_1 & t_2\psi(s'_2) + t_3 t'_2 & t_3 t'_3 \end{pmatrix}.$$

Let $\mathcal{C}$ be the category defined as follows:
The objects of $\mathcal{C}$ are quinteplets

$$\{A \xrightarrow{\gamma} B \xrightarrow{\delta} C \mid A : R\text{−module}, B : S\text{−module}, C : T\text{−module}, \gamma : \phi\text{−hom.}, \delta : \psi\text{−hom.}\}$$

and the morphisms of two objects $A \xrightarrow{\gamma} B \xrightarrow{\delta} C$ and $A' \xrightarrow{\gamma'} B' \xrightarrow{\delta'} C'$ of $\mathcal{C}$ are triples

$$\{\alpha : A \to A' : R\text{−hom.}, \ \beta : B \to B' : S\text{−hom.}, \ \text{and}\ \gamma : C \to C' : T\text{−hom.}\}$$

such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\gamma} & B & \xrightarrow{\delta} & C \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow \\ A' & \xrightarrow{\gamma'} & B' & \xrightarrow{\delta'} & C' \end{array} \tag{20}$$

commutes. On the other hand, let $\mathcal{C}'$ be the category of left $\mathcal{R}$-modules. Then $\mathcal{C}$ and $\mathcal{C}'$ are naturally equivalent:

For any given object $\{A \xrightarrow{\gamma} B \xrightarrow{\delta} C\}$ of $\mathcal{C}$, we define on $A \oplus B \oplus C$ a left $\mathcal{R}$-module structure by

$$\begin{pmatrix} r & 0 & 0 \\ s_1 & s_2 & 0 \\ t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} ra \\ s_1\gamma(a) + s_2 b \\ t_1\delta(\gamma(a)) + t_2\delta(b) + t_3 c \end{pmatrix}.$$

Let $\{\alpha, \beta, \gamma\}$ be a morphism from $A \xrightarrow{\gamma} B \xrightarrow{\delta} C$ to $A' \xrightarrow{\gamma'} B' \xrightarrow{\delta'} C'$. Then we associate $(\alpha, \beta, \gamma) : A \oplus B \oplus C \to A' \oplus B' \oplus C'$ to $\{\alpha, \beta, \gamma\}$. (α, β, γ) is readily seen to be a morphism of $\mathcal{C}'$ by using the commutative diagram (20).

Conversely, if E is a left $\mathcal{R}$-module, we set

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} E, \ B := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} E, \ C := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} E,$$

$$\gamma := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \delta := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\{A \xrightarrow{\gamma} B \xrightarrow{\delta} C\}$ becomes an object of $\mathcal{C}$. If $\Phi : E \to E'$ is a homomorphism of $\mathcal{R}$-module, then Φ naturally induces homomorphisms $A \to A', B \to B'$ and $C \to C'$ as R, S and T-modules respectively. Further, it is easily verified that the above correspondences are mutually converse. Thus, we have seen the equivalence of $\mathcal{C}$ and $\mathcal{C}'$.

Let $A_j \xrightarrow{\gamma_j} B_j \xrightarrow{\delta_j} C_j$ ($j = 0, 1$) be given objects of $\mathcal{C}$. Then for any $i \geq 0$, we define

$$\mathrm{Ext}^i((\gamma_1, \delta_1), (\gamma_0, \delta_0)) := \mathrm{Ext}^i_{\mathcal{R}}(A_1 \oplus B_1 \oplus C_1, A_0 \oplus B_0 \oplus C_0),$$

where the right-hand-side is the usual Ext-group of $\mathcal{R}$-modules.

Next we apply these algebraic constructions to a deformation theory of holomorphic maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, where X, Y and Z are reduced compact complex spaces. Associated to these objects, we firstly define the Grothendieck topology $G = G(f, g)$ as follows ([1, 20]): the open sets of G are the triples (W, V, U) where $U \subseteq X, V \subseteq Y$, and $W \subseteq Z$ are open sets satisfying $f(U) \subseteq V$ and $g(V) \subseteq W$. We then define the non-commutative structure sheaf $\mathcal{O}_G$ by

$$\mathcal{O}_G(W, V, U) := \left\{ \begin{pmatrix} r & 0 & 0 \\ s_1 & s_2 & 0 \\ t_1 & t_2 & t_3 \end{pmatrix} \;\middle|\; \begin{matrix} r \in \mathcal{O}_Z(W) \\ s_1, s_2 \in \mathcal{O}_Y(V) \\ t_1, t_2, t_3 \in \mathcal{O}_X(U) \end{matrix} \right\}$$

with the obvious multiplicative structure using the pull-backs of holomorphic functions.

