Title	On multiply transitive groups. XIII
Author(s)	Oyama, Tuyosi
Citation	Osaka Journal of Mathematics. 1976, 13(2), p. 367-383
Version Type	VoR
URL	https://doi.org/10.18910/9973
rights	
Note	

Oyama, T.
Osaka J Math.
13 (1976), 367-383

ON MULTIPLY TRANSITIVE GROUPS XIII

Dedicated to Professor Mutuo Takahasi on his 60th birthday

TUYOSI OYAMA

(Received March 26, 1975)

1. Introduction

In this paper we shall prove the following

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots ‘ n\}$. If the order of the stabilizer of four points in G is not divisible by three, then G is one of the following groups: S_4, S_5, S_6, A_6, M_{11} or M_{12}.*

In the proof of this theorem we shall use the following two lemmas, which will be proved in the section 3 and 4.

Lemma 1. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ satisfying the following two conditions.*

(i) *The order of the stabilizer of any four points in G is even and not divisible by three.*

(ii) *Any involution fixing at least four points fixes exactly four or six points.*

Then $G=S_6$ or M_{12}.

Lemma 2. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$ satisfying the following three conditions.*

(i) *The order of the stabilizer of any four points in G is even and not divisible by three.*

(ii) *Any involution fixing at least four points fixes exactly four or twelve points.*

(iii) *For any 2-subgroup X fixing exactly twelve points, $N(X)^{I(X)}\leq M_{12}$.*

Then $G=S_6$ or M_{12}.

We shall use the same notation as in [4].

2. Proof of the theorem

Let G be a group satisfying the assumption of the theorem. If the order of the stabilizer of four points in G is odd and not divisible by three, then G is S_4, S_5, A_6 or M_{11} by a theorem of M. Hall ([1], Theorem 5.8.1). Hence we may consider only the case in which the stabilizer of four points in G is of even order.

Let P be a Sylow 2-subgroup of G_{1234}. Then $P\neq 1$. If P is semiregular on $\Omega-I(P)$, then G is S_6 or M_{12} by Theorem of [3] and the assumption. Hence from now on we assume that P is not semiregular on $\Omega-I(P)$ and prove the theorem by way of contradiction.

By Corollary of [5] and Theorem of [7], $|I(P)|=4$ or 5. We treat these cases separately.

Case I. $|I(P)|=4$.

(1) *There is a point t in $\Omega-I(P)$ such that $|I(P_t)|=6$ or 12 and $N(P_t)^{I(P_t)}=S_6$ or M_{12} respectively. In particular if t is a point of a minimal P-orbit, then $N(P_t)^{I(P_t)}$ is one of the groups listed above.*

Proof. Since G has no element of order three fixing at least four points, this follows from Corollary of [6].

(2) *Any element of order three fixes no point or exactly three points.*

Proof. By (1), there is a point t in $\Omega-I(P)$ such that $N(P_t)^{I(P_t)}=S_6$ or M_{12}. Then $N(P_t)$ has a 3-element whose restriction on $I(P_t)$ has exactly three fixed points. Since any element of order three fixes at most three points, $|\Omega|\equiv 0$ (mod 3) and any element of order three fixes no point or exactly three points.

(3) *If G has a 2-subgroup Q such that $|I(Q)|=6$ and $N(Q)^{I(Q)}=S_6$, then there is no 2-subgroup R such that $|I(R)|=12$ and $N(R)^{I(R)}=M_{12}$.*

Proof. Suppose by way of contradiction that there are 2-subgroups Q and R such that $|I(Q)|=6$, $N(Q)^{I(Q)}=S_6$, $|I(R)|=12$ and $N(R)^{I(R)}=M_{12}$. Let $\bar{Q}$ be a Sylow 2-subgroup of $G_{I(Q)}$. Then $|I(\bar{Q})|=6$ and $N(\bar{Q})^{I(\bar{Q})}=S_6$. Similarly let $\bar{R}$ be a Sylow 2-subgroup of $G_{I(R)}$. Then $|I(\bar{R})|=12$ and $N(\bar{R})^{I(\bar{R})}\geq M_{12}$. If $N(\bar{R})^{I(\bar{R})}\neq M_{12}$, then $N(\bar{R})^{I(\bar{R})}\geq A_{12}$. Hence $N(\bar{R})^{I(\bar{R})}$ has an element which is of order three and fixes nine points, contrary to (2). Thus $N(\bar{R})^{I(\bar{R})}=M_{12}$. Hence we may assume that Q and R are Sylow 2-subgroups of $G_{I(Q)}$ and $G_{I(R)}$ respectively.

Since G is 4-fold transitive on Ω, we may assume that P contains Q and R. Then set $I(Q)=\{1, 2, 3, 4, i_1, i_2\}$ and $I(R)=\{1, 2, 3, 4, j_1, j_2, \cdots, j_8\}$. Since $N(Q)^{I(Q)}=S_6$, for any point i of $\{i_1, i_2\}$ $P_i=Q$ and Q is a Sylow 2-subgroup of G_{1234i}. Similarly since $N(R)^{I(R)}=M_{12}$, for any point j of $\{j_1, j_2, \cdots, j_8\}$ $P_j=R$ and R is a Sylow 2-subgroup of G_{1234j}. Hence the G_{1234}-orbit Δ containing i is different from the G_{1234}-orbit Γ containing j. Since $N(Q)^{I(Q)}=S_6$ and

$N(R)^{I(R)}=M_{12}$, $\{i_1, i_2\}\subseteq\Delta$ and $\{j_1, j_2, \cdots, j_8\}\subseteq\Gamma$.

Since $N(Q)^{I(Q)}=S_6$, there is an element

$$x=(1\ 2\ 3)\ (4)\ (i_1)\ (i_2)\cdots.$$

Then $x\in N(G_{1234})$. Hence x induces a permutation on the set of G_{1234}-orbits. Since $\{i_1, i_2\}\subseteq\Delta$ and $\{i_1, i_2\}^x=\{i_1, i_2,\}$ $\Delta^x=\Delta^x$. Since the order of G_{1234} is not divisible by three, the lengts of G_{1234}-orbits in $\Omega-\{1, 2, 3, 4\}$ are not divisible by three. By (2), $I(x)=\{4, i_1, i_2\}$ and so x has no fixed point in $\Omega-(\{1, 2, 3, 4\}\cup\Delta)$. Thus $\Gamma^x\neq\Gamma$. On the other hand since $N(R)^{I(R)}=M_{12}$, there is an element

$$y=(1\ 2\ 3)\ (4)\ (j_1)\ (j_2)\ (j_3\ j_4\ j_5)\ (j_6\ j_7\ j_8)\cdots.$$

Then $y\in N(G_{1234})$. Since $\{j_1, j_2, \cdots, j_8\}\subseteq\Gamma$ and $\{j_1, j_2, \cdots, j_8\}^y=\{j_1, j_2, \cdots, j_8\}$, $\Gamma^y=\Gamma$. Hence $\Gamma^{yx^{-1}}=\Gamma^{x^{-1}}\neq\Gamma$. This is a contrdiction since $yx^{-1}\in G_{1234}$ and Γ is a G_{1234}-orbit. Thus we complete the proof.

