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MODULI SPACES OF YANG-MILLS CONNECTIONS OVER QUATERNIONIC KÄHLER MANIFOLDS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction

The concept of anti-self-dual connections plays an important role in Yang-Mills theory for 4-manifolds (cf. Atiyah's monograph [1]). For instance, Atiyah, Hitchin and Singer [2] determined the moduli space of instantons on S^4 by differential geometric method, while Hartshorne [5] obtained the same result via twistor theory by showing that the moduli space of instantons over S^4 is the real part of the moduli space of null-correlation bundles over $P^3(\mathbb{C})$.

Now the purpose of this paper is to give a generalization of the result of Hartshorne [5] in the following way. We have the notion of B_2 -connections ∇ on vector bundles over quaternionic Kähler manifolds M as higher dimensional analogue of anti-self-dual connections over 4-manifolds (cf. [3], [11], [15]). Let $p: Z \rightarrow M$ be the twistor space. Then, to each B_2 -connection ∇ over M , we can associate in a unique way an Einstein-Hermitian connection $\tilde{\nabla} := p^*\nabla$ over Z . Our main result is:

Theorem. *The mapping $\nabla \mapsto \tilde{\nabla}$ naturally induces an embedding of the moduli space of B_2 -connections over M as a totally real submanifold of the moduli space of Einstein-Hermitian connections over Z .*

In a forthcoming paper, we shall give a compactification of the moduli space of Einstein-Hermitian connections for null-correlation bundles on $P^{2m+1}(\mathbb{C})$.

In concluding this introduction, I would like to express my sincere gratitude to Professors H. Ozeki, M. Takeuchi, M. Itoh for valuable suggestions and to Professor T. Mabuchi for constant encouragement.

1. Notation, conventions and preliminaries

For this section we refer to [6], [7], [8], [9], [10] and [11].

Let N be a compact complex manifold and (F, h_F) a Hermitian vector bundle over N where F is a C^∞ complex vector bundle and h_F is a Hermitian metric on F .

DEFINITION. A Hermitian connection D on (F, h_F) is said to be *integrable*, if the curvature R^D of D is an $\text{End}(F)$ -valued $(1, 1)$ -form. An integrable connection D on (F, h_F) is said to be *irreducible*, if the only parallel sections of $\text{End}(F)$ are constant multiples of the identity endomorphism id_F of F .

We denote by $U(F, h_F)$ the group of unitary gauge transformations of (F, h_F) and by $\mathcal{C}'_H(F, h_F)$ the set of all irreducible integrable connections D on (F, h_F) . The set of all equivalence classes in $\mathcal{C}'_H(F, h_F)$ modulo $U(F, h_F)$ is called the moduli space of irreducible integrable connections on (F, h_F) , which we denote by $\mathcal{H}'(F, h_F)$.

Now we assume that N admits a Kähler metric with Kähler form ω_N . The mapping $L: \wedge^p T^*N \ni \eta \mapsto L(\eta) \in \wedge^{p+2} T^*N$ being defined by $L(\eta) = \omega \wedge \eta$, we denote its adjoint operator by Λ . This induces the mapping

$$\text{id} \otimes \Lambda: \text{End}(F, h_F) \otimes \wedge^{p+2} T^*N \rightarrow \text{End}(F, h_F) \otimes \wedge^p T^*N.$$

When a connection D on F is given, R^D denotes the curvature tensor of the connection D . Put $\text{Ric}(D) := \sqrt{-1}(\text{id} \otimes \Lambda)R^D$, which is called the *Ricci curvature* of D .

DEFINITION. A Hermitian connection D on (F, h_F) is called an *Einstein-Hermitian connection* if the *Ricci curvature* $\text{Ric}(D)$ of D is a constant multiple of id_F .

Let $\mathcal{C}'_E(F, h_F)$ be the set of all irreducible Einstein-Hermitian connections on (F, h_F) . The set of all equivalence classes in $\mathcal{C}'_E(F, h_F)$ modulo the group of unitary gauge transformations $U(F, h_F)$ is called the moduli space of irreducible Einstein-Hermitian connections on (F, h_F) , which we denote by $\mathcal{E}'(F, h_F)$.

Let D be an irreducible integrable connection on (F, h_F) . Consider the connection, denoted also by D , on $\text{End}(F)$ induced by D . We then have a Dolbeaut complex

$$(A_D): 0 \rightarrow A^{0,0}(\text{End}(F)) \rightarrow A^{0,1}(\text{End}(F)) \rightarrow \cdots \rightarrow A^{0,n}(\text{End}(F)) \rightarrow 0$$

$$(n = \dim_{\mathbb{C}} N),$$

where $A^{0,i}(\text{End}(F))$ is the space of all $\text{End}(F)$ -valued $(0, i)$ -forms on N and $D'': A^{0,i}(\text{End}(F)) \rightarrow A^{0,i+1}(\text{End}(F))$ is the $(0, i+1)$ part of the covariant exterior derivative d^D . Recall that the moduli space $\mathcal{H}'(F, h_F)$ admits a non-Hausdorff complex analytic space structure (see [7; (0.2)], [8; Chapter 7, (3.35)] and [10; (2.7)]). As a neighborhood of the equivalence class $\langle D \rangle$ of D , we can take an open set (centered at 0) of a slice

$$S_H = \{\alpha \in A^{0,1}(\text{End}(F)); D''\alpha \wedge \alpha = 0, \quad D''^*\alpha = 0\}.$$