The above equivalence of categories is naturally generalized to the equivalence of the category of left $\mathcal{O}_G$-modules and the category

$$\{f^*\mathcal{G} \xrightarrow{\gamma} \mathcal{F},\ g^*\mathcal{H} \xrightarrow{\delta} \mathcal{G} \mid \mathcal{F} : \mathcal{O}_X\text{-module},\ \mathcal{G} : \mathcal{O}_Y\text{-module},\ \mathcal{H} : \mathcal{O}_Z\text{-module},$$

$$\gamma : \mathcal{O}_X\text{-hom},\ \delta : \mathcal{O}_Y\text{-hom}.\}$$

(The morphisms of the last category is defined in the obvious ways.)

Then using pull-backs of Kähler differentials and holomorphic functions, we define

$$\Theta^i_{f,g} := \mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus \Omega_Y \oplus \Omega_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X),$$

$$\boldsymbol{T}^i_{f,g} := \operatorname{Ext}^i_{\mathcal{O}_G}(\Omega_Z \oplus \Omega_Y \oplus \Omega_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X).$$

If both f and g are embeddings, we write

$$\Theta^i_{f,g} = \Theta^i_{Z,Y,X}, \quad \boldsymbol{T}^i_{f,g} = \boldsymbol{T}^i_{Z,Y,X}.$$

Then analogous to [20, Proposition 3.1], we have the following:

Proposition 5.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be as above. Then the versal family of deformations of $X \xrightarrow{f} Y \xrightarrow{g} Z$ exists. The Zariski tangent space at the reference point of the parameter space is naturally identified with $\boldsymbol{T}^1_{f,g}$ and the obstruction space is $\boldsymbol{T}^2_{f,g}$. In particular, if $\boldsymbol{T}^2_{f,g} = 0$ then any first order deformation of $X \xrightarrow{f} Y \xrightarrow{g} Z$ can be extended to an actual deformation.*

The rest of this section is devoted to prove Propositions 5.5 and 5.6 below. In the following, we assume that both f and g are embeddings and Y is a Cartier divisor on Z. If X is a topological space, 0_X denotes the zero-sheaf on X.

Lemma 5.2. *Let $X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} Z$ be as above, $(G, \mathcal{O}_G)$ the associated Grothendieck topology and the structure sheaf, and $(G', \mathcal{O}_{G'})$ the Grothendieck topology and the structure sheaf associated to $X \overset{f}{\hookrightarrow} Y$ (cf. [20]). Then for any $i \geq 0$, there exist canonical isomorphisms:*

$$\mathcal{E}xt^i_{\mathcal{O}_G}(0_Z \oplus \Omega_Y \oplus \Omega_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) \simeq \mathcal{E}xt^i_{\mathcal{O}_{G'}}(\Omega_Y \oplus \Omega_X, \mathcal{O}_Y \oplus \mathcal{O}_X),$$

$$\operatorname{Ext}^i_{\mathcal{O}_G}(0_Z \oplus \Omega_Y \oplus \Omega_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) \simeq \operatorname{Ext}^i_{\mathcal{O}_{G'}}(\Omega_Y \oplus \Omega_X, \mathcal{O}_Y \oplus \mathcal{O}_X).$$

Proof. Let $0 \to \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X \to \mathcal{F}^\cdot$ be an injective resolution of $\mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X$ as an $\mathcal{O}_G$-module which exists globally. By the above equivalence of categories, each $\mathcal{F}^n$, $n \geq 0$ can be written as

$$\mathcal{F}^n = \mathcal{I}^n \oplus \mathcal{J}^n \oplus \mathcal{K}^n,$$

where $\mathcal{I}^n, \mathcal{J}^n$ and $\mathcal{K}^n$ are $\mathcal{O}_Z, \mathcal{O}_Y$ and $\mathcal{O}_X$-modules respectively and we have homomorphisms

$$\gamma^n : \mathcal{I}^n|_Y \to \mathcal{J}^n \quad \text{and} \quad \delta^n : \mathcal{J}^n|_X \to \mathcal{K}^n$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{I}^n|_Y & \to & \mathcal{I}^{n+1}|_Y \\ \gamma^n \downarrow & & \downarrow \gamma^{n+1} \\ \mathcal{J}^n & \to & \mathcal{J}^{n+1}, \end{array} \qquad \begin{array}{ccc} \mathcal{J}^n|_X & \to & \mathcal{J}^{n+1}|_X \\ \delta^n \downarrow & & \downarrow \delta^{n+1} \\ \mathcal{K}^n & \to & \mathcal{K}^{n+1}. \end{array}$$

