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ON THE EQUIVARIANT HOMOTOPY OF STIEFEL MANIFOLDS

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1. Introduction and results

Throughout this paper G denotes a compact Lie group, and Λ denotes one of the real numbers $\boldsymbol{R}$, the complex numbers $\boldsymbol{C}$ and the quaternions $\boldsymbol{Q}$. Let E be a representation of G over Λ. All representations considered in this paper are orthogonal if $\Lambda=\boldsymbol{R}$, unitary if $\Lambda=\boldsymbol{C}$, and symplectic if $\Lambda=\boldsymbol{Q}$. For a positive integer $m\leq\dim_\Lambda E$, the *Stiefel manifold* $V_m^\Lambda(E)$ consists of all orthonormal m-frames in E, i.e.,

$$V_m^\Lambda(E)=\{(v_1,\cdots,v_m)\,|\,v_i\in E,\ \|v_i\|=1\quad\text{for}\quad i=1,\cdots,m,\\ \text{and}\quad v_i\perp v_j\quad\text{if}\quad i\neq j\}\,.$$

If $m=1$, then $V_m^\Lambda(E)$ is the unit sphere $S(E)$ in E. For any $g\in G$ and any orthonormal m-frame $(v_1,\cdots,v_m)$ in E, $(gv_1,\cdots,gv_m)$ is also an orthonormal m-frame in E. This induces a smooth G-action on $V_m^\Lambda(E)$.

Let E' be another representation of G over Λ. We are interested in the set of G-homotopy classes of G-maps from $S(E)$ to $V_m^\Lambda(E')$, $[S(E), V_m^\Lambda(E')]_G$. If $m=1$, this set is the set of G-homotopy classes of G-maps from sphere to sphere, $[S(E), S(E')]_G$, which was studied in Hauschild [1], Rubinsztein [3] and others. (I am grateful to the referee who informed me that there was a gap in the proof of Rubinsztein's main theorem [3; Theorem 7.2]. This information leads to an improvement of the presentation of this paper.)

For any positive integer n, let

$$\Lambda^n=\Lambda\oplus\cdots\oplus\Lambda\qquad(n\text{ summands})$$

be a representation with trivial G-action and with the standard inner product. We define a map

$$j\colon\ S(E')\to V_m^\Lambda(E'\oplus\Lambda^{m-1})$$

by $j(v)=(v,e_1,\cdots,e_{m-1})$ for $v\in S(E')$ and the canonical orthonormal $(m-1)$-frame $(e_1,\cdots,e_{m-1})$ in Λ^{m-1}. Then j is a G-embedding, and induces a transformation

$$j_*\colon [S(E), S(E')]_G \to [S(E), V_m^\Lambda(E'\oplus\Lambda^{m-1})]_G$$

between G-homotopy sets. We are also interested in this transformation j_*.

In the non-equivariant case we already know some facts about j_*. Clearly $S^{dn-1}=S(\Lambda^n)$ where $d=1$ if $\Lambda=\boldsymbol{R}$, $d=2$ if $\Lambda=\boldsymbol{C}$, and $d=4$ if $\Lambda=\boldsymbol{Q}$. The map

$$j\colon S^{dn-1} = S(\Lambda^n) \to V_m^\Lambda(\Lambda^n\oplus\Lambda^{m-1}) = V_m^\Lambda(\Lambda^{m+n-1})$$

defined above induces a group homomorphism

$$j_*\colon \pi_i(S^{dn-1}) \to \pi_i(V_m^\Lambda(\Lambda^{m+n-1}))$$

between the i-th homotopy groups for an integer $i\geq 0$. We collect known results about the homomorphism j_* in the following:

Proposition 1 (See for example [2; Chapter 7]). (a) *j_* is an isomorphism in each case of the followings*:

(i) $m=1$,

(ii) $0\leq i\leq dn-2$,

(iii) $\Lambda=R$, *and* $i=n-1$ *is even*,

(iv) $\Lambda=\boldsymbol{C}$ *or* $\boldsymbol{Q}$, *and* $i=dn-1$.