(4) *Suppose that P has a subgroup Q such that $|I(Q)|=6$ and $N(Q)^{I(Q)}=S_6$ ($|I(Q)|=12$ and $N(Q)^{I(Q)}=M_{12}$). Let $\bar{Q}$ be a subgroup of P such that the order of $\bar{Q}$ is maximal among all subgroups of P fixing more than six (twelve) points. Set $N=N(\bar{Q})^{I(\bar{Q})}$. Then M satisfies the following conditions.*

(i) *The order of the stabillzer of any four points in N is even and not divisiby three.*

(ii) *Any involution of N fixing at least four points fixes exactly four or six (twelve) points.*

(iii) *N has an involution fixing exactly six (twelve) points.*

(iv) *When P has a subgroup Q such that $|I(Q)|=12$ and $N(Q)^{I(Q)}=M_{12}$, for any 2-subgroup X of N fixing exactly twelve points, $N_N(X)^{I(X)}\leq M_{12}$.*

Proof. (i), (ii) and (iv) are obvious. (iii) follows immediatly from Theorem 1 in [6].

(5) By Lemma 1 and 2, which will be proved in the section 4, there is no such group N as in (4). Thus we complete the proof of Case I.

Case II. $|I(P)|=5$.

(1) *Let t be a point of a minimal P-orbit in $\Omega-I(P)$. Then $|I(P_t)|=7,9$ or 13. In particular if $|I(P_t)|=9$ or 13, then $N(P_t)^{I(P_t)}\leq A_8$ or $N(P_t)^{I(P_t)}=S_1\times M_{12}$ respectively.*

Proof. This is Theorem of [6].

(2) $|I(P_t)|\neq 7$.

Proof. If $|I(P_t)|=7$, then $N(P_t)^{I(P_t)}$ is one of the groups listed in (2) of Case II in the section 3 of [6]. But these groups have an element of order three fixing four points. Thus $|I(P_t)|\neq 7$.

(3) $|I(P_t)|\neq 9$.

Proof. Suppose by way of contradiction that $|I(P_t)|=9$. Then we may assume that $I(P_t)=\{1, 2, \cdots, 9\}$. Set $N=N(P_t)^{I(P_t)}$. Then for any four points i, j, k and l of $I(P_t)$, N_{ijkl} has an involution fixing exactly five points.

First assume that N is primitive. Then since N is a subgroup of A_9 and has an involution fixing five points, $N=A_9$ (see [9]). But this is a contradiction since N has no element which is of order three and fixes six points.

Next assume that N is transitive but imprimitive. Then N has three blocks $\{i_1, i_2, i_3\}$, $\{j_1, j_2, j_3\}$ and $\{k_1, k_2, k_3\}$ of length three. Let x be an inovlution fixing i_1, i_2, j_1 and j_2. Then x fixes i_3, j_3 and one more point of $\{k_1, k_2, k_3\}$. Thus x is a transposition. This is a contradiction since $N\leq A_9$.

Finally assume that N is intransitive. Then one of the N-orbits is of length less than five.

Suppose that N has an orbit of length one, say $\{1\}$. Then for any four point i, j, k and l of $\{2, 3, \cdots, 9\}$, there is an involution in N fixing exactly five points $1, i, j, k$ and l. Then by a lemma of *D.* Livingstone and *A.* Wagner [2], N_1 is 4-fold transitive on $\{2, 3, \cdots, 9\}$. Thus $N=S_1\times A_8$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length two, say $\{1, 2\}$. Then for any three points i, j and k of $\{3, 4, \cdots, 9\}$, there is an involution in N fixing exactly five points $1, 2, i, j$ and k. Thus by a lemma of *D.* Livingstone and *A.* Wagner [2], N_{12} is 3-fold transitive on $\{3, 4, \cdots, 9\}$. Hence by [9], $N_{12}=A_7$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length three, say $\{1, 2, 3\}$. Set $\Delta=\{4, 5, \cdots, 9\}$. Then for any four points of Δ, there is an involution in N^{Δ} fixing exactly these four points. Hence by a lemma of *D.* Livingstone and *A.* Wagner [2], N^{Δ} is 4-fold transitive on Δ and so $N^{\Delta}=S_6$. Thus N has an element

$$x=(4)\,(5\ 6)\,(7\ 8\ 9)\cdots.$$

Since $N\leq A_9$, x is an even permutation. Hence x has one more 2-cycle on $\{1, 2, 3\}$. Thus x^2 is of order three and fixes six points, which is a contradiction.

Suppose that N has an orbit of length four, say $\{1, 2, 3, 4\}$. Set $\Delta=\{5, 6, \cdots, 9\}$. Then for any three points i, j and k of Δ, N has an involution fixing i, j, k and two more points of $\{1, 2, 3, 4\}$. Thus by a lemma of D. Livingstone and A. Wagner [2], N^{Δ} is 3-fold transitive on Δ and so $N^{\Delta}=S_5$. Thus N has an element

$$x=(5\ 6)\ (7\ 8\ 9)\cdots.$$

Since $N\leq A_9$, x is an even permutation. Hence x has one 2-cycle and two fixed points, or one 4-cycle on $\{1, 2, 3, 4\}$. Thus x^4 is of order three and fixes six points, which is a contradiction.

Thus $|I(P_t)|\neq 9$.

(4) If $|I(P_t)|=13$, then $N(P_t)^{I(P_t)}=S_1\times M_{12}$. Hence $N(P_t)^{I(P_t)}$ has an element of order three fixing four points, which is a contradiction.

Thus we complete the proof of Case II and so complete the proof of Theorem.

3. Proof of Lemma 1

Let G be a permutation group satisfying the assumptions of Lemma 1. If G has no involution fixing six points, then $G=S_6$ or M_{12} by Theorem 1 in [6] and the assumptions. Hence from now on we assume that G has an involution fixing exactly six points and prove Lemma 1 by way of contradiction. Then we may assume that G has an involution a fixing exactly six points $1, 2, \cdots, 6$ and

$$\mathrm{a}=(1)\ (2)\cdots(6)\ (7\ 8)\cdots.$$

Set $\mathrm{T}=\mathrm{C}(\mathrm{a})_{7\,8}$.

(1) *For any two points i and j of $I(a)$, there is an involution in $T_{i\,j}$. Any involution of T is not the identity on $I(a)$.*

Proof. Since a normalizes $G_{7\,8\,i\,j}$ and $G_{7\,8\,i\,j}$ is of even order, $G_{7\,8\,i\,j}$ has an involution x commuting with a. Then $x\in T_{i\,j}$. Since $|I(a)|=6$ and $I(x)\supseteq\{7, 8\}$, any involution of T is not the identity on $I(a)$ by (ii).

(2) *Any element of order three of T has no fixed points in $I(a)$.*

Proof. If an element u of order three of T has fixed points in $I(a)$, then since $|I(a)|=6$, u fixes at least three points of $I(a)$. This contradicts (i) since $I(u)\supseteq\{7, 8\}$. Thus any element of order three of T has no fixed point in $I(a)$.

(3) *We may assume that $(T^{I(a)})_{1\,2\,3\,4}=1$.*

Proof. By (2), $T^{I(a)}\neq S_6$. Hence there is four points in $I(a)$ such that the stabilizer of these four points in $T^{I(a)}$ is the identity. Hence we may assume that $(T^{I(a)})_{1\,2\,3\,4}=1$.

(4) $T^{I(a)}$ *is one of the following groups.*
(a) $T^{I(a)}$ *is intransitive and one of the* $T^{I(a)}$*-orbits is of length one, two or three.*
(b) $T^{I(a)}$ *is a transitive but imprimitive group with three blocks of length two or two blocks of length three.*
(c) $T^{I(a)}$ *is primitive.*

Proof. This is clear.