Then each $\mathcal{J}^n \oplus \mathcal{K}^n$, $n \geq 0$ is an injective $\mathcal{O}_{G'}$-module. In fact, let $0 \to \mathcal{S}' \oplus \mathcal{T}' \xrightarrow{(\mu,\nu)} \mathcal{S} \oplus \mathcal{T}$ be any exact sequence of $\mathcal{O}_{G'}$-modules and $(\alpha', \beta') : \mathcal{S}' \oplus \mathcal{T}' \to \mathcal{J}^n \oplus \mathcal{K}^n$ any homomorphism of $\mathcal{O}_{G'}$-modules. Then the sequence $0 \to 0_Z \oplus \mathcal{S}' \oplus \mathcal{T}' \xrightarrow{(0,\mu,\nu)} 0_Z \oplus \mathcal{S} \oplus \mathcal{T}$ is clearly an exact sequence of $\mathcal{O}_G$ -modules and the map $(0, \alpha', \beta') : 0 \oplus \mathcal{S}' \oplus \mathcal{T}' \to \mathcal{I}^n \oplus \mathcal{J}^n \oplus \mathcal{K}^n$ is easily seen to be a homomorphism of $\mathcal{O}_G$-module. Hence by the injectivity of $\mathcal{I}^n \oplus \mathcal{J}^n \oplus \mathcal{K}^n$, there exists an $\mathcal{O}_G$-homomorphism $(0, \alpha, \beta) : 0 \oplus \mathcal{S} \oplus \mathcal{T} \to \mathcal{I}^n \oplus \mathcal{J}^n \oplus \mathcal{K}^n$ such that

$$(0, \alpha', \beta') = (0, \alpha, \beta) \cdot (0, \mu, \nu).$$

Then $(\alpha, \beta) : \mathcal{S} \oplus \mathcal{T} \to \mathcal{J}^n \oplus \mathcal{K}^n$ is the required homomorphism of $\mathcal{O}_{G'}$-modules.

Therefore, the exact sequence

$$0 \to \mathcal{O}_Y \oplus \mathcal{O}_X \to \mathcal{J}^0 \oplus \mathcal{K}^0 \to \mathcal{J}^1 \oplus \mathcal{K}^1 \to \cdots \tag{21}$$

is an injective resolution of $\mathcal{O}_Y \oplus \mathcal{O}_X$ as an $\mathcal{O}_{G'}$-module. Then the required long exact sequences of the lemma follow if we note that there exists canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}_G}(0 \oplus \Omega_Y \oplus \Omega_X, \mathcal{I}^\cdot \oplus \mathcal{J}^\cdot \oplus \mathcal{K}^\cdot) \simeq \mathcal{H}om_{\mathcal{O}_{G'}}(\Omega_Y \oplus \Omega_X, \mathcal{J}^\cdot \oplus \mathcal{K}^\cdot)$$

and the same for Hom. □

Lemma 5.3. *Let $(G, \mathcal{O}_G)$ and $(G', \mathcal{O}_{G'})$ be as in Lemma 5.2 and $\mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R}$ a projective $\mathcal{O}_G$-module. Then $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ and $\mathcal{Q} \oplus \mathcal{R}$ are projective $\mathcal{O}_Z, \mathcal{O}_Y, \mathcal{O}_X$ and $\mathcal{O}_{G'}$-modules respectively.*

Proof. First we show that $\mathcal{P}$ is a projective $\mathcal{O}_Z$-module. We consider the diagram

