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Osaka University

ON MULTIPLY TRANSITIVE GROUPS XIII

Dedicated to Professor Mutuo Takahasi on his 60th birthday

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1. Introduction

In this paper we shall prove the following

Theorem. *Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$. If the order of the stabilizer of four points in G is not divisible by three, then G is one of the following groups: S_4 , S_5 , S_6 , A_6 , M_{11} or M_{12} .*

In the proof of this theorem we shall use the following two lemmas, which will be proved in the section 3 and 4.

Lemma 1. *Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$ satisfying the following two conditions.*

- (i) *The order of the stabilizer of any four points in G is even and not divisible by three.*
- (ii) *Any involution fixing at least four points fixes exactly four or six points.*

Then $G = S_6$ or M_{12} .

Lemma 2. *Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$ satisfying the following three conditions.*

- (i) *The order of the stabilizer of any four points in G is even and not divisible by three.*
- (ii) *Any involution fixing at least four points fixes exactly four or twelve points.*
- (iii) *For any 2-subgroup X fixing exactly twelve points, $N(X)^{G(X)} \leq M_{12}$.*

Then $G = S_6$ or M_{12} .

We shall use the same notation as in [4].

2. Proof of the theorem

Let G be a group satisfying the assumption of the theorem. If the order

of the stabilizer of four points in G is odd and not divisible by three, then G is S_4 , S_5 , A_6 or M_{11} by a theorem of M. Hall ([1], Theorem 5.8.1). Hence we may consider only the case in which the stabilizer of four points in G is of even order.

Let P be a Sylow 2-subgroup of $G_{1_2 3_4}$. Then $P \neq 1$. If P is semiregular on $\Omega - I(P)$, then G is S_6 or M_{12} by Theorem of [3] and the assumption. Hence from now on we assume that P is not semiregular on $\Omega - I(P)$ and prove the theorem by way of contradiction.

By Corollary of [5] and Theorem of [7], $|I(P)| = 4$ or 5. We treat these cases separately.

Case I. $|I(P)| = 4$.

(1) *There is a point t in $\Omega - I(P)$ such that $|I(P_t)| = 6$ or 12 and $N(P_t)^{I(P_t)} = S_6$ or M_{12} respectively. In particular if t is a point of a minimal P -orbit, then $N(P_t)^{I(P_t)}$ is one of the groups listed above.*

Proof. Since G has no element of order three fixing at least four points, this follows from Corollary of [6].

(2) *Any element of order three fixes no point or exactly three points.*

Proof. By (1), there is a point t in $\Omega - I(P)$ such that $N(P_t)^{I(P_t)} = S_6$ or M_{12} . Then $N(P_t)$ has a 3-element whose restriction on $I(P_t)$ has exactly three fixed points. Since any element of order three fixes at most three points, $|\Omega| \equiv 0 \pmod{3}$ and any element of order three fixes no point or exactly three points.

(3) *If G has a 2-subgroup Q such that $|I(Q)| = 6$ and $N(Q)^{I(Q)} = S_6$, then there is no 2-subgroup R such that $|I(R)| = 12$ and $N(R)^{I(R)} = M_{12}$.*

Proof. Suppose by way of contradiction that there are 2-subgroups Q and R such that $|I(Q)| = 6$, $N(Q)^{I(Q)} = S_6$, $|I(R)| = 12$ and $N(R)^{I(R)} = M_{12}$. Let \bar{Q} be a Sylow 2-subgroup of $G_{I(Q)}$. Then $|I(\bar{Q})| = 6$ and $N(\bar{Q})^{I(\bar{Q})} = S_6$. Similarly let \bar{R} be a Sylow 2-subgroup of $G_{I(R)}$. Then $|I(\bar{R})| = 12$ and $N(\bar{R})^{I(\bar{R})} \geq M_{12}$. If $N(\bar{R})^{I(\bar{R})} \neq M_{12}$, then $N(\bar{R})^{I(\bar{R})} \geq A_{12}$. Hence $N(\bar{R})^{I(\bar{R})}$ has an element which is of order three and fixes nine points, contrary to (2). Thus $N(\bar{R})^{I(\bar{R})} = M_{12}$. Hence we may assume that Q and R are Sylow 2-subgroups of $G_{I(Q)}$ and $G_{I(R)}$ respectively.

Since G is 4-fold transitive on Ω , we may assume that P contains Q and R . Then set $I(Q) = \{1, 2, 3, 4, i_1, i_2\}$ and $I(R) = \{1, 2, 3, 4, j_1, j_2, \dots, j_8\}$. Since $N(Q)^{I(Q)} = S_6$, for any point i of $\{i_1, i_2\}$ $P_i = Q$ and Q is a Sylow 2-subgroup of $G_{1_2 3_4 i}$. Similarly since $N(R)^{I(R)} = M_{12}$, for any point j of $\{j_1, j_2, \dots, j_8\}$ $P_j = R$ and R is a Sylow 2-subgroup of $G_{1_2 3_4 j}$. Hence the $G_{1_2 3_4}$ -orbit Δ containing i is different from the $G_{1_2 3_4}$ -orbit Γ containing j . Since $N(Q)^{I(Q)} = S_6$ and

$$N(R)^{I(R)} = M_{12}, \{i_1, i_2\} \subseteq \Delta \text{ and } \{j_1, j_2, \dots, j_8\} \subseteq \Gamma.$$

Since $N(Q)^{I(Q)} = S_6$, there is an element

$$x = (1\ 2\ 3)\ (4)\ (i_1)\ (i_2)\ \dots.$$

Then $x \in N(G_{1\ 2\ 3\ 4})$. Hence x induces a permutation on the set of $G_{1\ 2\ 3\ 4}$ -orbits. Since $\{i_1, i_2\} \subseteq \Delta$ and $\{i_1, i_2\}^x = \{i_1, i_2\}$, $\Delta^x = \Delta^x$. Since the order of $G_{1\ 2\ 3\ 4}$ is not divisible by three, the lengths of $G_{1\ 2\ 3\ 4}$ -orbits in $\Omega - \{1, 2, 3, 4\}$ are not divisible by three. By (2), $I(x) = \{4, i_1, i_2\}$ and so x has no fixed point in $\Omega - (\{1, 2, 3, 4\} \cup \Delta)$. Thus $\Gamma^x \neq \Gamma$. On the other hand since $N(R)^{I(R)} = M_{12}$, there is an element

$$y = (1\ 2\ 3)\ (4)\ (j_1)\ (j_2)\ (j_3\ j_4\ j_5)\ (j_6\ j_7\ j_8)\ \dots.$$

Then $y \in N(G_{1\ 2\ 3\ 4})$. Since $\{j_1, j_2, \dots, j_8\} \subseteq \Gamma$ and $\{j_1, j_2, \dots, j_8\}^y = \{j_1, j_2, \dots, j_8\}$, $\Gamma^y = \Gamma$. Hence $\Gamma^{yx^{-1}} = \Gamma^{x^{-1}} \neq \Gamma$. This is a contradiction since $yx^{-1} \in G_{1\ 2\ 3\ 4}$ and Γ is a $G_{1\ 2\ 3\ 4}$ -orbit. Thus we complete the proof.

