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The purpose of this paper is the investigation of modules over Dedekind prime rings. In Section 1, we shall prove that the double centralizer of a $P$-primary module over a Dedekind prime ring $R$ is isomorphic to $\hat{R}_P$ or $\hat{R}_P/\hat{P}^n$, where $P$ is a nonzero prime ideal of $R$ and $\hat{R}_P$ is the $P$-adic completion of $R$ with unique maximal ideal $\hat{P}$. Using this result we shall determine the structure of the double centralizer of primary modules over bounded Dedekind prime rings. In Section 2, we shall give a characterization of quasi-injective modules over bounded Dedekind prime rings. This paper is a continuation of [7] and [8]. A number of concepts and results are needed from [7] and [8].

1. The double centralizer of torsion modules

Throughout this paper, $R$ will denote a Dedekind prime ring with the two-sided quotient ring $Q$, we denote the completion of $R$ with respect to $P$ by $\hat{R}_P$ and its maximal ideal by $\hat{P}$. By Theorem 1.1 of [6], $\hat{R}_P$ is a complete, $g$-discrete valuation ring in the sense of [8] and $\hat{R}_P=(\hat{L})_k$, where $\hat{L}$ is a complete, discrete valuation ring with unique maximal ideal $\hat{P}_o$. Further, $\hat{P}=p_o\hat{R}_P=\hat{R}_Pp_o$, where $p_o\in \hat{L}$ with $\hat{p}_o=p_o\hat{L}$. Since the proper ideals of $\hat{R}_P$ are only the powers of $\hat{P}$, we obtain $\hat{P}^n=\hat{R}_Pp^nP_\hat{P}$ for $n=0, 1, 2, \ldots$ (cf. the proof of Theorem 4.5 of [4]). In this section we denote the complete set of the matrix units of $\hat{R}_P=(\hat{L})_k$ by $e_{ij}$ ($i=1, 2, \ldots, k$).

Let $M$ be a $P$-primary module. Then, by the same way as in Lemma 3.14 of [7], $M$ is an $\hat{R}_P$-module by a natural way. It is evident that $\text{Hom}_R(M, M)=\text{Hom}_{\hat{R}_P}(M, M)$ and that $M$ is torsion as an $\hat{R}_P$-module. If $M$ is indecomposable, $P$-primary and divisible, then $M$ is isomorphic to $\lim_{\leftarrow} e_{11}\hat{R}_P/e_{11}\hat{P}^n$, and we denote it by $R(P^n)$. If $M$ is indecomposable, $P$-primary with $O(M)=P^n$, then $M$ is isomorphic to $e_{11}\hat{R}_P/e_{11}\hat{P}^n$, and we denote it by $R(P^n)$.

Lemma 1.1. Let $R$ be a Dedekind prime ring. Then the double centralizer $D_n$ of the module $R(P^n)$ is isomorphic to $\hat{R}_P/\hat{P}^n$.

Proof. By Lemma 3.20 of [7], $L_n=\text{Hom}_R(R(P^n), R(P^n))$, where $L_n=\hat{L}/\hat{P}^n$. Hence we have
\[ R(P^n) = L_n(e_{i1} + e_{i2} \hat{P}^n) + \cdots + L_n(e_{i_k} + e_{i1} \hat{P}^n). \]

From this the assertion is evident.

**Lemma 1.2.** Let \( R \) be a Dedekind prime ring. Then the double centralizer \( D \) of the module \( R(P^n) \) is isomorphic to \( \hat{R}_P \).

**Proof.** It is clear that \( R(P^n) \) is faithful as an \( \hat{R}_P \)-module. Hence \( D \cong \hat{R}_P \).

Let \( d \) be any nonzero element of \( D \). Then \( \tilde{d}^n[(e_{i1} \hat{R}_P/e_{i1} \hat{P}^n) d] = 0 \), because \( \text{Hom}_R(R(P^n), R(P^n)) = e_{i1} \hat{R}_P e_{i1} \) (cf. Theorem 3.21 of [7]). Therefore we may assume that \( d = d | e_{i1} \hat{R}_P/e_{i1} \hat{P}^n = r_n (r_n \in \hat{R}_P) \) by Lemma 1.1, where \( | \) means the restriction and \( r_n \) is unique up to mod \( \hat{P}^n \). Since \( R(P^n) \) is injective, the natural homomorphism \( e_{i1} \hat{R}_P/e_{i1} \hat{P}^{n+1} \to e_{i1} \hat{R}_P/e_{i1} \hat{P}^n \) can be extended to a map \( \varphi_n: R(P^n) \to R(P^n) \). Because

\[ (e_{i1} \hat{R}_P/e_{i1} \hat{P}^n) r_n = [\varphi_n(e_{i1} \hat{R}_P/e_{i1} \hat{P}^{n+1})] d = \varphi_n[(e_{i1} \hat{R}_P/e_{i1} \hat{P}^{n+1}) d] = (e_{i1} \hat{R}_P/e_{i1} \hat{P}^n) r_{n+1}, \]

we have \( r_n - r_{n+1} \in \hat{P}^n \). Therefore \( \hat{r} = (\cdots, r_n, \cdots) \in \hat{R}_P \) and it is easily seen that \( d = \hat{r} \).