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & & \downarrow \alpha' & & \\ \mathcal{F} & \to & \mathcal{F}' & \to & 0, \end{array}$$

where $\mathcal{F} \to \mathcal{F}' \to 0$ is any exact sequence of $\mathcal{O}_Z$-modules and α' is any homomorphism of $\mathcal{O}_Z$-modules. Then the naturally induced map $\mathcal{F} \oplus 0_Y \oplus 0_X \to \mathcal{F}' \oplus 0_Y \oplus 0_X$ (with the obvious $\mathcal{O}_Z$-module structures) is clearly a surjective homomorphism of $\mathcal{O}_G$-modules and

$$(\alpha', 0, 0) : \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} \to \mathcal{F}' \oplus 0_Y \oplus 0_X$$

is a homomorphism of $\mathcal{O}_G$-modules. Hence by the projectivity of $\mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R}$, there exists $(\alpha, 0, 0) : \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} \to \mathcal{F} \oplus 0_Y \oplus 0_X$ which makes the diagram

$$\begin{array}{ccccc} & & \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} & & \\ & (\alpha,0,0) \swarrow & \downarrow \ (\alpha',0,0) & & \\ \mathcal{F} \oplus 0_Y \oplus 0_X & \longrightarrow & \mathcal{F}' \oplus 0_Y \oplus 0_X & \to & 0 \end{array}$$

commute. Then $\alpha : \mathcal{P} \to \mathcal{F}$ is the required homomorphism.

Next we show that $\mathcal{Q}$ is a projective $\mathcal{O}_Y$-module. Let

$$\begin{array}{ccccc} & & \mathcal{Q} & & \\ & & \downarrow \beta' & & \\ \mathcal{G} & \xrightarrow{\pi} & \mathcal{G}' & \to & 0 \end{array}$$

be any exact diagram of $\mathcal{O}_Y$-modules. We regard $\mathcal{P} \oplus \mathcal{G}' \oplus 0_X$ as an $\mathcal{O}_G$-module by the commutative diagram

$$\begin{array}{ccc} \mathcal{P}|_Y & = & \mathcal{P}|_Y \\ \delta \downarrow & & \downarrow \epsilon' := \beta \cdot \delta \\ \mathcal{Q} & \xrightarrow{\beta'} & \mathcal{G}', \end{array}$$

where δ is the given $\mathcal{O}_Y$-homomorphism. Then

$$(id, \beta', 0) : \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} \to \mathcal{P} \oplus \mathcal{G}' \oplus 0_X$$

becomes a homomorphism of $\mathcal{O}_G$-modules by the commutativity of the above diagram. On the other hand, since we have seen that $\mathcal{P}$ is a projective $\mathcal{O}_Z$-module, there exists an $\mathcal{O}_Y$-homomorphism $\epsilon : \mathcal{P}|_Y \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccccc} \mathcal{P}|_Y & & & & \\ \epsilon \downarrow & \searrow \epsilon' & & & \\ \mathcal{G} & \xrightarrow{\pi} & \mathcal{G}' & \to & 0 \end{array}$$

commutes. Then

$$(id, \pi, 0) : \mathcal{P} \oplus \mathcal{G} \oplus 0_X \to \mathcal{P} \oplus \mathcal{G}' \oplus 0_X$$

is easily seen to be a surjective $\mathcal{O}_G$-homomorphism, where we regard $\mathcal{P} \oplus \mathcal{G} \oplus 0_X$ as an $\mathcal{O}_G$-module by using ϵ. Thus we get the following exact diagram of $\mathcal{O}_G$-modules:

$$\begin{array}{ccccc} & & \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} & & \\ & (id, \pi, 0) & \downarrow (id, \beta', 0) & & \\ \mathcal{P} \oplus \mathcal{G} \oplus 0_X & \longrightarrow & \mathcal{P} \oplus \mathcal{G}' \oplus 0_X & \to & 0. \end{array}$$

Hence by the projectivity of $\mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R}$, we have a homomorphism $(id, \beta, 0) : \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} \to \mathcal{P} \oplus \mathcal{G} \oplus 0_X$ such that $(id, \pi, 0) \cdot (id, \beta, 0) = (id, \beta', 0)$. Then β is the required homomorphism. Thus we have proved that $\mathcal{Q}$ is a projective $\mathcal{O}_Y$-module. The remaining claims are proved in a similar way. □

Lemma 5.4. *In the situation of Lemma 5.2, there exist the following canonical isomorphisms for any $i \geq 0$:*

$$\mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) \simeq \mathcal{E}xt^i_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z),$$

$$\mathrm{Ext}^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) \simeq \mathrm{Ext}^i_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z).$$