For the above Dolbeault complex (A_D) , we denote by G_H , K_H and H_H the Green

operator, the Kuranishi map and the orthogonal projection to the space $\mathcal{H}^1(N, A_D)$ of all $\text{End}(F)$ -valued harmonic 1-forms on N respectively. Then this open set of S_H is homeomorphic to an open set of a complex analytic space

$$O_H = \{\alpha \in \mathcal{H}^1(N, A_D); H_H(K_H(\alpha) \wedge K_H(\alpha)) = 0\}.$$

Let $\text{End}(F)_0$ be the subbundle $\{S \in \text{End}(F) | \text{trace}(S) = 0\}$ of $\text{End}(F)$. We then have the following subcomplex (\tilde{A}_D) of (A_D) :

$$(\tilde{A}_D): 0 \rightarrow A^{0,0}(\text{End}(F)_0) \rightarrow A^{0,1}(\text{End}(F)_0) \rightarrow \cdots \rightarrow A^{0,n}(\text{End}(F)_0) \rightarrow 0$$

$$(n = \dim_{\mathbb{C}} N),$$

where $A^{0,1}(\text{End}(F)_0)$ is the space of all $\text{End}(F)_0$ -valued $(0, i)$ -forms on N . Denote by $\mathcal{C}_H''(F, h_F)$ the set of all irreducible integrable connections D on (F, h_F) such that the second cohomology of the Dolbeaut complex (\tilde{A}_D) vanishes. Then the quotient space $\mathcal{H}''(F, h_F) := \mathcal{C}_H''(F, h_F) / G(F, h_F)$ is a (possibly non-Hausdorff) complex manifold (cf. [8]), where $G(F, h_F)$ denotes the group of automorphisms of (F, h_F) whose determinant is one at each point.

On the other hand, an irreducible Einstein-Hermitian connection D on (F, h_F) induces a connection on $\text{End}(F, h_F)$, denoted also by D . We denote by $A^i(\text{End}(F, h_F))$ the space of all $\text{End}(F, h_F)$ -valued i -forms. Then we have the following elliptic complex (B_D) due to Kim [7]:

$$(B_D): 0 \rightarrow A^0(\text{End}(F, h_F)) \xrightarrow{D} A^1(\text{End}(F, h_F)) \xrightarrow{D_+} A_+^2(\text{End}(F, h_F)) \xrightarrow{D_2}$$

$$A^{0,3}(\text{End}(F, h_F)) \xrightarrow{D''} \cdots \xrightarrow{D''} A^{0,n}(\text{End}(F, h_F)) \rightarrow 0,$$

where $A^p(\text{End}(F, h_F))$ is the space of all real C^∞ p -forms with values in $\text{End}(F, h_F)$, $A^{p,q}(\text{End}(F, h_F))$ is the space of C^∞ (p, q) -forms with values in $\text{End}(F, h_F)$ and

$$A_+^2(\text{End}(F, h_F)) =$$

$$A^2(\text{End}(F, h_F)) \cap (A^{2,0}(\text{End}(F, h_F)) + A^{0,2}(\text{End}(F, h_F)) + A^0(\text{End}(F, h_F)) \otimes \omega_N).$$

Moreover D_+ and D_2 are defined as $D_+ = p_+ \circ d^D$ and $D_2 = D'' \circ p^{0,2}$, where p_+ and $p^{0,2}$ are natural projections of $A^2(\text{End}(F, h_F))$ onto $A_+^2(\text{End}(F, h_F))$ and $A^{0,2}(\text{End}(F, h_F))$, respectively. Note that the moduli space $\mathcal{E}'(F, h_F)$ is a Hausdorff real analytic space (cf. [7], [8] and [10]). We can identify a neighborhood of $\langle D \rangle$ in $\mathcal{E}(F, h_F)$ with a small open subset (centered at 0) of a slice

$$S_E = \{\beta \in A^1(\text{End}(F, h_F)); D_+ \beta + p_+(\beta \wedge \beta) = 0, \quad D^* \beta = 0\}.$$

This open subset of S_E is homeomorphic to an open set (centered at 0) of the real analytic space

$$O_E = \{\beta \in \mathcal{H}^1(N, B_D); H_E(K_E(\beta) \wedge K_E(\beta)) = 0\},$$

where G_E , K_E and H_E are the operators of (B_D) , corresponding respectively to the Green operator, the Kuranishi map and the orthogonal projection to the space $\mathcal{H}^1(N, B_D)$ of all $\text{End}(F, h_F)$ -valued harmonic 1-forms of (B_D) . The moduli space $\mathcal{E}'(F, h_F)$ is naturally embedded in $\mathcal{H}'(F, h_F)$ as an open subset of $\mathcal{H}'(F, h_F)$ (cf. [7], [8] and [10]). Let $H^i(N, A_D)$ and $H^i(N, B_D)$ be the i -th cohomology groups of the complexes (A_D) and (B_D) respectively. Then $H^1(N, A_D) \cong H^1(N, B_D)$ (cf. [7], [8] and [10]). More precisely, we have

$$\mathcal{H}^1(N, A_D) + \overline{\mathcal{H}^1(N, A_D)} = \mathcal{H}^1(N, B_D)^c.$$

Let (\tilde{B}_D) be the subcomplex (B_D) consisting of the sections with trace 0, and let $\mathcal{C}_E''(F, h_F)$ be the set of all irreducible Einstein-Hermitian connections D on (F, h_F) such that the second cohomology of the complex (\tilde{B}_D) vanishes. We denote by $\mathcal{E}''(F, h_F)$ the quotient space $\mathcal{C}_E''(F, h_F)/(U(F, h_F) \cap G(F, h_F))$. Then $\mathcal{E}''(F, h_F)$ has a natural structure of Kähler manifold (cf. [8] and [10]) and is holomorphically embedded in $\mathcal{H}''(F, h_F)$ as an open subset.

