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STABLE HOMOTOPY TYPES OF THOM SPACES OF BUNDLES OVER ORBIT MANIFOLDS $(S^{2m+1}\times S^l)/D_p$

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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1. Introduction

Let $q\geqq 3$ be an integer, and D_q the dihedral group of order $2q$ generated by two elements a and b with relations $a^q=b^2=abab=1$. Let S^{2m+1} and S^l be the unit spheres in the complex $(m+1)$-space $\boldsymbol{C}^{m+1}$ and the real $(l+1)$-space $\boldsymbol{R}^{l+1}$ respectively. Then D_q operates on the product space $S^{2m+1}\times S^l$ by

$$\begin{cases} a\cdot(z,x)=(\exp(2\pi\sqrt{-1}/q)\cdot z,x) \\ b\cdot(z,x)=(\bar{z},-x) \end{cases}$$

for $(z,x)\in S^{2m+1}\times S^l$, where $\bar{z}$ is the conjugate of z. We set

$$\begin{cases} D(q)^{2m+1,l}=(S^{2m+1}\times S^l)/D_q, \\ D(q)^{2m,l}=\{[(z_0,\cdots,z_m,x)]\in D(q)^{2m+1,l}|z_m \text{ is real } \geqq 0\}, \\ D(q)^{m,l,i,j}=D(q)^{m,l}\cup D(q)^{i,l+1}\cup D(q)^{m+1,j}. \end{cases}$$

Then $D(q)^{m,0}$ is naturally identified with the space L_q^m defined in [6], and $D(q)^{m,l}\approx(L_q^m\times S^l)/(\boldsymbol{Z}/2)$, where the action of $\boldsymbol{Z}/2$ is given by $b\cdot([z],x)=([\bar{z}],-x)$. Complex K-rings $K(D(q)^{m,l})$ for odd q are studied in [9]. KO-groups $\widetilde{KO}(D(q)^{m,l})$ and J-groups $\tilde{J}(D(q)^{m,l})$ for odd q are studied in [8] and [16]. Let m, n, l, k, i, j, c and d be integers with $m\geqq n\geqq 0$, $l\geqq k\geqq 0$, $m+1\geqq i\geqq n-1$, $l+1\geqq j\geqq k-1$, $m+1\geqq c\geqq n$ and $l+1\geqq d\geqq k$. We set

$$\begin{cases} D(q)_{m,k}^{m,l}=D(q)^{m,l}/(D(q)^{m,k-1}\cup D(q)^{n-1,l}), \\ D(q)_{n,k,c,d}^{m,l,i,j}=D(q)^{m,l,i,j}/(D(q)^{m,k-1,c-1,k-1}\cup D(q)^{n-1,l,n-1,d-1}). \end{cases}$$

Let q be an odd integer. Then the group $\widetilde{KO}(S^jD(q)_{n,k}^{m,l})$ is decomposed to a direct sum of $\widetilde{KO}$-groups of suspensions of stunted lens spaces mod q or mod 2 (Theorem 1). J-groups $\tilde{J}(S^jD(q)_{n,k}^{m,l})$ of suspensions $S^jD(q)_{n,k}^{m,l}$ of the spaces $D(q)_{n,k}^{m,l}$ are determined for the case in which q is an odd prime (Theorems 2 and 3). Combining the results in [6] and [16], we obtain a sufficient condition for the spaces $D(q)^{m,l}_{2n,k}$ and $D(q)^{m+2s,l+t}_{2n+2s,k+t}$ to have the same stable homotopy type for the case $q\equiv 1 \pmod 2$ (Theorem 4). As an application of Theorems 1, 2 and 3, we obtain some necessary conditions for the spaces $D(q)^{2m+1,l}_{2n,k}$ and $D(q)^{2m+2s+1,l+t}_{2n+2s,k+t}$ to have the same stable homotopy type for the case in which q is an odd prime (Theorem 5).

The paper is organized as follows. In section 2 we state main theorems. In section 3 we prepare some lemmas and recall known results in [5], [10], [16] and [18]. The proofs of Theorems 1 and 2 are given in section 4. Theorem 3 is proved in section 5. We prove Theorems 4 and 5 in the final section.