**Lemma 1.3.** Let \( S \) be a \( g \)-discrete valuation ring with unique maximal ideal \( P \) (cf. [8]). Assume that \( B \) is a submodule of the torsion \( S \)-module \( M \) and that \( B = \sum_n \oplus B_n \), where \( B_n \) is a direct sum of cocyclic modules of order \( P^n \). Then \( B \) is a basic submodule of \( M \) if and only if

\[ M = B_1 \oplus \cdots \oplus B_n \oplus (B_n^+ + MP^n) \quad \text{for every } n, \]

where \( B_n^+ = B_{n+1} \oplus B_{n+2} \oplus \cdots \) (cf. Theorem 32.4 of [2]).

In the case of indecomposable, injective and \( P \)-primary modules the following theorem was proved by Kuzmanovich [6].

**Theorem 1.4.** Let \( R \) be a Dedekind prime ring, let \( M \) be a \( P \)-primary module and let \( D \) be the double centralizer of \( M \). Then

(a) If \( O(M) = P^n \), then \( D \cong \hat{R}_P/P^n \).

(b) If \( M \) is faithful, then \( D \cong \hat{R}_P \).

**Proof.** We may assume without loss of generality that \( R \) is a complete, \( g \)-discrete valuation ring with unique maximal ideal \( P \). Let \( H = \text{Hom}_R(M, M) \)

and \( D = \text{Hom}_R(M, M) \).

(a) It is evident that \( D \cong R/P^n \). By Theorems 3.7 and 3.38 of [7], \( M = \sum e_i M \), where \( e_i M \cong R(P^n) \) and \( e_i \) is an idempotent in \( \text{Hom}_R(M, M) \). Since \( O(M) = P^n \), there is \( e_i \in H \) such that \( O(e_i M) = P^n \). Let \( d \) be any element of \( D \). Then \( (e_i M) d = e_i (Md) \subseteq e_i M \). Thus, by Lemma 1.1, \( d = e_i M = r \), where \( r \in R \) and it is unique up to mod \( P^n \). Now, for any direct summand

\[ a R \]
e_iM, there exists \( \varphi_i \in H \) such that \( \varphi_i(e_i x) = e_i M \). Let \( u \) be any element of \( e_iM \). Then \( ud = \varphi_i(vd) = \varphi_i(vr) = ur \), and thus we obtain \( \text{d} = r \), as desired.

(b) It is evident that \( D \supseteq R \). To prove the converse inclusion, let \( d \) be any nonzero element of \( D \).

Case I. If \( M \) is divisible, then \( M = \sum \oplus M_i \), where \( M_i = R(P^{\ast_1}) \). Let \( \pi_i \) be the projection map from \( M \) to \( M \). Therefore, by Lemma 1.2, \( d_i = d | M_i = r_i \), where \( r_i \in R \). For any \( i, j \), there is an element \( \varphi_i \in H \) such that \( \varphi_i(M_i) = M_j \). Let \( y \) be any element of \( M_j \) and let \( \varphi_i(x) = y(x \in M_i) \). Then \( yr_i = yd = [\varphi_i(x)]d = \varphi_i(xd) = yr_i \). Thus we have \( r_i = r_j \), and so \( d = r \) for some \( r \in R \).

Case II. If \( M \) is reduced, then it is evident that \( B^* \neq 0 \) for every natural integer \( n \), where \( B^* \) is defined in Lemma 1.3. Hence we have submodules \( \{ M_i \} \) with the following properties:

1. \( M_i = R(P^{n_i}) \), where \( n_1 < n_2 < \cdots \).
2. \( M_i = e_i M \), where \( e_i \) is an idempotent element of \( H \). Then \( (e_i M) = (e_i M) = e_i M \) and \( H \supseteq \text{Hom}(e_i M, e_i M) \). Hence \( d_i = d | M = r_i \) by Lemma 1.2, where \( r_i \in R \).

