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## MODULI SPACES OF YANG-MILLS CONNECTIONS OVER QUATERNIONIC KÄHLER MANIFOLDS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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(Received January 13, 1988)

### Introduction

The concept of anti-self-dual connections plays an important role in Yang-Mills theory for 4-manifolds (cf. Atiyah's monograph [1]). For instance, Atiyah, Hitchin and Singer [2] determined the moduli space of instantons on  $S^4$  by differential geometric method, while Hartshorne [5] obtained the same result via twistor theory by showing that the moduli space of instantons over  $S^4$  is the real part of the moduli space of null-correlation bundles over  $P^3(\mathbb{C})$ .

Now the purpose of this paper is to give a generalization of the result of Hartshorne [5] in the following way. We have the notion of  $B_2$ -connections  $\nabla$  on vector bundles over quaternionic Kähler manifolds  $M$  as higher dimensional analogue of anti-self-dual connections over 4-manifolds (cf. [3], [11], [15]). Let  $p: Z \rightarrow M$  be the twistor space. Then, to each  $B_2$ -connection  $\nabla$  over  $M$ , we can associate in a unique way an Einstein-Hermitian connection  $\tilde{\nabla} := p^* \nabla$  over  $Z$ . Our main result is:

**Theorem.** *The mapping  $\nabla \mapsto \tilde{\nabla}$  naturally induces an embedding of the moduli space of  $B_2$ -connections over  $M$  as a totally real submanifold of the moduli space of Einstein-Hermitian connections over  $Z$ .*

In a forthcoming paper, we shall give a compactification of the moduli space of Einstein-Hermitian connections for null-correlation bundles on  $P^{2m+1}(\mathbb{C})$ .

In concluding this introduction, I would like to express my sincere gratitude to Professors H. Ozeki, M. Takeuchi, M. Itoh for valuable suggestions and to Professor T. Mabuchi for constant encouragement.

### 1. Notation, conventions and preliminaries

For this section we refer to [6], [7], [8], [9], [10] and [11].

Let  $N$  be a compact complex manifold and  $(F, h_F)$  a Hermitian vector bundle over  $N$  where  $F$  is a  $C^\infty$  complex vector bundle and  $h_F$  is a Hermitian metric on  $F$ .

**DEFINITION.** A Hermitian connection  $D$  on  $(F, h_F)$  is said to be *integrable*, if the curvature  $R^D$  of  $D$  is an  $\text{End}(F)$ -valued  $(1, 1)$ -form. An integrable connection  $D$  on  $(F, h_F)$  is said to be *irreducible*, if the only parallel sections of  $\text{End}(F)$  are constant multiples of the identity endomorphism  $\text{id}_F$  of  $F$ .

We denote by  $U(F, h_F)$  the group of unitary gauge transformations of  $(F, h_F)$  and by  $\mathcal{C}'_H(F, h_F)$  the set of all irreducible integrable connections  $D$  on  $(F, h_F)$ . The set of all equivalence classes in  $\mathcal{C}'_H(F, h_F)$  modulo  $U(F, h_F)$  is called the moduli space of irreducible integrable connections on  $(F, h_F)$ , which we denote by  $\mathcal{H}'(F, h_F)$ .

Now we assume that  $N$  admits a Kähler metric with Kähler form  $\omega_N$ . The mapping  $L: \wedge^p T^*N \ni \eta \mapsto L(\eta) \in \wedge^{p+2} T^*N$  being defined by  $L(\eta) = \omega \wedge \eta$ , we denote its adjoint operator by  $\Lambda$ . This induces the mapping

$$\text{id} \otimes \Lambda: \text{End}(F, h_F) \otimes \wedge^{p+2} T^*N \rightarrow \text{End}(F, h_F) \otimes \wedge^p T^*N.$$

When a connection  $D$  on  $F$  is given,  $R^D$  denotes the curvature tensor of the connection  $D$ . Put  $\text{Ric}(D) := \sqrt{-1}(\text{id} \otimes \Lambda)R^D$ , which is called the *Ricci curvature* of  $D$ .

**DEFINITION.** A Hermitian connection  $D$  on  $(F, h_F)$  is called an *Einstein-Hermitian connection* if the *Ricci curvature*  $\text{Ric}(D)$  of  $D$  is a constant multiple of  $\text{id}_F$ .

Let  $\mathcal{C}'_E(F, h_F)$  be the set of all irreducible Einstein-Hermitian connections on  $(F, h_F)$ . The set of all equivalence classes in  $\mathcal{C}'_E(F, h_F)$  modulo the group of unitary gauge transformations  $U(F, h_F)$  is called the moduli space of irreducible Einstein-Hermitian connections on  $(F, h_F)$ , which we denote by  $\mathcal{E}'(F, h_F)$ .

Let  $D$  be an irreducible integrable connection on  $(F, h_F)$ . Consider the connection, denoted also by  $D$ , on  $\text{End}(F)$  induced by  $D$ . We then have a Dolbeault complex

$$(A_D): 0 \rightarrow A^{0,0}(\text{End}(F)) \rightarrow A^{0,1}(\text{End}(F)) \rightarrow \cdots \rightarrow A^{0,n}(\text{End}(F)) \rightarrow 0$$

$$(n = \dim_{\mathbb{C}} N),$$

where  $A^{0,1}(\text{End}(F))$  is the space of all  $\text{End}(F)$ -valued  $(0, i)$ -forms on  $N$  and  $D'': A^{0,1}(\text{End}(F)) \rightarrow A^{0,i+1}(\text{End}(F))$  is the  $(0, i+1)$  part of the covariant exterior derivative  $d^D$ . Recall that the moduli space  $\mathcal{H}'(F, h_F)$  admits a non-Hausdorff complex analytic space structure (see [7; (0.2)], [8; Chapter 7, (3.35)] and [10; (2.7)]). As a neighborhood of the equivalence class  $\langle D \rangle$  of  $D$ , we can take an open set (centered at 0) of a slice

$$S_H = \{\alpha \in A^{0,1}(\text{End}(F)); D''\alpha \wedge \alpha = 0, D''^*\alpha = 0\}.$$