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2. Statement of results

In this section q denotes an odd integer with $q\geqq 3$. In order to state theorems, we set

$$(2.1)\qquad G_0(n)=\begin{cases} \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (n\equiv 1 \pmod 8)\\ \boldsymbol{Z}/2 & (n\equiv 0 \text{ or } 2 \pmod 8)\\ 0 & (\text{otherwise}).\end{cases}$$

$$(2.2)\qquad A(q,j,k)^m_n=\begin{cases}\widetilde{KO}(S^{j+k}(L^m_q/L^{n-1}_q)) & (j\equiv k+2 \pmod 4)\\ 0 & (\text{otherwise}).\end{cases}$$

$$(2.3)\qquad B(q,j,k)^m_n=\begin{cases}\tilde{J}(S^{j+k}(L^m_q/L^{n-1}_q)) & (j\equiv k+2 \pmod 4)\\ 0 & (\text{otherwise}).\end{cases}$$

$$(2.4)\qquad RP^l_k=RP(l)/RP(k-1).$$

Theorem 1. *Let m, n, l and k be integers with $m\geqq n\geqq 0$ and $l>k\geqq 0$. Then*

(1) $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\cong A(q,j-1,k+1)^{2m}_{2n+1}\oplus A(q,j,l)^{2m}_{2n+1}$.

(2) $\widetilde{KO}(S^jD(q)^{2m+1,l}_{2n+1,k})\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+m}RP^{m+l+1}_{m+k+1})$.

(3) $\widetilde{KO}(S^jD(q)^{2m+1,l}_{2n,k})\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\oplus\widetilde{KO}(S^{j+m}RP^{m+l+1}_{m+k+1})$.

(4) $\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})$.

Remark. (1) If $l=k$, then

$$S^jD(q)^{m,l}_{n,k}\approx S^{j+k}(L^m_q/L^{n-1}_q)$$

(Lemma 3.11), and groups $\widetilde{KO}(S^{j+k}(L^m_q/L^{n-1}_q))$ are studied in [19].

(2) The partial results for the case $j=n=k=0$ of this theorem have been obtained in [16] (Proposition 3.20 (1)).

(3) KO-groups of suspensions of stunted real projective spaces are determined completely in [7].

Let $v_p(s)$ denote the exponent of the prime p in the prime power decomposition of s, and $\mathfrak{m}(s)$ the function defined on positive integers as follows (cf. [3]):

$$v_p(\mathfrak{m}(s))=\begin{cases}(1+v_p(s))([s/(p-1)]-[(s-1)/(p-1)]) & (p\neq 2)\\ (1+v_2(s))([s/2]-[(s-1)/2])+1 & (p=2).\end{cases}$$

Theorem 2. *Let m, n, l and k be integers with $m\geqq n\geqq 0$ and $l>k\geqq 0$. Then*

(1) $\tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})\cong B(q,j-1,k+1)^{2m}_{2n+1}\oplus B(q,j,l)^{2m}_{2n+1}$.

(2) $\tilde{J}(S^jD(q)^{2m+1,l}_{2n+1,k})\cong\tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\tilde{J}(S^{j+m}RP^{m+l+1}_{m+k+1})$.

(3) $\tilde{J}(S^jD(q)^{2m+1,l}_{2n,k})\cong\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\oplus\tilde{J}(S^{j+m}RP^{m+l+1}_{m+k+1})$.

(4) *If* $(k-j, j+2n+k)\not\equiv(0,0)$ $(\mathrm{mod}\,4)$ *and* $(l+2-j, j+2n+l)\not\equiv(0,0)$ $(\mathrm{mod}\,4)$, *or* $(m-n)n=0$, *then*

$$\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\cong\tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\tilde{J}(S^{j+n}RP^{n+l}_{n+k}).$$

(5) *Suppose* $m>n>0$ *and* $j-l+2\equiv j+2n+l\equiv 0$ $(\mathrm{mod}\,4)$.

i) *If* $j+n\equiv 1$ $(\mathrm{mod}\ 4)$, *then*

$$\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\cong B(q,j-1,k+1)^{2m}_{2n+1}\oplus B(q,j,l)^{2m}_{2n}\oplus G_0(j+2n+k).$$

ii) *If* $j+n\equiv 3$ $(\mathrm{mod}\,4)$, *then*

$$\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\cong B(q,j-1,k+1)^{2m}_{2n+1}\oplus B(q,j,l)^{2m}_{2n}.$$

(6) *Suppose* $m>n>0$ *and* $j-k\equiv j+2n+k\equiv 0$ $(\mathrm{mod}\,4)$.

i) *If* $j+n\equiv 2$ $(\mathrm{mod}\,4)$, *then*

$$\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\cong B(q,j-1,k+1)^{2m}_{2n}\oplus B(q,j,l)^{2m}_{2n+1}\oplus G_0(j+2n+l+0^{l-k-1}).$$

ii) *If* $j+n\equiv 0$ $(\mathrm{mod}\,4)$, *then*

$$\tilde{J}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k})\cong\tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}).$$

iii) *If* $j+n\equiv 0$ $(\mathrm{mod}\,4)$ *and* $l\equiv j+2$ $(\mathrm{mod}\,4)$, *then*

$$\tilde{J}(S^jD(q)^{2m,l}_{2n,k})\cong\tilde{J}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\oplus B(q,j,l)^{2m}_{2n+1}$$

and $$\tilde{J}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n+1,k})\cong B(q,j-1,k+1)^{2m}_{2n+1}\oplus\tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}).$$

REMARK. The partial results for the case $j=n=k=0$ of this theorem have been obtained in [16] (Proposition 3.20 (2)).