Therefore

$$\pi_i(V_m^\Lambda(\Lambda^{m+n-1})) = \begin{cases} 0 & \textit{csae (ii)} \\ \boldsymbol{Z} & \textit{case (iii) or (iv).} \end{cases}$$

(b) *If $\Lambda=\boldsymbol{R}$, $m\geq 2$, and $i=n-1$ is odd, then j_* is an epimorphism and*

$$\pi_i(V_m^\Lambda(\Lambda^{m+n-1})) = \boldsymbol{Z}/2\boldsymbol{Z}\,.$$

To state our result in the equivariant case, let us define some notations. For any closed subgroup H of G, $N(H)$ denotes the normalizer of H in G, and (H) denotes the conjugacy class of H in G. Let X be a G-space. For any $x\in X$, G_x denotes the isotropy subgroup at x. The conjugacy class of an isotropy subgroup is called an orbit type. We put

$$\begin{aligned} X^H &= \{x\in X\mid H\subset G_x\}\,, \\ X_H &= \{x\in X\mid H = G_x\}\,, \qquad \text{and} \\ X_{(H)} &= \{x\in X\mid (H) = (G_x)\}\,. \end{aligned}$$

For a representation E of G, $\mathfrak{M}(E)$ denotes the set of orbit types appearing on $S(E)$. Choose a representative of each element of $\mathfrak{M}(E)$, and denote by $\mathfrak{M}_r(E)$ the set of those representatives. For any $H\in\mathfrak{M}_r(E)$ there is a transformation

$$r_H\colon [S(E), S(E')]_G \to [S(E^H), S(E'^H)]$$

restricting to the fixed point set by H, where [,] denotes the non-equivariant homotopy set.

Our result is

Theorem 2. *Let E, F be representations of a compact Lie group G over Λ. Let*

$$j_*: [S(E), S(E\oplus F)]_G \to [S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j: S(E\oplus F) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}).$$

Then

(a) *j_* is surjective,*

(b) *j_* is bijective in particular in each case of the followings* (i), (ii):

(i) $\Lambda=\boldsymbol{R}$, $\dim_{\boldsymbol{R}} E^H$ *is odd for any* $H\in\mathfrak{N}(E, F)=\{H\in\mathfrak{M}_r(E)\,|\,\dim_{\boldsymbol{R}} F^H=0\}$, *and*

$$r = \prod_{H\in\mathfrak{N}(E,F)} r_H: [S(E), S(E\oplus F)]_G \to \prod_{H\in\mathfrak{N}(E,F)}[S(E^H), S(E^H\oplus F^H)]$$

is injective,

(ii) $\Lambda=\boldsymbol{C}$ *or* $\boldsymbol{Q}$, *and r is injective,*

(c) *if* $\dim_{\boldsymbol{R}} E^G\geq 2$ *then* $[S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$ *has a group structure and j_* is a group homomorphism.*

NOTE. The injectivity of r is studied by several authors, e.g., Hauschild [1; Satz 4.5].

In the subsequent sections we prove Theorem 2. Section 2 is devoted to preliminary lemmas. Section 3 is devoted to proving the surjectivity of j_*, and section 4 is devoted to proving the injectivity of j_*. In section 5 we give a group structure to $[S(E), V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$ so that j_* is a group homomorphism.

2. Preliminary lemmas

Let E, F be representations of a compact Lie group G over Λ, and let M be a compact, smooth, free G-manifold with $\dim M\leq\dim S(E\oplus F)$. Consider the fibre bundles

$$B = M\times_G V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}) \to M/G$$

with fibre $V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$, and

$$B' = M\times_G S(E\oplus F) \to M/G$$

with fibre $S(E\oplus F)$. The G-map

$$Id\times i\colon M\times S(E\oplus F)\to M\times V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

induces a bundle embedding $\tilde{j}\colon B'\to B$. There is a one-to-one correspondence between G-maps from M to $V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$ and cross sections of B, and there is also a one-to-one correspondence between their homotopies. This shows that the following two lemmas are equivalent:

Lemma 3. *Let*

$$f\colon M\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-map, and let

$$P\colon \partial M\times[0, 1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-homotopy with $P_0=f|\partial M$ *and* $P_1(\partial M)\subset i(S(E\oplus F))$. *Then P extends to a G-homotopy*

$$Q\colon M\times[0, 1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

with $Q_0=f$ *and* $Q_1(M)\subset j(S(E\oplus F))$.