Let M be a compact quaternionic Kähler manifold and $p: Z \rightarrow M$ the associated twistor space. The vector bundle $\wedge^2 T^*M$ over M formed by covectors of degree 2 is expressed as a direct sum of three holonomy invariant vector subbundles A'_2 , A''_2 and B_2 (cf. [14]). Fix an arbitrary C^∞ vector bundle V over M . Then a connection D on V is called a B_2 -connection, if the curvature R^D of D is an $\text{End}(V)$ -valued B_2 -form. We now assume that V is a complex vector bundle over M , and choose a Hermitian metric h_V on V . Recall that Z has a natural real structure, i.e., an involutive antiholomorphic mapping $\tau: Z \rightarrow Z$ (cf. [11; (2.8)]). Let $\mathcal{C}_B(V, h_V)$ be the set of all Hermitian B_2 -connections on (V, h_V) and let $\tilde{\mathcal{C}}_H(p^*V, p^*h_V)$ be the set of all integrable connections on (p^*V, p^*h_V) satisfying the conditions: (a) D is trivial on each fibre $p^{-1}(x)$ ($x \in M$), and (b) the connection form associated with D is fixed by the pull-back τ^* (for more details see [11; Introduction]). Then we have the following:

Theorem 1.1 ([11]). *The pull-back $D \mapsto p^*D$ of connections induces a natural bijective correspondence: $\mathcal{C}_B(V, h_V) \cong \tilde{\mathcal{C}}_H(p^*V, p^*h_V)$. Furthermore, if the scalar curvature σ_M of M is positive, then $\tilde{\mathcal{C}}_H(p^*V, p^*h_V)$ is the set of all Einstein-Hermitian connections on (p^*V, p^*h_V) satisfying the conditions (a) and (b).*

2. Moduli spaces of Hermitian B_2 -connections

Let $\text{End}(V, h_V)_0$ be the subbundle consisting of $S \in \text{End}(V, h_V)$ such that $\text{trace}(S) = 0$. Let D be a Hermitian B_2 -connection on (V, h_V) . Then D induces B_2 -connection on $\text{End}(V, h_V)$ and $\text{End}(V, h_V)_0$, which we denote also by D . Using the B_2 -connection D on $\text{End}(V, h_V)$, we have an $\text{End}(V, h_V)$ -valued elliptic complex $C_D = \{(A^i, d_i), 0 \leq i \leq 2m\}$ ($\dim M = 4m$) (cf. [11; (3.5.)]), where A^1

is the space of all $\text{End}(V, h_V)$ -valued 1-forms on M . Furthermore, the B_2 -connection D on $\text{End}(V, h_V)_0$ induces an $\text{End}(V, h_V)_0$ -valued elliptic complex $\tilde{C}_D = \{(\tilde{A}^i, \tilde{d}_i)\}$ (cf. [11; (3.5)]), where in this case \tilde{A}^1 is the space of all $\text{End}(V, h_V)_0$ -valued 1-forms on M . We denote the i -th cohomology groups of C_D and \tilde{C}_D by $H^i(M, C_D)$ and $H^i(M, \tilde{C}_D)$ respectively. The spaces of the i -th harmonic elements for C_D and \tilde{C}_D are denoted by $\mathcal{H}^i(M, C_D)$ and $\mathcal{H}^i(M, \tilde{C}_D)$ respectively.

Now we denote by $U(V, h_V)$ the group of unitary gauge transformations of (V, h_V) . Let $\mathcal{C}'_B(V, h_V)$ be the set of all Hermitian B_2 -connections D on (V, h_V) such that $H^0(M, \tilde{C}_D) = \{0\}$, namely the set of all irreducible Hermitian B_2 -connections on (V, h_V) . We denote by $\mathcal{B}'(V, h_V)$ the quotient space $\mathcal{C}'_B(V, h_V)/U(V, h_V)$, which is called the moduli space of irreducible Hermitian B_2 -connections on (V, h_V) . Furthermore, let $\mathcal{C}''_B(V, h_V)$ be the set of Hermitian B_2 -connections D on (V, h_V) such that $H^0(M, \tilde{C}_D) = H^2(M, \tilde{C}_D) = \{0\}$. We then put $\mathcal{B}''(V, h_V) := \mathcal{C}''_B(V, h_V)/U(V, h_V)$. In the complex C_D , let $H_S: A^* \rightarrow \mathcal{H}^*(M, C_D)$ be the orthogonal projection to harmonic part and let G_S be the Green operator for $\Delta_S = \sum_{i=1}^{2m} (d_i \circ d_{i-1}^* + d_i^* \circ d_i)$. Note that $id = H_S + G_S \circ \Delta_S$.

Lemma 2.1. *Given a connection D in $\mathcal{C}_B(V, h_V)$, we denote by φ_D the set of forms $\alpha \in A^1$ such that $d_1\alpha + \pi_2(\alpha \wedge \alpha) = 0$ and $d_0^*\alpha = 0$, where π_2 denotes the natural projection of $\Gamma(M, \text{End}(V, h_V) \otimes \wedge^2 T^*M)$ onto A^2 . Then the mapping: $\varphi_D \ni \alpha \mapsto [D + \alpha] \in \mathcal{B}'$ is a homeomorphism of an open neighborhood of the origin in φ_D to an open set in \mathcal{B}' around $[D]$.*

Proof. This is proved by the same argument as in the proof of the slice lemma in [7; (1.7)].