Let p be an odd prime. In order to state next theorem, we set

(2.5) $\varphi(m)=[m/4]+[(m+7)/8]+[(m+6)/8]$.

(2.6) $a_2(m,n)=\varphi(m)-[(n+1)/4]-[(n+7)/8]-[(n+6)/8]$.

(2.7) $a_p(m,n)=[m/2(p-1)]-[(n+1)/2(p-1)]$.

(2.8) $$b_2(j,m,n)=\begin{cases} a_2(m,n) & (j=0) \\ \min\{v_2(j)+1, a_2(m+j,n+j)\} & (j>0). \end{cases}$$

(2.9) $$b_p(j,m,n)=\begin{cases} a_p(m,n) & (j=0) \\ \min\{v_p(j)+1, a_p(m+j,n+j)\} & (j>0). \end{cases}$$

Theorem 3. *Let p be an odd prime. Suppose $m>n>0$, $l>k\geqq 0$, $j\equiv k \pmod 4$ and $j+n\equiv 0 \pmod 4$. Then*

$$\tilde{J}(S^jD(p)_{2n,k}^{2m,l})\cong B(p,j,l)_{2n+1}^{2m}\oplus \boldsymbol{Z}/2^{b_2-i_2}p^{b_p-i_p}M\oplus \boldsymbol{Z}/2^{i_2}\oplus \boldsymbol{Z}/p^{i_p},$$

where $M=\mathrm{m}((j+2n+k)/2)$, $b_2=b_2(j+n,n+l,n+k)$, $b_p=b_p(j+k,2m,2n)$, $i_2=\min\{b_2, v_2(n+k)\}$ *and* $i_p=\min\{b_p, v_p(n), v_p(M)\}$.

Remark. Combining Theorem 2, Theorem 3, [13] and [14], we obtain complete results of groups $\tilde{J}(S^jD(p)_{n,k}^{m,l})$.

Considering the $(\boldsymbol{Z}/q)$-action on $S^{2m+1}\times \boldsymbol{C}$ given by

$$\exp(2\pi\sqrt{-1}/q)\cdot(z,v)=(\exp(2\pi\sqrt{-1}/q)\cdot z,\ \exp(2\pi\sqrt{-1}/q)v)$$

for $(z,v)\in S^{2m+1}\times \boldsymbol{C}$, we have a complex line bundle

$$\eta_q\colon (S^{2m+1}\times \boldsymbol{C})/(\boldsymbol{Z}/q)\to L_q^{2m+1}.$$

We denote the restriction of η_q to L_q^n by η_q $(0\leqq n\leqq 2m+1)$. Let $h(q,k)$ denotes the order of $J(r(\eta_q)-2)\in\tilde{J}(L_q^k)$, which has been determined completely (cf. [6]). Spaces X and Y are said to have the same stable homotopy type ($X\underset{S}{\simeq}Y$) if there exist non-negative integers c and d such that S^cX and S^dY have the same homotopy type ($S^cX\simeq S^dY$).

Theorem 4. *If $s\equiv 0 \pmod{h(q,m)}$ and $t\equiv -s \pmod{2^{\varphi(l)}}$, then $D(q)_{2n,k}^{2n+m,k+l}$ and $D(q)_{2n+2s,k+t}^{2n+2s+m,k+t+l}$ have the same stable homotopy type.*

Remark. (1) The partial results for the case in which q is an odd prime, and $m\equiv 1 \pmod 2$, $n=s=0$ or $k=t=0$, $m\equiv l\equiv 7 \pmod 8$ of this theorem have been obtained in [8].
(2) Let q be an odd prime. Then $h(q,m)=q^{[m/2(q-1)]}$ (cf. [11]).

In order to state the next theorem, we prepare functions β and γ defined by

(2.10) $\beta(k,n)$ is equal to the corresponding integer in the following table:

n (mod 4) \ k (mod 8)	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	1	1	0	1	0	0	0	1
2	1	0	0	0	0	0	1	1
3	0	0	0	1	0	1	1	1

(2.11) $\gamma(q,k,n)=[(n+k-2[n/2]-2)/(q-1)]$.