(5) $T^{I(a)}$ *has no orbit of length one.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length one.

First assume that a $T^{I(a)}$-orbit of length one is contained in $\{1, 2, 3, 4\}$. Then we may assume that $\{1\}$ is a $T^{I(a)}$-orbit of length one. By (1), T_{23} has an involution x_1. By (3), we may assume that

$$x_1 = (1)(2)(3)(5)(4\ 6)\cdots .$$

Similarly T_{24} has an involution x_2 of the form

$$x_2 = (1)(2)(4)(5)(3\ 6)\cdots \text{ or } (1)(2)(4)(6)(3\ 5)\cdots .$$

If x_2 is of the first from, then $x_1 x_2=(1)(2)(5)(3\ 6\ 4)\cdots$, contrary to (2). Thus x_2 is of the second form. Similarly T_{34} has an invloution x_3 of the form

$$x_3 = (1)(3)(4)(5)(2\ 6)\cdots \text{ or } (1)(3)(4)(6)(2\ 5)\cdots .$$

If x_3 is of the first form, then $x_1 x_3=(1)(3)(5)(2\ 6\ 4)\cdots$, contrary to (2). If x_3 is of the second form, then $x_2 x_3=(1)(4)(6)(2\ 5\ 3)\cdots$, contrary to (2).

Let $\{i\}$ be a $T^{I(a)}$-orbit of length one. Then as is shown above, for any three points j, k and l of $I(a)-\{i\}$ $(T^{I(a)})_{ijkl}\neq 1$. Hence by a lemma of *D*. Livingstone and *A*. Wagner [2], $(T^{I(a)})_i$ is 3-fold transitive on $I(a)-\{i\}$. Hence $(T^{I(a)})_i=S_5$. Then T has an element which is of order three and has fixed points in $I(a)$, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

(6) $T^{I(a)}$ *has neither orbit of length two nor block of length two.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length two or three blocks of length two.

First assume that $\{1, 2, 3, 4\}$ contains an orbit of length two or a block of length two. Then we may assume that $\{1, 2\}$ is an orbit or a block. By (1), T_{13} has an involution x_1. By (3), we may assume that

$$x_1 = (1)(2)(3)(5)(4\ 6)\cdots .$$

Let x_2 be an involution of T_{14}. Then similarly

$$x_2=(1)\,(2)\,(4)\,(5)\,(3\ 6)\cdots \text{ or } (1)\,(2)\,(4)\,(6)\,(3\ 5)\cdots.$$

If x_2 is of the first form, then $x_1x_2=(1)\,(2)\,(5)\,(3\ 6\ 4)\cdots$, contrary to (2). Thus x_2 is of the second form. Hence when $T^{I(a)}$ is imprimitive, $\{1, 2\}$, $\{3, 5\}$ and $\{4, 6\}$ form a complete block system. Let x_3 be an involution of T_{34}. When $T^{I(a)}$ is imprimitive

$$x_3=(1\ 2)\,(3)\,(4)\,(5)\,(6)\cdots.$$

When $T^{I(a)}$ has an orbit $\{1, 2\}$, x_3 is of this form or $x_3=(1\ 2)\,(3)\,(4)\,(5\ 6)\cdots$. But if $x_3=(1\ 2)\,(3)\,(4)\,(5\ 6)\cdots$, then $(x_1x_3)^2=(1)\,(2)\,(3)\,(4\ 6\ 5)\cdots$, contrary to (2). Thus in any case x_3 is of the same form on $I(a)$.

Set $\Delta=\{1, 2, \cdots, 8\}$. Let Q be a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3\rangle$. Then $a\in Z(Q)$, $Q^\Delta=\langle a, x_1, x_2, x_3\rangle^\Delta$ and $Q_\Delta=1$. Hence $Q=\langle a, \bar{x}_1, \bar{x}_2, \bar{x}_3\rangle$, where $\bar{x}_i^\Delta=x_i^\Delta$ and $\bar{x}_i$ is conjugate to x_i, $i=1, 2, 3$. Thus we may assume that $\langle a, x_1, x_2, x_3\rangle$ is a 2-group. Then $\langle a, x_1, x_2, x_3\rangle$ is elementary abelian. Since $|I(ax_1)|\leq 6$, $\langle a, x_1\rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length four.

Suppose that $\langle a, x_1\rangle$ has an orbit of length four. Then we may assume that $\{9, 10, 11, 12\}$ is an orbit of length four and

$$a=(1)\,(2)\cdots(6)\,(7\ 8)\,(9\ 10)\,(11\ 12)\cdots,$$
$$x_1=(1)\,(2)\,(3)\,(5)\,(4\ 6)\,(7)\,(8)\,(9\ 11)\,(10\ 12)\cdots.$$

Suppose that x_2 fixes $\{9, 10, 11, 12\}$. Then since $|I(ax_2)|\leq 6$ and $|I(x_1x_2)|\leq 6$, $x_2=(9\ 12)(10\ 11)$ on $\{9, 10, 11, 12\}$. Hence $\langle a, x_1, x_2\rangle_{9\,10\,11\,12}=\langle ax_1x_2\rangle$ and $I(ax_1x_2)=\{1, 2, 9, 10, 11, 12\}$. Thus $\langle a, x_1, x_2\rangle$ has exactly one orbit $\{9, 10, 11, 12\}$ of length four. Then since x_3 normalizes $\langle a, x_1, x_2\rangle$, x_3 fixes $\{9, 10, 11, 12\}$. Then by the same argument as is used for x_2, x_3 is of the same form as x_2 on $\{9, 10, 11, 12\}$. Hence $I(x_2x_3)\geq\{4, 6, 7, 8, 9, 10, 11, 12\}$, contrary to (ii). Thus x_2 does not fix any $\langle a, x_1\rangle$-orbit of length four. Hence $\langle a, x_1, x_2\rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length eight. Hence $\langle a, x_1, x_2, x_3\rangle$-orbits whose lengths are not two are of length eight or sixteen. If $\langle a, x_1, x_2, x_3\rangle$ has an orbit of length eight, then $\langle a, x_1, x_2, x_3\rangle$ has an involution fixing at least eight points of this orbit, contrary to (ii). Thus $\langle a, x_1, x_2, x_3\rangle^{\Omega-\Delta}$ has at most one orbit of lenght two and is semiregular on the set consisting of the remaining points. Since $\langle a, x_1\rangle$ nomralizes $G_{9\,10\,11\,12}$ and $G_{9\,10\,11\,12}$ is of even order, there is an involution y in $G_{9\,10\,11\,12}$ commuting with a and x_1. Then y fixes $\{1, 2, 3, 5\}$, $\{4, 6\}$ and $\{7, 8\}$. Suppose that $y^\Delta\in\langle a, x_1, x_2, x_3\rangle^\Delta$. Then since $\langle a, x_1, x_2, x_3, y\rangle_\Delta$ is of odd order, $\langle a, x_1, x_2, x_3\rangle$ is a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3, y\rangle$. Hence $\langle a, x_1, x_2, x_3\rangle$ has an element which is conjugate to y in $\langle a, x_1, x_2, x_3, y\rangle$. This is a contradiction since any involution of $\langle a, x_1, x_2, x_3\rangle$ fixes at most two points of $\Omega-\Delta$. Thus $y^{\Delta}\notin \langle a, x_1, x_2, x_3\rangle^{\Delta}$. Hence $\{1, 2\}^{y}=\{3, 5\}$. On the other hand since y fixes $\{7, 8\}$, y or ya is contained in T. Thus $\{1, 2\}$ is not a T-orbit. Then $T^{I(a)}$ is imprimitive and we may assume that $y=(1\ 3)\ (2\ 5)$ on $\{1, 2, 3, 5\}$. Then $x_2 y$ is of order $4m$, where m is odd. Set $z=(x_2 y)^{2m}$. Then