The mapping $K_S: A^1 \ni \alpha \mapsto \alpha + (d_2^* \circ G_S \circ \pi_2)(\alpha \wedge \alpha) \in A^1$, called the Kuranishi map of C_D . The restriction of K_S defines a diffeomorphism between two small open neighborhoods of the origin on A^1 . Let K_S^{-1} be its inverse. Then we have:

Lemma 2.2. *Put*

$$\mathcal{V}_D = \{a \in \mathcal{H}^1(M, C_D); (H_S \circ \pi_2)(K_S^{-1}(\alpha) \wedge K_S^{-1}(\alpha)) = 0\}.$$

Then the restriction of the Kuranishi map defines a local homeomorphism between certain small neighborhoods of the origin of φ_D and \mathcal{V}_D .

We here observe that if $H^2(M, \tilde{C}_D) = \{0\}$, then \mathcal{V}_D is equal to $\mathcal{H}^1(M, C_D)$. Now by Lemmas 2.1 and 2.2, the following theorems follows immediately:

Theorem 2.3. *The moduli space $\mathcal{B}'(V, h_V)$ of irreducible Hermitian B_2 -connections has a natural real analytic structure.*

Theorem 2.4. *The quotient space $\mathcal{B}''(V, h_V)$ is a smooth manifold. The*

dimension of the connected component containing $[D]$ is $\dim_{\mathbb{R}} H^1(M, C_D)$. Moreover, by identifying the tangent space $T_{[D]}\mathcal{B}''(V, h_V)$ with $\mathcal{H}^1(M, C_D)$, the L^2 -inner product of $\mathcal{H}^1(M, C_D)$ defines a Riemannian metric on $\mathcal{B}''(V, h_V)$.

Theorems 2.3 and 2.4 are valid also for the case where the holonomy group of connections is a closed subgroup of $SO(r)$ or $U(r)$. Furthermore, by the same argument as in Kim [7], it is easily checked that both the spaces $\mathcal{B}'(V, h_V)$ and $\mathcal{B}''(V, h_V)$ are Hausdorff.

3. B_2 -connections and Einstein-Hermitian connections

From now on, we fix a compact connected quaternionic Kähler manifold M and a Hermitian vector bundle (V, h_V) over M . In the subsequent sections we use the notations introduced in Section 2. We prove the following:

Theorem 3.1. *If M has positive scalar curvature, $\mathcal{B}''(V, h_V)$ is embedded in $\mathcal{E}''(p^*V, p^*h_V)$ as a totally real submanifold.*

Given a Hermitian connection D on (V, h_V) , we denote by p^*D the pull-back of D by p .

Lemma 3.2. *If $D \in \mathcal{C}_B(V, h_V)$ is irreducible, then so is $p^*D \in \mathcal{C}'_H(p^*V, p^*h_V)$. In particular, if the scalar curvature σ_M of M is positive, then we have $p^*(\mathcal{C}'_B(V, h_V)) \subset \mathcal{C}'_E(p^*V, p^*h_V)$, where $p^*(\mathcal{C}'_B(V, h_V)) := \{p^*D \mid D \in \mathcal{C}'_B(V, h_V)\}$ (cf. Theorem 1.1).*

Proof. Fix an arbitrary $D \in \mathcal{C}'_B(V, h_V)$ and suppose that $(p^*D)\mathfrak{s} = 0$ for some $\mathfrak{s} \in \Gamma(Z, p^*\text{End}(V, h_V))$. Let (v_1, \dots, v_r) be a local unitary frame for (V, h_V) over an open set U of M . Let $\omega = (\omega_{ij})$ be the connection form of D defined by $Dv_j = \sum_{i=1}^r v_i \omega_{ij}$. Then by setting $\tilde{v}_i := p^*v_i$, we can express \mathfrak{s} as $\mathfrak{s} = \sum_{1 \leq i, j \leq r} \mathfrak{s}_{ij} \tilde{v}_i \otimes \tilde{v}_j^*$. In terms of the frame $(\tilde{v}_1, \dots, \tilde{v}_r)$, the assumption $(p^*D)\mathfrak{s} = 0$ is written as

$$(1) \quad (d\mathfrak{s}_{ij}) + [p^*\omega, (\mathfrak{s}_{ij})] = 0.$$

By (1), the restriction of the form $d\mathfrak{s}_{ij}$ to each fibre of p is zero, which means that the function \mathfrak{s}_{ij} is constant along the fibres of p . Hence there exists a global section $s \in \Gamma(M, \text{End}(V, h_V))$ such that $p^*s = \mathfrak{s}$. By the irreducibility of D , s is a constant multiple of id_V . Thus \mathfrak{s} is a constant multiple of id_{p^*V} , as required.

Lemma 3.3. *Let $D_1, D_2 \in \mathcal{C}_B(V, h_V)$. Then $[D_1] = [D_2]$ if and only if $\langle p^*D_1 \rangle = \langle p^*D_2 \rangle$, where $[D_\alpha]$ (resp. $\langle \tilde{D}_\alpha \rangle$) ($\alpha = 1, 2$) denotes the equivalence class of D_α (resp. \tilde{D}_α) modulo the unitary gauge groups on (V, h_V) (resp. (p^*V, p^*h_V)).*

Proof. It suffices to show $[D_1] = [D_2]$ when $\langle p^*D_1 \rangle = \langle p^*D_2 \rangle$. In this case, there exists a gauge transformation \tilde{g} for (p^*V, p^*h_V) such that $p^*D_1 = \tilde{g} \cdot p^*D_2$.