Theorem 5. *Suppose $D(q)^{2m+1,l}_{2n,k}$ and $D(q)^{2m+2s+1,l+t}_{2n+2s,k+t}$ have the same stable homotopy type, where m, n, l, k, s and t are integers with $m\geqq n\geqq 0$, $l>k\geqq 0$, $s\geqq 0$ and $k+t\geqq 0$. Then*

(1) *Set $v=v_2(|s+t|+2^l)$ and $v_2=v_2(n+k+2^l)$. Then*

i) $v\geqq[\log_2(l-k)]+1$.

ii) $v\geqq\varphi(l-k)-1+\max\{\beta(l-k,n+k),\ \beta(l-k,m+k+1)\}$.

iii) *If* $\max\{v_2,\ v_2(n+l+1),\ v_2(m+k+1),\ v_2(m+l+2)\}\geqq\varphi(l-k)-1$, *then* $v\geqq\varphi(l-k)$.

(2) *Let q be an odd prime. Set $v_q=v_q(n+q^m)$. Then*

i) $v_q(s+q^m)\geqq\max\{\gamma(q,m-n,n+k),\ \gamma(q,m-n,n+l+1)\}$.

ii) *If* $\max\{(-1)^{(n+k)(n+l+1)}v_q,\ (-1)^{(m+l)(m+k+1)}v_q(m+1)\}\geqq[(m-n)/(q-1)]$, *then* $v_q(s+q^m)\geqq[(m-n)/(q-1)]$.

REMARK. Let q be an odd prime. It follows from Theorems 4 and 5 that we have obtained the necessary and sufficient condition for spaces $D(q)^{2m+1,l}_{2n,k}$ and $D(q)^{2m+2s+1,l+t}_{2n+2s,k+t}$ to have the same stable homotopy type if following conditions (1) and (2) are satisfied.

(1) One of the following conditions:

i) $k<l\leqq k+8$,

ii) $\max\{\beta(l-k,n+k),\ \beta(l-k,m+k+1)\}=1$,

iii) $\max\{v_2(n+k+2^l),\ v_2(n+l+1),\ v_2(m+k+1),\ v_2(m+l+2)\} \geqq \varphi(l-k)-1$.

(2) One of the following conditions:

i) $n \leqq m < n+q-1$,

ii) $\max\{\gamma(q,m-n,n+k),\ \gamma(q,m-n,n+l+1)\} = [(m-n)/(q-1)]$,

iii) $\max\{(-1)^{(n+k)(n+l+1)}v_q(n+q^m),\ (-1)^{(m+l)(m+k+1)}v_q(m+1)\} \geqq [(m-n)/(q-1)]$.

3. Preliminaries

We begin by recalling some notation in [18]. Let $\alpha_i(u,v)$ $(1 \leqq i \leqq 8)$ be the integers defined by

$$(3.1)\quad \begin{cases} (1) & \alpha_1(u,v) = \binom{2u}{u-v}(-1)^{u-v}, \\ (2) & \alpha_4(u,v) = \binom{u+v-1}{u-v}, \\ (3) & \alpha_6(u,v) = \binom{2u-v-1}{u-v}(-1)^{u-v}, \\ (4) & \alpha_7(u,v) = \binom{v-1}{u-v}, \\ (5) & \alpha_3(u,v) = \alpha_1(u-1,v-1) - \alpha_1(u-1,v+1), \\ (6) & \alpha_2(u,v) = \alpha_4(u+1,v+1) - \alpha_4(u-1,v+1), \\ (7) & \alpha_5(u,v) = \alpha_7(u+1,v+1) + \alpha_7(u-1,v), \\ (8) & \alpha_8(u,v) = \alpha_6(u-1,v-1) + \alpha_6(u,v+1). \end{cases}$$

We set elements $a_i^{2j,m}(q)$, $b_i^{2j,m}(q)$ and $c_i^{2j,m}(q)$ of $\widetilde{KO}(S^{2j}L_q^m)$ by

$$(3.2)\quad \begin{cases} a_i^{2j,m}(q) = r(I^j((\eta_q)^i - 1)) \\ b_i^{2j,m}(q) = \begin{cases} \Sigma_{u=1}^i \alpha_1(i,u) a_u^{2j,m}(q) & (j \equiv 0 \pmod 2) \\ \Sigma_{u=1}^i \alpha_3(i,u) a_u^{2j,m}(q) & (j \equiv 1 \pmod 2) \end{cases} \\ c_i^{2j,m}(q) = r(I^j((\eta_q - 1)^i)), \end{cases}$$

where $r: K \to KO$ denotes the real restriction and $I: \tilde{K}(X) \to \tilde{K}(S^2X)$ is the Bott periodicity isomorphism.