Lemma 4. *Let* $N=M/G$. *Let* $s\colon N\to B$ *be a cross section of B, and let* $P\colon \partial N\times[0, 1]\to B|\partial N$ *be a homotopy of cross section of* $B|\partial N$ *with* $P_0=s|\partial N$ *and* $P_1(\partial N)\subset\tilde{j}(B')$. *Then P extends to a homotopy of cross section of B,* $Q\colon N\times[0, 1]\to B$, *with* $Q_0=s$ *and* $Q_1(N)\subset\tilde{j}(B')$.

We give a proof of Lemma 4 making use of the obstruction theory. Refer to [4; Part III] for the obstruction theory.

Proof of Lemma 4. Since N is a smooth manifold, we obtain a triangulation of N. Let $n=\dim S(E\oplus F)$. Then $\dim N\leq n$, and $S(E\oplus F)$, which is the fibre of B', is $(n-1)$-connected. So the cross section $\tilde{j}^{-1}P_1$ of $B'|\partial N$ extends to a cross section $s_1; N\to B'$ of B'. We see from Proposition 1 that $V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$ is also $(n-1)$-connected. Let N^{n-1} denote the $(n-1)$-skeleton of N, which contains ∂N. Then P extends to a homotopy of cross section,

$$R\colon N^{n-1}\times[0, 1]\to B|N^{n-1},$$

with $R_0=s|N^{n-1}$ and $R_1=\tilde{j}s_1|N^{n-1}$. So, if $\dim N<n$, the lemma is proved.

Now let $\dim N=n$. Let $B(\pi_n)$ and $B'(\pi_n)$ be the bundles of coefficients associated with the bundles B and B' by the n-th homotopy group, respectively. Also let $C^n(N; B(\pi_n))$ and $C^n(N; B'(\pi_n))$ be the groups of n-cochains of N with coefficients in $B(\pi_n)$ and $B'(\pi_n)$, respectively. The bundle embedding $\tilde{j}\colon B'\to B$ induces a group homomorphism

$$\tilde{j}_*\colon C^n(N; B'(\pi_n))\to C^n(N; B(\pi_n)).$$

We see from Proposition 1 that $\tilde{j}_*$ is an epimorphism. Let

$$d = d(s, R, \tilde{j}s_1) \in C^n(N; B(\pi_n))$$

be the deformation n-cochain. (See [4; p 172].) There is $d' \in C^n(N; B'(\pi_n))$ with $\tilde{j}_*(d') = d$. By [4; 33.9] there is a cross section s_2 of B' such that s_2 agrees with s_1 on N^{n-1} and $d(s_1, s_2) = -d'$, where $d(s_1, s_2)$ is the difference n-cochain. (Also see [4; p 172].) We see

$$d(\tilde{j}s_1, \tilde{j}s_2) = \tilde{j}_*(d(s_1, s_2)) = -d .$$

We define a homotopy of cross section of $B|N^{n-1}$,

$$S: N^{n-1} \times [0, 1] \to B|N^{n-1} ,$$

by

$$S(x, t) = \begin{cases} R(x, 2t) & \text{if } 0 \leq t \leq 1/2 \\ \tilde{j}s_2(x) & \text{if } 1/2 \leq t \leq 1 . \end{cases}$$

By [4; 33.7],

$$\begin{aligned} d(s, S, \tilde{j}s_2) &= d(s, R, \tilde{j}s_1) + d(\tilde{j}s_1, \tilde{j}s_2) \\ &= d - d \\ &= 0 . \end{aligned}$$

By this and [4; 33.8], S extends to a homotopy of cross section of B, $T: N \times [0, 1] \to B$, with $T_0 = s$ and $T_1 = \tilde{j}s_2$. Let $\partial N \times [0, 1] \subset N$ be a collar of ∂N in N. Then we define a homotopy $Q: N \times [0, 1] \to B$ as, for $x \in N$ and $t \in [0, 1]$,

$$Q(x, t) = T(x, t) \quad \text{if} \quad x \in N - \partial N \times [0, 1) ,$$
$$Q(x, t) = T(x, t/(2-r)) \quad \text{if} \quad x = (y, r) \in \partial N \times [0, 1] \quad \text{and} \quad 2t + r \leq 2 ,$$
$$Q(x, t) = T(x, (t+r-1)/r) \quad \text{if} \quad x = (y, r) \in \partial N \times [0, 1] , \quad 2t + r \geq 2 \quad \text{and} \quad r \neq 0 .$$

Q is well-defined, and is an extension of P with the desired property.