(4) Suppose that P has a subgroup Q such that $|I(Q)| = 6$ and $N(Q)^{I(Q)} = S_6$ ($|I(Q)| = 12$ and $N(Q)^{I(Q)} = M_{12}$). Let \bar{Q} be a subgroup of P such that the order of \bar{Q} is maximal among all subgroups of P fixing more than six (twelve) points. Set $N = N(\bar{Q})^{I(\bar{Q})}$. Then N satisfies the following conditions.

- (i) The order of the stabilizer of any four points in N is even and not divisible by three.
- (ii) Any involution of N fixing at least four points fixes exactly four or six (twelve) points.
- (iii) N has an involution fixing exactly six (twelve) points.
- (iv) When P has a subgroup Q such that $|I(Q)| = 12$ and $N(Q)^{I(Q)} = M_{12}$, for any 2-subgroup X of N fixing exactly twelve points, $N_N(X)^{I(X)} \leq M_{12}$.

Proof. (i), (ii) and (iv) are obvious. (iii) follows immediately from Theorem 1 in [6].

(5) By Lemma 1 and 2, which will be proved in the section 4, there is no such group N as in (4). Thus we complete the proof of Case I.

Case II. $|I(P)| = 5$.

(1) Let t be a point of a minimal P -orbit in $\Omega - I(P)$. Then $|I(P_t)| = 7, 9$ or 13 . In particular if $|I(P_t)| = 9$ or 13 , then $N(P_t)^{I(P_t)} \leq A_8$ or $N(P_t)^{I(P_t)} = S_1 \times M_{12}$ respectively.

Proof. This is Theorem of [6].

(2) $|I(P_t)| \neq 7$.

Proof. If $|I(P_t)|=7$, then $N(P_t)^{I(P_t)}$ is one of the groups listed in (2) of Case II in the section 3 of [6]. But these groups have an element of order three fixing four points. Thus $|I(P_t)| \neq 7$.

$$(3) \quad |I(P_t)| \neq 9.$$

Proof. Suppose by way of contradiction that $|I(P_t)|=9$. Then we may assume that $I(P_t)=\{1, 2, \dots, 9\}$. Set $N=N(P_t)^{I(P_t)}$. Then for any four points i, j, k and l of $I(P_t)$, $N_{i j k l}$ has an involution fixing exactly five points.

First assume that N is primitive. Then since N is a subgroup of A_9 and has an involution fixing five points, $N=A_9$ (see [9]). But this is a contradiction since N has no element which is of order three and fixes six points.

Next assume that N is transitive but imprimitive. Then N has three blocks $\{i_1, i_2, i_3\}$, $\{j_1, j_2, j_3\}$ and $\{k_1, k_2, k_3\}$ of length three. Let x be an involution fixing i_1, i_2, j_1 and j_2 . Then x fixes i_3, j_3 and one more point of $\{k_1, k_2, k_3\}$. Thus x is a transposition. This is a contradiction since $N \leq A_9$.

Finally assume that N is intransitive. Then one of the N -orbits is of length less than five.

Suppose that N has an orbit of length one, say $\{1\}$. Then for any four points i, j, k and l of $\{2, 3, \dots, 9\}$, there is an involution in N fixing exactly five points $1, i, j, k$ and l . Then by a lemma of D. Livingstone and A. Wagner [2], N_1 is 4-fold transitive on $\{2, 3, \dots, 9\}$. Thus $N=S_1 \times A_8$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length two, say $\{1, 2\}$. Then for any three points i, j and k of $\{3, 4, \dots, 9\}$, there is an involution in N fixing exactly five points $1, 2, i, j$ and k . Thus by a lemma of D. Livingstone and A. Wagner [2], $N_{1,2}$ is 3-fold transitive on $\{3, 4, \dots, 9\}$. Hence by [9], $N_{1,2}=A_7$. This is a contradiction since N has no element which is of order three and fixes six points.

Suppose that N has an orbit of length three, say $\{1, 2, 3\}$. Set $\Delta=\{4, 5, \dots, 9\}$. Then for any four points of Δ , there is an involution in N^Δ fixing exactly these four points. Hence by a lemma of D. Livingstone and A. Wagner [2], N^Δ is 4-fold transitive on Δ and so $N^\Delta=S_6$. Thus N has an element

$$x = (4) (5\ 6) (7\ 8\ 9) \dots$$

Since $N \leq A_9$, x is an even permutation. Hence x has one more 2-cycle on $\{1, 2, 3\}$. Thus x^2 is of order three and fixes six points, which is a contradiction.

Suppose that N has an orbit of length four, say $\{1, 2, 3, 4\}$. Set $\Delta=\{5, 6, \dots, 9\}$. Then for any three points i, j and k of Δ , N has an involution

fixing i, j, k and two more points of $\{1, 2, 3, 4\}$. Thus by a lemma of *D. Livingstone* and *A. Wagner* [2], N^Δ is 3-fold transitive on Δ and so $N^\Delta = S_6$. Thus N has an element

$$x = (5\ 6)(7\ 8\ 9) \dots$$

Since $N \leq A_9$, x is an even permutation. Hence x has one 2-cycle and two fixed points, or one 4-cycle on $\{1, 2, 3, 4\}$. Thus x^4 is of order three and fixes six points, which is a contradiction.

Thus $|I(P_t)| \neq 9$.

(4) If $|I(P_t)| = 13$, then $N(P_t)^{I(P_t)} = S_1 \times M_{12}$. Hence $N(P_t)^{I(P_t)}$ has an element of order three fixing four points, which is a contradiction.

Thus we complete the proof of Case II and so complete the proof of Theorem.

3. Proof of Lemma 1

Let G be a permutation group satisfying the assumptions of Lemma 1. If G has no involution fixing six points, then $G = S_6$ or M_{12} by Theorem 1 in [6] and the assumptions. Hence from now on we assume that G has an involution fixing exactly six points and prove Lemma 1 by way of contradiction. Then we may assume that G has an involution a fixing exactly six points $1, 2, \dots, 6$ and

$$a = (1\ 2) \dots (6\ 7\ 8) \dots$$

Set $T = C(a)_{7,8}$.

(1) For any two points i and j of $I(a)$, there is an involution in T_{ij} . Any involution of T is not the identity on $I(a)$.

Proof. Since a normalizes $G_{7,8ij}$ and $G_{7,8ij}$ is of even order, $G_{7,8ij}$ has an involution x commuting with a . Then $x \in T_{ij}$. Since $|I(a)| = 6$ and $I(x) \supseteq \{7, 8\}$, any involution of T is not the identity on $I(a)$ by (ii).

(2) Any element of order three of T has no fixed points in $I(a)$.

Proof. If an element u of order three of T has fixed points in $I(a)$, then since $|I(a)| = 6$, u fixes at least three points of $I(a)$. This contradicts (i) since $I(u) \supseteq \{7, 8\}$. Thus any element of order three of T has no fixed point in $I(a)$.

(3) We may assume that $(T^{I(a)})_{1\ 2\ 3\ 4} = 1$.

Proof. By (2), $T^{I(a)} \neq S_6$. Hence there is four points in $I(a)$ such that the stabilizer of these four points in $T^{I(a)}$ is the identity. Hence we may assume that $(T^{I(a)})_{1\ 2\ 3\ 4} = 1$.