Let (v_1, \dots, v_r) be a local unitary frame for (V, h_V) . Each $D_\alpha (\alpha=1, 2)$ defines the connection form $\omega^{(\alpha)} = (\omega_{ij}^{(\alpha)})_{1 \leq i, j \leq r}$ by $D_\alpha v_j = \sum_{i=1}^r v_i \omega_{ij}^{(\alpha)}$. Write \tilde{g} as $\sum_{1 \leq i, j \leq r} \tilde{g}_{ij} \tilde{v}_i \otimes \tilde{v}_j^*$, where $\tilde{v}_k = p^* v_k$, $1 \leq k \leq r$. Then the condition $p^* D_1 = \tilde{g} \cdot p^* D_2$ is locally expressed in the form

$$(2) \quad p^* \omega^{(1)} = p^* \omega^{(2)} + \tilde{G}^{-1} d\tilde{G},$$

where \tilde{G} denotes the $r \times r$ matrix (\tilde{g}_{ij}) . From (3.3.1) the restriction of $d\tilde{G}$ to each fibre of p is zero, and so every \tilde{g}_{kl} is constant along the fibres of p . Hence, there exists a gauge transformation g for (V, h_V) such that $\tilde{g} = p^* g$. Thus $D_1 = g \cdot D_2$, i.e., $[D_1] = [D_2]$.

Theorem 3.4. *The mapping $p^*: C'_B(V, h_V) \rightarrow C'_H(p^*V, p^*h_V)$, induced from the projection $p: Z \rightarrow M$, gives rise to an injection: $\mathcal{B}'(V, h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V)$ (which is also denoted by p^*).*

Proof. This follows immediately from Lemmas 3.2 and 3.3.

REMARK 3.5. If $\sigma_M > 0$, then the image of $p^*: \mathcal{B}'(V, h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V)$ is contained in $\mathcal{E}'(p^*V, p^*h_V)$ (cf. Theorem 1.1).

We denote by $(\tilde{C}_D)^C$ the complexification of the elliptic complex (\tilde{C}_D) . Then by Carpi and Salamon [4; Theorem 3] the i -th cohomology group of the complex $(\tilde{C}_D)^C$ on M is embedded, via p^* , as a subgroup in the corresponding cohomology group of the Dolbeault complex (A_{p^*D}) on Z , and this embedding is an isomorphism for $i \geq 1$. It follows the following:

Corollary 3.6. *The mapping p^* maps $C'_B(V, h_V)$ to $C'_H(p^*V, p^*h_V)$ injectively. Moreover, this mapping induces an injection: $\mathcal{B}''(V, h_V) \rightarrow \mathcal{H}''(p^*V, p^*h_V)$ (denoted also by p^*). In particular, if $\sigma_M > 0$, the image of $\mathcal{B}''(V, h_V)$ under the injection $p^*: \mathcal{B}''(V, h_V) \rightarrow \mathcal{H}''(p^*V, p^*h_V)$ is contained in $\mathcal{E}''(p^*V, p^*h_V)$.*

Since p^*V is trivial on each fibre of $p: Z \rightarrow M$, τ induces a bundle automorphism $\tau^\sharp: p^*V \rightarrow p^*V$ such that the following diagram is commutative:

$$\begin{array}{ccc} p^*V & \xrightarrow{\tau^\sharp} & p^*V \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\tau} & Z \end{array}.$$

Let $C_H(p^*V, p^*h_V)$ be the set of all Hermitian integrable connections on (p^*V, p^*h_V) . Then the bundle automorphism τ^\sharp induces the mapping $\tilde{\tau}$ defined as follows:

$$C_H(p^*V, p^*h_V) \ni \tilde{D} \mapsto \tilde{\tau}(\tilde{D}) := \tau^\sharp \circ \tilde{D} \circ \tau^\sharp \in C_H(p^*V, p^*h_V).$$

We shall now write $\tilde{\tau}$ explicitly in terms of local frames. Choose an open

cover $\{U_\omega\}$ of M with a local unitary frame $(v_1^\omega, \dots, v_r^\omega)$ for (V, h_V) over U_ω . Then $\{p^{-1}(U_\omega)\}$ is an open cover of Z with local unitary frame $(p^*v_1^\omega, \dots, p^*v_r^\omega)$ for (p^*V, p^*h_V) over $p^{-1}(U_\omega)$. Given a Hermitian integrable connection \tilde{D} on (p^*V, p^*h_V) , we denote by (ω_{ij}^ω) the connection form for \tilde{D} on $p^{-1}(U_\omega)$ with respect to the frame $(p^*v_1^\omega, \dots, p^*v_r^\omega)$, (i.e. $\tilde{D}(p^*v_j^\omega) = \sum (p^*v_i^\omega) \omega_{ij}^\omega$). Then $(\tau^*\omega_{ij}^\omega)$ is just the connection form for $\tilde{\tau}(\tilde{D})$ with respect to the same frame on $p^{-1}(U_\omega)$. Since τ is antiholomorphic, $\tilde{\tau}(\tilde{D})$ is also integrable. Note that if \tilde{D} is irreducible, then $\tilde{\tau}(\tilde{D})$ is also irreducible, and that \tilde{D} is fixed by $\tilde{\tau}$ if and only if \tilde{D} satisfies the condition (b) in Section 1. Hence, by $\tilde{\tau}^2 = id$, the mapping $\tilde{\tau}$ is a bijection of $\mathcal{C}_H(p^*V, p^*h_V)$ onto itself. Since τ is an isometry of Z , the same argument is applied also to $\mathcal{C}_E(p^*V, p^*h_V)$. Given a unitary transformation $\tilde{s} \in U(p^*V, p^*h_V)$ and an integrable connection $\tilde{D} \in \mathcal{C}_H(p^*V, p^*h_V)$, we have the identity