For any \( i, j \), \( i < j \), there is an element \( e_{ij} \in H \) such that \( e_{ij}(M_j) = M_i \). Now let \( x \) be any element of \( e_i M \). Then we have

\[
(e_{ij}x)r_i = (e_{ij}x)d = e_{ij}(xd) = e_{ij}(xr_j) = (e_{ij}x)r_j.
\]

Hence \( r_i - r_j \in P^{n_i} \), and so \( r_i = r_j \) for every \( i \). Let \( u \) be any uniform element of \( M \). Then \( uR = R(P^l) \) for some \( l \) by Lemma 3.37 of [7]. So there is \( \theta_i \in H \) such that \( \theta_i \) maps \( e_i M \) onto \( uR \). Let \( \theta_i(e_i y) = u_y \), where \( y \in M \). Then we obtain

\[
ud = [\theta_i(e_i y)]d = \theta_i[(e_i y)d] = \theta_i[(e_i y)r] = u^r.
\]

Let \( m \) be any element of \( M \). Then, by Theorem 3.38 of [7], \( mR \) is a direct sum of a finite number of reduced cocyclic modules, and so \( md = mr \), as desired.

Case III. If \( M \) is not reduced, then there are idempotent elements \( e_1, e_2 \in H \) such that \( M = e_1 M \oplus e_2 M \), where \( e_1 M \) is divisible and \( e_2 M \) is reduced. First we assume that \( e_1 M \) is not bounded, then, by Cases I, II, there exist \( r_1, r_2 \in R \) such that \( d_i = r_i \), where \( d_i = d | e_i M \) (i = 1, 2). Let \( u \) be any uniform element in \( e_i M \). Then there is \( \varphi \in H \) such that \( \varphi(e_2 M) = uR \), because \( e_2 M \) contains a reduced, cocyclic direct summand \( U \) such that \( O(U) \subseteq O(uR) \). Let \( \varphi(x) = u_y \), where \( x \in e_2 M \). Then we have

\[
ur_1 = ud = [\varphi(x)]d = \varphi(xd) = \varphi(xr_2) = ur_2.
\]

Therefore \( r_1 = r_2 \). Second assume that \( e_2 M \) is of bounded order. By Case I, there is \( r_1 \in R \) such that \( d_1 = d | e_1 M = r_1 \) and \( e_2 M = \sum \oplus N_i \) by Theorem 3.7 of [7], where \( N_i = R(P^{n_i}) \). For each \( i \), there is \( \theta_i \in H \) such that it induces a mono-
morphism from $N_i$ to $e_iM$. Let $u$ be any element of $N_i$ and let $\theta_i(u)=x\in e_iM$. Then we obtain

$$\theta_i(ur_i) = xd = [\theta_i(u)]d = \theta_i(ud).$$

Hence $ur_i=ud$, and thus we have $r_i=d$. This completes the proof of Theorem 1.4.

**Corollary 1.5.** Let $R$ be a bounded Dedekind prime ring, let $M$ be a torsion module and let $M=\sum \oplus M_p$ be the primary decomposition of $M$ (cf. Theorem 3.2 of [7]). Then the double centralizer $D$ of $M$ is isomorphic to $\Pi \hat{R}_p|\hat{P}^n_p$, where $O(M_p)=P^n_p$, $n_p$ is a natural integer or $\infty$ and $\hat{P}^\infty=0$.

**Proof.** Let $\alpha=(r_p+\hat{P}^n_p)$ be any element of $\Pi \hat{R}_p|\hat{P}^n_p$, where $r_p\in \hat{R}_p$ and let $m=\sum p^i m_{pi}$ be any element of $M$, where $m_{pi}\in M_p$. Define $ma=\sum p^i m_{pi}$. By Theorem 1.4, it is easily seen that $\alpha \in D$. Conversely let $d$ be any element of $D$. Since $M_d \subseteq M_p$, we have $d_p=r_p+\hat{P}^n_p$, where $d_p=d|M_p$. Then it is evident that $d=(r_p+\hat{P}^n_p)$.

2. Quasi-injective modules

Let $R$ be a bounded Dedekind prime ring and let $Q$ be the quotient ring of $R$. In [7], the author proved that any injective module is a direct sum of minimal right ideals of $Q$ and modules of type $P^\infty$ for various prime ideals $P$.

In this section, we shall characterize quasi-injective modules. By virtue of Goldie’s theorem, $Q=(F)_k$, where $F$ is a division ring. Throughout this section we denote a complete matrix units of $Q=(F)_k$ by $e_i$.

**Lemma 2.1.** If a module $M=\sum \oplus M_\alpha$ and if $N$ is a fully invariant submodule of $M$, then $N=\sum \oplus (M_\alpha \cap N)$ (cf. Lemma 9.3 of [3]).