For the above Dolbeault complex  $(A_D)$ , we denote by  $G_H$ ,  $K_H$  and  $H_H$  the Green

operator, the Kuranishi map and the orthogonal projection to the space  $\mathcal{H}^1(N, A_D)$  of all  $\text{End}(F)$ -valued harmonic 1-forms on  $N$  respectively. Then this open set of  $S_H$  is homeomorphic to an open set of a complex analytic space

$$O_H = \{\alpha \in \mathcal{H}^1(N, A_D); H_H(K_H(\alpha) \wedge K_H(\alpha)) = 0\}.$$

Let  $\text{End}(F)_0$  be the subbundle  $\{S \in \text{End}(F) \mid \text{trace}(S) = 0\}$  of  $\text{End}(F)$ . We then have the following subcomplex  $(\mathcal{A}_D)$  of  $(A_D)$ :

$$(\mathcal{A}_D): 0 \rightarrow A^{0,0}(\text{End}(F)_0) \rightarrow A^{0,1}(\text{End}(F)_0) \rightarrow \cdots \rightarrow A^{0,n}(\text{End}(F)_0) \rightarrow 0$$

$$(n = \dim_{\mathbb{C}} N),$$

where  $A^{0,1}(\text{End}(F)_0)$  is the space of all  $\text{End}(F)_0$ -valued  $(0, i)$ -forms on  $N$ . Denote by  $\mathcal{C}'_H(F, h_F)$  the set of all irreducible integrable connections  $D$  on  $(F, h_F)$  such that the second cohomology of the Dolbeaut complex  $(\mathcal{A}_D)$  vanishes. Then the quotient space  $\mathcal{H}''(F, h_F) := \mathcal{C}'_H(F, h_F)/G(F, h_F)$  is a (possibly non-Hausdorff) complex manifold (cf. [8]), where  $G(F, h_F)$  denotes the group of automorphisms of  $(F, h_F)$  whose determinant is one at each point.

On the other hand, an irreducible Einstein-Hermitian connection  $D$  on  $(F, h_F)$  induces a connection on  $\text{End}(F, h_F)$ , denoted also by  $D$ . We denote by  $A^i(\text{End}(F, h_F))$  the space of all  $\text{End}(F, h_F)$ -valued  $i$ -forms. Then we have the following elliptic complex  $(B_D)$  due to Kim [7]:

$$(B_D): 0 \rightarrow A^0(\text{End}(F, h_F)) \xrightarrow{D} A^1(\text{End}(F, h_F)) \xrightarrow{D_+} A_+^2(\text{End}(F, h_F)) \xrightarrow{D_2}$$

$$A^{0,3}(\text{End}(F, h_F)) \xrightarrow{D''} \cdots \xrightarrow{D''} A^{0,n}(\text{End}(F, h_F)) \rightarrow 0,$$

where  $A^p(\text{End}(F, h_F))$  is the space of all real  $C^\infty$   $p$ -forms with values in  $\text{End}(F, h_F)$ ,  $A^{p,q}(\text{End}(F, h_F))$  is the space of  $C^\infty$   $(p, q)$ -forms with values in  $\text{End}(F, h_F)$  and

$$A_+^2(\text{End}(F, h_F)) =$$

$$A^2(\text{End}(F, h_F)) \cap (A^{2,0}(\text{End}(F, h_F)) + A^{0,2}(\text{End}(F, h_F)) + A^0(\text{End}(F, h_F)) \otimes \omega_N).$$

Moreover  $D_+$  and  $D_2$  are defined as  $D_+ = p_+ \circ d^p$  and  $D_2 = D'' \circ p^{0,2}$ , where  $p_+$  and  $p^{0,2}$  are natural projections of  $A^2(\text{End}(F, h_F))$  onto  $A_+^2(\text{End}(F, h_F))$  and  $A^{0,2}(\text{End}(F, h_F))$ , respectively. Note that the moduli space  $\mathcal{E}'(F, h_F)$  is a Hausdorff real analytic space (cf. [7], [8] and [10]). We can identify a neighborhood of  $\langle D \rangle$  in  $\mathcal{E}(F, h_F)$  with a small open subset (centered at 0) of a slice

$$S_E = \{\beta \in A^1(\text{End}(F, h_F)); D_+ \beta + p_+(\beta \wedge \beta) = 0, D^* \beta = 0\}.$$

This open subset of  $S_E$  is homeomorphic to an open set (centered at 0) of the real analytic space

$$O_E = \{\beta \in \mathcal{H}^1(N, B_D); H_E(K_E(\beta) \wedge K_E(\beta)) = 0\},$$

where  $G_E$ ,  $K_E$  and  $H_E$  are the operators of  $(B_D)$ , corresponding respectively to the Green operator, the Kuranishi map and the orthogonal projection to the space  $\mathcal{H}^1(N, B_D)$  of all  $\text{End}(F, h_F)$ -valued harmonic 1-forms of  $(B_D)$ . The moduli space  $\mathcal{E}'(F, h_F)$  is naturally embedded in  $\mathcal{H}'(F, h_F)$  as an open subset of  $\mathcal{H}'(F, h_F)$  (cf. [7], [8] and [10]). Let  $H^i(N, A_D)$  and  $H^i(N, B_D)$  be the  $i$ -th cohomology groups of the complexes  $(A_D)$  and  $(B_D)$  respectively. Then  $H^1(N, A_D) \simeq H^1(N, B_D)$  (cf. [7], [8] and [10]). More precisely, we have

$$\mathcal{H}^1(N, A_D) + \overline{\mathcal{H}^1(N, A_D)} = \mathcal{H}^1(N, B_D)^C.$$

Let  $(\tilde{B}_D)$  be the subcomplex  $(B_D)$  consisting of the sections with trace 0, and let  $\mathcal{C}'_E(F, h_F)$  be the set of all irreducible Einstein-Hermitian connections  $D$  on  $(F, h_F)$  such that the second cohomology of the complex  $(\tilde{B}_D)$  vanishes. We denote by  $\mathcal{E}''(F, h_F)$  the quotient space  $\mathcal{C}'_E(F, h_F)/(U(F, h_F) \cap G(F, h_F))$ . Then  $\mathcal{E}''(F, h_F)$  has a natural structure of Kähler manifold (cf. [8] and [10]) and is holomorphically embedded in  $\mathcal{H}''(F, h_F)$  as an open subset.

