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4-FOLD TRANSITIVE PERMUTATION GROUPS IN WHICH THE STABILIZER OF FOUR POINTS IN G HAS AN ORBIT OF LENGTH THREE

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1. Introduction

Let G be a 4-fold transitive permutation group on Ω. If the stabilizer of four points i, j, k and l in G has an orbit of length one in $\Omega-\{i, j, k, l\}$, then G is S_5, A_6 or M_{11} by a theorem of H. Nagao [4]. If the stabilizer of four points in G has an orbit of length two, then G is S_6 by a theorem of T. Oyama [12].

We now consider the case in which the stabilizer of four points in G has an orbit of length three and have the following results.

Theorem. *Let G be a 4-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If the stabilizer of four points in G has an orbit of length three, then G is S_7, A_7 or M_{23}.*

In the proof of this theorem we shall use the following lemma, which will be proved in the section 3.

Lemma. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ satisfying the following condition:*

For any four points i, j, k and l in Ω, there exist three points i_1, i_2 and i_3 in $\Omega-\{i, j, k, l\}$ such that any involution in G_{ijkl} fixes exactly seven points i, j, k, l, i_1, i_2 and i_3.

Then G is M_{23}.

The theorem implies the following corollaiy.

Corollary. *Let D be a 4-$(v, k, 1)$ design, where $k=5$, 6 or 7. If an automorphism group G of D is a 4-fold transitive permutation group on the set of points of D, then D is a 4-(11, 5, 1) design, a 4-(23, 7, 1) design or a trivial design: a 4-(5, 5, 1) design, a 4-(6, 6, 1) design or a 4-(7, 7, 1) design.*

The case $k=5$ has been proved by H. Nagao [4] and the case $k=6$ by T. Oyama [12]. Hence in this paper we shall prove the remaining case $k=7$ in the section 4.

We shall use the same notations as in [6].

2. Proof of Theorem

Let G be a group satisfying the assumption of Theorem. Let P be a Sylow 2-subgroup of G_{1234}.

If $P=1$, then G is A_7 by a theorem of M. Hall ([2] Theorem 5.8.1) and the assumption.

Since P fixes a G_{1234}-orbit of length three as a set, $|I(P)|\geq 5$. If $|I(P)|>5$ and $P\neq 1$, then G is M_{23} by a theorem of T. Oyama ([6], [7] and [9]) and the assumption.

If P is semiregular on $\Omega-I(P)$, $P\neq 1$ and $|I(P)|=5$, then G is S_7 by a theorem of H. Nagao [5] and the assumption.

Hence from now on we assume that $P\neq 1$, $|I(P)|=5$ and P is not semiregular on $\Omega-I(P)$ and prove the theorem by way of contradiction.

(1) *G_{1234} has exactly one orbit of length three.*

Proof. Suppose by way of contradiction that G_{1234} has two orbits $\{i_1, i_2, i_3\}$ and $\{i'_1, i'_2, i'_3\}$ of length three. Since P fixes $\{i_1, i_2, i_3\}$ and $\{i'_1, i'_2, i'_3\}$ as a set, P fixes at least six points, which is a contradiction since $|I(P)|=5$.

We may assume that $I(P)=\{1, 2, 3, 4, 5\}$ and $\{5, 6, 7\}$ is the unique G_{1234}-orbit of length three. Then $\{6, 7\}$ is a P-orbit of length two. Hence a minimal P-orbit in $\Omega-I(P)$ is of length two.

(2) *Let t be a point of a minimal P-orbit in $\Omega-I(P)$. Then a Sylow 2-subgroup of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ is of order two.*

Proof. Let P' be a Sylow 2-subgroup of G_{ijkl} containing P_t for any four points i, j, k and l in $I(P_t)$. Since P_t is a normal subgroup of index two in P', $N_{P'}(P_t)^{I(P_t)}=P'^{I(P_t)}$ is a Sylow 2-subgroup of $N_G(P_t)^{I(P_t)}_{ijkl}$ and is of order two.

(3) $|I(P_t)|=7$, 9 *or* 13. *In particular, if* $|I(P_t)|=9$ *or* 13, *then* $N_G(P_t)^{I(P_t)}\leq A_9$ *or* $N_G(P_t)^{I(P_t)}=S_1\times M_{12}$, *respectively.*

Proof. A Sylow 2-subgroup of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ is a nonidentity semiregular group and fixes exactly five points. Thus this follows from Theorem 1 of [8].