Lemma 3.3 (Tamamura [18]). *The elements $a_i^{2j,m}(q)$, $b_i^{2j,m}(q)$ and $c_i^{2j,m}(q)$ satisfy following relations.*

(1) $a_1^{2j,m}(q) = b_1^{2j,m}(q) = c_1^{2j,m}(q)$.

(2) $a_i^{2j,m}(q) = \begin{cases} \Sigma_{u=1}^i \alpha_2(i,u) b_u^{2j,m}(q) & (j \equiv 0 \pmod 2) \\ \Sigma_{u=1}^i \alpha_4(i,u) b_u^{2j,m}(q) & (j \equiv 1 \pmod 2). \end{cases}$

(3) $a_i^{2j,m}(q)=\Sigma_{u=1}^i\binom{i}{u}c_u^{2j,m}(q)$.

(4) $c_i^{2j,m}(q)=\Sigma_{u=1}^i\binom{i}{u}(-1)^{i-u}a_u^{2j,m}(q)$.

(5) $c_i^{2j,m}(q)=\begin{cases}\Sigma_{u=1}^i\alpha_5(i,u)b_u^{2j,m}(q) & (j\equiv 0 \pmod 2)\\ \Sigma_{u=1}^i\alpha_7(i,u)b_u^{2j,m}(q) & (j\equiv 1 \pmod 2).\end{cases}$

(6) $b_i^{2j,m}(q)=\begin{cases}\Sigma_{u=1}^i\alpha_6(i,u)c_u^{2j,m}(q) & (j\equiv 0 \pmod 2)\\ \Sigma_{u=1}^i\alpha_8(i,u)c_u^{2j,m}(q) & (j\equiv 1 \pmod 2).\end{cases}$

Lemma 3.4 (Tamamura [18]). *Let $q\geqq 3$ be an odd integer and $d=(q-1)/2$. Then*

$$b_{d+1+u}^{2j,m}(q)=-\Sigma_{i=1}^d\alpha_5(q,d+i)b_{i+u}^{2j,m}(q),$$

where $u\geqq 0$ is an integer.

By Lemmas 3.3 and 3.4, we obtain

Lemma 3.5. *Let p be an odd prime, and $d=(p-1)/2$. Then*

$$\widetilde{KO}(S^{2j}L_p^m)=\langle\{c_{2i-j+2[j/2]}^{2j,m}(p)|1\leqq i\leqq d\}\rangle.$$

For each integer n with $0\leqq n<m$, we denote the inclusion map of L_q^n into L_q^m by i_n^m, and the kernel of the homomorphism

$$(i_n^m)^!:\widetilde{KO}(S^{2j}L_q^m)\to\widetilde{KO}(S^{2j}L_q^n)$$

by $VO_{m,n}^{2j}(q)$, and set

$$(3.6)\qquad UO_{m,n}^{2j}(q)=\sum_k\left(\bigcap_e k^e(\psi^k-1)VO_{m,n}^{2j}(q)\right).$$

Proposition 3.7 (Tamamura [18]). *Let p be an odd prime, and $d=(p-1)/2$. Then the group $VO_{2m,2n}^{2j}(p)$ is isomorphic to the direct sum of cyclic groups of order*

$$p^{a_p(2m-4i+2j-4[j/2],\,2n-4i+2j-4[j/2])}$$

generated by $p^{a_p(2n-4i+2j-4[j/2],0)+1}b_i^{2j,2m}(p)\qquad(1\leqq i\leqq d)$.

Proposition 3.8 ([14]). *Let p be an odd prime. Then*

$$\begin{aligned}\tilde{J}(S^{2j}(L_p^{2m}/L_p^{2n}))&\cong VO_{2m,2n}^{2j}(p)/UO_{2m,2n}^{2j}(p)\\ &=\langle[p^{[(n-v)/(p-1)]+1}c_v^{2j,2m}(p)]\rangle\cong \boldsymbol{Z}/p^{b_p(2j,2m,2n)},\end{aligned}$$

where $v=p-1-j+(p-1)[j/(p-1)]$.