$$z = (1\ 2)\ (3\ 5)\ (4)\ (6)\ (7)\ (8) \cdots$$

and z centralizes $\langle a, x_1, x_2, y\rangle$. Since $|I(y)|\leq 6$, y fixes exactly four points 9, 10, 11 and 12 in $\Omega-\Delta$. Hence z fixes $\{9, 10, 11, 12\}$. Thus the $\langle a, x_1, x_2, z\rangle$-orbit containing $\{9, 10, 11, 12\}$ is of length eight. Since $\langle a, x_1, x_2, z\rangle$ is abelian and of order sixteen, there is an involution fixing this $\langle a, x_1, x_2, z\rangle$-orbit of length eight pointwise, contrary to (ii). Thus $\langle a, x_1\rangle$ has no orbit of length four. Since $|I(ax_1)|\leq 6$, $|\Omega|=8$ or 10.

Suppose that $|\Omega|=8$. Then by (i), there is an involution x in G fixing 1, 3, 4 and 7. If x fixes 8, then $x\in T$. Hence x fixes 2. Then $x^{I(a)}\in (T^{I(a)})_{1234}$ and $x^{I(a)}\neq 1$, contrary to (3). Hence $x=(1)\ (3)\ (4)\ (7)\ (8\ i) \cdots$, $i\in \{2, 5, 6\}$. Then $(ax)^2=(7\ 8\ i)$, contrary to (i).

Suppose that $|\Omega|=10$. Then

$$\begin{aligned} a &= (1)\ (2) \cdots (6)\ (7\ 8)\ (9\ 10)\,, \\ x_1 &= (1)\ (2)\ (3)\ (5)\ (4\ 6)\ (7)\ (8)\ (9\ 10)\,, \\ x_2 &= (1)\ (2)\ (3\ 5)\ (4)\ (6)\ (7)\ (8)\ (9\ 10)\,. \end{aligned}$$

By (i), there is an involution x in G fixing 1, 3, 4 and 7. Assume that x fixes 8. If x commutes with a, then $x\in T$. Hence x fixes 2. Then $x^{I(a)} \in (T^{I(a)})_{1234}$ and $x^{I(a)}\neq 1$, contrary to (3). Thus x does not commute with a and so $\{9, 10\}^{x}\neq \{9, 10\}$. If x fixes 9, then $x=(9)\ (10\ i) \cdots$, $i\in \{2, 5, 6\}$. Hence $(ax)^2=(9\ 10\ i)$, contrary to (i). Similarly x does not fix 10. Thus $x=(9\ i)\ (10\ j)$, $\{i, j\}\subset \{2, 5, 6\}$. Then $(x_1 x_2 x)^2$ is of order three and fixes at lesat four points, contrary to (i). Thus x does not fix 8. Hence $x=(1)\ (3)\ (4)\ (7)\ (8\ i) \cdots$, $i\in \{2, 5, 6, 9, 10\}$. If $i\in \{2, 5, 6\}$, then $ax=(1)\ (3)\ (4)\ (8\ 7\ i)\cdots$. Since $|\Omega|=10$, a suitable power of ax is of order three and fixes at least four points, contrary to (i). If $i\in \{9, 10\}$, then $ax_1x=(1)\ (3)\ (8\ 7\ i)\cdots$. Then similarly we have a contradiction. Hence $\{1, 2\}$ is neither orbit nor block.

Let $\{i, j\}$ be an orbit or a block of $T^{I(a)}$. Then by what we have proved above, for any two points k and l of $\{1, 2, \cdots, 6\} - \{i, j\}$ there is an involution in $(T^{I(a)})_{ijkl}$. Hence by a lemma of D. Livingstone and A. Wagner [2], $(T^{I(a)})_{ij}$ is doubly transitive on $I(a)-\{i, j\}$. Hence $(T^{I(a)})_{ij}=S_4$. Then $(T^{I(a)})_{ij}$ has an element of order three, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length two nor block of length two.

(7) *$T^{I(a)}$ has neither orbit of length three nor block of length three.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or two blocks of length three. When $T^{I(a)}$ is intransitive, $T^{I(a)}$ has two orbits of length three by (5) and (6). Let $\{i_1, i_2, i_3\}$ and $\{j_1, j_2, j_3\}$ be the two orbits or the two blocks. Then $T_{i_1 i_2}$ has an involution

$$x = (i_1)\,(i_2)\,(i_3)\,(j_1)\,(j_2 j_3)\cdots .$$

Since $\{j_1, j_2, j_3\}$ is an orbit or a block and $x \in T_{i_1 i_2 i_3}$, $(T^{I(a)})_{i_1 i_2 i_3} = S_3$. Thus $(T^{I(a)})_{i_1 i_2 i_3}$ has an element of order three, contrary to (2). Hence $T^{I(a)}$ has neither orbit of length three nor block of length three.

(8) *We show that $T^{I(a)}$ is not primitive and complete the proof.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ is primitive. Then since any element of order three in $T^{I(a)}$ has no fixed point, $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$ (see [9]). Let u be an element of order three of T. Since u commutes with a, if u has a fixed point in $\Omega - (I(a) \cup \{7, 8\})$, then u fixes at least two points of $\Omega - (I(a) \cup \{7, 8\})$, contrary to (i). Thus $I(u) = \{7, 8\}$ and so $|\Omega| \equiv 2 \pmod 3$. Furthermore this shows that any element of order three fixes exactly two points of Ω. Hence $N(G_{I(a)})^{I(a)}$ has no element consisting of exactly one 3-cycle. Thus $N(G_{I(a)})^{I(a)} \not\geq A_6$. Then since $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$, $N(G_{I(a)})^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$. Furthermore this shows that for any involution v fixing exactly six points, $N(G_{I(v)})^{I(v)} = PSL(2, 5)$ or $PGL(2, 5)$.

Suppose that G has an involution x fixing exactly four points. Then x is of the form

$$x = (i_1)\,(i_2)\,(i_3)\,(i_4)\,(j_1 j_2)\cdots .$$

For any two points i_r and i_s of $\{i_1, i_2, i_3, i_4\}$ x normalizes $G_{j_1 j_2 i_r i_s}$. Hence by (i), $G_{j_1 j_2 i_r i_s}$ has an involution y commuting with x. If y fixes $I(x)$ pointwise, then $I(y) = I(x) \cup \{j_1, j_2\}$. Thus $|I(y)| = 6$ and $x^{I(y)} = (j_1 j_2)$. This is a contradiction since $N(G_{I(y)})^{I(y)} = PSL(2, 5)$ or $PGL(2, 5)$. Hence y fixes exactly two points i_r and i_s in $I(x)$. Hence by a lemma of D. Levingstone and A. Wagner [2], $(C(x)_{j_1 j_2})^{I(x)} = S_4$. Thus $C(x)_{j_1 j_2}$ has a 3-element of the form $(i_1 i_2 i_3)(i_4)(j_1)(j_2)\cdots$. This is a contradiction since every element of order treee fixes exactly two points. Thus G has no involution fixing exactly four points.