- (4) $T^{I(a)}$ is one of the following groups.
 (a) $T^{I(a)}$ is intransitive and one of the $T^{I(a)}$ -orbits is of length one, two or three.
 (b) $T^{I(a)}$ is a transitive but imprimitive group with three blocks of length two or two blocks of length three.
 (c) $T^{I(a)}$ is primitive.

Proof. This is clear.

- (5) $T^{I(a)}$ has no orbit of length one.

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length one.

First assume that a $T^{I(a)}$ -orbit of length one is contained in $\{1, 2, 3, 4\}$. Then we may assume that $\{1\}$ is a $T^{I(a)}$ -orbit of length one. By (1), $T_{2,3}$ has an involution x_1 . By (3), we may assume that

$$x_1 = (1) (2) (3) (5) (4\ 6) \dots$$

Similarly $T_{2,4}$ has an involution x_2 of the form

$$x_2 = (1) (2) (4) (5) (3\ 6) \dots \text{ or } (1) (2) (4) (6) (3\ 5) \dots$$

If x_2 is of the first form, then $x_1 x_2 = (1) (2) (5) (3\ 6\ 4) \dots$, contrary to (2). Thus x_2 is of the second form. Similarly $T_{3,4}$ has an involution x_3 of the form

$$x_3 = (1) (3) (4) (5) (2\ 6) \dots \text{ or } (1) (3) (4) (6) (2\ 5) \dots$$

If x_3 is of the first form, then $x_1 x_3 = (1) (3) (5) (2\ 6\ 4) \dots$, contrary to (2). If x_3 is of the second form, then $x_2 x_3 = (1) (4) (6) (2\ 5\ 3) \dots$, contrary to (2).

Let $\{i\}$ be a $T^{I(a)}$ -orbit of length one. Then as is shown above, for any three points j, k and l of $I(a) - \{i\}$ ($T^{I(a)}_{j,kl} \neq 1$). Hence by a lemma of D. Livingstone and A. Wagner [2], $(T^{I(a)})_i$ is 3-fold transitive on $I(a) - \{i\}$. Hence $(T^{I(a)})_i = S_3$. Then T has an element which is of order three and has fixed points in $I(a)$, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

- (6) $T^{I(a)}$ has neither orbit of length two nor block of length two.

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length two or three blocks of length two.

First assume that $\{1, 2, 3, 4\}$ contains an orbit of length two or a block of length two. Then we may assume that $\{1, 2\}$ is an orbit or a block. By (1), $T_{1,3}$ has an involution x_1 . By (3), we may assume that

$$x_1 = (1) (2) (3) (5) (4\ 6) \dots$$

Let x_2 be an involution of $T_{1,4}$. Then similarly

$$x_2 = (1) (2) (4) (5) (3\ 6) \dots \text{ or } (1) (2) (4) (6) (3\ 5) \dots .$$

If x_2 is of the first form, then $x_1 x_2 = (1) (2) (5) (3\ 6\ 4) \dots$, contrary to (2). Thus x_2 is of the second form. Hence when $T^{I(a)}$ is imprimitive, $\{1, 2\}$, $\{3, 5\}$ and $\{4, 6\}$ form a complete block system. Let x_3 be an involution of $T_{3,4}$. When $T^{I(a)}$ is imprimitive

$$x_3 = (1\ 2) (3) (4) (5) (6) \dots .$$

When $T^{I(a)}$ has an orbit $\{1, 2\}$, x_3 is of this form or $x_3 = (1\ 2) (3) (4) (5\ 6) \dots$. But if $x_3 = (1\ 2) (3) (4) (5\ 6) \dots$, then $(x_1 x_3)^2 = (1) (2) (3) (4\ 6\ 5) \dots$, contrary to (2). Thus in any case x_3 is of the same form on $I(a)$.

Set $\Delta = \{1, 2, \dots, 8\}$. Let Q be a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3 \rangle$. Then $a \in Z(Q)$, $Q^\Delta = \langle a, x_1, x_2, x_3 \rangle^\Delta$ and $Q_\Delta = 1$. Hence $Q = \langle a, \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle$, where $\bar{x}_i^\Delta = x_i^\Delta$ and \bar{x}_i is conjugate to x_i , $i=1, 2, 3$. Thus we may assume that $\langle a, x_1, x_2, x_3 \rangle$ is a 2-group. Then $\langle a, x_1, x_2, x_3 \rangle$ is elementary abelian. Since $|I(ax_1)| \leq 6$, $\langle a, x_1 \rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length four.

Suppose that $\langle a, x_1 \rangle$ has an orbit of length four. Then we may assume that $\{9, 10, 11, 12\}$ is an orbit of length four and

$$a = (1) (2) \dots (6) (7\ 8) (9\ 10) (11\ 12) \dots ,$$

$$x_1 = (1) (2) (3) (5) (4\ 6) (7) (8) (9\ 11) (10\ 12) \dots .$$

Suppose that x_2 fixes $\{9, 10, 11, 12\}$. Then since $|I(ax_2)| \leq 6$ and $|I(x_1 x_2)| \leq 6$, $x_2 = (9\ 12)(10\ 11)$ on $\{9, 10, 11, 12\}$. Hence $\langle a, x_1, x_2 \rangle_{9\ 10\ 11\ 12} = \langle ax_1 x_2 \rangle$ and $I(ax_1 x_2) = \{1, 2, 9, 10, 11, 12\}$. Thus $\langle a, x_1, x_2 \rangle$ has exactly one orbit $\{9, 10, 11, 12\}$ of length four. Then since x_3 normalizes $\langle a, x_1, x_2 \rangle$, x_3 fixes $\{9, 10, 11, 12\}$. Then by the same argument as is used for x_2 , x_3 is of the same form as x_2 on $\{9, 10, 11, 12\}$. Hence $I(x_2 x_3) \geq \{4, 6, 7, 8, 9, 10, 11, 12\}$, contrary to (ii). Thus x_2 does not fix any $\langle a, x_1 \rangle$ -orbit of length four. Hence $\langle a, x_1, x_2 \rangle^{\Omega-\Delta}$ has at most one orbit of length two and the remaining orbits are of length eight. Hence $\langle a, x_1, x_2, x_3 \rangle$ -orbits whose lengths are not two are of length eight or sixteen. If $\langle a, x_1, x_2, x_3 \rangle$ has an orbit of length eight, then $\langle a, x_1, x_2, x_3 \rangle$ has an involution fixing at least eight points of this orbit, contrary to (ii). Thus $\langle a, x_1, x_2, x_3 \rangle^{\Omega-\Delta}$ has at most one orbit of length two and is semiregular on the set consisting of the remaining points. Since $\langle a, x_1 \rangle$ normalizes $G_{9\ 10\ 11\ 12}$ and $G_{9\ 10\ 11\ 12}$ is of even order, there is an involution y in $G_{9\ 10\ 11\ 12}$ commuting with a and x_1 . Then y fixes $\{1, 2, 3, 5\}$, $\{4, 6\}$ and $\{7, 8\}$. Suppose that $y^\Delta \in \langle a, x_1, x_2, x_3 \rangle^\Delta$. Then since $\langle a, x_1, x_2, x_3, y \rangle_\Delta$ is of odd order, $\langle a, x_1, x_2, x_3 \rangle$ is a Sylow 2-subgroup of $\langle a, x_1, x_2, x_3, y \rangle$. Hence $\langle a, x_1, x_2, x_3 \rangle$ has an element which is conjugate to y in $\langle a, x_1, x_2, x_3, y \rangle$. This is a contradiction since any