(4) $|I(P_t)|\neq 13$.

Proof. If $|I(P_t)|=13$, then $N_G(P_t)^{I(P_t)}=S_1\times M_{12}$. Hence a Sylow 2-subgroup of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ is of order eight. This is contrary to (2). Thus $|I(P_t)|\neq 13$.

(5) $|I(P_t)| \neq 9$.

Proof. Suppose by way of contradiction that $|I(P_t)|=9$. Then by (2) for any four points i, j, k and l in $I(P_t)$, any involution in $N_G(P_t)_{ijkl}^{I(P_t)}$ fixes exactly five points.

First assume that $N_G(P_t)^{I(P_t)}$ is primitive. Then since $N_G(P_t)^{I(P_t)}$ is a subgroup of A_9 and has an involution fixing five points, $N_G(P_t)^{I(P_t)}=A_9$ (see [13]). This is contrary to (2).

Next assume that $N_G(P_t)^{I(P_t)}$ is imprimitive. Then $N_G(P_t)^{I(P_t)}$ has three blocks $\{i_1, i_2, i_3\}$, $\{j_1, j_2, j_3\}$ and $\{k_1, k_2, k_3\}$ of length three. Let x be an involution fixing i_1, i_2, j_1 and j_2. Then x fixes i_3, j_3 and one more point in $\{k_1, k_2, k_3\}$. Thus x is a transposition. This is a contradiction.

Finally assume that $N_G(P_t)^{I(P_t)}$ is intransitive. Then one of $N_G(P_t)^{I(P_t)}$-orbits is of length less than five.

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1\}$ of length one. Then for any four points i, j, k and l in $I(P_t)-\{i_1\}$, there exists an involution in $N_G(P_t)^{I(P_t)}$ fixing exactly five points i_1, i, j, k and l. Thus by a lemma of D. Livingstone and A. Wagner ([3], Lemma 6), $N_G(P_t)_{i_1}^{I(P_t)-\{i_1\}}$ is 4-fold transitive on $I(P_t)-\{i_1\}$. Hence by (3) $N_G(P_t)^{I(P_t)}=S_1\times A_8$. This is contrary to (2).

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1, i_2\}$ of length two. Then for any three points i, j and k in $I(P_t)-\{i_1, i_2\}$, there exists an involution in $N_G(P_t)^{I(P_t)}$ fixing exactly five points i_1, i_2, i, j and k. Thus by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)_{i_1 i_2}^{I(P_t)-\{i_1,i_2\}}$ is 3-fold transitive on $I(P_t)-\{i_1, i_2\}$. Hence by (3) $N_G(P_t)_{i_1 i_2}^{I(P_t)-\{i_1,i_2\}}=A_7$. This is contrary to (2).

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1, i_2, i_3\}$ of length three. Set $\Delta=I(P_t)-\{i_1, i_2, i_3\}=\{i_4, i_5, \cdots, i_9\}$. Then for any four points in Δ, there exists an involution in $N_G(P_t)^{\Delta}$ fixing exactly these four points. Hence by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)^{\Delta}$ is 4-fold transitive on Δ, and so $N_G(P_t)^{\Delta}=S_6$.

Thus $N_G(P_t)^{I(P_t)}$ has two elements

$$x=(i_4)(i_5)(i_6 i_7)(i_8 i_9)\cdots \quad \text{and}$$
$$y=(i_4)(i_5)(i_6 i_8)(i_7 i_9)\cdots .$$

Since by (3) $N_G(P_t)^{I(P_t)}\leq A_9$, x and y have three fixed points or one 3-cycle on $\{i_1, i_2, i_3\}$. Thus x^3 and y^3 fix five points i_1, i_2, i_3, i_4 and i_5 and $\langle x^3, y^3\rangle$ is an elementary abelian group of order four. This is contrary to (2).

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1, i_2, i_3, i_4\}$ of length four. Set $\Delta=I(P_t)-\{i_1, i_2, i_3, i_4\}=\{i_5, i_6, \cdots, i_9\}$. Then for any three points i, j and k in Δ, $N_G(P_t)^{I(P_t)}$ has an involution fixing i_4, i, j, k and one more point in $\{i_1, i_2, i_3\}$. Thus by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)_{i_4}^{\Delta}$ is 3-fold transitive on Δ, and so $N_G(P_t)_{i_4}^{\Delta}=S_5$.