Q.E.D.

3. The surjectivity of j_*

Let E, F be representations of a compact Lie group G over Λ, and let

$$j_*: [S(E), S(E \oplus F)]_G \to [S(E), V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j: S(E \oplus F) \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1}) .$$

The purpose of this section is to prove the surjectivity of j_*. Since j is an embedding, it suffices to prove the following fact:

Lemma 5. *Let*

$$f\colon S(E)\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-map, and let N be a compact smooth G-submanifold of S(E) with $\dim N=\dim S(E)$. *Let*

$$T\colon N\times[0,1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-homotopy with $T_0=f|N$ *and* $T_1(N)\subset j(S(E\oplus F))$. *Then T extends to a G-homotopy*

$$R\colon S(E)\times[0,1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

with $R_0=f$ *and* $R_1(S(E))\subset i(S(E\oplus F))$.

Proof. $\mathfrak{M}(E)$ is a finite set. Let us number its elements

$$\mathfrak{M}(E)=\{(H_1),(H_2),\cdots,(H_a)\}$$

in such a way that if $i<k$ then $(H_i)\not\subset(H_k)$. Consider the following Assertion:

Assertion. *There are compact smooth G-submanifolds* $M_0, M_1,\cdots,M_a$ *of* $S(E)$ *such that*

$$\dim M_i=\dim S(E)\quad\text{for}\quad i=0,1,\cdots,a\,,\quad M_0\supset N\,,\quad\textit{and}$$
$$\operatorname{Int}M_i\supset M_{i-1}\cup S(E)_{(H_i)}\quad\textit{for}\quad i=1,2,\cdots,a\,.$$

Furthermore there are G-homotopies $R^{(0)}, R^{(1)},\cdots,R^{(a)}$ *such that*

$$R^{(i)}\colon M_i\times[0,1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})\qquad\textit{for}\quad i=0,1,\cdots,a\,,$$
$$R_0^{(i)}=f|M_i\qquad\textit{for}\quad i=0,1,\cdots,a\,,$$
$$R_1^{(i)}(M_i)\subset j(S(E\oplus F))\qquad\textit{for}\quad i=0,1,\cdots,a\,,$$
$$R^{(0)}|N\times[0,1]=T\,,\qquad\textit{and}$$
$$R^{(i)}|M_{i-1}\times[0,1]=R^{(i-1)}\qquad\textit{for}\quad i=1,2,\cdots,a\,.$$

Lemma 5 follows from the Assertion since $M_a=S(E)$. In the following we prove the Assertion.

N and T satisfy the conditions for M_0 and $R^{(0)}$, respectively. Suppose that $M_0,\cdots,M_{i-1}$, and $R^{(0)},\cdots,R^{(i-1)}$ are constructed. Put

$$M=(S(E)-\operatorname{Int}M_{i-1})^{H_i}=S(E^{H_i})-\operatorname{Int}M_{i-1}^{H_i}\,.$$

Then M is a compact smooth manifold with boundary $\partial M=M\cap\partial M_{i-1}$. Moreover M is $N(H_i)$-invariant, and all isotropy subgroups on M are H_i. So M becomes a free $N(H_i)/H_i$-manifold. Regard E^{H_i} and F^{H_i} as representations of $N(H_i)/H_i$. By Lemma 3 there is an $N(H_i)/H_i$-homotopy

$$Q:\ M\times[0, 1]\to V_m^\Lambda(E^{H_i}\oplus F^{H_i}\oplus\Lambda^{m-1})$$

such that

$$Q_0=f|M,$$
$$Q_1(M)\subset i(S(E^{H_i}\oplus F^{H_i})),\qquad\text{and}$$
$$Q|\partial M\times[0, 1]=R^{(i-1)}|\partial M\times[0, 1].$$