Let x be an involution of $T_{1\,2}$. Then we may assume that

$$a = (1)\,(2)\cdots(6)\,(7\;8)\,(9\;10)\cdots ,$$
$$x = (1)\,(2)\,(3\;4)\,(5\;6)\,(7)\,(8)\,(9)\,(10)\cdots .$$

Let $(i\ j)$ be any 2-cycle of a. Then $(C(a)_{i\ j})^{I(a)}=PSL(2,5)$ or $PGL(2,5)$. Since $N(G_{I(a)})^{I(a)}$ is also $PSL(2,5)$ or $PGL(2,5)$, $T^{I(a)}=(C(a)_{i\ j})^{I(a)}$ or one of these two groups is a subgroup of the other. Hence there are 3-elements u and u' in T and $C(a)_{i\ j}$ respectively such that $u^{I(a)}=u'^{I(a)}$. Then u and u' normalize $G_{I(a)}$, $I(u)=\{7,8\}$ and $I(u')=\{i,j\}$. Let Γ be the $G_{I(a)}$-orbit containing $\{7,8\}$. Then since $\{7,8\}^u=\{7,8\}$, $\Gamma^u=\Gamma$. Suppose that $\{i,j\}$ is contained in a $G_{I(a)}$-orbit different form Γ. Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{u'}\neq\Gamma$. Thus $\Gamma^{uu'^{-1}}=\Gamma^{u'^{-1}}\neq\Gamma$. This is a contradiction since $uu'^{-1}\in G_{I(a)}$. Thus $\{i,j\}\subset\Gamma$. Since $(i\ j)$ is any 2-cycle of a, $G_{I(a)}$ is transitive on $\Omega-I(a)$. From the same reason, $G_{I(x)}$ is transitive on $\Omega-I(x)$. Then since $I(\langle G_{I(a)}, G_{I(x)}\rangle)=\{1,2\}$, $G_{1\,2}$ is transitive on $\Omega-\{1,2\}$. Since $N(G_{I(a)})$ is doubly transitive on $I(a)$, G is 3-fold transitive on Ω.

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)}=N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{1\,2\,3}=1$. Hence Q is a Sylow 2-subgroup of $G_{1\,2\,3}$. Since $|I(Q)|=6$, G is not 4-fold transitive by Theorem of [4]. On the other hand $G_{I(a)}$ is transitive on $\Omega-I(a)$. Hence there is a point i_1 in $\{4,5,6\}$ such that i_1 does not belong to the $G_{1\,2\,3}$-orbit containing $\Omega-I(a)$. Since Q is a Sylow 2-subgroup of $G_{1\,2\,3}$, the length of the $G_{1\,2\,3}$-orbit containing i_1 is not two. Moreover the length of the $G_{1\,2\,3}$-orbit containing i_1 is not three since $G_{1\,2\,3}$ has no element of order three. Thus $G_{1\,2\,3}$ fixes i_1. Since Q is a Sylow 2-subgroup of $G_{1\,2\,3}$, $\{4,5,6,\}-\{i_1\}$ is not a $G_{1\,2\,3}$-orbit. Similarly since $|\{4,5,\cdots,n\}-\{i_1\}|$ is even, $\{4,5,\cdots,n\}-\{i_1\}$ is not a $G_{1\,2\,3}$-orbit. Hence $G_{1\,2\,3}$-orbits on $\Omega-\{1,2,3\}$ are $\{4\}$, $\{5\}$, $\{6\}$ and $\{7,8,\cdots,n\}$ or $\{i_1\}$, $\{i_2\}$ and $\{i_3,7,8,\cdots,n\}$, where $\{i_1,i_2,i_3\}=\{4,5,6\}$. First assume that $\{4\}$, $\{5\}$, $\{6\}$ and $\{7,8,\cdots,n\}$ are $G_{1\,2\,3}$-orbits. By (i), $G_{1\,2\,3\,7}$ has an involution y. Then $y\in G_{1\,2\,3}$. Hence $I(y)\supset\{1,2,\cdots,7\}$, contraty to (ii). Next assume that $\{i_1\}$, $\{i_2\}$ and $\{i_3,7,8,\cdots,n\}$ are $G_{1\,2\,3}$-orbits. Since G is 3-fold transitive on Ω, $G_{1\,2\,3}=G_{1\,2\,i_1}=G_{1\,2\,i_2}$ and $G_{1\,2\,3}\neq G_{1\,2\,i_3}$. Thus $G_{1\,2\,i_3}$ fixes exactly two points of $\Omega-\{1,2,\cdots,6\}$. This is a contradiction since $a\in G_{1\,2\,i_3}$ and a has no fixed point in $\Omega-\{1,2,\cdots,6\}$.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

The proof of Lemma 2 is similar to the proof of Lemma 1. Let G be a permutation group satisfying the assumptions of Lemma 2. If G has no involution fixing twelve points, then $G=S_6$ or M_{12} by Theorem 1 and the assumptions. Hence from now on we assume that G has a involution fixing exactly twelve points and prove Lemma 2 by way of contradiction. Then we may assume that G has an involution a fixing exactly twelve points $1,2,\cdots,12$ and

$$a=(1)(2)\cdots(12)(13\ 14)\cdots.$$

Set $T=C(a)_{13\,14}$.

(1) *For any two points i and j of $I(a)$, there is an involution in $T_{i\,j}$. Any involution of T is not the identity on $I(a)$.*

(2) *Any element of order three in T has no fixed point on $I(a)$.*

The proofs of (1) and (2) are similar to the proofs of (3.1) and (3.2) respectively.

(3) *$T^{I(a)}$ is one of the following groups.*
(a) *$T^{I(a)}$ is intransitive and one of the $T^{I(a)}$-orbits is of length one, two, three, four, five or six.*
(b) *$T^{I(a)}$ is a transitive but imprimitive group with six blocks of length two, four blocks of length three, three blocks of length four or two blocks of length six.*
(c) *$T^{I(a)}$ is primitive.*

Proof. This is clear.

(4) *$T^{I(a)}$ is not primitive.*

Proof. If $T^{I(a)}$ is primitive, then by (iii) $T^{I(a)}$ is $PSL(2, 11)$, M_{11} or M_{12}, which are of degree twelve (see [9]). But since $T^{I(a)}$ has an involution fixing at least two points by (1), $T^{I(a)} \neq PSL(2, 11)$. Furthermore since any element of order three of $T^{I(a)}$ has no fixed point by (2), $T^{I(a)} \neq M_{11}$, M_{12}. Thus $T^{I(a)}$ is not primitive.

(5) *$T^{I(a)}$ has no orbit of length one.*

Proof. If $T^{I(a)}$ has an orbit $\{i\}$ of length one, then $T^{I(a)-\{i\}}$ is one of the groups of (4) of Lemma 4 in [5]. But all these groups have an element of order three which has fixed points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

(6) *$T^{I(a)}$ has neither orbit of length three nor block of length three.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or a block of length three, say $\{1, 2, 3\}$. Let x_1 be an involution of $T_{1\,2}$. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

When $T^{I(a)}$ is transitive but imprimitive, we may assume that the block containing 4 is $\{4, 5, 6\}$. Assume that $T^{I(a)}$ is intransitive. If the length of the orbit containing 4 is not divisible by three, then $(T^{I(a)})_4$ has an element of order three, contrary to (2). If the length of the orbit containing 4 is nine, then $(T^{I(a)})_1$ has an element of order three, contrary to (2). Thus the length of the orbit containing 4 is three or six. On the other hand x_1 fixes exactly one point 4 in the orbit containing 4. Hence the length of the orbit containing 4 is three. Thus we may assume that $\{4, 5, 6\}$ is an orbit.