Let  $M$  be a compact quaternionic Kähler manifold and  $p: Z \rightarrow M$  the associated twistor space. The vector bundle  $\wedge^2 T^*M$  over  $M$  formed by covectors of degree 2 is expressed as a direct sum of three holonomy invariant vector subbundles  $A'_2$ ,  $A''_2$  and  $B_2$  (cf. [14]). Fix an arbitrary  $C^\infty$  vector bundle  $V$  over  $M$ . Then a connection  $D$  on  $V$  is called a  $B_2$ -connection, if the curvature  $R^D$  of  $D$  is an  $\text{End}(V)$ -valued  $B_2$ -form. We now assume that  $V$  is a complex vector bundle over  $M$ , and choose a Hermitian metric  $h_V$  on  $V$ . Recall that  $Z$  has a natural real structure, i.e., an involutive antiholomorphic mapping  $\tau: Z \rightarrow Z$  (cf. [11; (2.8)]). Let  $\mathcal{C}_B(V, h_V)$  be the set of all Hermitian  $B_2$ -connections on  $(V, h_V)$  and let  $\tilde{\mathcal{C}}_H(p^*V, p^*h_V)$  be the set of all integrable connections on  $(p^*V, p^*h_V)$  satisfying the conditions: (a)  $D$  is trivial on each fibre  $p^{-1}(x)$  ( $x \in M$ ), and (b) the connection form associated with  $D$  is fixed by the pull-back  $\tau^*$  (for more details see [11; Introduction]). Then we have the following:

**Theorem 1.1** ([11]). *The pull-back  $D \mapsto p^*D$  of connections induces a natural bijective correspondence:  $\mathcal{C}_B(V, h_V) \simeq \tilde{\mathcal{C}}_H(p^*V, p^*h_V)$ . Furthermore, if the scalar curvature  $\sigma_M$  of  $M$  is positive, then  $\tilde{\mathcal{C}}_H(p^*V, p^*h_V)$  is the set of all Einstein-Hermitian connections on  $(p^*V, p^*h_V)$  satisfying the conditions (a) and (b).*

## 2. Moduli spaces of Hermitian $B_2$ -connections

Let  $\text{End}(V, h_V)_0$  be the subbundle consisting of  $S \in \text{End}(V, h_V)$  such that  $\text{trace}(S) = 0$ . Let  $D$  be a Hermitian  $B_2$ -connection on  $(V, h_V)$ . Then  $D$  induces  $B_2$ -connection on  $\text{End}(V, h_V)$  and  $\text{End}(V, h_V)_0$ , which we denote also by  $D$ . Using the  $B_2$ -connection  $D$  on  $\text{End}(V, h_V)$ , we have an  $\text{End}(V, h_V)$ -valued elliptic complex  $C_D = \{(A^i, d_i), 0 \leq i \leq 2m\}$  ( $\dim M = 4m$ ) (cf. [11; (3.5.)]), where  $A^1$

is the space of all  $\text{End}(V, h_V)$ -valued 1-forms on  $M$ . Furthermore, the  $B_2$ -connection  $D$  on  $\text{End}(V, h_V)_0$  induces an  $\text{End}(V, h_V)_0$ -valued elliptic complex  $\tilde{C}_D = \{(\tilde{A}^i, \tilde{d}_i)\}$  (cf. [11; (3.5)]), where in this case  $\tilde{A}^1$  is the space of all  $\text{End}(V, h_V)_0$ -valued 1-forms on  $M$ . We denote the  $i$ -th cohomology groups of  $C_D$  and  $\tilde{C}_D$  by  $H^i(M, C_D)$  and  $H^i(M, \tilde{C}_D)$  respectively. The spaces of the  $i$ -th harmonic elements for  $C_D$  and  $\tilde{C}_D$  are denoted by  $\mathcal{H}^i(M, C_D)$  and  $\mathcal{H}^i(M, \tilde{C}_D)$  respectively.

Now we denote by  $U(V, h_V)$  the group of unitary gauge transformations of  $(V, h_V)$ . Let  $\mathcal{C}'_B(V, h_V)$  be the set of all Hermitian  $B_2$ -connections  $D$  on  $(V, h_V)$  such that  $H^0(M, \tilde{C}_D) = \{0\}$ , namely the set of all irreducible Hermitian  $B_2$ -connections on  $(V, h_V)$ . We denote by  $\mathcal{B}'(V, h_V)$  the quotient space  $\mathcal{C}'_B(V, h_V)/U(V, h_V)$ , which is called the moduli space of irreducible Hermitian  $B_2$ -connections on  $(V, h_V)$ . Furthermore, let  $\mathcal{C}''_B(V, h_V)$  be the set of Hermitian  $B_2$ -connections  $D$  on  $(V, h_V)$  such that  $H^0(M, \tilde{C}_D) = H^2(M, \tilde{C}_D) = \{0\}$ . We then put  $\mathcal{B}''(V, h_V) := \mathcal{C}''_B(V, h_V)/U(V, h_V)$ . In the complex  $C_D$ , let  $H_s: A^* \rightarrow \mathcal{H}^*(M, C_D)$  be the orthogonal projection to harmonic part and let  $G_s$  be the Green operator for  $\Delta_s = \sum_{i=1}^{2m} (d_i \circ d_{i-1}^* + d_i^* \circ d_i)$ . Note that  $id = H_s + G_s \circ \Delta_s$ .

**Lemma 2.1.** *Given a connection  $D$  in  $\mathcal{C}_B(V, h_V)$ , we denote by  $\varphi_D$  the set of forms  $\alpha \in A^1$  such that  $d_1\alpha + \pi_2(\alpha \wedge \alpha) = 0$  and  $d_0^*\alpha = 0$ , where  $\pi_2$  denotes the natural projection of  $\Gamma(M, \text{End}(V, h_V) \otimes \wedge^2 T^*M)$  onto  $A^2$ . Then the mapping:  $\varphi_D \ni \alpha \mapsto [D + \alpha] \in \mathcal{B}'$  is a homeomorphism of an open neighborhood of the origin in  $\varphi_D$  to an open set in  $\mathcal{B}'$  around  $[D]$ .*

*Proof.* This is proved by the same argument as in the proof of the slice lemma in [7; (1.7)].