Considering the D_q-action on $S^{2m+1}\times S^l\times \boldsymbol{R}$ and $S^{2m+1}\times S^l\times \boldsymbol{C}$ given by

$$\begin{cases} a\cdot(z,x,y)=(\exp(2\pi\sqrt{-1}/q)\cdot z,x,y)\\ b\cdot(z,x,y)=(\bar{z},-x,-y)\end{cases}$$

for $(z,x,y)\in S^{2m+1}\times S^l\times \boldsymbol{R}$ and

$$\begin{cases} a\cdot(z,x,w)=(\exp(2\pi\sqrt{-1}/q)\cdot z,x,\exp(2\pi\sqrt{-1}/q)w)\\ b\cdot(z,x,w)=(\bar{z},-x,\bar{w})\end{cases}$$

for $(z,x,w)\in S^{2m+1}\times S^l\times \boldsymbol{C}$, we have a real line bundle

$$\xi(q)\colon (S^{2m+1}\times S^l\times \boldsymbol{R})/D_q\to D(q)^{2m+1,l}$$

and a real 2-plane bundle

$$\eta(q)\colon (S^{2m+1}\times S^l\times \boldsymbol{C})/D_q\to D(q)^{2m+1,l}.$$

We denote the restriction of $\xi(q)$ (resp. $\eta(q)$) to $D(q)^{n,k}$ $(0\leqq n\leqq 2m+1,\ 0\leqq k\leqq l)$ by $\xi(q)$ (resp. $\eta(q)$). Then we have following elements of $\widetilde{KO}(D(q)^{m,l})$:

(3.9) $\alpha(q)=\eta(q)-\xi(q)-1.$

We denote by X^γ the Thom complex of a vector bundle γ over a finite CW-complex X. Define a map

$$f\colon S^{2m+1}\times S^l\times D^{2n}\times D^k\to S^{2m+2n+1}\times S^{l+k}$$

by setting

$$f((z,x,v,w))=((v,(1-\|v\|^2)^{1/2}z),\ (w,(1-\|w\|^2)^{1/2}x)).$$

Then f induces homeomorphisms

$$\bar{f}\colon (D(q)^{2m+1,l})^{n\eta(q)\oplus k\xi(q)}\to D(q)^{2m+2n+1,l+k}_{2n,k}$$

and $\bar{f}|D(q)^{2m,l}\colon (D(q)^{2m,l})^{n\eta(q)\oplus k\xi(q)}\to D(q)^{2m+2n,l+k}_{2n,k}$. Thus we obtain

Lemma 3.10. *$(D(q)^{m,l})^{n\eta(q)\oplus k\xi(q)}$ is homeomorphic to $D(q)^{2n+m,k+l}_{2n,k}$.*

REMARK. The partial results for the case in which q is an odd prime and $m\equiv 1 \pmod 2$ have been obtained in [8].

Lemma 3.11. *There are following homeomorphisms*:

(1) $D(q)^{2m,l}_{2m,k}\approx S^mRP^{m+l}_{m+k}$,

(2) $D(q)^{2m+1,l}_{2m+1,k}\approx S^mRP^{m+l+1}_{m+k+1}$,

(3) $D(q)^{m,l}_{n,l}\approx S^l(L^m_q/L^{n-1}_q)$.

Proof. By Lemma 3.10, we obtain

$$D(q)^{2m+1,l}_{2m,k} \approx (D(q)^{1,l-k})^{m\eta(q)\oplus k\xi(q)}.$$

Define a map

$$h\colon S^1 \times S^{l-k} \times \boldsymbol{C} \to S^1 \times S^{l-k} \times \boldsymbol{C}$$

by setting $h((z,x,v))=(z,x,z^{q-1}v)$. Then h induces a bundle isomorphism $\bar{h}\colon\eta(q) \to 1\oplus\xi(q)$ over $D(q)^{1,l-k}$. This implies

$$D(q)^{2m+1,l}_{2m,k} \approx (D(q)^{1,l-k})^{m\oplus(m+k)\xi(q)} \approx S^m(D(q)^{1,l-k})^{(m+k)\xi(q)},$$

$$D(q)^{2m,l}_{2m,k} \approx S^m(D(q)^{0,l-k})^{(m+k)\xi(q)} \approx S^m RP(l-k)^{(m+k)\xi(q)} \approx S^m RP^{m+l}_{m+k}$$

and

$$\begin{aligned} D(q)^{2m+1,l}_{2m+1,k} &\approx S^m(D(q)^{1,l-k})^{(m+k)\xi(q)} / S^m(D(q)^{0,l-k})^{(m+k)\xi(q)} \\ &\approx S^m(((S^{l-k} \times D^{m+k+1}) / (S^{l-k} \times S^{m+k})) / (\boldsymbol{Z}/2)) \\ &\approx S^m RP(l-k)^{(m+k+1)\xi(q)} \approx S^m RP^{m+l+1}_{m+k+1}. \end{aligned}$$

By the homemorphism $D(q)^{m,l} \approx (L^m_q \times S^l)/(\boldsymbol{Z}/2)$,

$$\begin{aligned} D(q)^{m,l}_{n,l} &\approx (L^m_q \times D^l_+) / ((L^m_q \times S^{l-1}) \cup (L^{n-1}_q \times D^l_+)) \\ &\approx (L^m_q \times S^l) / ((L^m_q \times *) \cup (L^{n-1}_q \times S^l)) \\ &\approx ((L^m_q / L^{n-1}_q) \times S^l) / (((L^m_q / L^{n-1}_q) \times *) \cup (* \times S^l)) \\ &\approx S^l(L^m_q / L^{n-1}_q). \end{aligned}$$

q.e.d.