Proof. We first consider the following exact sequence of $\mathcal{O}_G$-modules:

(22) $$0 \to \mathcal{O}_Z(-Y) \oplus 0_Y \oplus 0_X \to \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X \to 0.$$

(Here, we note that since the diagram

$$\begin{array}{ccccc} 0 & \to & \mathcal{O}_Z(-Y) & \to & \mathcal{O}_Z \\ & & \downarrow & & \downarrow \\ & & 0_Y & \to & \mathcal{O}_Y \end{array}$$

commutes, the second arrow of (22) is a homomorphism of $\mathcal{O}_G$-modules. On the other hand, the $\mathcal{O}_G$-module structure of $\mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X$ is given by $id : \mathcal{O}_Y \to \mathcal{O}_Y$ and the restriction of holomorphic functions.) It is easy to show that for any $i \geq 0$, there exist canonical isomorphisms

$$\mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z(-Y) \oplus 0_Y \oplus 0_X) \simeq \mathcal{E}xt^i_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z(-Y)), \tag{23}$$

and

$$\mathrm{Ext}^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z(-Y) \oplus 0_Y \oplus 0_X) \simeq \mathrm{Ext}^i_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z(-Y)). \tag{24}$$

Now we show that

$$\mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) = 0 \tag{25}$$

for any $i \geq 0$. Let

$$\mathcal{P}_i \oplus \mathcal{Q}_i \oplus \mathcal{R}_i \overset{(\partial_i, \partial_i', \partial_i'')}{\longrightarrow} \mathcal{P}_{i-1} \oplus \mathcal{Q}_{i-1} \oplus \mathcal{R}_{i-1} \to \cdots \to \Omega_Z \oplus 0_Y \oplus 0_X \to 0$$

be a projective resolution of $\Omega_Z \oplus 0_Y \oplus 0_X$ which exists at least locally. An element of

$$\mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X)$$

is represented by $\alpha_i : \mathcal{P}_i \to \mathcal{O}_Y, \beta_i : \mathcal{Q}_i \to \mathcal{O}_Y$ and $\gamma_i : \mathcal{R}_i \to \mathcal{O}_X$ such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{P}_i|_Y & \to & \mathcal{Q}_i \\ \alpha_i|_Y \downarrow & & \downarrow \beta_i \\ \mathcal{O}_Y & = & \mathcal{O}_Y, \end{array} \qquad \begin{array}{ccc} \mathcal{Q}_i|_X & \to & \mathcal{R}_i \\ \beta_i|_X \downarrow & & \downarrow \gamma_i \\ \mathcal{O}_X & = & \mathcal{O}_X \end{array}$$

and satisfying $\alpha_{i+1} \cdot \partial_{i+1} = 0$, $\beta_{i+1} \cdot \partial'_{i+1} = 0$ and $\gamma_{i+1} \cdot \partial''_{i+1} = 0$. (These are cocycle conditions.) Then noting that $\mathcal{Q}. \oplus \mathcal{R}. \to 0_Y \oplus 0_X \to 0$ gives a projective resolution of $0_Y \oplus 0_X$ as an $\mathcal{O}_{G'}$-module by Lemma 5.3 and hence $0 \to \mathcal{H}om_{\mathcal{O}_{G'}}(\mathcal{Q}. \oplus \mathcal{R}., \mathcal{O}_Y \oplus \mathcal{O}_X)$ is an exact sequence, there exist $\beta_{i-1} : \mathcal{Q}_{i-1} \to \mathcal{O}_Y$ and $\gamma_{i-1} : \mathcal{R}_{i-1} \to \mathcal{O}_X$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{Q}_i|_X & & \longrightarrow & & \mathcal{R}_i \\ & \beta_i|_X \searrow & & \swarrow \gamma_i & \\ \partial'_i|_X \downarrow & & \mathcal{O}_X & & \downarrow \partial''_i \\ & \beta_{i-1}|_X \nearrow & & \nwarrow \gamma_{i-1} & \\ \mathcal{Q}_{i-1}|_X & & \longrightarrow & & \mathcal{R}_{i-1}. \end{array}$$