Thus $N_G(P_t)^{I(P_t)}$ has two elements

$$x=(i_4)(i_5)(i_6i_7)(i_8i_9)\cdots \quad \text{and}$$
$$y=(i_4)(i_5)(i_6i_8)(i_7i_9)\cdots .$$

By the same argument as is shown above, we have a contradiction.

Thus $|I(P_t)|\neq 9$.

(6) *$N_G(P_t)^{I(P_t)}$ is one of the following groups.*

(*a*) $N_G(P_t)^{I(P_t)}=S_7$

(*b*) $N_G(P_t)^{I(P_t)}=S_1\times S_6$

(*c*) $N_G(P_t)^{I(P_t)}=S_2\times S_5$

(*d*) $N_G(P_t)^{I(P_t)}=S_3\times S_4$

Proof. First assume that $N_G(P_t)^{I(P_t)}$ is transitive on $I(P_t)$. Since by (2) $N_G(P_t)^{I(P_t)}$ has a transposition, $N_G(P_t)^{I(P_t)}=S_7$.

Next assume that $N_G(P_t)^{I(P_t)}$ is intransitive. Then one of $N_G(P_t)^{I(P_t)}$-orbits is of length less than four.

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1\}$ of length one. Then for any four points i, j, k and l in $I(P_t)-\{i_1\}$, there exists an involution in $N_G(P_t)^{I(P_t)}$ fixing exactly five points i_1, i, j, k and l. Thus by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)_{i_1}^{I(P_t)-\{i_1\}}$ is 4-fold transitive on $I(P_t)-\{i_1\}$, and so $N_G(P_t)^{I(P_t)}=S_1\times S_6$.

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1, i_2\}$ of length two. Then for any three points i, j and k in $I(P_t)-\{i_1, i_2\}$, there exists an involution in $N_G(P_t)^{I(P_t)}$ fixing exactly five points i_1, i_2, i, j and k. Thus by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)_{i_1i_2}^{I(P_t)-\{i_1,i_2\}}$ is 3-fold transitive on $I(P_t)-\{i_1, i_2\}$, and so $N_G(P_t)_{i_1i_2}^{I(P_t)-\{i_1,i_2\}}=S_5$.

On the other hand $N_G(P_t)^{I(P_t)}$ has an involution

$$x=(i_1i_2)(i_3)(i_4)(i_5)(i_6)(i_7)\,.$$

Hence $N_G(P_t)^{I(P_t)}=S_2\times S_5$.

Suppose that $N_G(P_t)^{I(P_t)}$ has an orbit $\{i_1, i_2, i_3\}$ of length three. Set $\Delta=I(P_t)-\{i_1, i_2, i_3\}=\{i_4, i_5, i_6, i_7\}$. Then for any two points i and j in Δ, there exists an involution in $N_G(P_t)^{I(P_t)}$ fixing exactly five points i_1, i_2, i_3, i and j. Thus by a lemma of D. Livingstone and A. Wagner, $N_G(P_t)_{i_1i_2i_3}^{\Delta}$ is doubly transitive on Δ, and so $N_G(P_t)_{i_1i_2i_3}^{\Delta}=S_4$.

On the other hand $N_G(P_t)^{I(P_t)}$ has two involutions

$$x_1=(i_1i_2)(i_3)(i_4)(i_5)(i_6)(i_7) \quad \text{and}$$
$$x_2=(i_1)(i_2i_3)(i_4)(i_5)(i_6)(i_7)\,.$$

Hence $N_G(P_t)^{I(P_t)}=S_3\times S_4$.

(7) $N_G(P_t)^{I(P_t)}=S_7$ *and* $t=6$ *or* 7. *For four points* i, j, k *and* l *in* Ω, *let* $\{i_1, i_2, i_3\}$ *be the* G_{ijkl}*-orbit of length three. Set* $\Delta(i, j, k, l)=\{i, j, k, l, i_1, i_2, i_3\}$. *Then* $\{\Delta(i, j, k, l) \mid i, j, k, l \in \Omega\}$ *forms a* 4-$(n, 7, 1)$ *design on* Ω.