Let x_2 be an involution of $T_{1\,5}$. Then x_2 fixes $\{1, 2, 3\}$ and $\{4, 5, 6\}$. If $x_2=(1)\,(5)\,(4\;6)\cdots$, then $x_1x_2=(1)\,(4\;6\;5)\cdots$, contrary to (2). Hence x_2 fixes $\{4, 5, 6\}$ pointwise. Since $|I(x_2^{I(a)})|=4$,

$$x_2 = (1)\,(2\;3)\,(4)\,(5)\,(6)\cdots .$$

Let x_3 be an involution of $T_{2\,5}$. Then by the same argument as is used for x_2,

$$x_3 = (2)\,(1\;3)\,(4)\,(5)\,(6)\cdots .$$

Then $x_2\,x_3=(1\;3\;2)\,(4)\,(5)\,(6)\cdots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) $T^{I(a)}$ *has no subgroup which is isomorphic to the following group* $\langle x_1, x_2, x_3\rangle$ *as a permutation group.*

$$\begin{aligned} x_1 &= (1)\;(2)\;(3)\;(4)\;(5\;\;6)\;(7\;\;8)\;(9\;\;10)\;(11\;\;12),\\ x_2 &= (1)\;(2)\;(3\;\;4)\;(5)\;(6)\;(7\;\;8)\;(9\;\;11)\;(10\;\;12),\\ x_3 &= (1\;\;2)\;(3\;\;4)\;(5)\;(6)\;(7)\;(8)\;(9\;\;10)\;(11\;\;12). \end{aligned}$$

Proof. This follows from the same argument as in the proof of (3.3) in [8].

(8) $T^{I(a)}$ *has neither orbit of length four nor block of length four.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length four or a block of length four, say $\{1, 2, 3, 4\}$.

First assume that T has an involution x_1 fixing $\{1, 2, 3, 4\}$ pointwise. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\;6)(7\;8)(9\;10)(11\;12)\cdots .$$

Let x_2 be an involution of $T_{1\,5}$. Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2 = (1)(2)(3\;4)(5)(6)(7\;8)(9\;11)(10\;12)\cdots .$$

Let x_3 be an involution of $T_{3\,5}$. Then similarly $x_3^{I(a)}$ commutes with $x_1^{I(a)}$. Hence $x_3^{I(a)}$ fixes 3, 5 and 6. Since $|I(x_3^{I(a)})|=4$, $x_3^{I(a)}$ fixes one more point of $\{1, 2, 4\}$. If x_3 fixes 1 or 2, then $x_2\,x_3=(1)\,(2\;4\;3)\,(5)\,(6)\cdots$ or $(2)\,(1\;4\;3)\,(5)\,(6)\cdots$ respectively, contrary to (2). Thus x_3 fixes 4. Then $x_3^{I(a)}$ commutes with $x_2^{I(a)}$ and so

$$x_3 = (1\;2)(3)(4)(5)(6)(7\;8)(9\;12)(10\;11)\cdots .$$

This is a contradiction since $T^{I(a)}$ has no such subgroup as $\langle x_1, x_2, x_3\rangle^{I(a)}$ by (7).

Next assume that T has no involution fixing $\{1, 2, 3, 4\}$ pointwise. Let x_1 be an involution of $T_{1\,2}$. Then

$$x_1 = (1)\,(2)\,(3\ 4) \cdots .$$

Let x_2 be an involution of $T_{1\,3}$. Then

$$x_2 = (1)\,(3)\,(2\ 4) \cdots .$$

Then $x_1 x_2 = (1)\ (2\ 4\ 3) \cdots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length four nor block of length four.

(9) *$T^{I(a)}$ has no orbit of length five.*

Proof. If $T^{I(a)}$ has an orbit Δ of length five, then $T^{I(a)}$ has an involution fixing exactly three points of Δ. Thus $T^{\Delta} = S_5$ (see [9]). Then T^{Δ} has an element of order three fixing two points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length five.

(10) *$T^{I(a)}$ has no orbit of length two. If $T^{I(a)}$ is a transitive but imprimitive group with six blocks of length two, then $T^{I(a)}$ is also a transitive but imprimitive group with two blocks of length six.*

Proof. Suppose that $T^{I(a)}$ has an orbit of length two or a block of length two, say $\{1, 2\}$, Since $(T^{I(a)})_{1\,2}$ is a subgroup of M_{10} and has no element of order three, the order of $(T^{I(a)})_{1\,2}$ is $2^r 5^s$, where $4 \geq r \geq 1$ and $s = 0$ or 1.

Assume that $s = 0$. Then the subgroup H of T fixing $\{1, 2\}$ as a set is a 2-group on $I(a)$. Since $(T^{I(a)})_{1\,2}$ is a normal subgroup of $H^{I(a)}$, $T_{1\,2}$ has an involution x_1 whose restriction on $I(a)$ is a central involution of $H^{I(a)}$. Then we may assume that

$$x_1 = (1)\,(2)\,(3)\,(4)\,(5\ 6)\,(7\ 8)\,(9\ 10)\,(11\ 12) \cdots .$$

When $T^{I(a)}$ is imprimitive, $\{3, 4\}$ is a block of $T^{I(a)}$ since $I(x_1{}^{I(a)}) = \{1, 2, 3, 4\}$.

Let x_2 be an involution of $T_{1\,5}$. Then $x_2 \in T_{1\,2}$. Hence $x_2{}^{I(a)}$ commutes with $x_1{}^{I(a)}$. Hence we may assume that

$$x_2 = (1)\,(2)\,(3\ 4)\,(5)\,(6)\,(7\ 8)\,(9\ 11)\,(10\ 12) \cdots .$$

Let x_3 be an involution of $T_{3\,5}$. When $\{1, 2\}$ is a T-orbit, $x_3{}^{I(a)}$ commutes with $x_1{}^{I(a)}$. Hence $x_3 = (1\ 2)\ (3)\ (4)\ (5)\ (6) \cdots$. Hence $x_3{}^{I(a)}$ commutes with $x_2{}^{I(a)}$. When $T^{I(a)}$ is imprimitive, $\{5, 6\}$ is a block of $T^{I(a)}$ since $I(x_2{}^{I(a)}) = \{1, 2, 5, 6\}$. Hence x_3 fixes $\{3, 4, 5, 6\}$ pointwise. Hence $x_3{}^{I(a)}$ commutes with $x_1{}^{I(a)}$ and $x_2{}^{I(a)}$. Thus in any case

$$x_3 = (1\,2)(3)(4)(5)(6)(7\,8)(9\,12)(10\,11)\cdots.$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3\rangle$ by (7), we have a contradiction.

Thus $s=1$. Since the order of $(T^{I(a)})_{1\,2}$ is $2^r\,5$, $(T^{I(a)})_{1\,2}$ is solvable. Let N be a minimal normal subgroup of $(T^{I(a)})_{1\,2}$. Then N is elementary abelian. Let u be an element of $T_{1\,2}$ such that the order of $u^{I(a)}$ is five. Suppose that N is a 2-group. Since N is an elementary abelian subgroup of $M_{1\,0}$, the order of N is two or four. Hence $u^{I(a)}$ centralizes N. This is a contradiction since $u^{I(a)}$ consists of two 5-cycles on $I(a)-\{1, 2\}$ and any involution of N has exactly two fixed points in $I(a)-\{1, 2\}$. Thus N is a 5-group. Hence $\langle u\rangle^{I(a)}$ is normal in $(T^{I(a)})_{1\,2}$ and so the unique Sylow 5-subgroup of $(T^{I(a)})_{1\,2}$.