Then we have the following diagram

$$\begin{tikzcd}
\mathcal{P}_i|_Y \arrow[rr, "\psi_i"] \arrow[dr, "\alpha_i|_Y"] \arrow[dd, "\partial_i|_Y"'] & & \mathcal{Q}_i \arrow[dl, "\beta_i"'] \arrow[dd, "\partial'_i"] \\
& \mathcal{O}_Y & \\
\mathcal{P}_{i-1}|_Y \arrow[rr, "\psi_{i-1}"] & & \mathcal{Q}_{i-1} \arrow[ul, "\beta_{i-1}"]
\end{tikzcd}$$

with $\partial'_i \cdot \psi_i = \psi_{i-1} \cdot (\partial_i|_Y)$, $\alpha_i|_Y = \beta_i \cdot \psi_i$ and $\beta_i = \beta_{i-1} \cdot \partial'_i$. Now we set $\alpha_{i-1} := \beta_{i-1} \cdot \psi_{i-1}$. Then we have

$$\alpha_{i-1} \cdot (\partial_i|_Y) = \beta_{i-1} \cdot \psi_{i-1} \cdot (\partial_i|_Y) = \beta_{i-1} \cdot \partial'_i \cdot \psi_i = \beta_i \cdot \psi_i = \alpha_i|_Y.$$

Therefore, if we let α_{i-1} also denote the naturally induced map $\mathcal{P}_{i-1} \to \mathcal{O}_Y$, we have $\alpha_i = \alpha_{i-1} \cdot \partial_i$. This means that $(\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1})$ gives an $\mathcal{O}_G$-homomorphism $\mathcal{P}_{i-1} \oplus \mathcal{Q}_{i-1} \oplus \mathcal{R}_{i-1} \to \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X$ with $(\alpha_i, \beta_i, \gamma_i) = (\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}) \cdot (\partial_i, \partial'_i, \partial''_i)$, which implies that

$$[(\alpha_i, \beta_i, \gamma_i)] = 0 \in \mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X).$$

Thus we have shown that (25) holds. Then the local to global spectral sequence shows that

$$\mathrm{Ext}^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) = 0 \tag{26}$$

for any $i \geq 0$.

Therefore by (25), the long exact sequence of local-Ext's associated to (22) and isomorphism (23) induces the isomorphism

$$\begin{aligned}
\mathcal{E}xt^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) &\simeq \mathcal{E}xt^i_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z(-Y)) \\
&\simeq \mathcal{E}xt^i_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z),
\end{aligned}$$

for any $i \geq 0$ and by (26), the long exact sequence of global-Ext's associated to (22) and isomorphism (24) induces

$$\begin{aligned}
\mathrm{Ext}^i_{\mathcal{O}_G}(\Omega_Z \oplus 0_Y \oplus 0_X, \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) &\simeq \mathrm{Ext}^i_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z(-Y)) \\
&\simeq \mathrm{Ext}^i_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z).
\end{aligned}$$

for any $i \geq 0$. □

Proposition 5.5. *Let X, Y and Z be compact complex spaces and $f : X \hookrightarrow Y, g : Y \hookrightarrow Z$ closed embeddings and assume that Y is a Cartier divisor on Z. Then we have the following two long exact sequences:*

(27)
$$\cdots \to \Theta^i_Z(-Y) \to \Theta^i_{Z,Y,X} \to \Theta^i_{Y,X} \to \Theta^{i+1}_Z(-Y) \to \cdots,$$
$$\cdots \to \mathrm{Ext}^i_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z) \to \boldsymbol{T}^i_{Z,Y,X} \to \boldsymbol{T}^i_{Y,X} \to \mathrm{Ext}^{i+1}_{\mathcal{O}_Z}(\Omega_Z(Y), \mathcal{O}_Z) \to \cdots.$$

Proof. We consider the following short exact sequence of $\mathcal{O}_G$-modules:

$$0 \to 0_Z \oplus \Omega_Y \oplus \Omega_X \to \Omega_Z \oplus \Omega_Y \oplus \Omega_X \to \Omega_Z \oplus 0_Y \oplus 0_X \to 0.$$

Then the required exact sequences immediately follow from this and Lemmas 5.2 and 5.4. □