Proof. Suppose by way of contradiction that $N_G(P_t)^{I(P_t)}$ is not S_7. Set $I(P_t)=\{i_1, i_2, \cdots, i_7\}$.

First assume that $N_G(P_t)^{I(P_t)}=S_1\times S_6$ and $\{i_1\}$ is an orbit of length one. For four points i_1, i_2, i_3 and i_4 in $I(P_t)$, $N_G(P_t)_{i_1i_2i_3i_4}^{I(P_t)-\{i_1,i_2,i_3,i_4\}}=S_3$. Thus $\{i_5, i_6, i_7\}$ is the unique $G_{i_1i_2i_3i_4}$-orbit of length three, and so $N_G(G_{i_1i_2i_3i_4})\leq N_G(G_{I(P_t)})$. Since P_t is a Sylow 2-subgroup of $G_{I(P_t)}$, by Frattini argument $N_G(P_t)^{I(P_t)}=N_G(G_{I(P_t)})^{I(P_t)}$. Thus $N_G(P_t)^{I(P_t)}\geq N_G(G_{i_1i_2i_3i_4})^{I(P_t)}$.

On the other hand $N_G(G_{i_1i_2i_3i_4})^{\{i_1,i_2,i_3,i_4\}}=S_4$ by a theorem of H. Nagao [4] and $N_G(P_t)^{I(P_t)}$ has an orbit containing four points i_1, i_2, i_3 and i_4. This is a contradiction.

Next assume that $N_G(P_t)^{I(P_t)}=S_2\times S_5$ and $\{i_1, i_2\}$ is an orbit of length two. For four points i_1, i_2, i_3 and i_4 in $I(P_t)$, $N_G(P_t)_{i_1i_2i_3i_4}^{I(P_t)-\{i_1,i_2,i_3,i_4\}}=S_3$. Thus by the same argument as is shown above, we have a contradiction.

Finally assume that $N_G(P_t)^{I(P_t)}=S_3\times S_4$ and $\{i_1, i_2, i_3\}$ is an orbit of length three. For four points i_1, i_2, i_3 and i_4 in $I(P_t)$, $N_G(P_t)_{i_1i_2i_3i_4}^{I(P_t)-\{i_1,i_2,i_3,i_4\}}=S_3$. Thus by the same argument as is shown above, we have a contradiction. Thus $N_G(P_t)^{I(P_t)}=S_7$.

Let $\{t, t'\}$ be a P-orbit of length two. Thus $I(P_t)=\{1, 2, 3, 4, 5, t, t'\}$. Since $N_G(P_t)^{I(P_t)}=S_7$, $N_G(P_t)_{1234}^{I(P_t)-\{1,2,3,4\}}=S_3$. Therefore $\{5, t, t'\}$ is the unique G_{1234}-orbit of length three, and so $t=6$ or 7.

(8) *Let* Q *be a subgroup of* P *fixing exactly seven points. Then* $I(Q)=\{1, 2, \cdots, 7\}$.

Proof. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing the same seven points. Since $|I(Q)|=7$, a Sylow 2-subgroup of the stabilizer of any four points in $N_G(Q)^{I(Q)}$ is of order two. By the same argument as is shown in (6), $N_G(Q)^{I(Q)}=S_7$, $S_1\times S_6$, $S_2\times S_5$ or $S_3\times S_4$. Thus for some four points i_1, i_2, i_3 and i_4 in $I(Q)$, $N_G(Q)_{i_1i_2i_3i_4}^{I(Q)-\{i_1,i_2,i_3,i_4\}}=S_3$. Therefore $I(Q)-\{i_1, i_2, i_3, i_4\}$ is the unique $G_{i_1i_2i_3i_4}$-orbit of length three. Thus $N_G(Q)^{I(Q)}=S_7$ and $I(Q)=\Delta(j_1, j_2, j_3, j_4)$ for any four points j_1, j_2, j_3 and j_4 in $I(Q)$. Since $I(Q)\supseteq\{1, 2, 3, 4, 5\}$, by (7) $I(Q)=\{1, 2, \cdots, 7\}$.

Let $\bar{Q}$ be a subgroup of P such that $|I(\bar{Q})|$ is minimal among all subgroups of P fixing more than seven points. Moreover choose $\bar{Q}$ so that the order of $\bar{Q}$ is maximal among all such subgroups.