involution of $\langle a, x_1, x_2, x_3 \rangle$ fixes at most two points of $\Omega - \Delta$. Thus $y^A \notin \langle a, x_1, x_2, x_3 \rangle^A$. Hence $\{1, 2\}^y = \{3, 5\}$. On the other hand since y fixes $\{7, 8\}$, y or ya is contained in T . Thus $\{1, 2\}$ is not a T -orbit. Then $T^{I(a)}$ is imprimitive and we may assume that $y = (1\ 3)(2\ 5)$ on $\{1, 2, 3, 5\}$. Then $x_2 y$ is of order $4m$, where m is odd. Set $z = (x_2 y)^{2m}$. Then

$$z = (1\ 2)(3\ 5)(4)(6)(7)(8) \dots$$

and z centralizes $\langle a, x_1, x_2, y \rangle$. Since $|I(y)| \leq 6$, y fixes exactly four points 9, 10, 11 and 12 in $\Omega - \Delta$. Hence z fixes $\{9, 10, 11, 12\}$. Thus the $\langle a, x_1, x_2, z \rangle$ -orbit containing $\{9, 10, 11, 12\}$ is of length eight. Since $\langle a, x_1, x_2, z \rangle$ is abelian and of order sixteen, there is an involution fixing this $\langle a, x_1, x_2, z \rangle$ -orbit of length eight pointwise, contrary to (ii). Thus $\langle a, x_1 \rangle$ has no orbit of length four. Since $|I(ax_1)| \leq 6$, $|\Omega| = 8$ or 10 .

Suppose that $|\Omega| = 8$. Then by (i), there is an involution x in G fixing 1, 3, 4 and 7. If x fixes 8, then $x \in T$. Hence x fixes 2. Then $x^{I(a)} \in (T^{I(a)})_{1\ 2\ 3\ 4}$ and $x^{I(a)} \neq 1$, contrary to (3). Hence $x = (1)(3)(4)(7)(8\ i) \dots$, $i \in \{2, 5, 6\}$. Then $(ax)^2 = (7\ 8\ i)$, contrary to (i).

Suppose that $|\Omega| = 10$. Then

$$\begin{aligned} a &= (1)(2) \dots (6)(7\ 8)(9\ 10), \\ x_1 &= (1)(2)(3)(5)(4\ 6)(7)(8)(9\ 10), \\ x_2 &= (1)(2)(3\ 5)(4)(6)(7)(8)(9\ 10). \end{aligned}$$

By (i), there is an involution x in G fixing 1, 3, 4 and 7. Assume that x fixes 8. If x commutes with a , then $x \in T$. Hence x fixes 2. Then $x^{I(a)} \in (T^{I(a)})_{1\ 2\ 3\ 4}$ and $x^{I(a)} \neq 1$, contrary to (3). Thus x does not commute with a and so $\{9, 10\}^x \neq \{9, 10\}$. If x fixes 9, then $x = (9)(10\ i) \dots$, $i \in \{2, 5, 6\}$. Hence $(ax)^2 = (9\ 10\ i)$, contrary to (i). Similarly x does not fix 10. Thus $x = (9\ i)(10\ j)$, $\{i, j\} \subset \{2, 5, 6\}$. Then $(x_1 x_2 x)^2$ is of order three and fixes at least four points, contrary to (i). Thus x does not fix 8. Hence $x = (1)(3)(4)(7)(8\ i) \dots$, $i \in \{2, 5, 6, 9, 10\}$. If $i \in \{2, 5, 6\}$, then $ax = (1)(3)(4)(8\ 7\ i) \dots$. Since $|\Omega| = 10$, a suitable power of ax is of order three and fixes at least four points, contrary to (i). If $i \in \{9, 10\}$, then $ax_1 x = (1)(3)(8\ 7\ i) \dots$. Then similarly we have a contradiction. Hence $\{1, 2\}$ is neither orbit nor block.

Let $\{i, j\}$ be an orbit or a block of $T^{I(a)}$. Then by what we have proved above, for any two points k and l of $\{1, 2, \dots, 6\} - \{i, j\}$ there is an involution in $(T^{I(a)})_{ijkl}$. Hence by a lemma of D. Livingstone and A. Wagner [2], $(T^{I(a)})_{i\ j}$ is doubly transitive on $I(a) - \{i, j\}$. Hence $(T^{I(a)})_{i\ j} = S_4$. Then $(T^{I(a)})_{i\ j}$ has an element of order three, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length two nor block of length two.

(7) $T^{I(a)}$ has neither orbit of length three nor block of length three.

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or two blocks of length three. When $T^{I(a)}$ is intransitive, $T^{I(a)}$ has two orbits of length three by (5) and (6). Let $\{i_1, i_2, i_3\}$ and $\{j_1, j_2, j_3\}$ be the two orbits or the two blocks. Then $T_{i_1 i_2}$ has an involution

$$x = (i_1) (i_2) (i_3) (j_1) (j_2 j_3) \dots$$

Since $\{j_1, j_2, j_3\}$ is an orbit or a block and $x \in T_{i_1 i_2 i_3}$, $(T^{I(a)})_{i_1 i_2 i_3} = S_3$. Thus $(T^{I(a)})_{i_1 i_2 i_3}$ has an element of order three, contrary to (2). Hence $T^{I(a)}$ has neither orbit of length three nor block of length three.

(8) We show that $T^{I(a)}$ is not primitive and complete the proof.

Proof. Suppose by way of contradiction that $T^{I(a)}$ is primitive. Then since any element of order three in $T^{I(a)}$ has no fixed point, $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$ (see [9]). Let u be an element of order three of T . Since u commutes with a , if u has a fixed point in $\Omega - (I(a) \cup \{7, 8\})$, then u fixes at least two points of $\Omega - (I(a) \cup \{7, 8\})$, contrary to (i). Thus $I(u) = \{7, 8\}$ and so $|\Omega| \equiv 2 \pmod{3}$. Furthermore this shows that any element of order three fixes exactly two points of Ω . Hence $N(G_{I(a)})^{I(a)}$ has no element consisting of exactly one 3-cycle. Thus $N(G_{I(a)})^{I(a)} \not\cong A_6$. Then since $T^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$, $N(G_{I(a)})^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$. Furthermore this shows that for any involution v fixing exactly six points, $N(G_{I(v)})^{I(v)} = PSL(2, 5)$ or $PGL(2, 5)$.

Suppose that G has an involution x fixing exactly four points. Then x is of the form

$$x = (i_1) (i_2) (i_3) (i_4) (j_1 j_2) \dots$$

For any two points i_r and i_s of $\{i_1, i_2, i_3, i_4\}$ x normalizes $G_{j_1 j_2 i_r i_s}$. Hence by (i), $G_{j_1 j_2 i_r i_s}$ has an involution y commuting with x . If y fixes $I(x)$ pointwise, then $I(y) = I(x) \cup \{j_1, j_2\}$. Thus $|I(y)| = 6$ and $x^{I(y)} = (j_1 j_2)$. This is a contradiction since $N(G_{I(y)})^{I(y)} = PSL(2, 5)$ or $PGL(2, 5)$. Hence y fixes exactly two points i_r and i_s in $I(x)$. Hence by a lemma of D. Livingstone and A. Wagner [2], $(C(x)_{j_1 j_2})^{I(x)} = S_4$. Thus $C(x)_{j_1 j_2}$ has a 3-element of the form $(i_1 i_2 i_3) (i_4) (j_1) (j_2) \dots$. This is a contradiction since every element of order three fixes exactly two points. Thus G has no involution fixing exactly four points.