Finally, we prove a proposition which is needed in Section 4. Let Z_1, Z_2 be compact complex manifolds of dimension n and $W_i \subseteq Z_i$ $(i = 1, 2)$ be irreducible non-singular divisors. We assume that there exists a biholomorphic map $\phi : W_1 \to W_2$ and let $Z = Z_1 \cup_\phi Z_2$ be the normal crossing variety obtained by identifying W_1 and W_2 by ϕ. Let $Y_i \subseteq Z_i$ $(i = 1, 2)$ be irreducible non-singular divisors which intersect W_i transversally and assume that ϕ induces a biholomorphic map between $W_1 \cap Y_1$ and $W_2 \cap Y_2$. We set $Y := Y_1 \cup Y_2$. Further, we suppose that X_i $(i = 1, 2)$ are irreducible non-singular divisors on Y_i, X_i intersect $W_i \cap Y_i$ transversally and ϕ induces a biholomorphic map between $X_1 \cap W_1$ and $X_2 \cap W_2$. We set $X := X_1 \cup X_2$. We get a triple of normal crossing varieties $Z \supseteq Y \supseteq X$.

Proposition 5.6. *Under this situation, we have $\Theta^1_{Z,Y,X} \simeq \Theta^1_Z$.* (The right-hand-side is isomorphic to $N_{W_1/Z_1} \otimes N_{W_2/Z_2}$ [4].)

Proof. We prove this isomorphism by explicitly giving a projective resolution of $\Omega_Z \oplus \Omega_Y \oplus \Omega_X$ as an $\mathcal{O}_G$-module. Let $p \in W_1 \simeq W_2$ be any point. Then in a sufficiently small neighborhood of p, Z, Y and X can be written as

$$\begin{aligned} Z &= \{(z_1, \cdots, z_{n+1}) \in U \subseteq \boldsymbol{C}^{n+1} \mid z_1 z_2 = 0\}, \\ Y &= \{(z_1, \cdots, z_{n+1}) \in U \subseteq \boldsymbol{C}^{n+1} \mid z_1 z_2 = z_{n+1} = 0\}, \\ X &= \{(z_1, \cdots, z_{n+1}) \in U \subseteq \boldsymbol{C}^{n+1} \mid z_1 z_2 = z_n = z_{n+1} = 0\}, \end{aligned}$$

where $(z_1, \cdots, z_{n+1})$ are coordinates on $\boldsymbol{C}^{n+1}$ and U is a sufficiently small open neighborhood of 0 in $\boldsymbol{C}^{n+1}$. Then the sequence

(28)
$$\begin{aligned} 0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2 &\to \Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X \\ &\to \Omega_Z \oplus \Omega_Y \oplus \Omega_X \to 0 \end{aligned}$$

is easily checked to be an exact sequence of $\mathcal{O}_G$-modules, where $\mathcal{I}_Z, \mathcal{I}_Y$ and $\mathcal{I}_X$ denote the ideal sheaves of Z, Y and X in U respectively. (Here, $\mathcal{O}_G$-module structures on each sheaf are given in natural ways.) Further, both $\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X$ and $\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2$ are projective $\mathcal{O}_G$-modules. In fact, since Ω_U is isomorphic to ${\mathcal{O}_U}^{\oplus(n+1)}$, we have

$$\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X \simeq \mathcal{F}^{\oplus(n+1)}, \tag{29}$$

where we put $\mathcal{F} := \mathcal{O}_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X$ for simplicity. Hence $\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X$ is a projective $\mathcal{O}_G$-module because $\mathcal{O}_G$ (which is of course a free $\mathcal{O}_G$-module) has $\mathcal{F}$ as a direct summand. On the other hand, we have an isomorphism

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2 \simeq \mathcal{F} \oplus (0_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X) \oplus (0_Z \oplus 0_Y \oplus \mathcal{O}_X) \tag{30}$$

as $\mathcal{O}_G$-module, where the $\mathcal{O}_G$-module structure on the right-hand-side is given by restricting holomorphic functions. (If one embeds $\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2$ in $\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X$, the above isomorphism (30) is explicitly described as follows: We have isomorphisms

$$\begin{aligned}
\mathcal{I}_Z/\mathcal{I}_Z^2 &\simeq \mathcal{O}_Z(z_1dz_2 + z_2dz_1), \\
\mathcal{I}_Y/\mathcal{I}_Y^2 &\simeq \mathcal{O}_Y(z_1dz_2 + z_2dz_1) \oplus \mathcal{O}_Y(dz_{n+1}), \\
\mathcal{I}_X/\mathcal{I}_X^2 &\simeq \mathcal{O}_X(z_1dz_2 + z_2dz_1) \oplus \mathcal{O}_X(dz_n) \oplus \mathcal{O}_X(dz_{n+1}).
\end{aligned}$$