Since $G(M)=G\times_{N(H_i)}M$, we may extend Q to a G-homotopy

$$Q':\ G(M)\times[0, 1]\to G(V_m^\Lambda(E^{H_i}\oplus F^{H_i}\oplus\Lambda^{m-1}))\subset V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

such that

$$Q_0'=f|G(M),$$
$$Q_1'(G(M))\subset i(S(E\oplus F)),\qquad\text{and}$$
$$Q'|\partial G(M)\times[0, 1]=R^{(i-1)}|\partial G(M)\times[0, 1].$$

Applying [3; Lemma 1.1] to the G-manifold $A=S(E)-\operatorname{Int}M_{i-1}$ and the submanifold $G(M)$ of A, we obtain compact G-submanifolds K, L of A such that

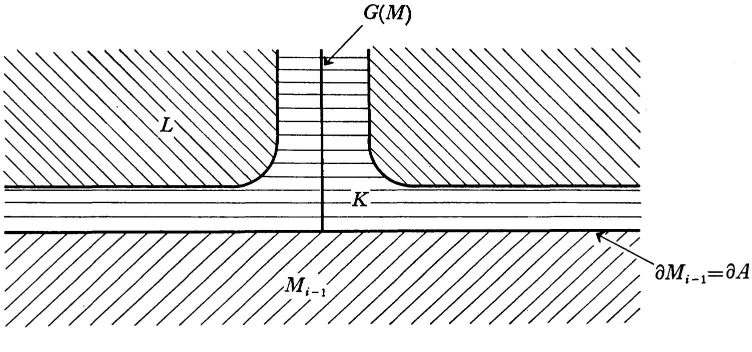
(i) $K\cup L=A$,

(ii) $\partial L=L\cap K$,
$\partial K=\partial L\cup\partial A=\partial L\cup\partial M_{i-1}$,
$\partial M_{i-1}\cap\partial L=\phi$,

(iii) $\partial M_{i-1}\cup G(M)\subset K$, and

(iv) K is a mapping cylinder of some G-map

$$\psi:\ \partial L\to\partial M_{i-1}\cup G(M).$$



Put $M_i=M_{i-1}\cup K$ in $S(E)$. Then M_i is a compact smooth G-submanifold of $S(E)$ with $\dim M_i=\dim S(E)$, and with $\operatorname{Int}M_i\supset M_{i-1}\cup S(E)_{(H_i)}$. According to (iv), let us denote a point of K by the form $[y, s]$, where $y\in\partial L$ and $s\in[0, 1]$. Under this form $[y, 1]=y$ and $[y, 0]=\psi(y)$. We define a G-homotopy

$$R^{(i)}:\ M_i\times[0, 1]\to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

as the following: For $(x, t)\in M_i\times[0, 1]$,

$$\begin{aligned}
&R^{(i)}(x, t) = R^{(i-1)}(x, t) \quad \text{if} \quad x\in M_{i-1}\,,\\
&R^{(i)}(x, t) = f([y, s-2t]) \quad \text{if} \quad x=[y, s]\in K \quad \text{and} \quad 2t\leq s\,,\\
&R^{(i)}(x, t) = R^{(i-1)}(\psi(y), (2t-s)/(2-s)) \quad \text{if} \quad x=[y, s]\in K\,,\\
&\qquad \psi(y)\in\partial M_{i-1} \quad \text{and} \quad s\leq 2t\,,\\
&R^{(i)}(x, t) = Q'(\psi(y), \ (2t-s)/(2-s)) \quad \text{if} \quad x=[y, s]\in K\,,\\
&\qquad \psi(y)\in G(M) \quad \text{and} \quad s\leq 2t\,.
\end{aligned}$$

M_i and $R^{(i)}$ constructed above satisfy the conditions in the Assertion. Thus this completes the proof. Q.E.D.