The mapping  $K_s: A^1 \ni \alpha \mapsto \alpha + (d_2^* \circ G_s \circ \pi_2)(\alpha \wedge \alpha)$  ( $\alpha \wedge \alpha \in A^1$ ), called the Kuranishi map of  $C_D$ . The restriction of  $K_s$  defines a diffeomorphism between two small open neighborhoods of the origin on  $A^1$ . Let  $K_s^{-1}$  be its inverse. Then we have:

**Lemma 2.2.** *Put*

$$\mathcal{CV}_D = \{a \in \mathcal{H}^1(M, C_D); (H_s \circ \pi_2)(K_s^{-1}(\alpha) \wedge K_s^{-1}(\alpha)) = 0\}.$$

*Then the restriction of the Kuranishi map defines a local homeomorphism between certain small neighborhoods of the origin of  $\varphi_D$  and  $\mathcal{CV}_D$ .*

We here observe that if  $H^2(M, \tilde{C}_D) = \{0\}$ , then  $\mathcal{CV}_D$  is equal to  $\mathcal{H}^1(M, C_D)$ . Now by Lemmas 2.1 and 2.2, the following theorems follows immediately:

**Theorem 2.3.** *The moduli space  $\mathcal{B}'(V, h_V)$  of irreducible Hermitian  $B_2$ -connections has a natural real analytic structure.*

**Theorem 2.4.** *The quotient space  $\mathcal{B}''(V, h_V)$  is a smooth manifold. The*

dimension of the connected component containing  $[D]$  is  $\dim_R H^1(M, C_D)$ . Moreover, by identifying the tangent space  $T_{[D]} \mathcal{B}'(V, h_V)$  with  $\mathcal{H}^1(M, C_D)$ , the  $L^2$ -inner product of  $\mathcal{H}^1(M, C_D)$  defines a Riemannian metric on  $\mathcal{B}'(V, h_V)$ .

Theorems 2.3 and 2.4 are valid also for the case where the holonomy group of connections is a closed subgroup of  $SO(r)$  or  $U(r)$ . Furthermore, by the same argument as in Kim [7], it is easily checked that both the spaces  $\mathcal{B}'(V, h_V)$  and  $\mathcal{B}''(V, h_V)$  are Hausdorff.

### 3. $\mathcal{B}_2$ -connections and Einstein-Hermitian connections

From now on, we fix a compact connected quaternionic Kähler manifold  $M$  and a Hermitian vector bundle  $(V, h_V)$  over  $M$ . In the subsequent sections we use the notations introduced in Section 2. We prove the following:

**Theorem 3.1.** *If  $M$  has positive scalar curvature,  $\mathcal{B}''(V, h_V)$  is embedded in  $\mathcal{E}''(p^*V, p^*h_V)$  as a totally real submanifold.*

Given a Hermitian connection  $D$  on  $(V, h_V)$ , we denote by  $p^*D$  the pull-back of  $D$  by  $p$ .

**Lemma 3.2.** *If  $D \in \mathcal{C}_B(V, h_V)$  is irreducible, then so is  $p^*D \in \mathcal{C}'_H(p^*V, p^*h_V)$ . In particular, if the scalar curvature  $\sigma_M$  of  $M$  is positive, then we have  $p^*(\mathcal{C}'_B(V, h_V)) \subset \mathcal{C}'_E(p^*V, p^*h_V)$ , where  $p^*(\mathcal{C}'_B(V, h_V)) := \{p^*D \mid D \in \mathcal{C}'_B(V, h_V)\}$  (cf. Theorem 1.1).*

**Proof.** Fix an arbitrary  $D \in \mathcal{C}'_B(V, h_V)$  and suppose that  $(p^*D)\tilde{s} = 0$  for some  $\tilde{s} \in \Gamma(Z, p^* \text{End}(V, h_V))$ . Let  $(v_1, \dots, v_r)$  be a local unitary frame for  $(V, h_V)$  over an open set  $U$  of  $M$ . Let  $\omega = (\omega_{ij})$  be the connection form of  $D$  defined by  $Dv_j = \sum_{i=1}^r v_i \omega_{ij}$ . Then by setting  $\tilde{v}_i := p^*v_i$ , we can express  $\tilde{s}$  as  $\tilde{s} = \sum_{1 \leq i, j \leq r} \tilde{s}_{ij} \tilde{v}_i \otimes \tilde{v}_j^*$ . In terms of the frame  $(\tilde{v}_1, \dots, \tilde{v}_r)$ , the assumption  $(p^*D)\tilde{s} = 0$  is written as

$$(1) \quad (d\tilde{s}_{ij}) + [p^*\omega, (\tilde{s}_{ij})] = 0.$$

By (1), the restriction of the form  $d\tilde{s}_{ij}$  to each fibre of  $p$  is zero, which means that the function  $\tilde{s}_{ij}$  is constant along the fibres of  $p$ . Hence there exists a global section  $s \in \Gamma(M, \text{End}(V, h_V))$  such that  $p^*s = \tilde{s}$ . By the irreducibility of  $D$ ,  $s$  is a constant multiple of  $id_V$ . Thus  $\tilde{s}$  is a constant multiple of  $id_{p^*V}$ , as required.

**Lemma 3.3.** *Let  $D_1, D_2 \in \mathcal{C}_B(V, h_V)$ . Then  $[D_1] = [D_2]$  if and only if  $\langle p^*D_2 \rangle = \langle p^*D_1 \rangle$ , where  $[D_\alpha]$  (resp.  $\langle D_\alpha \rangle$ ) ( $\alpha = 1, 2$ ) denotes the equivalence class of  $D_\alpha$  (resp.  $\tilde{D}_\alpha$ ) modulo the unitary gauge groups on  $(V, h_V)$  (resp.  $(p^*V, p^*h_V)$ ).*

**Proof.** It suffices to show  $[D_1] = [D_2]$  when  $\langle p^*D_1 \rangle = \langle p^*D_2 \rangle$ . In this case, there exists a gauge transformation  $\tilde{g}$  for  $(p^*V, p^*h_V)$  such that  $p^*D_1 = \tilde{g} \cdot p^*D_2$ .