Let x be an involution of $T_{1,2}$. Then we may assume that

$$\begin{aligned} a &= (1) (2) \dots (6) (7\ 8) (9\ 10) \dots, \\ x &= (1) (2) (3\ 4) (5\ 6) (7) (8) (9) (10) \dots. \end{aligned}$$

Let (ij) be any 2-cycle of a . Then $(C(a)_{ij})^{I(a)} = PSL(2, 5)$ or $PGL(2, 5)$. Since $N(G_{I(a)})^{I(a)}$ is also $PSL(2, 5)$ or $PGL(2, 5)$, $T^{I(a)} = (C(a)_{ij})^{I(a)}$ or one of these two groups is a subgroup of the other. Hence there are 3-elements u and u' in T and $C(a)_{ij}$ respectively such that $u^{I(a)} = u'^{I(a)}$. Then u and u' normalize $G_{I(a)}$, $I(u) = \{7, 8\}$ and $I(u') = \{i, j\}$. Let Γ be the $G_{I(a)}$ -orbit containing $\{7, 8\}$. Then since $\{7, 8\}^u = \{7, 8\}$, $\Gamma^u = \Gamma$. Suppose that $\{i, j\}$ is contained in a $G_{I(a)}$ -orbit different from Γ . Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{u'} \neq \Gamma$. Thus $\Gamma^{uu'^{-1}} = \Gamma^{u'^{-1}} \neq \Gamma$. This is a contradiction since $uu'^{-1} \in G_{I(a)}$. Thus $\{i, j\} \subset \Gamma$. Since (ij) is any 2-cycle of a , $G_{I(a)}$ is transitive on $\Omega - I(a)$. From the same reason, $G_{I(x)}$ is transitive on $\Omega - I(x)$. Then since $I(\langle G_{I(a)}, G_{I(x)} \rangle) = \{1, 2\}$, $G_{1,2}$ is transitive on $\Omega - \{1, 2\}$. Since $N(G_{I(a)})$ is doubly transitive on $I(a)$, G is 3-fold transitive on Ω .

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)} = N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{1,2,3} = 1$. Hence Q is a Sylow 2-subgroup of $G_{1,2,3}$. Since $|I(Q)| = 6$, G is not 4-fold transitive by Theorem of [4]. On the other hand $G_{I(a)}$ is transitive on $\Omega - I(a)$. Hence there is a point i_1 in $\{4, 5, 6\}$ such that i_1 does not belong to the $G_{1,2,3}$ -orbit containing $\Omega - I(a)$. Since Q is a Sylow 2-subgroup of $G_{1,2,3}$, the length of the $G_{1,2,3}$ -orbit containing i_1 is not two. Moreover the length of the $G_{1,2,3}$ -orbit containing i_1 is not three since $G_{1,2,3}$ has no element of order three. Thus $G_{1,2,3}$ fixes i_1 . Since Q is a Sylow 2-subgroup of $G_{1,2,3}$, $\{4, 5, 6\} - \{i_1\}$ is not a $G_{1,2,3}$ -orbit. Similarly since $|\{4, 5, \dots, n\} - \{i_1\}|$ is even, $\{4, 5, \dots, n\} - \{i_1\}$ is not a $G_{1,2,3}$ -orbit. Hence $G_{1,2,3}$ -orbits on $\Omega - \{1, 2, 3\}$ are $\{4\}$, $\{5\}$, $\{6\}$ and $\{7, 8, \dots, n\}$ or $\{i_1\}$, $\{i_2\}$ and $\{i_3, 7, 8, \dots, n\}$, where $\{i_1, i_2, i_3\} = \{4, 5, 6\}$. First assume that $\{4\}$, $\{5\}$, $\{6\}$ and $\{7, 8, \dots, n\}$ are $G_{1,2,3}$ -orbits. By (i), $G_{1,2,3,7}$ has an involution y . Then $y \in G_{1,2,3}$. Hence $I(y) \supset \{1, 2, \dots, 7\}$, contrary to (ii). Next assume that $\{i_1\}$, $\{i_2\}$ and $\{i_3, 7, 8, \dots, n\}$ are $G_{1,2,3}$ -orbits. Since G is 3-fold transitive on Ω , $G_{1,2,3} = G_{1,2,i_1} = G_{1,2,i_2}$ and $G_{1,2,3} \neq G_{1,2,i_3}$. Thus $G_{1,2,i_3}$ fixes exactly two points of $\Omega - \{1, 2, \dots, 6\}$. This is a contradiction since $a \in G_{1,2,i_3}$ and a has no fixed point in $\Omega - \{1, 2, \dots, 6\}$.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

The proof of Lemma 2 is similar to the proof of Lemma 1. Let G be a permutation group satisfying the assumptions of Lemma 2. If G has an involution fixing twelve points, then $G = S_6$ or M_{12} by Theorem 1 and the assumptions. Hence from now on we assume that G has an involution fixing exactly twelve points and prove Lemma 2 by way of contradiction. Then we may assume that G has an involution a fixing exactly twelve points $1, 2, \dots, 12$ and

$$a = (1)(2) \cdots (12)(13\ 14) \cdots .$$

Set $T=C(a)_{13\ 14}$.

(1) For any two points i and j of $I(a)$, there is an involution in $T_{i\ j}$. Any involution of T is not the identity on $I(a)$.

(2) Any element of order three in T has no fixed point on $I(a)$.

The proofs of (1) and (2) are similar to the proofs of (3.1) and (3.2) respectively.

(3) $T^{I(a)}$ is one of the following groups.

(a) $T^{I(a)}$ is intransitive and one of the $T^{I(a)}$ -orbits is of length one, two, three, four, five or six.

(b) $T^{I(a)}$ is a transitive but imprimitive group with six blocks of length two, four blocks of length three, three blocks of length four or two blocks of length six.

(c) $T^{I(a)}$ is primitive.

Proof. This is clear.

(4) $T^{I(a)}$ is not primitive.

Proof. If $T^{I(a)}$ is primitive, then by (iii) $T^{I(a)}$ is $PSL(2, 11)$, M_{11} or M_{12} , which are of degree twelve (see [9]). But since $T^{I(a)}$ has an involution fixing at least two points by (1), $T^{I(a)} \neq PSL(2, 11)$. Furthermore since any element of order three of $T^{I(a)}$ has no fixed point by (2), $T^{I(a)} \neq M_{11}, M_{12}$. Thus $T^{I(a)}$ is not primitive.

(5) $T^{I(a)}$ has no orbit of length one.

Proof. If $T^{I(a)}$ has an orbit $\{i\}$ of length one, then $T^{I(a)-(i)}$ is one of the groups of (4) of Lemma 4 in [5]. But all these groups have an element of order three which has fixed points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length one.