Now let X, Y be G-spaces, and let $x_0\in X^G$, $y_0\in Y^G$. Denote by

$$[(X, x_0), (Y, y_0)]_G$$

the set of G-homotopy classes rel. x_0 of G-maps $f\colon X\to Y$ with $f(x_0)=y_0$.

The following Proposition is required in section 5.

Proposition 6. *Let E, F be representations of G, and let $x_0\in S(E^G)$, $y_0\in S(E^G\oplus F^G)$. Then*

$$j_*\colon\ [(S(E), x_0), (S(E\oplus F), y_0)]_G \to [(S(E), x_0), (V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}), j(y_0))]_G$$

is surjective.

Proof. Let

$$f\colon\ S(E) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

be a G-map with $f(x_0)=j(y_0)$. Let D be a G-invariant, top-dimensional, small disc in $S(E)$ with x_0 as its center. We may deform f to a G-map f' such that $f'(D)=j(y_0)$ and $f'\simeq f$ rel. x_0. By Lemma 5 there is a G-homotopy

$$R\colon\ S(E)\times[0, 1] \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})$$

such that

$$\begin{aligned}
&R_0 = f'\,,\\
&R_1(S(E))\subset j(S(E\oplus F))\,, \qquad \text{and}\\
&R(D\times[0, 1]) = j(y_0)\,.
\end{aligned}$$

Then f' is G-homotopic to R_1 rel. x_0. This proves the Proposition. Q.E.D.

4. The injectivity of j_*

Let E, F be representations of a compact Lie group G over Λ, and let

$$j_*\colon [S(E),\ S(E\oplus F)]_G \to [S(E),\ V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

be the transformation induced from the G-map

$$j\colon S(E\oplus F) \to V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1}).$$

The purpose of this section is to prove the injectivity of j_* under the assumption (i) or (ii) in Theorem 2.

For any closed subgroup H of G, let

$$j^H = j\,|\,S(E^H\oplus F^H)\colon\ S(E^H\oplus F^H) \to V_m^\Lambda(E^H\oplus F^H\oplus\Lambda^{m-1})\,.$$

The following diagram is commutative:

$$\begin{array}{ccc} [S(E),\ S(E\oplus F)]_G & \xrightarrow{j_*} & [S(E),\ V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G \\ \downarrow{\scriptstyle r_H} & & \downarrow{\scriptstyle r'_H} \\ [S(E^H),\ S(E^H\oplus F^H)] & \xrightarrow[j_*^H]{} & [S(E^H),\ V_m^\Lambda(E^H\oplus F^H\oplus\Lambda^{m-1})] \end{array}$$

where r_H and r'_H are the transformations restricting to the fixed point set by H.

Now suppose

$$j_*(\alpha) = j_*(\beta)$$

for α, $\beta\in[S(E),\ S(E\oplus F)]_G$. Then, by the commutativity of the above diagram,

$$j_*^H r_H(\alpha) = j_*^H r_H(\beta)$$

for any closed subgroup H of G. Proposition 1 implies that j_*^H is an isomorphism under the assumption (i) or (ii) in Theorem 2. Thus

$$r_H(\alpha) = r_H(\beta)$$

for any H. Hence $r(\alpha)=r(\beta)$. By the assumption r is injective, hence $\alpha=\beta$. Thus j_* is injective.

5. The group structure

Let E, F be representations of a compact Lie group G over Λ. Suppose $\dim_R E^G\geq 2$. Then, according to [3; Section 6], $[S(E),\ S(E\oplus F)]_G$ has a group structure. In the similar way we may give a group structure to

$$[S(E),\ V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

so that

$$j_*\colon [S(E),\ S(E\oplus F)]_G \to [S(E),\ V_m^\Lambda(E\oplus F\oplus\Lambda^{m-1})]_G$$

is a group homomorphism. To show this is the purpose of this section.