Let $\tau(q)^{2m+1,l}\colon TD(q)^{2m+1,l} \to D(q)^{2m+1,l}$ be the tangent bundle of $D(q)^{2m+1,l}$. Then we have

Lemma 3.12. *$\tau(q)^{2m+1,l}\oplus 2$ is isomorphic to $(m+1)\eta(q)\oplus(l+1)\xi(q)$.*

Proof. There exists an equivariant isomorphism

$$h\colon T(S^{2m+1} \times S^l) \times \boldsymbol{R}^2 \to S^{2m+1} \times S^l \times \boldsymbol{C}^{m+1} \times \boldsymbol{R}^{l+1},$$

which induces a bundle isomorphism

$$\bar{h}\colon (T(S^{2m+1} \times S^l)/D_q) \times \boldsymbol{R}^2 \to (S^{2m+1} \times S^l \times \boldsymbol{C}^{m+1} \times \boldsymbol{R}^{l+1})/D_q$$

from $\tau(q)^{2m+1,l}\oplus 2$ to $(m+1)\eta(q)\oplus(l+1)\xi(q)$. q.e.d.

Lemma 3.13. *Let N and M be integers with $N\equiv 0 \pmod{h(q,2m-2n+1)}$, $M\equiv 0 \pmod{2^{\varphi(l-k)}}$, $N>m+1$ and $M>N+l+2$. Then the S-dual of $D(q)^{2m+1,l}_{2n,k}$ is*

$D(q)^{2N-2n-1,M-N-k-1}_{2N-2m-2,M-N-l-1}$.

Proof. By Lemma 3.10, Lemma 3.12 and [5, Proposition (2.6) and Theorem (3.5)], the S-dual of

$$D(q)^{2m+1,l}_{2n,k} \approx (D(q)^{2m-2n+1,l-k})^{n\eta(q)\oplus k\xi(q)}$$

is

$$\begin{aligned}
&(D(q)^{2m-2n+1,l-k})^{(N-n)\eta(q)\oplus(M-N-k)\xi(q)-\tau(q)^{2m-2n+1,l-k}}\\
&\underset{S}{\simeq}(D(q)^{2m-2n+1,l-k})^{(N-n)\eta(q)\oplus(M-N-k)\xi(q)-((m-n+1)\eta(q)\oplus(l-k+1)\xi(q))}\\
&\approx(D(q)^{2m-2n+1,l-k})^{(N-m-1)\eta(q)\oplus(M-N-l-1)\xi(q)}\\
&\approx D(q)^{2N-2n-1,M-N-k-1}_{2N-2m-2,M-N-l-1}.
\end{aligned}$$

q.e.d.

According to [10], $D(q)^{m,l}$ has a cellular decomposition

$$\{(C_i,D_j)|0\leqq i\leqq m;\ 0\leqq j\leqq l\},$$

where $\dim (C_i, D_j)=i+j$ and boundary operations are given by

$$(3.14)\quad \begin{cases}\partial(C_{2i},D_j)=q(C_{2i-1},D_j)+((-1)^i+(-1)^j)(C_{2i},D_{j-1}),\\ \partial(C_{2i+1},D_j)=((-1)^i+(-1)^{j+1})(C_{2i+1},D_{j-1}).\end{cases}$$

We denote by (c^i, d^j) the dual cochain of (C_i, D_j).

Lemma 3.15. *Suppose* $q\equiv 1 \pmod 2$.

(1) $\tilde{H}^*(D(q)^{2m,l}_{2n+1,k})\cong(\oplus_{2n<4i-2k\leqq 2m}(\boldsymbol{Z}/q)[(c^{4i-2k},d^k)])$

$$\oplus(\oplus_{2n<4i-2l-2\leqq 2m}(\boldsymbol{Z}/q)[(c^{4i-2l-2},d^l)]).$$

(2) $\tilde{H}^*(D(q)^{2m,l}_{2n+1,k};\ \boldsymbol{Z}/2)\cong 0$.

Lemma 3.16. *Suppose* $q\equiv 1 \pmod 2$ *and* $l>k$. *Then there exists a split short exact sequence*

$$(3.17)\quad 0\to A(q,j,l)^{2m}_{2n+1}\to \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\to A(q,j-1,k+1)^{2m}_{2n+1}\to 0$$

of ψ-groups.