Then the isomorphism (30) is given by the following isomorphisms:

$$\begin{aligned}
\mathcal{F} &\simeq \mathcal{O}_Z(z_1dz_2 + z_2dz_1) \oplus \mathcal{O}_Y(z_1dz_2 + z_2dz_1) \oplus \mathcal{O}_X(z_1dz_2 + z_2dz_1), \\
0_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X &\simeq 0_Z \oplus \mathcal{O}_Y(dz_{n+1}) \oplus \mathcal{O}_X(dz_{n+1}) \\
0_Z \oplus 0_Y \oplus \mathcal{O}_X &\simeq 0_Z \oplus 0_Y \oplus \mathcal{O}_X(dz_n).)
\end{aligned}$$

Then since the right-hand-side of (30) is isomorphic to $\mathcal{O}_G$, $\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2$ is also a projective $\mathcal{O}_G$-module.

Therefore, (28) gives a projective resolution of $\Omega_Z \oplus \Omega_Y \oplus \Omega_X$ as an $\mathcal{O}_G$-module. On the other hand, the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_U|_Z \to \Omega_Z \to 0$$

is clearly a projective resolution of Ω_Z as an $\mathcal{O}_Z$-module. Then by definition of $\mathcal{E}xt^1$, we have only to prove that the cokernel of the natural map

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{O}_G}(\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X, \mathcal{F}) \to \\
&\qquad\qquad \mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2, \mathcal{F})
\end{aligned} \tag{31}$$

is canonically isomorphic to the cokernel of

$$\mathcal{H}om_{\mathcal{O}_Z}(\Omega_U|_Z, \mathcal{O}_Z) \to \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z). \tag{32}$$

In fact by (29),we have

$$\mathcal{H}om_{\mathcal{O}_G}(\Omega_U|_Z \oplus \Omega_U|_Y \oplus \Omega_U|_X, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{O}_G}(\mathcal{F}, \mathcal{F})^{\oplus(n+1)}$$

and the latter is easily seen to be isomorphic to $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{O}_Z)^{\oplus(n+1)}$, which is canonically isomorphic to $\mathcal{H}om_{\mathcal{O}_Z}(\Omega_U|_Z, \mathcal{O}_Z)$. On the other hand, by (30), we have

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_G}(\mathcal{I}_Z/\mathcal{I}_Z^2 \oplus \mathcal{I}_Y/\mathcal{I}_Y^2 \oplus \mathcal{I}_X/\mathcal{I}_X^2, \mathcal{F}) &\simeq \mathcal{H}om_{\mathcal{O}_G}(\mathcal{F}, \mathcal{F}) \\ &\quad \oplus\mathcal{H}om_{\mathcal{O}_G}(0_Z \oplus \mathcal{O}_Y \oplus \mathcal{O}_X, \mathcal{F}) \\ &\quad \oplus\mathcal{H}om_{\mathcal{O}_G}(0_Z \oplus 0_Y \oplus \mathcal{O}_X, \mathcal{F}) \\ &\simeq \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) \\ &\quad \oplus\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \oplus \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)\end{aligned}$$

and it is obvious that the map (31) is surjective on $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y)\oplus$ $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$-factors. Further it is obvious that under the above isomorphisms, the map (31) and (32) are naturally identified. Thus we have proved the required isomorphism. □

REMARK 5.A. Let $X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} Z$ be as in Proposition 5.5 and further assume that X is a Cartier divisor on Y. Then it is actualy desirable to show the existence of the follwoing exact sequences:

$$\cdots \to \Theta^i_{Z,Y,X} \to \Theta^i_{Z,Y} \to \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}, \mathcal{O}_X) \to \cdots, \tag{33}$$

and

$$\cdots \to \boldsymbol{T}^i_{Z,Y,X} \to \boldsymbol{T}^i_{Z,Y} \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}, \mathcal{O}_X) \to \cdots, \tag{34}$$

where $\mathcal{I}_{X/Y}$ denotes the ideal sheaf of X in Y. In fact, Proposition 5.6 easily follows from (33).

REMARK 5.B. The local to global spectral sequence exists even if the structure sheaf is a sheaf of non-commutative ring, as proved by Grothendieck [6, Théorème 4.2.1].

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