Lemma 7. *Suppose* $\dim_R E^G = 2$ *and* $x_0 \in S(E^G)$. *Let*

$$\omega\colon [0,1] \to S(E^G) \subset S(E)$$

be a path with $\omega(0) = \omega(1) = x_0$. *Then there is a G-homotopy*

$$H\colon S(E) \times [0, 1] \to S(E)$$

such that

$$H_0 = H_1 = Id, \quad \text{and}$$
$$H(x_0, t) = \omega(t) \quad \text{for any} \quad t \in [0, 1].$$

Proof. Choose a homotopy

$$J\colon S(E^G) \times [0, 1] \to S(E^G) \subset S(E)$$

such that

$$J_0 = J_1 = \text{the inclusion}, \quad \text{and}$$
$$J(x_0, t) = \omega(t) \quad \text{for any} \quad t \in [0, 1].$$

Denote by $(E^G)^\perp$ the orthogonal complement of E^G in E, and denote a point of E by the form $x+y$ where $x \in E^G$ and $y \in (E^G)^\perp$. Define

$$H\colon S(E) \times [0, 1] \to S(E)$$

as

$$H(x+y, t) = \|x\| J(x/\|x\|, t) + y \quad \text{if} \quad x \neq 0, \qquad \text{and}$$
$$H(x+y, t) = y \quad \text{if} \quad x = 0.$$

Then H is a G-homotopy with the desired property. Q.E.D.

Lemma 8. *Suppose* $\dim_R E^G \geq 2$, $x_0 \in S(E^G)$ *and* $y_0 \in S(E^G \oplus F^G)$. *Then the natural transformations*

$$\psi_1\colon [(S(E), x_0), (S(E \oplus F), y_0)]_G \to [S(E), S(E \oplus F)]_G$$

and

$$\psi_2\colon [(S(E), x_0), (V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1}), j(y_0))]_G \to [S(E), V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})]_G$$

are bijective.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} [(S(E), x_0), (S(E \oplus F), y_0)]_G & \xrightarrow{\psi_1} & [S(E), S(E \oplus F)]_G \\ \downarrow j_* & & \downarrow j_* \\ [(S(E), x_0), (V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1}), j(y_0))]_G & \xrightarrow[\psi_2]{} & [S(E), V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})]_G \end{array}$$

In [3; Section 6], ψ_1 is already seen to be bijective. The two j_* are surjective by the arguments in section 3. So it follows that ψ_2 is surjective.

It only remains to show that ψ_2 is injective. Suppose

$$\psi_2(\alpha) = \psi_2(\beta)$$

for $\alpha, \beta \in [(S(E), x_0), (V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1}), j(y_0))]_G$. Since j_* is surjective, there are G-maps

$$f, g\colon S(E) \to S(E \oplus F)$$

such that $f(x_0)=y_0$, $g(x_0)=y_0$, and jf, jg are representatives of α, β, respectively. There also is a G-homotopy

$$K\colon S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

with $K_0 = jf$ and $K_1 = jg$. Define a path

$$\omega\colon [0, 1] \to V_m^\Lambda(E^G \oplus F^G \oplus \Lambda^{m-1})$$

by $\omega(t) = K(x_0, t)$ for $t \in [0, 1]$. Then

$$\omega(0) = \omega(1) = j(y_0)\,.$$

By Proposition 1 there is a path

$$\omega'\colon [0, 1] \to S(E^G \oplus F^G)$$

such that

$$\omega'(0) = \omega'(1) = y_0, \qquad \text{and}$$
$$\omega \simeq j\omega' \ \text{rel.}\ \{0, 1\}\,.$$

Let D be a G-invariant, top-dimensional, small disc in $S(E)$ with x_0 as its center, and let $D' = \frac{1}{2}D$. By radius contraction we may deform K to a G-homotopy

$$K'\colon S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

such that $K'(x, t) = j\omega'(t)$ for $x \in D'$ and $t \in [0, 1]$. Moreover, if we put $f' = K'_0$ and $g' = K'_1$, then

$$f'(S(E)) \subset i(S(E \oplus F))\,,$$
$$g'(S(E)) \subset i(S(E \oplus F))\,,$$

and f', g' are G-homotopic to jf, jg rel. x_0, respectively.

So, to show $\alpha = \beta$ we must show that f' is G-homotopic to g' rel. x_0.