Suppose that $\{1, 2\}$ is a T-orbit. Then $(T^{I(a)})_{1\,2}$ is normal in $T^{I(a)}$. Since $\langle u\rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{1\,2}$, $\langle u\rangle^{I(a)}$ is normal in $T^{I(a)}$. Let Δ be a $\langle u\rangle^{I(a)}$-orbit of length five. Then for any two points i and j of Δ, $T_{i\,j}$ has an involution x, which fixes Δ. Since $|I(x^{I(a)})|=4$ and $|\Delta|=5$, $|I(x)\cap\Delta|=3$. Thus the subgroup of T fixing Δ as a set is S_5 on Δ. Hence T has an element of order three fixing two points of Δ, contrary to (2). Thus $T^{I(a)}$ has no orbit of length two.

Suppose that $T^{I(a)}$ is imprimitive. Let x_1 be an involution of $T_{1\,3}$. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

Since $(\langle u\rangle^{x_1})^{I(a)}=\langle u\rangle^{I(a)}$ and x_1 is of order two, $(u^{x_1})^{I(a)}=u^{I(a)}$ or $(u^{-1})^{I(a)}$. Since x_1 fixes exactly two points of $I(a)-\{1, 2\}$ and u has no fixed point in $I(a)-\{1, 2\}$, $(u^{x_1})^{I(a)}\neq u^{I(a)}$. Thus $(u^{x_1})^{I(a)}=(u^{-1})^{I(a)}$. Hence we may assume that

$$u = (1)(2)(3\,5\,7\,8\,6)(4\,9\,11\,12\,10)\cdots.$$

Since $T^{I(a)}$ is an imprimitive group with blocks of length two and x_1 fixes a block containing 3, $\{3, 4\}$ is a block. Then $\{3, 4\}^{u^i}$, $0\leq i\leq 4$, is also a block. Thus $\{1, 2\}$, $\{3, 4\}$ $\{5, 9\}$, $\{7, 11\}$, $\{8, 12\}$ and $\{6, 10\}$ are a complete block system of $T^{I(a)}$.

Since $u\in T_{1\,2}$, $(T^{I(a)})_{1\,2}$ is transitive or has two orbits of length five on $I(a)-\{1, 2\}$. Suppose that $(T^{I(a)})_{1\,2}$ is transitive on $I(a)-\{1, 2\}$. Then since $\langle u\rangle^{I(a)}$ is a normal subgroup of $(T^{I(a)})_{1\,2}$, $T_{1\,2}$ has a 2-element x such that $\{3, 5, 7, 8, 6\}^x=\{4, 9, 11, 12, 10\}$. Then $|I(x)\cap I(a)|=2$ and so $x^{I(a)}$ is of order eight. Then $(x^4)^{I(a)}$ is of order two and fixes exactly two points of $I(a)-\{1, 2\}$. Hence $(u^{x^4})^{I(a)}=(u^{-1})^{I(a)}$. Hence $x^{I(a)}$ induces an automorphism of order eight of $\langle u\rangle^{I(a)}$ by conjugation. This is a contradiction since the order of $\langle u\rangle^{I(a)}$ is five. Hence $(T^{I(a)})_{1\,2}$ has two orbits of length five on $I(a)-\{1, 2\}$. Then since $(T^{I(a)})_1=(T^{I(a)})_{1\,2}$, $(T^{I(a)})_1$ has three orbits $\{2\}$, $\{3, 5, 6, 7, 8\}$ and

$\{4, 9, 10, 11, 12\}$ on $I(a)-\{1\}$.

Let x_2 be an involution of T_{56}. Since $\{5, 9\}$ and $\{6, 10\}$ are blocks of $T^{I(a)}$, x_2 fixes 5, 9, 6 and 10. Hence $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Then x_2 fixes $\{1, 2, 3, 4\}$. If $x_2=(1\ 2)(3\ 4)(5)(6)(9)(10)\cdots$, then x_2 normalizes T_{12} and $(\langle u\rangle^{x_2})^{I(a)}\neq\langle u\rangle^{I(a)}$. This is a contradiction since $\langle u\rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{12}$. Hence we may assume that

$$x_2=(1\,3)(2\,4)(5)(6)(9)(10)(7\,8)(11\,12)\cdots.$$

Then $\langle T_1, x_2\rangle^{I(a)}$ has two orbits $\{1, 3, 5, 6, 7, 8\}$ and $\{2, 4, 9, 10, 11, 12\}$. Thus $T^{I(a)}$ is also an imprimitive group with blocks of length six.

(11) *We show that $T^{I(a)}$ has neither orbit of length six nor block of length six and complete the proof.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length six or a block of length six, say $\{1, 2, \cdots, 6\}$. Set $\Delta=\{1, 2, \cdots, 6\}$.

Assume that T has an involution fixing exactly four points of Δ. Then we may assume that T has an involution

$$x_1=(1)(2)(3)(4)(5\,6)(7\,8)(9\,10)(11\,12)\cdots.$$

Let x_2 be an involution of T_{15}. Then x_2 fixes Δ. If $x_2=(1)(5)(6\ i)\cdots, i\in\{2, 3, 4\}$, then $x_1x_2=(1)(5\ i\ 6)\cdots$, contrary to (2). Hence x_2 fixes 6. Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2=(1)(2)(3\,4)(5)(6)(7\,8)(9\,11)(10\,12)\cdots.$$

Let x_3 be an involution of T_{35}. Then by the same argument as is used for x_2, $x_3^{I(a)}$ commutes with $x_1^{I(a)}$ and $x_3=(1\ 2)(3)(4)(5)(6)\cdots$. Hence $x_3^{I(a)}$ commutes with $x_2^{I(a)}$. Hence

$$x_3=(1\,2)(3)(4)(5)(6)(7\,8)(9\,12)(10\,11)\cdots.$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3\rangle$ by (7), we have a contradiction.

Thus T has no involution fixing four points of Δ. Then we may assume that T has an involution

$$x_1=(1)(2)(3\,4)(5\,6)(7)(8)(9\,10)(11\,12)\cdots.$$

Since $I(x_1)\supset\{1, 2, 7, 8, 13, 14\}$, $|I(x_1)|=12$ by (i). Hence we may assume that

$$a=(1)\,(2)\cdots(12)\,(13\,14)\,(15\,16)\,(17\,18)\,(19\,20)\cdots,$$
$$x_1=(1)\,(2)\,(3\,4)\,(5\,6)\,(7)\,(8)\,(9\,10)\,(11\,12)\,(13)\,(14)\cdots(20)\cdots.$$

Let x_2 be an involution of T_{13}. Then x_2 fixes Δ and $I(x_2)\cap\Delta=\{1, 3\}$. If $x_2=(1)\ (3)\ (2\ 4)\cdots$, then $x_1 x_2=(1)\ (2\ 4\ 3)\cdots$, contrary to (2). Hence we may assume that $x_2=(1)\ (3)\ (2\ 5)\ (4\ 6)\cdots$. Then $x_1 x_2=(1)\ (2\ 5\ 4\ 3\ 6)\cdots$. Thus $(x_1 x_2)^{I(a)}$ is of order five and so $(x_1 x_2)^{I(a)}$ has one more fixed points in $I(a)-\Delta$. Hence we may assume that

$$x_2=(1)\ (3)\ (2\ 5)\ (4\ 6)\ (7)\ (8\ 11)\ (10\ 12)\cdots.$$

Hence

$$x_1 x_2=(1)\ (2\ 5\ 4\ 3\ 6)\ (7)\ (8\ 11\ 10\ 9\ 12)\cdots.$$

Thus the subgroup of T fixing Δ as a set is doubly transitive on Δ and on $I(a)-\Delta$.