(6) $T^{I(a)}$ has neither orbit of length three nor block of length three.

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length three or a block of length three, say $\{1, 2, 3\}$. Let x_1 be an involution of $T_{1\ 2}$. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\ 6)(7\ 8)(9\ 10)(11\ 12) \dots$$

When $T^{I(a)}$ is transitive but imprimitive, we may assume that the block containing 4 is $\{4, 5, 6\}$. Assume that $T^{I(a)}$ is intransitive. If the length of the orbit containing 4 is not divisible by three, then $(T^{I(a)})_4$ has an element of order three, contrary to (2). If the length of the orbit containing 4 is nine,

then $(T^{I(a)})_1$ has an element of order three, contrary to (2). Thus the length of the orbit containing 4 is three or six. On the other hand x_1 fixes exactly one point 4 in the orbit containing 4. Hence the length of the orbit containing 4 is three. Thus we may assume that $\{4, 5, 6\}$ is an orbit.

Let x_2 be an involution of T_{15} . Then x_2 fixes $\{1, 2, 3\}$ and $\{4, 5, 6\}$. If $x_2 = (1)(5)(4\ 6)\dots$, then $x_1 x_2 = (1)(4\ 6\ 5)\dots$, contrary to (2). Hence x_2 fixes $\{4, 5, 6\}$ pointwise. Since $|I(x_2^{I(a)})| = 4$,

$$x_2 = (1)(2\ 3)(4)(5)(6)\dots$$

Let x_3 be an involution of T_{25} . Then by the same argument as is used for x_2 ,

$$x_3 = (2)(1\ 3)(4)(5)(6)\dots$$

Then $x_2 x_3 = (1\ 3\ 2)(4)(5)(6)\dots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) $T^{I(a)}$ has no subgroup which is isomorphic to the following group $\langle x_1, x_2, x_3 \rangle$ as a permutation group.

$$x_1 = (1)(2)(3)(4)(5\ 6)(7\ 8)(9\ 10)(11\ 12),$$

$$x_2 = (1)(2)(3\ 4)(5)(6)(7\ 8)(9\ 11)(10\ 12),$$

$$x_3 = (1\ 2)(3\ 4)(5)(6)(7)(8)(9\ 10)(11\ 12).$$

Proof. This follows from the same argument as in the proof of (3.3) in [8].

(8) $T^{I(a)}$ has neither orbit of length four nor block of length four.

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length four or a block of length four, say $\{1, 2, 3, 4\}$.

First assume that T has an involution x_1 fixing $\{1, 2, 3, 4\}$ pointwise. Then we may assume that

$$x_1 = (1)(2)(3)(4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)\dots$$

Let x_2 be an involution of T_{15} . Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2 = (1)(2)(3\ 4)(5)(6)(7\ 8)(9\ 11)(10\ 12)\dots$$

Let x_3 be an involution of T_{35} . Then similarly $x_3^{I(a)}$ commutes with $x_1^{I(a)}$. Hence $x_3^{I(a)}$ fixes 3, 5 and 6. Since $|I(x_3^{I(a)})| = 4$, $x_3^{I(a)}$ fixes one more point of $\{1, 2, 4\}$. If x_3 fixes 1 or 2, then $x_2 x_3 = (1)(2\ 4\ 3)(5)(6)\dots$ or $(2)(1\ 4\ 3)(5)(6)\dots$ respectively, contrary to (2). Thus x_3 fixes 4. Then $x_3^{I(a)}$ commutes with $x_2^{I(a)}$ and so

$$x_3 = (1\ 2)(3)(4)(5)(6)(7\ 8)(9\ 12)(10\ 11)\dots$$

This is a contradiction since $T^{I(a)}$ has no such subgroup as $\langle x_1, x_2, x_3 \rangle^{I(a)}$ by (7).

Next assume that T has no involution fixing $\{1, 2, 3, 4\}$ pointwise. Let x_1 be an involution of $T_{1,2}$. Then

$$x_1 = (1) (2) (3\ 4) \dots$$

Let x_2 be an involution of $T_{1,3}$. Then

$$x_2 = (1) (3) (2\ 4) \dots$$

Then $x_1 x_2 = (1) (2\ 4\ 3) \dots$, contrary to (2). Thus $T^{I(a)}$ has neither orbit of length four nor block of length four.

(9) $T^{I(a)}$ has no orbit of length five.

Proof. If $T^{I(a)}$ has an orbit Δ of length five, then $T^{I(a)}$ has an involution fixing exactly three points of Δ . Thus $T^\Delta = S_5$ (see [9]). Then T^Δ has an element of order three fixing two points, contrary to (2). Thus $T^{I(a)}$ has no orbit of length five.

(10) $T^{I(a)}$ has no orbit of length two. If $T^{I(a)}$ is a transitive but imprimitive group with six blocks of length two, then $T^{I(a)}$ is also a transitive but imprimitive group with two blocks of length six.

Proof. Suppose that $T^{I(a)}$ has an orbit of length two or a block of length two, say $\{1, 2\}$. Since $(T^{I(a)})_{1,2}$ is a subgroup of M_{10} and has no element of order three, the order of $(T^{I(a)})_{1,2}$ is $2^r 5^s$, where $4 \geq r \geq 1$ and $s = 0$ or 1 .

Assume that $s = 0$. Then the subgroup H of T fixing $\{1, 2\}$ as a set is a 2-group on $I(a)$. Since $(T^{I(a)})_{1,2}$ is a normal subgroup of $H^{I(a)}$, $T_{1,2}$ has an involution x_1 whose restriction on $I(a)$ is a central involution of $H^{I(a)}$. Then we may assume that

$$x_1 = (1) (2) (3) (4) (5\ 6) (7\ 8) (9\ 10) (11\ 12) \dots$$

When $T^{I(a)}$ is imprimitive, $\{3, 4\}$ is a block of $T^{I(a)}$ since $I(x_1^{I(a)}) = \{1, 2, 3, 4\}$.

Let x_2 be an involution of $T_{1,5}$. Then $x_2 \in T_{1,2}$. Hence $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2 = (1) (2) (3\ 4) (5) (6) (7\ 8) (9\ 11) (10\ 12) \dots$$

Let x_3 be an involution of $T_{3,5}$. When $\{1, 2\}$ is a T -orbit, $x_3^{I(a)}$ commutes with $x_1^{I(a)}$. Hence $x_3 = (1\ 2) (3) (4) (5) (6) \dots$. Hence $x_3^{I(a)}$ commutes with $x_2^{I(a)}$. When $T^{I(a)}$ is imprimitive, $\{5, 6\}$ is a block of $T^{I(a)}$ since $I(x_2^{I(a)}) = \{1, 2, 5, 6\}$. Hence x_3 fixes $\{3, 4, 5, 6\}$ pointwise. Hence $x_3^{I(a)}$ commutes with $x_1^{I(a)}$ and $x_2^{I(a)}$. Thus in any case

$$x_3 = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 12)(10\ 11) \dots$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3 \rangle$ by (7), we have a contradiction.