Set $M=N_G(\bar{Q})^{I(\bar{Q})}$.

(9) *A Sylow* 2*-subgroup of the stabilizer of any four points in* M *is noniden-*

tity and any nonidentity 2-subgroup of M fixing at least four points fixes exactly five or seven points.

Proof. Let P_0 be a Sylow 2-subgroup of $N_G(\bar{Q})_{ijkl}$ for any four points i, j, k and l in $I(\bar{Q})$ and P' be a Sylow 2-subgroup of G_{ijkl} containing P_0. Then $P_0=N_{P'}(\bar{Q})$. Since $P'>Q$, $N_{P'}(\bar{Q})>\bar{Q}$, and so $P_0^{I(\bar{Q})}=N_{P'}(\bar{Q})^{I(\bar{Q})}\neq 1$.

Let Q_0 be a 2-subgroup of $N_G(\bar{Q})_{ijkl}$ such that $Q_0>\bar{Q}$, P_0 be a Sylow 2-subgroup of $N_G(\bar{Q})_{ijkl}$ containing Q_0 and P' be a Sylow 2-subgroup of G_{ijkl} containing P_0. Then since $P'\geq N_{P'}(\bar{Q})=P_0\geq Q_0>\bar{Q}$, by the maximality of $|\bar{Q}|$ $I(P')\subseteq I(Q_0)\subset I(\bar{Q})$, and so $|I(Q_0^{I(\bar{Q})})|=|I(Q_0)|=5$ or 7.

(10) *Let $\{i_1, i_2, i_3\}$ be the unique G_{ijkl}-orbit of length three for any four points i, j, k and l in $I(\bar{Q})$.*

There exists an involution in M_{ijkl} fixing seven points if and only if $I(\bar{Q})$ contains three points i_1, i_2 and i_3.

Then an involution in M_{ijkl} fixing seven points fixes seven points i, j, k, l, i_1, i_2 and i_3.

Proof. If there exists an involution in M_{ijkl} fixing seven points, then there exists a 2-subgroup Q of G_{ijkl} such that $Q>\bar{Q}$ and $|I(Q)|=7$. By (8) $I(Q)=\{i, j, k, l, i_1, i_2, i_3\}$, and so $I(\bar{Q})$ contains three points i_1, i_2 and i_3.

Conversely $I(\bar{Q})$ contains three points i_1, i_2 and i_3. Let P' be a Sylow 2-subgroup of G_{ijkl} containing $\bar{Q}$ and $I(P')=\{i, j, k, l, i_1\}$. Then $P'>P'_{i_2}\geq\bar{Q}$ and $I(P'_{i_2})=\{i, j, k, l, i_1, i_2, i_3\}$. Thus $P'_{i_2}>\bar{Q}$, and so $N_{P'_{i_2}}(\bar{Q})>\bar{Q}$. By the maximality of $|\bar{Q}|$ there exists an involution in M_{ijkl} fixing seven points.

(11) *Let $\{i_1, i_2, i_3\}$ be the unique G_{ijkl}-orbit of length three for any four points i, j, k and l in $I(\bar{Q})$. Then $I(\bar{Q})$ contains three points i_1, i_2 and i_3 and any involution in M_{ijkl} fixes exactly seven points i, j, k, l, i_1, i_2 and i_3.*

Proof. Suppose by way of contradiction that for some four points i, j, k and l in $I(\bar{Q})$, there exists an involution x in M_{ijkl} fixing exactly five points. Since $|I(x)\cap\{i_1, i_2, i_3\}|\geq 1$ and $|I(\bar{Q})|\geq 9$, we may assume that

$$x=(i)(j)(k)(l)(i_1)(j_1j_2)\cdots,$$

where $\{j_1, j_2\}\neq\{i_2, i_3\}$. Set $C=C_M(x)_{j_1j_2}$ and we consider $C^{I(x)}$.

For any two points k_1 and k_2 in $I(x)$, x normalizes $M_{j_1j_2k_1k_2}$. Since $M_{j_1j_2k_1k_2}$ is of even order, $M_{j_1j_2k_1k_2}$ has an involution y commuting with x. Then $y\in C_M(x)_{j_1j_2k_1k_2}$. Since $|I(x)|=5$, y fixes one more point k in $I(x)$. If $y^{I(x)}=1$, then y fixes i, j, k, l, i_1, j_1 and j_2, which is contrary to (10). Hence $y^{I(x)}$ is a transposition.