$$\tilde{s} \cdot \tilde{\tau}(\tilde{D}) = \tilde{\tau}(s' \cdot \tilde{D}),$$

where $s' := \tau^\sharp \cdot \tilde{s} \circ \tau^\sharp$. Hence, $\tilde{\tau}$ naturally induces a bijection of the moduli space $\mathcal{H}'(p^*V, p^*h_V)$ onto itself, denoted by $\tau' : \mathcal{H}'(p^*V, p^*h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V)$, and the restriction of τ' to \mathcal{E}' gives a bijection of \mathcal{E}' onto itself (denoted also by $\tau' : \mathcal{E}'(p^*V, p^*h_V) \rightarrow \mathcal{E}'(p^*V, p^*h_V)$). Recall that the complex structure of Z induces those of $\mathcal{H}'(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h_V)$. Since τ is antiholomorphic, we have

Theorem 3.7. *Both the mappings*

$$\begin{aligned} \tau' : \mathcal{H}'(p^*V, p^*h_V) &\rightarrow \mathcal{H}'(p^*V, p^*h_V) \quad \text{and} \\ \tau' : \mathcal{E}'(p^*V, p^*h_V) &\rightarrow \mathcal{E}'(p^*V, p^*h_V) \end{aligned}$$

*are antiholomorphic bijection. Therefore τ defines real structures of $\mathcal{H}'(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h_V)$.*

Given an integrable connection \tilde{D} on (p^*V, p^*h_V) , we obtain the elliptic complex $(\tilde{A}_{\tilde{\tau}(\tilde{D})})$ from the complex $\tau^*(\tilde{A}_{\tilde{D}})$ by taking complex conjugation. Similarly, for any Einstein-Hermitian connection \tilde{D} , we obtain $(\tilde{B}_{\tilde{\tau}(\tilde{D})})$ from $\tau^*(\tilde{B}_{\tilde{D}})$ by complex conjugation. Hence the restrictions of the bijections

$$\begin{aligned} \tau' : \mathcal{H}'(p^*V, p^*h_V) &\rightarrow \mathcal{H}'(p^*V, p^*h_V) \quad \text{and} \\ \tau' : \mathcal{E}'(p^*V, p^*h_V) &\rightarrow \mathcal{E}'(p^*V, p^*h_V) \end{aligned}$$

on $\mathcal{H}''(p^*V, p^*h_V)$ and $\mathcal{E}''(p^*V, p^*h_V)$ define the bijections

$$\begin{aligned} \tau'' : \mathcal{H}''(p^*V, p^*h_V) &\rightarrow \mathcal{H}''(p^*V, p^*h_V) \quad \text{and} \\ \tau'' : \mathcal{E}''(p^*V, p^*h_V) &\rightarrow \mathcal{E}''(p^*V, p^*h_V) \end{aligned}$$

respectively. The Kähler metric of $\mathcal{E}''(p^*V, p^*h_V)$ is defined by the L^2 -inner product on $\mathcal{H}^1(Z, B_{\tilde{D}})$, which identified with the tangent space of $\mathcal{E}''(p^*V, p^*h_V)$

at $\langle \tilde{D} \rangle$. Since τ is isometry on Z , the real structure $\tau'': \mathcal{E}''(p^*V, p^*h_V) \rightarrow \mathcal{E}''(p^*V, p^*h_V)$ is an isometry.

Now we fix an arbitrary element \tilde{D} of $p^*(\mathcal{C}'_B(V, h_V))$. Put

$$\begin{aligned} \eta_H(\alpha) &= H_H(K_H^{-1}(\alpha) \wedge K_H^{-1}(\alpha)) \quad \text{for } \alpha \in \mathcal{H}^1(Z, A_{\tilde{D}}), \quad \text{and} \\ \eta_E(\beta) &= H_E(K_E^{-1}(\beta) \wedge K_E^{-1}(\beta)) \quad \text{for } \beta \in \mathcal{H}^1(Z, B_{\tilde{D}}). \end{aligned}$$

Since \tilde{D} is fixed by $\tilde{\tau}$ (cf. Section 1) we immediately obtain:

$$(3) \quad \eta_H(\tau^*\alpha) = \tau^*\eta_H(\alpha), \quad \alpha \in \mathcal{H}^1(Z, A_{\tilde{D}}),$$

$$(4) \quad \eta_E(\tau^*\beta) = \tau^*\eta_E(\beta), \quad \beta \in \mathcal{H}^1(Z, B_{\tilde{D}}).$$

Let $(\mathcal{H}'(p^*V, p^*h_V))_{\mathbf{R}}$, $(\mathcal{E}'(p^*V, p^*h_V))_{\mathbf{R}}$, $(\mathcal{H}''(p^*V, p^*h_V))_{\mathbf{R}}$, $(\mathcal{E}''(p^*V, p^*h_V))_{\mathbf{R}}$ be the subsets of $\mathcal{H}'(p^*V, p^*h_V)$, $\mathcal{E}'(p^*V, p^*h_V)$, $\mathcal{H}''(p^*V, p^*h_V)$, $\mathcal{E}''(p^*V, p^*h_V)$, respectively consisting of all elements fixed by the real structures defined above. Then by Theorem 1.1, $p^*(\mathcal{B}'(V, h_V))$ is embedded in $(\mathcal{E}'(p^*V, p^*h_V))_{\mathbf{R}} \subset (\mathcal{H}'(p^*V, p^*h_V))_{\mathbf{R}}$ and $p^*(\mathcal{B}''(V, h_V)) \subset (\mathcal{E}''(p^*V, p^*h_V))_{\mathbf{R}} \subset (\mathcal{H}''(p^*V, p^*h_V))_{\mathbf{R}}$.