Thus $s=1$. Since the order of $(T^{I(a)})_{1,2}$ is $2^r 5$, $(T^{I(a)})_{1,2}$ is solvable. Let N be a minimal normal subgroup of $(T^{I(a)})_{1,2}$. Then N is elementary abelian. Let u be an element of $T_{1,2}$ such that the order of $u^{I(a)}$ is five. Suppose that N is a 2-group. Since N is an elementary abelian subgroup of $M_{1,0}$, the order of N is two or four. Hence $u^{I(a)}$ centralizes N . This is a contradiction since $u^{I(a)}$ consists of two 5-cycles on $I(a) - \{1, 2\}$ and any involution of N has exactly two fixed points in $I(a) - \{1, 2\}$. Thus N is a 5-group. Hence $\langle u \rangle^{I(a)}$ is normal in $(T^{I(a)})_{1,2}$ and so the unique Sylow 5-subgroup of $(T^{I(a)})_{1,2}$.

Suppose that $\{1, 2\}$ is a T -orbit. Then $(T^{I(a)})_{1,2}$ is normal in $T^{I(a)}$. Since $\langle u \rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{1,2}$, $\langle u \rangle^{I(a)}$ is normal in $T^{I(a)}$. Let Δ be a $\langle u \rangle^{I(a)}$ -orbit of length five. Then for any two points i and j of Δ , $T_{i,j}$ has an involution x , which fixes Δ . Since $|I(x^{I(a)})|=4$ and $|\Delta|=5$, $|I(x) \cap \Delta|=3$. Thus the subgroup of T fixing Δ as a set is S_5 on Δ . Hence T has an element of order three fixing two points of Δ , contrary to (2). Thus $T^{I(a)}$ has no orbit of length two.

Suppose that $T^{I(a)}$ is imprimitive. Let x_1 be an involution of $T_{1,3}$. Then we may assume that

$$x_1 = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12) \dots$$

Since $(\langle u \rangle^{x_1})^{I(a)} = \langle u \rangle^{I(a)}$ and x_1 is of order two, $(u^{x_1})^{I(a)} = u^{I(a)}$ or $(u^{-1})^{I(a)}$. Since x_1 fixes exactly two points of $I(a) - \{1, 2\}$ and u has no fixed point in $I(a) - \{1, 2\}$, $(u^{x_1})^{I(a)} \neq u^{I(a)}$. Thus $(u^{x_1})^{I(a)} = (u^{-1})^{I(a)}$. Hence we may assume that

$$u = (1\ 2)(3\ 5\ 7\ 8\ 6)(4\ 9\ 11\ 12\ 10) \dots$$

Since $T^{I(a)}$ is an imprimitive group with blocks of length two and x_1 fixes a block containing 3, $\{3, 4\}$ is a block. Then $\{3, 4\}^{u^i}$, $0 \leq i \leq 4$, is also a block. Thus $\{1, 2\}$, $\{3, 4\}$, $\{5, 9\}$, $\{7, 11\}$, $\{8, 12\}$ and $\{6, 10\}$ are a complete block system of $T^{I(a)}$.

Since $u \in T_{1,2}$, $(T^{I(a)})_{1,2}$ is transitive or has two orbits of length five on $I(a) - \{1, 2\}$. Suppose that $(T^{I(a)})_{1,2}$ is transitive on $I(a) - \{1, 2\}$. Then since $\langle u \rangle^{I(a)}$ is a normal subgroup of $(T^{I(a)})_{1,2}$, $T_{1,2}$ has a 2-element x such that $\{3, 5, 7, 8, 6\}^x = \{4, 9, 11, 12, 10\}$. Then $|I(x) \cap I(a)| = 2$ and so $x^{I(a)}$ is of order eight. Then $(x^4)^{I(a)}$ is of order two and fixes exactly two points of $I(a) - \{1, 2\}$. Hence $(u^{x^4})^{I(a)} = (u^{-1})^{I(a)}$. Hence $x^{I(a)}$ induces an automorphism of order eight of $\langle u \rangle^{I(a)}$ by conjugation. This is a contradiction since the order of $\langle u \rangle^{I(a)}$ is five. Hence $(T^{I(a)})_{1,2}$ has two orbits of length five on $I(a) - \{1, 2\}$. Then since $(T^{I(a)})_1 = (T^{I(a)})_{1,2}$, $(T^{I(a)})_1$ has three orbits $\{2\}$, $\{3, 5, 6, 7, 8\}$ and

$\{4, 9, 10, 11, 12\}$ on $I(a) - \{1\}$.

Let x_2 be an involution of T_{56} . Since $\{5, 9\}$ and $\{6, 10\}$ are blocks of $T^{I(a)}$, x_2 fixes 5, 9, 6 and 10. Hence $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Then x_2 fixes $\{1, 2, 3, 4\}$. If $x_2 = (1\ 2)(3\ 4)(5\ 6)(9\ 10)\dots$, then x_2 normalizes T_{12} and $(\langle u \rangle^{x_2})^{I(a)} \neq \langle u \rangle^{I(a)}$. This is a contradiction since $\langle u \rangle^{I(a)}$ is the unique Sylow 5-subgroup of $(T^{I(a)})_{12}$. Hence we may assume that

$$x_2 = (1\ 3)(2\ 4)(5\ 6)(9\ 10)(7\ 8)(11\ 12)\dots$$

Then $\langle T_1, x_2 \rangle^{I(a)}$ has two orbits $\{1, 3, 5, 6, 7, 8\}$ and $\{2, 4, 9, 10, 11, 12\}$. Thus $T^{I(a)}$ is also an imprimitive group with blocks of length six.

(11) *We show that $T^{I(a)}$ has neither orbit of length six nor block of length six and complete the proof.*

Proof. Suppose by way of contradiction that $T^{I(a)}$ has an orbit of length six or a block of length six, say $\{1, 2, \dots, 6\}$. Set $\Delta = \{1, 2, \dots, 6\}$.

Assume that T has an involution fixing exactly four points of Δ . Then we may assume that T has an involution

$$x_1 = (1)(2)(3)(4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)\dots$$

Let x_2 be an involution of T_{15} . Then x_2 fixes Δ . If $x_2 = (1)(5)(6\ i)\dots$, $i \in \{2, 3, 4\}$, then $x_1 x_2 = (1)(5\ i\ 6)\dots$, contrary to (2). Hence x_2 fixes 6. Then x_2 fixes $\{1, 2, 3, 4\}$ and so $x_2^{I(a)}$ commutes with $x_1^{I(a)}$. Hence we may assume that

$$x_2 = (1)(2)(3\ 4)(5)(6)(7\ 8)(9\ 11)(10\ 12)\dots$$

Let x_3 be an involution of T_{35} . Then by the same argument as is used for x_2 , $x_3^{I(a)}$ commutes with $x_1^{I(a)}$ and $x_3 = (1\ 2)(3)(4)(5)(6)\dots$. Hence $x_3^{I(a)}$ commutes with $x_2^{I(a)}$. Hence

$$x_3 = (1\ 2)(3)(4)(5)(6)(7\ 8)(9\ 12)(10\ 11)\dots$$

Then since T has no such subgroup as $\langle x_1, x_2, x_3 \rangle$ by (7), we have a contradiction.