Let  $(v_1, \dots, v_r)$  be a local unitary frame for  $(V, h_V)$ . Each  $D_\alpha$  ( $\alpha=1, 2$ ) defines the connection form  $\omega^{(\alpha)}=(\omega_{ij}^\alpha)_{1 \leq i, j \leq r}$  by  $D_\alpha v_j = \sum_{i=1}^r v_i \omega_{ij}^\alpha$ . Write  $\tilde{g}$  as  $\sum_{1 \leq i, j \leq r} \tilde{g}_{ij} \tilde{v}_i \otimes \tilde{v}_j^*$ , where  $\tilde{v}_k = p^* v_k$ ,  $1 \leq k \leq r$ . Then the condition  $p^* D_1 = \tilde{g} \cdot p^* D_2$  is locally expressed in the form

$$(2) \quad p^* \omega^{(1)} = p^* \omega^{(2)} + \tilde{G}^{-1} d\tilde{G},$$

where  $\tilde{G}$  denotes the  $r \times r$  matrix  $(\tilde{g}_{ij})$ . From (3.3.1) the restriction of  $d\tilde{G}$  to each fibre of  $p$  is zero, and so every  $\tilde{g}_{kl}$  is constant along the fibres of  $p$ . Hence, there exists a gauge transformation  $g$  for  $(V, h_V)$  such that  $\tilde{g} = p^* g$ . Thus  $D_1 = g \cdot D_2$ , i.e.,  $[D_1] = [D_2]$ .

**Theorem 3.4.** *The mapping  $p^*: \mathcal{C}'_B(V, h_V) \rightarrow \mathcal{C}'_H(p^* V, p^* h_V)$ , induced from the projection  $p: Z \rightarrow M$ , gives rise to an injection:  $\mathcal{B}'(V, h_V) \rightarrow \mathcal{H}'(p^* V, p^* h_V)$  (which is also denoted by  $p^*$ ).*

**Proof.** This follows immediately from Lemmas 3.2 and 3.3.

**REMARK 3.5.** If  $\sigma_M > 0$ , then the image of  $p^*: \mathcal{B}'(V, h_V) \rightarrow \mathcal{H}'(p^* V, p^* h_V)$  is contained in  $\mathcal{E}'(p^* V, p^* h_V)$  (cf. Theorem 1.1).

We denote by  $(\tilde{\mathcal{C}}_D)^\mathbf{C}$  the complexification of the elliptic complex  $(\tilde{\mathcal{C}}_D)$ . Then by Carpia and Salamon [4; Theorem 3] the  $i$ -th cohomology group of the complex  $(\tilde{\mathcal{C}}_D)^\mathbf{C}$  on  $M$  is embedded, via  $p^*$ , as a subgroup in the corresponding cohomology group of the Dolbeault complex  $(A_{p^* D})$  on  $Z$ , and this embedding is an isomorphism for  $i \geq 1$ . It follows the following:

**Corollary 3.6.** *The mapping  $p^*$  maps  $\mathcal{C}'_B(V, h_V)$  to  $\mathcal{C}'_H(p^* V, p^* h_V)$  injectively. Moreover, this mapping induces an injection:  $\mathcal{B}''(V, h_V) \rightarrow \mathcal{H}''(p^* V, p^* h_V)$  (denoted also by  $p^*$ ). In particular, if  $\sigma_M > 0$ , the image of  $\mathcal{B}''(V, h_V)$  under the injection  $p^*: \mathcal{B}''(V, h_V) \rightarrow \mathcal{H}''(p^* V, p^* h_V)$  is contained in  $\mathcal{E}''(p^* V, p^* h_V)$ .*

Since  $p^* V$  is trivial on each fibre of  $p: Z \rightarrow M$ ,  $\tau$  induces a bundle automorphism  $\tau^*: p^* V \rightarrow p^* V$  such that the following diagram is commutative:

$$\begin{array}{ccc} p^* V & \xrightarrow{\tau^*} & p^* V \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\tau} & Z \end{array}.$$

Let  $\mathcal{C}_H(p^* V, p^* h_V)$  be the set of all Hermitian integrable connections on  $(p^* V, p^* h_V)$ . Then the bundle automorphism  $\tau^*$  induces the mapping  $\tilde{\tau}$  defined as follows:

$$\mathcal{C}_H(p^* V, p^* h_V) \ni \tilde{D} \mapsto \tilde{\tau}(\tilde{D}) := \tau^* \circ \tilde{D} \circ \tau^* \in \mathcal{C}_H(p^* V, p^* h_V).$$

We shall now write  $\tilde{\tau}$  explicitly in terms of local frames. Choose an open

cover  $\{U_\alpha\}$  of  $M$  with a local unitary frame  $(v_1^\alpha, \dots, v_r^\alpha)$  for  $(V, h_V)$  over  $U_\alpha$ . Then  $\{p^{-1}(U_\alpha)\}$  is an open cover of  $Z$  with local unitary frame  $(p^*v_1^\alpha, \dots, p^*v_r^\alpha)$  for  $(p^*V, p^*h_V)$  over  $p^{-1}(U_\alpha)$ . Given a Hermitian integrable connection  $\tilde{D}$  on  $(p^*V, p^*h_V)$ , we denote by  $(\omega_{ij}^\alpha)$  the connection form for  $\tilde{D}$  on  $p^{-1}(U_\alpha)$  with respect to the frame  $(p^*v_1^\alpha, \dots, p^*v_r^\alpha)$ , (i.e.,  $\tilde{D}(p^*v_i^\alpha) = \sum (p^*v_j^\alpha) \omega_{ij}^\alpha$ ). Then  $(\tau^* \omega_{ij}^\alpha)$  is just the connection form for  $\tilde{\tau}(\tilde{D})$  with respect to the same frame on  $p^{-1}(U_\alpha)$ . Since  $\tau$  is antiholomorphic,  $\tilde{\tau}(\tilde{D})$  is also integrable. Note that if  $\tilde{D}$  is irreducible, then  $\tilde{\tau}(\tilde{D})$  is also irreducible, and that  $\tilde{D}$  is fixed by  $\tilde{\tau}$  if and only if  $\tilde{D}$  satisfies the condition (b) in Section 1. Hence, by  $\tilde{\tau}^2 = id$ , the mapping  $\tilde{\tau}$  is a bijection of  $\mathcal{C}'_H(p^*V, p^*h_V)$  onto itself. Since  $\tau$  is an isometry of  $Z$ , the same argument is applied also to  $\mathcal{C}'_E(p^*V, p^*h_V)$ . Given a unitary transformation  $\tilde{s} \in U(p^*V, p^*h_V)$  and an integrable connection  $\tilde{D} \in \mathcal{C}'_H(p^*V, p^*h_V)$ , we have the identity