Thus for any two points k_1 and k_2 in $I(x)$, there exists an involution fixing exactly three points k_1, k_2 and exactly one more point k in $I(x)-\{k_1, k_2\}$.

First assume that $C^{I(x)}$ is transitive on $I(x)$. Since $C^{I(x)}$ has a transposition, $C^{I(x)}=S_5$.

Next assume that $C^{I(x)}$ is intransitive on $I(x)$. Then one of the $C^{I(x)}$-orbits is of length less than three.

Suppose that $C^{I(x)}$ has an orbit $\{l_1\}$ of length one. Then for any two points k_1 and k_2 in $I(x)-\{l_1\}$, there exists an involution in $C^{I(x)}$ fixing exactly three points l_1, k_1 and k_2. Then by a lemma of D. Livingstone and A. Wagner, $C_{l_1}^{I(x)-\{l_1\}}$ is doubly transitive on $I(x)-\{l_1\}$, and so $C_{l_1}^{I(x)-\{l_1\}}=S_4$. Thus $C^{I(x)}=S_1\times S_4$.

Suppose that $C^{I(x)}$ has an orbit $\{l_1, l_2\}$ of length two. Then for any point k_1 in $I(x)-\{l_1, l_2\}$, there exists an involution in $C^{I(x)}$ fixing exactly three points l_1, l_2 and k_1. Then by a lemma of D. Livingstone and A. Wagner, $C_{l_1l_2}^{I(x)-\{l_1,l_2\}}$ is transitive on $I(x)-\{l_1, l_2\}$, and so $C_{l_1l_2}^{I(x)-\{l_1,l_2\}}=S_3$.

On the other hand $C^{I(x)}$ has an involution

$$x' = (l_1l_2)(l_3)(l_4)(l_5)\,.$$

Thus $C^{I(x)}=S_2\times S_3$.

Hence $C^{I(x)}=S_5$, $S_1\times S_4$ or $S_2\times S_3$. In any cases for some two points l_1 and l_2 in $I(x)$, $C_{l_1l_2}^{I(x)-\{l_1,l_2\}}=S_3$. Then $I(x)-\{l_1, l_2\}=\{l_3, l_4, l_5\}$ is the unique $G_{j_1j_2l_1l_2}$-orbit of length three. Since $I(\bar{Q})\supseteq\{j_1, j_2, l_1, l_2, l_3, l_4, l_5\}\supseteq I(x)=\{i, j, k, l, i_1\}$, by (7) $\{i, j, k, l, i_1, i_2, i_3\}=\{i, j, k, l, i_1, j_1, j_2\}$, which is a contradiction.

Thus for any four points i, j, k and l in $I(\bar{Q})$, $I(\bar{Q})$ contains all the points of G_{ijkl}-orbit $\{i_1, i_2, i_3\}$ of length three and any involution in M_{ijkl} fixes exactly seven points i, j, k, l, i_1, i_2 and i_3.

(12) *$M=M_{23}$ and $\{\Delta(i, j, k, l)\,|\,i, j, k, l\in I(\bar{Q})\}$ forms a 4-(23, 7, 1) design on $I(\bar{Q})$.*

Proof. By (9) and (11) M satisfies the condition of Lemma. By Lemma $M=M_{23}$, and so $\{\Delta(i, j, k, l)\,|\,i, j, k, l\in I(\bar{Q})\}$ forms a 4-(23, 7, 1) design on $I(\bar{Q})$.

Let s be a point of a minimal $\bar{Q}$-orbit in $\Omega-I(\bar{Q})$. Set $R=\bar{Q}_s$ and $N=N_G(R)^{I(R)}$.