Since the order $T^{I(a)}$ is divisible by three, T has an element u of order three. Then by (2), u has no fixed point in $I(a)$. Thus u fixes exactly two points 13 and 14 in $I(a)\cup\{13, 14\}$. Since u commutes with a, if u has fixed points in $\Omega-(I(a)\cup\{13, 14\})$, then u fixes at least two points of $\Omega-(I(a)\cup\{13, 14\})$, contrary to (ii). Thus u has no fixed point in $\Omega-(I(a)\cup\{13, 14\})$ and so $I(u)=\{13, 14\}$. This shows that $|\Omega|\equiv 2 \pmod 3$. Hence any element of order three has exactly two fixed points.

Now we consider $N(G_{I(a)})$. Let H be the subgroup of $N(G_{I(a)})$ fixing Δ as a set and $\bar{H}$ the subgroup of T fixing Δ as a set. Since $\bar{H}$ is doubly transitive on Δ, H is doubly transitive on Δ. Hence $H^{\Delta}=S_6$, A_6, $PGL(2, 5)$ or $PSL(2, 5)$ (see [9]). Since any element of order three fixes exactly two points and $|I(a)|=12$, any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$. Hence $H^{\Delta}=PGL(2, 5)$ or $PSL(2, 5)$. Thus $\bar{H}^{I(a)}=H^{I(a)}$ or the index of $\bar{H}^{I(a)}$ in $H^{I(a)}$ is two. If $N(G_{I(a)})$ is transitive on $I(a)$, then by the same argument as is used in the proof of (4) $N(G_{I(a)})^{I(a)}$ is imprimitive. Then $(N(G_{I(a)})^{I(a)})_1$ is not transitive on $I(a)-\{1\}$. Moreover since any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$, $(N(G_{I(a)})^{I(a)})_1$ has no orbit of length six. Hence $(N(G_{I(a)})^{I(a)})_1$-orbits are $\{7\}$, $\Delta-\{1\}$ and $I(a)-(\Delta\cup\{7\})$ on $I(a)-\{1\}$, which are $(T^{I(a)})_1$-orbits. Thus when $N(G_{I(a)})^{I(a)}$ is imprimitive, $N(G_{I(a)})^{I(a)}$ has two blocks of length six, which are orbits or blocks of $T^{I(a)}$. This implies that for any involution x fixing exactly twelve points $N(G_{I(x)})^{I(x)}$ satisfies the same condition as $N(G_{I(a)})^{I(a)}$.

Let $(i\ j)$ be any 2-cycle of a. Then $T^{I(a)}$ and $(C(a)_{i\,j})^{I(a)}$ are subgroups of $N(G_{I(a)})^{I(a)}$. Hence there are 3-elements v and v' in T and $C(a)_{i\,j}$ respectively such that $v^{I(a)}=v'^{I(a)}$. Then v and v' normalizes $G_{I(a)}$, $I(v)=\{13, 14\}$ and $I(v')=\{i, j\}$. Let Γ be the $G_{I(a)}$-orbit containing $\{13, 14\}$. Then since $\{13, 14\}^v=\{13, 14\}$, $\Gamma^v=\Gamma$. Suppose that $\{i, j\}$ is contained in a $G_{I(a)}$-orbit different from Γ. Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{v'}\neq\Gamma$. Thus $\Gamma^{vv'^{-1}}=\Gamma^{v'^{-1}}\neq\Gamma$. This is a contradiction since $vv'^{-1} \in G_{I(a)}$. Thus $\{i, j\} \subset \Gamma$. Since (ij) is any 2-cycle of a, $G_{I(a)}$ is transitive on $\Omega - I(a)$. From the same reason, $G_{I(x_1)}$ is transitive on $\Omega - I(x_1)$. Then since $I(\langle G_{I(a)}, G_{I(x_1)} \rangle) = \{1, 2, 7, 8\}$, G_{1278} is transitive on $\Omega - \{1, 2, 7, 8\}$.

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)} = N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{12783} = 1$. Hence Q is a Sylow 2-subgroup of G_{12783}. Then since G_{1278} is transitive on $\Omega - \{1, 2, 7, 8\}$, $(N(Q)^{I(a)})_{1278}$ is transitive on $I(a) - \{1, 2, 7, 8\}$ by a lemma of E. Witt [10]. This is a contradiction since $N(Q)^{I(a)} = N(G_{I(Q)})^{I(a)}$ and $(N(G_{I(Q)})^{I(a)})_{1278}$ is intransitive on $I(a) - \{1, 2, 7, 8\}$.

Thus we complete the proof of Lemma 2.

Appendix

In Theorem of [8] we assumed that Q was a Sylow 2-subgroup of $G_{I(Q)}$. But this assumption is not necessary since if there is a 2-subgroup R satisfying $|I(R)| = t$ and $N(R)^{I(R)} = A_t$ or S_t, then a Sylow 2-subgroup of $G_{I(R)}$ satsifies the assumption of Theorem of [8]. Hence we have the following

Theorem. *Let G be a 4-fold transitive group on $\Omega = \{1, 2, \cdots, n\}$ and t be the maximal number of fixed points of involutions of G. Assume that G has a 2-subgroup Q such that $|I(Q)| = t$ and $N(Q)^{I(Q)} = S_t$ or A_t, then G is one of the following groups: S_n $(n \geq 4)$, A_n $(n \geq 6)$ or M_n $(n = 11, 12, 23, 24)$.*

OSAKA UNIVERSITY OF EDUCATION

References

[1] M. Hall: The Theory of Groups, Macmillan, New York, 1959.

[2] D. Livingstone and A. Wagner: *Transitivity of finite permutation groups on unordered sets,* Math. Z. **90** (1965), 393–403.

[3] T. Oyama: *On multiply transitive groups* VII, Osaka J. Math. **5** (1968), 319–326.

[4] T. Oyama: *On multiply trasnitive groups* VIII, Osaka J. Math. **6** (1969), 315–319.

[5] T. Oyama: *On multiply transitive groups* IX, Osaka J. Math. **7** (1970), 41–56.

[6] T. Oyama: *On multiply trasitive groups* X, Osaka J. Math. **8** (1971), 99–130.

[7] T. Oyama: *On multiplu trasnitive groups* XI, Osaka J. Math. **10** (1973), 379–439.

[8] T. Oyama: *On multiply transitive groups* XII, Osaka J. Math. **11** (1974), 595–636.

[9] C.C. Sims: *Computational methods in the study of permutation groups,* (in Computational Problems in Abstract Algebra), Pergamon Press, London, 1970, 169–183.

[10] E. Witt: *Die 5-fach transitiven Gruppen von Mathieu,* Abh. Math. Sem. Univ. Hamburg **12** (1937), 256–264.