Thus T has no involution fixing four points of Δ . Then we may assume that T has an involution

$$x_1 = (1)(2)(3\ 4)(5\ 6)(7)(8)(9\ 10)(11\ 12)\dots$$

Since $I(x_1) \supset \{1, 2, 7, 8, 13, 14\}$, $|I(x_1)| = 12$ by (i). Hence we may assume that

$$a = (1)(2)\dots(12)(13\ 14)(15\ 16)(17\ 18)(19\ 20)\dots,$$

$$x_1 = (1)(2)(3\ 4)(5\ 6)(7)(8)(9\ 10)(11\ 12)(13)(14)\dots(20)\dots$$

Let x_2 be an involution of $T_{1,3}$. Then x_2 fixes Δ and $I(x_2) \cap \Delta = \{1, 3\}$. If $x_2 = (1)(3)(2\ 4)\dots$, then $x_1 x_2 = (1)(2\ 4\ 3)\dots$, contrary to (2). Hence we may assume that $x_2 = (1)(3)(2\ 5)(4\ 6)\dots$. Then $x_1 x_2 = (1)(2\ 5\ 4\ 3\ 6)\dots$. Thus $(x_1 x_2)^{I(a)}$ is of order five and so $(x_1 x_2)^{I(a)}$ has one more fixed points in $I(a) - \Delta$. Hence we may assume that

$$x_2 = (1)(3)(2\ 5)(4\ 6)(7)(8\ 11)(10\ 12)\dots$$

Hence

$$x_1 x_2 = (1)(2\ 5\ 4\ 3\ 6)(7)(8\ 11\ 10\ 9\ 12)\dots$$

Thus the subgroup of T fixing Δ as a set is doubly transitive on Δ and on $I(a) - \Delta$.

Since the order $T^{I(a)}$ is divisible by three, T has an element u of order three. Then by (2), u has no fixed point in $I(a)$. Thus u fixes exactly two points 13 and 14 in $I(a) \cup \{13, 14\}$. Since u commutes with a , if u has fixed points in $\Omega - (I(a) \cup \{13, 14\})$, then u fixes at least two points of $\Omega - (I(a) \cup \{13, 14\})$, contrary to (ii). Thus u has no fixed point in $\Omega - (I(a) \cup \{13, 14\})$ and so $I(u) = \{13, 14\}$. This shows that $|\Omega| \equiv 2 \pmod{3}$. Hence any element of order three has exactly two fixed points.

Now we consider $N(G_{I(a)})$. Let H be the subgroup of $N(G_{I(a)})$ fixing Δ as a set and \bar{H} the subgroup of T fixing Δ as a set. Since \bar{H} is doubly transitive on Δ , H is doubly transitive on Δ . Hence $H^\Delta = S_6, A_6, PGL(2, 5)$ or $PSL(2, 5)$ (see [9]). Since any element of order three fixes exactly two points and $|I(a)| = 12$, any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$. Hence $H^\Delta = PGL(2, 5)$ or $PSL(2, 5)$. Thus $\bar{H}^{I(a)} = H^{I(a)}$ or the index of $\bar{H}^{I(a)}$ in $H^{I(a)}$ is two. If $N(G_{I(a)})$ is transitive on $I(a)$, then by the same argument as is used in the proof of (4) $N(G_{I(a)})^{I(a)}$ is imprimitive. Then $(N(G_{I(a)})^{I(a)})_1$ is not transitive on $I(a) - \{1\}$. Moreover since any element of order three of $N(G_{I(a)})$ has no fixed point in $I(a)$, $(N(G_{I(a)})^{I(a)})_1$ has no orbit of length six. Hence $(N(G_{I(a)})^{I(a)})_1$ -orbits are $\{7\}, \Delta - \{1\}$ and $I(a) - (\Delta \cup \{7\})$ on $I(a) - \{1\}$, which are $(T^{I(a)})_1$ -orbits. Thus when $N(G_{I(a)})^{I(a)}$ is imprimitive, $N(G_{I(a)})^{I(a)}$ has two blocks of length six, which are orbits or blocks of $T^{I(a)}$. This implies that for any involution x fixing exactly twelve points $N(G_{I(x)})^{I(x)}$ satisfies the same condition as $N(G_{I(a)})^{I(a)}$.

Let (ij) be any 2-cycle of a . Then $T^{I(a)}$ and $(C(a)_i, j)^{I(a)}$ are subgroups of $N(G_{I(a)})^{I(a)}$. Hence there are 3-elements v and v' in T and $C(a)_i, j$ respectively such that $v^{I(a)} = v'^{I(a)}$. Then v and v' normalizes $G_{I(a)}$, $I(v) = \{13, 14\}$ and $I(v') = \{i, j\}$. Let Γ be the $G_{I(a)}$ -orbit containing $\{13, 14\}$. Then since $\{13, 14\}^v = \{13, 14\}$, $\Gamma^v = \Gamma$. Suppose that $\{i, j\}$ is contained in a $G_{I(a)}$ -orbit different from Γ . Since the order of $G_{I(a)}$ is not divisible by three, $|\Gamma|$ is not divisible by three. Hence $\Gamma^{v'} \neq \Gamma$. Thus $\Gamma^{vv'^{-1}} = \Gamma^{v'^{-1}} \neq \Gamma$. This is a contra-

diction since $vv'^{-1} \in G_{I(a)}$. Thus $\{i, j\} \subset \Gamma$. Since (ij) is any 2-cycle of a , $G_{I(a)}$ is transitive on $\Omega - I(a)$. From the same reason, $G_{I(x_1)}$ is transitive on $\Omega - I(x_1)$. Then since $I(\langle G_{I(a)}, G_{I(x_1)} \rangle) = \{1, 2, 7, 8\}$, $G_{1,2,7,8}$ is transitive on $\Omega - \{1, 2, 7, 8\}$.

Let Q be a Sylow 2-subgroup of $G_{I(a)}$. Since $N(Q)^{I(a)} = N(G_{I(a)})^{I(a)}$, $(N(Q)^{I(a)})_{1,2,7,8,3} = 1$. Hence Q is a Sylow 2-subgroup of $G_{1,2,7,8,3}$. Then since $G_{1,2,7,8}$ is transitive on $\Omega - \{1, 2, 7, 8\}$, $(N(Q)^{I(a)})_{1,2,7,8}$ is transitive on $I(a) - \{1, 2, 7, 8\}$ by a lemma of E. Witt [10]. This is a contradiction since $N(Q)^{I(a)} = N(G_{I(Q)})^{I(a)}$ and $(N(G_{I(Q)})^{I(a)})_{1,2,7,8}$ is intransitive on $I(a) - \{1, 2, 7, 8\}$.

Thus we complete the proof of Lemma 2.

Appendix

In Theorem of [8] we assumed that Q was a Sylow 2-subgroup of $G_{I(Q)}$. But this assumption is not necessary since if there is a 2-subgroup R satisfying $|I(R)| = t$ and $N(R)^{I(R)} = A_t$ or S_t , then a Sylow 2-subgroup of $G_{I(R)}$ satisfies the assumption of Theorem of [8]. Hence we have the following

Theorem. *Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$ and t be the maximal number of fixed points of involutions of G . Assume that G has a 2-subgroup Q such that $|I(Q)| = t$ and $N(Q)^{I(Q)} = S_t$ or A_t , then G is one of the following groups: S_n ($n \geq 4$), A_n ($n \geq 6$) or M_n ($n = 11, 12, 23, 24$).*

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