(13) *Let u be an involution in N such that $I(u)=I(\bar{Q})$, and let (i_1i_2) be a 2-cycle of u. For any two points i and j in $I(u)$, set $\Delta(i, j)=I(u)\cap\Delta(i_1, i_2, i, j)$. Then $|\Delta(i, j)|=3$.*

Proof. Let $\bar{u}$ be a 2-element of $N_G(R)$ such that $\bar{u}^{I(R)}=u$. For any two points i and j in $I(u)$, $\langle\bar{u}, R\rangle$ fixes $\Delta(i_1, i_2, i, j)$ as a set. Since u fixes two points i and j, u fixes one more point k in $\Delta(i_1, i_2, i, j)-\{i_1, i_2, i, j\}$. Thus $|\Delta(i, j)|\geq 3$.

Suppose that $|\Delta(i,j)|=5$ and set $\Delta(i,j)=\{i,j,k,l,m\}$. Then $\Delta(i_1,i_2,i,j)=\{i_1,i_2,i,j,k,l,m\}$. By (12) $I(u)\supseteq\Delta(i_1,i_2,i,j)\ni i_1,i_2$, which is a contradiction. Hence $|\Delta(i,j)|=3$.

(14) *$\{\Delta(i,j)\mid i,j\in I(u)\}$ forms a 2-(23, 3, 1) design on $I(u)$. Thus we have a contradiction and complete the proof of Theorem.*

Proof. For any two points i and j in $I(u)$, $\Delta(i,j)$ is a subset of $I(u)$.

Suppose that $\Delta(i,j)\ni i',j'$. Set $\Delta(i,j)=\{i,j,k\}$ and $\Delta(i_1,i_2,i,j)=\{i,j,k,i_1,i_2,j_1,j_2\}$. Since $i',j'\in\{i,j,k\}$, $\Delta(i_1,i_2,i,j)\ni i_1,i_2,i',j'$, and so $\Delta(i_1,i_2,i,j)=\Delta(i_1,i_2,i',j')$. Thus $\Delta(i,j)=\Delta(i',j')$.

Hence $\{\Delta(i,j)\mid i,j\in I(u)\}$ forms a 2-(23, 3, 1) design on $I(u)$. Then the number of blocks is

$$\frac{\binom{23}{2}}{\binom{3}{2}}=\frac{23\cdot 22}{3\cdot 2}=\frac{253}{3},$$

which is a contradiction.

Thus we complete the proof of Theorem.

3. Proof of Lemma

Let G be a group satisfying the assumption of Lemma.

(1) *For any four points i, j, k and l in Ω, let $\{i,j,k,l,i_1,i_2,i_3\}$ be the set of the fixed points of an involution in G_{ijkl}. Set $\Delta(i,j,k,l)=\{i,j,k,l,i_1,i_2,i_3\}$. Then $\{\Delta(i,j,k,l)\mid i,j,k,l\in\Omega\}$ forms a 4-(n, 7, 1) design on Ω.*

Proof. Suppose that $\Delta(i,j,k,l)\ni i',j',k',l'$. Then there exists an involution x in G_{ijkl} fixing i', j', k' and l'. Thus x is an involution in $G_{i'j'k'l'}$, and so $\Delta(i,j,k,l)=\Delta(i',j',k',l')$. Hence $\{\Delta(i,j,k,l)\mid i,j,k,l\in\Omega\}$ forms a 4-(n, 7, 1) design on Ω.

We may assume that $\Delta(1,2,3,4)=\{1,2,3,4,5,6,7\}$.

Let a be an involution in G_{1234}. Then we may assume that

$$a=(1)(2)\cdots(7)(8\ 9)\cdots.$$

Set $T=C_G(a)_{89}$.

(2) *For any two points i and j in $I(a)$, set $\Delta(i,j)=\Delta(1,2,3,4)\cap\Delta(8,9,i,j)$. Then $\{\Delta(i,j)\mid i,j\in I(a)\}$ forms a 2-(7, 3, 1) design on $I(a)$ and $T^{I(a)}\leq PGL(3,2)$.*

Proof. Since a normalizes G_{89ij} and G_{89ij} is of even order, G_{89ij} has an involution x commuting with a. Thus $x\in T_{ij}$. Since $|I(a)|=7$, x fixes one more point in $I(a)$, and so $|\Delta(i,j)|\geq 3$.

If $|\Delta(i, j)| \geq 4$, then by (1) $\Delta(1, 2, 3, 4)=\Delta(8, 9, i, j)$, which is a contradiction. Thus $|\Delta(i, j)|=3$.