$$\tilde{s} \cdot \tilde{\tau}(\tilde{D}) = \tilde{\tau}(s' \cdot \tilde{D}),$$

where  $s' := \tau^* \cdot \tilde{s} \circ \tau^*$ . Hence,  $\tilde{\tau}$  naturally induces a bijection of the moduli space  $\mathcal{H}'(p^*V, p^*h_V)$  onto itself, denoted by  $\tau'$ :  $\mathcal{H}'(p^*V, p^*h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V)$ , and the restriction of  $\tau'$  to  $\mathcal{E}'$  gives a bijection of  $\mathcal{E}'$  onto itself (denoted also by  $\tau'$ :  $\mathcal{E}'(p^*V, p^*h_V) \rightarrow \mathcal{E}'(p^*V, p^*h_V)$ ). Recall that the complex structure of  $Z$  induces those of  $\mathcal{H}'(p^*V, p^*h_V)$  and  $\mathcal{E}'(p^*V, p^*h_V)$ . Since  $\tau$  is antiholomorphic, we have

**Theorem 3.7.** *Both the mappings*

$$\begin{aligned} \tau' &: \mathcal{H}'(p^*V, p^*h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V) \quad \text{and} \\ \tau' &: \mathcal{E}'(p^*V, p^*h_V) \rightarrow \mathcal{E}'(p^*V, p^*h_V) \end{aligned}$$

are antiholomorphic bijection. Therefore  $\tau$  defines real structures of  $\mathcal{H}'(p^*V, p^*h_V)$  and  $\mathcal{E}'(p^*V, p^*h_V)$ .

Given an integrable connection  $\tilde{D}$  on  $(p^*V, p^*h_V)$ , we obtain the elliptic complex  $(\tilde{A}_{\tilde{\tau}(\tilde{D})})$  from the complex  $\tau^*(\tilde{A}_{\tilde{D}})$  by taking complex conjugation. Similarly, for any Einstein-Hermitian connection  $\tilde{D}$ , we obtain  $(\tilde{B}_{\tilde{\tau}(\tilde{D})})$  from  $\tau^*(\tilde{B}_{\tilde{D}})$  by complex conjugation. Hence the restrictions of the bijections

$$\begin{aligned} \tau' &: \mathcal{H}'(p^*V, p^*h_V) \rightarrow \mathcal{H}'(p^*V, p^*h_V) \quad \text{and} \\ \tau' &: \mathcal{E}'(p^*V, p^*h_V) \rightarrow \mathcal{E}'(p^*V, p^*h_V) \end{aligned}$$

on  $\mathcal{H}''(p^*V, p^*h_V)$  and  $\mathcal{E}''(p^*V, p^*h_V)$  define the bijections

$$\begin{aligned} \tau'' &: \mathcal{H}''(p^*V, p^*h_V) \rightarrow \mathcal{H}''(p^*V, p^*h_V) \quad \text{and} \\ \tau'' &: \mathcal{E}''(p^*V, p^*h_V) \rightarrow \mathcal{E}''(p^*V, p^*h_V) \end{aligned}$$

respectively. The Kähler metric of  $\mathcal{E}''(p^*V, p^*h_V)$  is defined by the  $L^2$ -inner product on  $\mathcal{H}^1(Z, B_{\tilde{D}})$ , which identified with the tangent space of  $\mathcal{E}''(p^*V, p^*h_V)$

at  $\langle \tilde{D} \rangle$ . Since  $\tau$  is isometry on  $Z$ , the real structure  $\tau'': \mathcal{E}''(p^*V, p^*h_v) \rightarrow \mathcal{E}''(p^*V, p^*h_v)$  is an isometry.

Now we fix an arbitrary element  $\tilde{D}$  of  $p^*(\mathcal{C}'_B(V, h_v))$ . Put

$$\begin{aligned}\eta_H(\alpha) &= H_H(K_H^{-1}(\alpha) \wedge K_H^{-1}(\alpha)) \quad \text{for } \alpha \in \mathcal{H}^1(Z, A_{\tilde{D}}), \quad \text{and} \\ \eta_E(\beta) &= H_E(K_E^{-1}(\beta) \wedge K_E^{-1}(\beta)) \quad \text{for } \beta \in \mathcal{H}^1(Z, B_{\tilde{D}}).\end{aligned}$$

Since  $\tilde{D}$  is fixed by  $\tilde{\tau}$  (cf. Section 1) we immediately obtain:

$$(3) \quad \eta_H(\tau^*\alpha) = \tau^*\eta_H(\alpha), \quad \alpha \in \mathcal{H}^1(Z, A_{\tilde{D}}),$$

$$(4) \quad \eta_E(\tau^*\beta) = \tau^*\eta_E(\beta), \quad \beta \in \mathcal{H}^1(Z, B_{\tilde{D}}).$$

Let  $(\mathcal{H}'(p^*V, p^*h_v))_R$ ,  $(\mathcal{E}'(p^*V, p^*h_v))_R$ ,  $(\mathcal{H}''(p^*V, p^*h_v))_R$ ,  $(\mathcal{E}''(p^*V, p^*h_v))_R$  be the subsets of  $\mathcal{H}'(p^*V, p^*h_v)$ ,  $\mathcal{E}'(p^*V, p^*h_v)$ ,  $\mathcal{H}''(p^*V, p^*h_v)$ ,  $\mathcal{E}''(p^*V, p^*h_v)$ , respectively consisting of all elements fixed by the real structures defined above. Then by Theorem 1.1,  $p^*(\mathcal{B}'(V, h_v))$  is embedded in  $(\mathcal{E}'(p^*V, p^*h_v))_R$  ( $\subset (\mathcal{H}'(p^*V, p^*h_v))_R$ ) and  $p^*(\mathcal{B}''(V, h_v)) \subset (\mathcal{E}''(p^*V, p^*h_v))_R$  ( $\subset (\mathcal{H}''(p^*V, p^*h_v))_R$ ).