4. Proof of Theorem 3.1

Let g_M denote the given metric on M and let g_Z denote the induced metric by g_M on Z . Then $g_V := g_Z - p^*g_M$ is an indefinite metric which is positive definite on each fibre of the submersion $p: (Z, g_Z) \rightarrow (M, g_M)$. Let J_Z be the complex structure on Z . We define a 2-form ω_V on Z by

$$\omega_V(v_1, v_2) := g_V(v_1, J_Z v_2), \quad v_1, v_2 \in T_z Z \quad (z \in Z).$$

Recall that Salamon [14; p. 144] introduced (locally defined) vector bundles H and E on M such that the complexification $T^*M^{\mathbb{C}}$ of the cotangent bundle T^*M is nothing but $H \otimes_{\mathbb{C}} E$. Let (h_1, h_2) and (e_1, \dots, e_{2m}) be symplectic local frames of H and E respectively, and (z^1, z^2) the dual coordinate of H . (We follow [11; (3.2.2)] for definition of symplectic frames.) Moreover H and E have natural connections induced by Riemannian connection of M (cf. [14]). Let (ω_j^i) be the connection form on H with respect to the frame (h_1, h_2) . Then ω_V is written as $c(|z^1|^2 + 1)^{-2} \theta \wedge \bar{\theta}$, where $\theta := dz^1 + z^1 p^*\omega_1^1 + p^*\omega_1^2 - (z^1)^2 p^*\omega_2^1 - z^1 p^*\omega_2^2$ and c is a constant depending only on the scalar curvature of M and the dimension of M (cf. [14] for more details).

Then we have

Lemma 4.1. *Put*

$$\begin{aligned} u_i &= (|z^1|^2 + 1)^{-1/2} (z^1 p^*(e_i \otimes h_1) + p^*(e_i \otimes h_2)) \quad (1 \leq i \leq 2m), \quad \text{and} \\ \theta_V &= (|z^1|^2 + 1)^{-1} \theta. \end{aligned}$$

Then we have

$$d\omega_V = -2c(\sum_{i=1}^m u_i \wedge u_{m+1} \wedge \bar{\theta}_V + u_i \wedge u_{m+1} \wedge \theta_V).$$

$$\begin{aligned} \text{Proof. } d\omega_V &= c\{-2(|z^1|^2+1)^{-2}(z^1 d\bar{z}^1 + \bar{z}^1 dz^1) \wedge \theta \wedge \bar{\theta} \\ &+ (|z^1|^2+1)^{-2}(dz^1 \wedge p^*\omega_1^1 + z^1 p^*d\omega_1^1 + p^*d\omega_2^1 - 2z^1 dz^1 \wedge p^*\omega_1^2 - (z^1)^2 p^*d\omega_1^2 \\ &- dz^1 \wedge p^*\omega_2^2 - z^1 p^*d\omega_2^2) \wedge \bar{\theta} - \\ &(|z^1|^2+1)^{-2} \theta \wedge (-d\bar{z}^1 \wedge p^*\omega_1^1 - \bar{z}^1 p^*d\omega_1^1 + 2\bar{z}^1 d\bar{z}^1 \wedge p^*\omega_2^1 + (\bar{z}^1)^2 p^*d\omega_2^1 + d\bar{z}^1 \wedge p^*\omega_2^2 \\ &+ \bar{z}^1 p^*d\omega_2^2)\} \\ &= c(|z^1|^2+1)^{-2}\{z^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^1) + p^*(d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) \\ &- (z^1)^2 p^*(d\omega_1^2 + \omega_1^2 \wedge \omega_1^1 + \omega_1^2 \wedge \omega_1^2) - z^1 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1)\} \wedge \bar{\theta} \\ &+ c(|z^1|^2+1)^{-2} \theta \wedge \{\bar{z}^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^1) + p^*(d\omega_1^2 + \omega_1^2 \wedge \omega_1^1 + \omega_2^2 \wedge \omega_1^2) \\ &- (\bar{z}^1)^2 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1 + \omega_2^2 \wedge \omega_2^2) - \bar{z}^1 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1)\}. \end{aligned}$$

We denote by (Ω_j^i) the curvature form of the vector bundle H with respect to (h_1, h_2) :

$$\Omega_j^i = d\omega_j^1 + \sum_{k=1}^2 \omega_k^i \wedge \omega_j^k.$$

We have the following formula due to Salamon [14; Proposition 3.2].

$$\begin{aligned} \Omega_1^1 &= -\sum_{i=1}^m ((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2) + (e_i \otimes h_2) \wedge (e_{m+i} \otimes h_1)), \\ \Omega_1^2 &= -2\sum_{i=1}^m ((e_i \otimes h_2) \wedge (e_{m+i} \otimes h_2)), \\ \Omega_2^1 &= 2\sum_{i=1}^m ((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_1)), \\ \Omega_2^2 &= \sum_{i=1}^m ((e_i \otimes h_2) \wedge (e_{m-i} \otimes h_1) + (e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2)). \end{aligned}$$

Using this we get:

$$\begin{aligned} d\omega_V &= c(|Z^1|^2+1)^{-2}\{(z^1 p^*\Omega_1^1 + p^*\Omega_2^1 - (z^1)^2 p^*\Omega_1^2 - z^1 p^*\Omega_2^2) \wedge \bar{\theta} + \\ &(\bar{z}^1 p^*\Omega_1^1 + p^*\Omega_2^1 - (\bar{z}^1)^2 p^*\Omega_2^2 - \bar{z}^1 p^*\Omega_2^2) \wedge \theta\} \\ &= -2c(\sum_{i=1}^m (u_i \wedge u_{m+i} \wedge \bar{\theta}_V + u_i \wedge u_{m+i} \wedge \theta_V)), \end{aligned}$$

which proves Lemma 4.1.