Suppose that $\Delta(i, j) \ni i', j'$. Then $\Delta(8, 9, i, j) \ni 8, 9, i', j'$, and so by (1) $\Delta(8, 9, i, j)=\Delta(8, 9, i', j')$. Thus $\Delta(i, j)=\Delta(i', j')$.

Hence $\{\Delta(i, j) \mid i, j \in I(a)\}$ forms a 2-(7, 3, 1) design on $I(a)$. Since $T^{I(a)}$ is an automorphism group of this design, $T^{I(a)} \leq PGL(3, 2)$.

(3) $|\Omega|=23$ *and* $G \leq M_{23}$.

Proof. Let $\{i_1, i_2\}$ be a subset of $I(a)$ consisting of two points. Since a normalizes $G_{89i_1i_2}$ and $G_{89i_1i_2}$ is of even order, a centralizes an involution x in $G_{89i_1i_2}$, and so $x \in C_G(a)_{89}$. By (2) $x^{I(a)} \in C_G(a)_{89}^{I(a)} \leq PGL(3, 2)$. Thus $I(x^{I(a)}) = \{i_1, i_2, i_3\}$ and x fixes two points of a 2-cycle $(\neq(8\ 9))$ of a. Thus a subset $\{i_1, i_2\}$ of $I(a)$ determines uniquely a 2-cycle $(k\ l)$ $(\neq(8\ 9))$ of a.

If a subset $\{j_1, j_2\}$ of $I(a)$ determines the same 2-cycle $(k\ l)$ of a, then an involution x' in G_{89kl} is contained in $G_{89j_1j_2}$. Thus $\{j_1, j_2\} \subseteq \Delta(8, 9, k, l) \cap I(a)=\{i_1, i_2, i_3\}$. Hence just three subsets $\{i_\mu, i_\nu\}$ of $I(a)$ determines the same 2-cycle $(k\ l)$ of a.

Now suppose that a 2-cycle $(k\ l)$ $(\neq(8\ 9))$ of a is given. Then since a normalizes G_{89kl} and G_{89kl} is of even order, a centralizes an involution x in G_{89kl}, and so $x^{I(a)} \in C_G(a)_{89}^{I(a)} \leq PGL(3, 2)$. Thus $I(x^{I(a)})=\{i_1, i_2, i_3\} \subseteq I(a)$. Since $x \in G_{89i_1i_2}$, $\{i_1, i_2\}$ determines $(k\ l)$ in the above sence.

Thus we have that the number of 2-cycles of a other than $(8\ 9)$ is equal to $\frac{1}{3}{}_7C_2=7$. Hence $|\Omega|=2+7+2\cdot 7=23$. Thus $\{\Delta(i, j, k, l) \mid i, j, k, l \in \Omega\}$ forms a 4-(23, 7, 1) design. Hence $G \leq M_{23}$.

(4) $G=M_{23}$ *and we complete the proof.*

Proof. Let P be a Sylow 2-subgroup of G_{ijkl} for any four points i, j, k and l in Ω. By the assumption $P \neq 1$, $|I(P)| \geq 4$ and $P \leq M_{23}$ by (3). Thus $|I(P)|=7$ and $N_G(P)^{I(P)} \leq A_7$. Since P is nonidentity semiregular by the assumption, $G=M_{23}$ by Theorem 1 in [8].

Thus we complete the proof of Lemma.

4. Proof of Corollary

Let D be a 4-$(v, 7, 1)$ design. Let $\{1, 2, 3, 4, i, j, k\}$ be a block containing $\{1, 2, 3, 4\}$. Then G_{1234} fixes $\{i, j, k\}$ as a set. If G_{1234} has an orbit of length one in $\{i, j, k\}$, then $G=S_5$, A_6 or M_{11} by a theorem of H. Nagao [4]. Hence D is a 4-(11, 7, 1) design. Then the number of blocks is

$$\frac{\binom{11}{4}}{\binom{7}{4}}=\frac{11\cdot 10\cdot 9\cdot 8}{7\cdot 6\cdot 5\cdot 4}=\frac{66}{7},$$

which is a contradiction. If $\{i, j, k\}$ is a G_{1234}-orbit, then $G=S_7$, A_7 or M_{23} by Theorem. Hence D is a 4-(7, 7, 1) design or a 4-(23, 7, 1) design. Thus we complete the proof of Corollary.

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