#### 4. Proof of Theorem 3.1

Let  $g_M$  denote the given metric on  $M$  and let  $g_z$  denote the induced metric by  $g_M$  on  $Z$ . Then  $g_V := g_z - p^*g_M$  is an indefinite metric which is positive definite on each fibre of the submersion  $p: (Z, g_z) \rightarrow (M, g_M)$ . Let  $J_z$  be the complex structure on  $Z$ . We define a 2-form  $\omega_V$  on  $Z$  by

$$\omega_V(v_1, v_2) := g_V(v_1, J_z v_2), \quad v_1, v_2 \in T_z Z \quad (z \in Z).$$

Recall that Salamon [14; p. 144] introduced (locally defined) vector bundles  $H$  and  $E$  on  $M$  such that the complexification  $T^*M^C$  of the cotangent bundle  $T^*M$  is nothing but  $H \otimes_C E$ . Let  $(h_1, h_2)$  and  $(e_1, \dots, e_{2m})$  be symplectic local frames of  $H$  and  $E$  respectively, and  $(z^1, z^2)$  the dual coordinate of  $H$ . (We follow [11; (3.2.2)] for definition of symplectic frames.) Moreover  $H$  and  $E$  have natural connections induced by Riemannian connection of  $M$  (cf. [14]). Let  $(\omega_i^j)$  be the connection form on  $H$  with respect to the frame  $(h_1, h_2)$ . Then  $\omega_V$  is written as  $c(|z^1|^2 + 1)^{-2} \theta \wedge \bar{\theta}$ , where  $\theta := dz^1 + z^1 p^* \omega_1^1 + p^* \omega_1^2 - (z^1)^2 p^* \omega_2^1 - z^1 p^* \omega_2^2$  and  $c$  is a constant depending only on the scalar curvature of  $M$  and the dimension of  $M$  (cf. [14] for more details).

Then we have

**Lemma 4.1.** *Put*

$$\begin{aligned}u_i &= (|z^1|^2 + 1)^{-1/2} (z^1 p^*(e_i \otimes h_1) + p^*(e_i \otimes h_2)) \quad (1 \leq i \leq 2m), \quad \text{and} \\ \theta_V &= (|z^1|^2 + 1)^{-1} \theta.\end{aligned}$$

*Then we have*

$$d\omega_V = -2c(\sum_{i=1}^m u_i \wedge u_{m+i} \wedge \bar{\theta}_V + \bar{u}_i \wedge \bar{u}_{m+i} \wedge \theta_V).$$

$$\begin{aligned}
\text{Proof. } d\omega_V &= c\{-2(|z^1|^2+1)^{-2}(z^1 d\bar{z}^1 + \bar{z}^1 dz^1) \wedge \theta \wedge \bar{\theta} \\
&+ (|z^1|^2+1)^{-2}(dz^1 \wedge p^* \omega_1^1 + z^1 p^* d\omega_1^1 + p^* d\omega_2^1 - 2z^1 dz^1 \wedge p^* \omega_1^2 - (z^1)^2 p^* d\omega_1^2 \\
&- dz^1 \wedge p^* \omega_2^2 - z^1 p^* d\omega_2^2) \wedge \bar{\theta} - \\
&(|z^1|^2+1)^{-2} \theta \wedge (-d\bar{z}^1 \wedge p^* \omega_1^1 - \bar{z}^1 p^* d\omega_1^2 + 2\bar{z}^1 d\bar{z}^1 \wedge p^* \omega_2^1 + (\bar{z}^1)^2 p^* d\omega_2^1 + d\bar{z}^1 \wedge p^* \omega_2^2 \\
&+ \bar{z}^1 p^* d\omega_2^2)\} \\
&= c(|z^1|^2+1)^{-2}\{z^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^2) + p^*(d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) \\
&- (z^1)^2 p^*(d\omega_1^2 + \omega_1^2 \wedge \omega_1^1 + \omega_1^2 \wedge \omega_2^1) - z^1 p^*(d\omega_2^2 + \omega_2^2 \wedge \omega_2^1)\} \wedge \bar{\theta} \\
&+ c(|z^1|^2+1)^{-2} \theta \wedge \{z^1 p^*(d\omega_1^1 + \omega_2^1 \wedge \omega_1^2) + p^*(d\omega_2^1 + \omega_2^2 \wedge \omega_1^1 + \omega_2^2 \wedge \omega_2^1) \\
&- (\bar{z}^1)^2 p^*(d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2) - \bar{z}^1 p^*(d\omega_2^2 + \omega_1^2 \wedge \omega_2^1)\}.
\end{aligned}$$

We denote by  $(\Omega_j^i)$  the curvature form of the vector bundle  $H$  with respect to  $(h_1, h_2)$ :

$$\Omega_j^i = d\omega_j^i + \sum_{k=1}^2 \omega_k^i \wedge \omega_j^k.$$

We have the following formula due to Salamon [14; Proposition 3.2].

$$\begin{aligned}
\Omega_1^1 &= -\sum_{i=1}^m ((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2) + (e_i \otimes h_2) \wedge (e_{m+i} \otimes h_1)), \\
\Omega_1^2 &= -2\sum_{i=1}^m ((e_i \otimes h_2) \wedge (e_{m+i} \otimes h_2)), \\
\Omega_2^1 &= 2\sum_{i=1}^m ((e_i \otimes h_1) \wedge (e_{m+i} \otimes h_1)), \\
\Omega_2^2 &= \sum_{i=1}^m ((e_i \otimes h_2) \wedge (e_{m-i} \otimes h_1) + (e_i \otimes h_1) \wedge (e_{m+i} \otimes h_2)).
\end{aligned}$$