(i) Suppose $\dim_R E^G \oplus F^G > 2$. By Proposition 1, $j\omega'$ is homotopic to the constant path at $j(y_0)$ rel. $\{0, 1\}$. From this we may deform K' to a G-homotopy

$$K''\colon S(E) \times [0, 1] \to V_m^\Lambda(E \oplus F \oplus \Lambda^{m-1})$$

such that

$$K''_0 = f', \quad K''_1 = g', \qquad \text{and}$$
$$K''(x_0, t) = j(y_0) \qquad \text{for any} \quad t \in [0, 1].$$

Therefore f' is G-homotopic to g' rel. x_0.

(ii) Suppose $\dim_R E^G \oplus F^G = 2$. Define a path

$$\omega'' : [0, 1] \to S(E^G \oplus F^G)$$

by $\omega'' = (\omega')^{-1}$, i.e., $\omega''(t) = \omega'(1-t)$. Applying Lemma 7 to the path ω'', there is a G-homotopy

$$H: S(E \oplus F) \times [0, 1] \to S(E \oplus F)$$

such that

$$H_0 = H_1 = Id, \qquad \text{and}$$
$$H(y_0, t) = \omega''(t) \qquad \text{for any} \quad t \in [0, 1].$$

Define a G-homotopy

$$L: S(E) \times [0, 1] \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1})$$

as, foı $x \in S(E)$ and $t \in [0, 1]$,

$$L(x, t) = K'(x, 2t) \quad \text{if} \quad 0 \le t \le 1/2, \qquad \text{and}$$
$$L(x, t) = jH(j^{-1}g'(x), 2t-1) \quad \text{if} \quad 1/2 \le t \le 1.$$

Then

$$L_0 = f', \quad L_1 = g', \qquad \text{and}$$
$$L(x, t) = j\omega' * j\omega''(t) \quad \text{for} \quad x \in D' \quad \text{and} \quad t \in [0, 1] \cdot$$

$j\omega' * j\omega''$ is homotopic to the constant path at $j(y_0)$ rel. $\{0, 1\}$. So we may deform L to a G-homotopy

$$L': S(E) \times [0, 1] \to V_m^{\Lambda}(E \oplus F \oplus \Lambda^{m-1})$$

such that

$$L'_0 = f', \quad L'_1 = g', \qquad \text{and}$$
$$L'(x_0, t) = j(y_0) \quad \text{for any} \quad t \in [0, 1].$$

Therefore f' is G-homotopic to g' rel. x_0. Q.E.D.

Now suppose $\dim_R E^G \ge 2$, $x_0 \in S(E^G)$ and $y_0 \in S(E^G \oplus F^G)$. Let λ be the real one-dimensional subspace of E spanned by x_0, and let $\lambda^\perp$ be the orthogonal complement of λ in E. We may identify $S(E)$ with a nonreduced suspension

$$\Sigma S(\lambda^\perp) = [0, 1] \times S(\lambda^\perp)/\sim.$$

Under this identification $x_0=[0, x]$ and $-x_0=[1, x]$ for $x\in S(\lambda^{\perp})$. Let Y be one of $S(E\oplus F)$ and $V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})$. Put $z_0=y_0$ if Y is the former, and $z_0=j(y_0)$ if Y is the latter. Then we may give a group structure to $[S(E), Y]_G$ as follows. Let $[f]$, $[g]\in[S(E), Y]_G$. By Lemma 8 we may choose f and g in such a way that $f(-x_0)=z_0$ and $g(x_0)=z_0$. Define $h\colon S(E)\to Y$ as, for $[t, x]\in \Sigma S(\lambda^{\perp})=S(E)$,

$$h([t, x]) = f([2t, x]) \quad \text{if} \quad 0\leq t\leq 1/2\,, \qquad \text{and}$$
$$h([t, x)] = g([2t-1, x]) \quad \text{if} \quad 1/2\leq t\leq 1\,.$$

Define $[f]+[g]=[h]$. This gives a group structure to $[S(E), Y]_G$, and the transformation

$$j_*\colon [S(E), S(E\oplus F)]_G \to [S(E), V_m^{\Lambda}(E\oplus F\oplus\Lambda^{m-1})]_G$$

becomes a group homomorphism. We note that this group structure does not depend on the choice of $x_0\in S(E^G)$ and $y_0\in S(E^G\oplus F^G)$.

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