**Theorem 2.2** Let $R$ be a bounded Dedekind prime ring and let $M$ be a module. Then $M$ is quasi-injective if and only if it is;

(i) injective, or

(ii) a torsion module such that every $P$-primary component $M_p$ is a direct sum of isomorphic cocyclic modules.

**Proof.** The sufficiency easily follows from Theorem 1.1 of [5] and Proposition 1.1 of [8].

Conversely assume that $M$ is quasi-injective. Then the injective envelope $E(M)$ of $M$ is isomorphic to $\sum \oplus \bar{M}_\alpha$, where $\bar{M}_\alpha$ is a minimal right ideal of $Q$ or a module of type $P^\infty$. By Lemma 2.1 and Theorem 1.1 of [5], we have $M=\sum \oplus M_\alpha$, where $M_\alpha=\bar{M}_\alpha \cap M$.

Case 1. If $M$ is torsion-free then we may assume that $\bar{M}_\alpha=e_{11}Q$ for all $\alpha$. Assume that $M$ is not injective, then there is $M_\alpha$ such that $M_\alpha \subseteq \bar{M}_\alpha=e_{11}Q$. By
virtue of Faith-Utumi’s Theorem (cf. Theorem 6 of [1], p. 91) there is an Ore domain $D$ such that

$$S = \sum_{i,j} D e_{ij} \subseteq R \subseteq Q = (F)_h,$$

and $F$ is the quotient division ring of $D$. Now let

$$U = \left\{ \begin{pmatrix} d_{11} & \cdots & d_{1k} \\ 0 & \ddots & \vdots \\ 0 & \cdots & d_{kk} \end{pmatrix} \mid d_{ij} \in D \right\}.$$

Since $U$ is a uniform right ideal of $S$ and $Q$ is a quotient ring of $S$, we have $\theta \neq M_a U$. Hence there exists an element $u_a \in M_a$ such that $\theta = u_a U \approx U$ as an $S$-module. Let $q$ be any element of $\bar{M}_a (= e_{11}Q)$. Then there is an element $d \in D$ such that $d q = v \in U$, because $D$ is an Ore domain. It is clear that $O(v) = O(q)$.

Since $u_a U \approx U$, there exists an element $u \in U$ such that $O(u_a u) = O(v)$. The map $\theta : u_a u R \to q R$ defined by $u_a u r \to qr$, for $r \in R$, can be extended to the map $\theta : \bar{M}_a \to \bar{M}_a$. Since $\theta(u) \subseteq M$ and $\theta(u_a u) = q \in M$, we have $\bar{M}_a = M_a$, which is a contradiction. Therefore $M$ is injective.

Case II. If $M$ is torsion, then $M = \sum P_m$, where $M_m$ is the $P$-primary part of $M$ and $M_m$ is also quasi-injective. Hence we may assume that $M$ is $P$-primary, quasi-injective and that $M = \sum M_m$ where $M_m = R(P^n)$ ($n = 1, 2, \ldots$ or $\infty$). If $M_m = R(P^n)$ and $M_m = R(P^m)$ for $\alpha \neq \beta$, where $\infty \geq n > m$, then there exists a monomorphism $\phi : M_m \to \bar{M}_m$ ($= R(P^m)$), and it can be extended to an isomorphism $\phi : \bar{M}_m \to \bar{M}_m$. It is clear that $\phi(M_m) \subseteq \bar{M}_m \cap M = M_m$. This is a contradiction, and thus $m = n$.

Case III. If $M$ is mixed, then since $E(M) = C \oplus \bar{T}$, where $C$ is torsion-free and $\bar{T}$ is the torsion part of $E(M)$, we obtain $M = C \oplus T$, where $C = \overline{C} \cap M$ and $T = \overline{T} \cap M$. By Case I, $C = \sum \oplus C_{1q}$ and, by Case II, $T = \sum \oplus T_{p}$, $T_p = \sum \oplus R(P^n)$ for fixed $n_p$, where $T_p$ is the $P$-primary part of $T$ and $n_p$ is a natural integer or $\infty$. Now assume that $M$ is not injective, then there exists a prime ideal $P$ such that $T_P$ is not injective, i.e., $n_p$ is a natural integer. Consider the module $e_{11}R/e_{11}P^m$ for a fixed $m (> n_p)$. By Theorem 3.7 of [7], $e_{11}R/e_{11}P^m$ contains $R(P^m)$ as a direct summand. Hence there exists a map $\eta$ such that $e_{11}R \to R(P^m) \to 0$ is exact. It can be extended to a map $\eta : e_{11}Q \to R(P^m)$. Thus we have $R(P^m) \subseteq \eta(e_{11}Q) \subseteq M$, which is a contradiction.

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References