Let D be a Hermitian B_2 -connection on (V, h_V) on M . Then we have a morphism q between the complexes (C_D) and (A_{p^*D}) defined as follows:

$$C^i(\text{End}(V, h_V)) \ni d \mapsto (pr^{(0,i)} \circ p^*)(d) \in A^i(\text{End}(p^*V)),$$

where $pr^{(i,j)}: \Gamma(Z, \text{End}(p^*V) \otimes_{\mathcal{C}} \wedge^i T^*Z) \rightarrow \Gamma(Z, \text{End}(p^*V) \otimes_{\mathcal{C}} \wedge^{(i,j)} T^*Z)$ is the natural projection. Let $\tilde{\mathcal{D}}''$ and \mathcal{D}_i be the formal adjoint of $(p^*D)''$ and d_i in the complexes A_{p^*D} and C_D respectively. Then we obtain:

Lemma 4.2. Denoting by $*_M$ and $*_Z$ the star operators for vector bundles on M and Z , we have

$$\tilde{\mathcal{D}}''qv = q(\mathcal{D}_{i-1}v) - (*_Z \circ pr^{(2m-1, 2m)} \circ *_M)v \wedge (-2c \sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_V)$$

for all $v \in C^i(\text{End}(V, h_V))$.

Proof. Write the volume forms on M and Z as dv_M and dv_Z respectively. Then $dv_Z = p^*(dv_M) \wedge \omega_V$. Hence, for any $v \in C^i(\text{End}(V, h_V))$,

$$\begin{aligned} \tilde{\mathcal{D}}''qv &= -(*_Z \circ (d^{p^*D})' \circ *_Z \circ q)(v) = -(*_Z \circ (d^{p^*D})' \circ *_Z \circ pr^{(0,1)} \circ p^*)(v) \\ &= -(*_Z \circ (d^{p^*D})' \circ pr^{(2m+1-i, 2m+1)})(p^*(*_M v) \wedge \omega_V) \\ &= -(*_Z \circ (d^{p^*D})')((pr^{(2m-i, 2m)}(*_M v)) \wedge \omega_V) \\ &= -*_Z \{ (d^{p^*D})'((pr^{(2m-i, 2m)}(p^*(*_M v))) \wedge \omega_V) + pr^{(2m+i, 2m)}(p^*(*_M v)) \wedge d'\omega_V \} \\ &= -*_Z \{ (pr^{(2m-i, 2m)}(p^*(d^D(*_M v)))) \wedge \omega_V - (pr^{(2m-1, 2m)}(p^*(*_M v))) \wedge d'\omega_V \} \\ &= -pr^{(0,1)}((p^* \circ *_M \circ d^D \circ *_M)v) - *_Z \{ (pr^{(m-i, 2m)}(p^*(*_M v))) \wedge d'\omega_V \}. \end{aligned}$$

By using Lemma 4.1, it follows:

$$\tilde{\mathcal{D}}''qv = -q\mathcal{D}_{i-1}v - (*_Z \circ pr^{(2m-i, 2m)} \circ *_M)v \wedge (-2c \sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_V),$$

which proves Lemma 4.2.

In view of Lemma 4.2, we have $q(\mathcal{H}^1(M, C_D) \subset \mathcal{H}^1(Z, A_{p^*D})$. From [4; Theorem 3], it follows that $\dim_{\mathbb{C}} \mathcal{H}^1(Z, A_{p^*D}) = \dim_{\mathbb{C}} \mathcal{H}^1(M, (C_D)^{\mathcal{C}}) = \dim_{\mathbb{R}} \mathcal{H}^1(M, C_D)$. Together with the argument used by Kim [7; (1.3)], we have $\mathcal{H}^1(Z, A_{p^*D}) + \overline{\mathcal{H}^1(Z, A_{p^*D})} = (\mathcal{H}^1(Z, B_{p^*D}))^{\mathcal{C}}$. Hence

$$(1) \quad p^* \mathcal{H}^1(M, C_D) + J_Z p^* \mathcal{H}^1(M, C_D) = \mathcal{H}^1(Z, B_{p^*D}).$$

The tangent space of $\mathcal{B}''(V, h_V)$ at $[D]$ is $\mathcal{H}^1(M, C_D)$ and the tangent space of $\mathcal{E}''(p^*V, p^*h_V)$ at $\langle p^*D \rangle$ is $\mathcal{H}^1(Z, B_{p^*D})$. By (1), $\mathcal{B}''(V, h_V)$ is of dimension $\dim_{\mathbb{R}} \mathcal{H}^1(M, C_D)$ at $[D]$, which is equal to the complex dimension of $\mathcal{E}''(p^*V, p^*h_V)$ at $\langle p^*D \rangle$.

REMARKS. Capria and Salamon [4] constructed interesting families of B_2 -connections for some vector bundles over $P^*\mathbf{H}$. In a forthcoming paper [12], as an application of Theorem 3.1, we shall clarify the relationship between such families of B_2 -connections and the moduli space of Einstein-Hermitian connections on null-correlation bundles over odd dimensional complex projective spaces.

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