Using this we get:

$$\begin{aligned}
d\omega_V &= c(|Z^1|^2+1)^{-2}\{(z^1 p^* \Omega_1^1 + p^* \Omega_2^1 - (z^1)^2 p^* \Omega_1^2 - z^1 p^* \Omega_2^2) \wedge \bar{\theta} + \\
&(\bar{z}^1 p^* \Omega_1^1 + p^* \Omega_2^1 - (\bar{z}^1)^2 p^* \Omega_1^2 - \bar{z}^1 p^* \Omega_2^2) \wedge \theta\} \\
&= -2c(\sum_{i=1}^m (u_i \wedge u_{m+i} \wedge \bar{\theta}_V + \bar{u}_i \wedge \bar{u}_{m+i} \wedge \theta_V)),
\end{aligned}$$

which proves Lemma 4.1.

Let  $D$  be a Hermitian  $B_2$ -connection on  $(V, h_V)$  on  $M$ . Then we have a morphism  $q$  between the complexes  $(C_D)$  and  $(A_{p^*D})$  defined as follows:

$$C^i(\text{End}(V, h_V)) \ni d \mapsto (pr^{(0,i)} \circ p^*)(d) \in A^i(\text{End}(p^*V)),$$

where  $pr^{(i,j)}: \Gamma(Z, \text{End}(p^*V) \otimes_{\mathcal{C}} \wedge^i T^*Z) \rightarrow \Gamma(Z, \text{End}(p^*V) \otimes_{\mathcal{C}} \wedge^{(i,j)} T^*Z)$  is the natural projection. Let  $\tilde{\mathcal{D}}''$  and  $\mathcal{D}_i$  be the formal adjoint of  $(p^*D)''$  and  $d_i$  in the complexes  $A_{p^*D}$  and  $C_D$  respectively. Then we obtain:

**Lemma 4.2.** *Denoting by  $*_M$  and  $*_Z$  the star operators for vector bundles on  $M$  and  $Z$ , we have*

$$\tilde{\mathcal{D}}''qv = q(\mathcal{D}_{i-1}v) - (*_z \circ pr^{(2m-1, 2m)} \circ *_M) v \wedge (-2c \sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_v)$$

for all  $v \in C^i(\text{End}(V, h_v))$ .

Proof. Write the volume forms on  $M$  and  $Z$  as  $dv_M$  and  $dv_Z$  respectively. Then  $dv_Z = p^*(dv_M) \wedge \omega_V$ . Hence, for any  $v \in C^i(\text{End}(V, h_v))$ ,

$$\begin{aligned} \tilde{\mathcal{D}}''qv &= -(*_z \circ (d^{p^*D})' \circ *_z \circ q)(v) = -(*_z \circ (d^{p^*D})' \circ *_z \circ pr^{(0,1)} \circ p^*)(v) \\ &= -(*_z \circ (d^{p^*D})' \circ pr^{(2m+1-i, 2m+1)})(p^*(*_M v) \wedge \omega_V) \\ &= -(*_z \circ (d^{p^*D})')((pr^{(2m-i, 2m)}(*_M v)) \wedge \omega_V) \\ &= -*_z \{(d^{p^*D})'((pr^{(2m-i, 2m)}(p^*(*_M v))) \wedge \omega_V) + pr^{(2m+i, 2m)}(p^*(*_M v)) \wedge d' \omega_V\} \\ &= -*_z \{(pr^{(2m-i, 2m)}(p^*(d^D(*_M v)))) \wedge \omega_V - (pr^{(2m-1, 2m)}(p^*(*_M v))) \wedge d' \omega_V\} \\ &= -pr^{(0,1)}((p^* \circ *_M \circ d^D \circ *_M)v) - *_z \{(pr^{(m-i, 2m)}(p^*(*_M v))) \wedge d' \omega_V\}. \end{aligned}$$

By using Lemma 4.1, it follows:

$$\tilde{\mathcal{D}}''qv = -q\mathcal{D}_{i-1}v - (*_z \circ pr^{(2m-i, 2m)} \circ *_M) v \wedge (-2c \sum_{i=1}^m u_i \wedge u_{m+i} \wedge \theta_v),$$

which proves Lemma 4.2.

In view of Lemma 4.2, we have  $q(\mathcal{H}^1(M, C_D) \subset \mathcal{H}^1(Z, A_{p^*D}))$ . From [4; Theorem 3], it follows that  $\dim_C \mathcal{H}^1(Z, A_{p^*D}) = \dim_C \mathcal{H}^1(M, (C_D)^C) = \dim_R \mathcal{H}^1(M, C_D)$ . Together with the argument used by Kim [7; (1.3)], we have  $\mathcal{H}^1(Z, A_{p^*D}) + \overline{\mathcal{H}^1(Z, A_{p^*D})} = (\mathcal{H}^1(Z, B_{p^*D}))^C$ . Hence

$$(1) \quad p^* \mathcal{H}^1(M, C_D) + J_Z p^* \mathcal{H}^1(M, C_D) = \mathcal{H}^1(Z, B_{p^*D}).$$

The tangent space of  $\mathcal{B}''(V, h_v)$  at  $[D]$  is  $\mathcal{H}^1(M, C_D)$  and the tangent space of  $\mathcal{E}''(p^*V, p^*h_v)$  at  $\langle p^*D \rangle$  is  $\mathcal{H}^1(Z, B_{p^*D})$ . By (1),  $\mathcal{B}''(V, h_v)$  is of dimension  $\dim_R \mathcal{H}^1(M, C_D)$  at  $[D]$ , which is equal to the complex dimension of  $\mathcal{E}''(p^*V, p^*h_v)$  at  $\langle p^*D \rangle$ .

REMARKS. Capria and Salamon [4] constructed interesting families of  $B_2$ -connections for some vector bundles over  $P^*H$ . In a forthcoming paper [12], as an application of Theorem 3.1, we shall clarify the relationship between such families of  $B_2$ -connections and the moduli space of Einstein-Hermitian connections on null-correlation bundles over odd dimensional complex projective spaces.

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