Proof. It follows from Lemma 3.15 and the Atiyah-Hirzebruch spectral sequence for KO-theory that the order of the group $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$ is a divisor of $q^{a(j,m,n,l,k)}$, where

$$a(j,m,n,l,k)=\begin{cases}[(m+k)/2]-[(n+k)/2])+[(m+l+1)/2]-[(n+l+1)/2] & \\ & (j\equiv k\equiv l+2 \pmod 4)\\ [(m+k)/2]-[(n+k)/2] & (j\equiv k\not\equiv l+2 \pmod 4)\\ [(m+l+1)/2]-[(n+l+1)/2] & (j\equiv l+2\not\equiv k \pmod 4)\\ 0 & (\text{otherwise}).\end{cases}$$

In the case $k\equiv l \pmod 2$, the order of the group $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$ is equal to $q^{a(j,m,n,l,k)}$. By Lemma 3.11, we obtain a sequence

$$\widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n}))\xrightarrow{h_1}\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\xrightarrow{h_2}\widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n}))$$

of ψ-groups with $h_2\circ h_1=0$. It follows from [19] that

$$\widetilde{KO}(S^{2j+1}(L_q^{2m}/L_q^{2n}))\cong 0$$

and the order of $\widetilde{KO}(S^{2j}(L_q^{2m}/L_q^{2n}))$ is equal to $q^{[(m+j)/2]-[(n+j)/2]}$. Inspect the commutative diagram

$$\begin{tikzcd}\to \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) \arrow[r, "g_1"] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \arrow[r, "h_2"] & \widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n})) \to \\ \to \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n})) \arrow[u] \arrow[r, "h_1"] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \arrow[u, equal] \arrow[r, "g_2"] & \widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k}) \arrow[u] \to\end{tikzcd}$$

with exact rows. Suppose $j\equiv k\equiv l \pmod 4$. Then $g_1=0$. This implies that $h_1=0$, h_2 is an isomorphism and g_2 is an isomorphism. Suppose $j-2\equiv k\equiv l \pmod 4$. Then $g_2=0$. This implies that $h_2=0$, h_1 is an isomorphism and g_1 is an isomorphism. Thus we obtain the lemma for the case $k\equiv l \pmod 4$ and it is shown that the order of $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$ is equal to $q^{a(j,m,n,l,k)}$ if $k\equiv l-3 \pmod 4$. Suppose $j\equiv k\equiv l-3 \pmod 4$. Then $g_1=h_1=0$. This implies that h_2 is an isomorphism and g_2 is a monomorphism. Suppose $j-1\equiv k\equiv l-3 \pmod 4$. Then $h_2=g_2=0$. This implies that g_1 is an epimorphism and h_1 is an isomorphism. Thus we obtain the lemma for the case $k\equiv l-3 \pmod 4$. Suppose $j\equiv k\equiv l-2 \pmod 4$. Then h_1 is a monomorphism and h_2 is an epimorphism. This implies that $\operatorname{Im} h_1=\operatorname{Ker} h_2$. Using the isomorphism

$$\widetilde{KO}(S^jD(q)^{2m,l+2}_{2n+1,k})\xrightarrow{\cong}\widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n}))$$

we obtain a ψ-map

$$h_3\colon \widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n}))\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$$

with $h_2\circ h_3=1$. Thus we obtain the lemma for the case $k\equiv l-2 \pmod 4$ and it is shown that the order of $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$ is equal to $q^{a(j,m,n,l,k)}$ if $k\equiv l-1 \pmod 4$. Suppose $j\equiv k\equiv l-1 \pmod 4$. Then $g_1=h_1=0$. This implies that h_2 and

g_2 are isomorphisms. Suppose $j+1\equiv k\equiv l-1 \pmod 4$. Then $h_2=g_2=0$. This implies that g_1 and h_1 are isomorphisms. Thus we obtain the lemma for the case $k\equiv l-1 \pmod 4$. q.e.d.

We consider the following maps

$$(3.18)\quad \begin{cases} i_1: L_q^{2m+1}\to D(q)^{2m+1,l},\ i_2: RP(l)\to D(q)^{2m+1,l} \\ p_0: D(q)^{2m+1,l}\to RP(l),\ p_1: D(q)^{2m+1,l}\to S^l L_q^{2m+1}, \\ p_2: D(q)^{2m+1,l}\to S^m RP_{m+1}^{m+l+1}. \end{cases}$$

We set the following homomorphisms

$$(3.19)\quad \begin{cases} f_1: \widetilde{KO}(L_q^{2m})\to \widetilde{KO}(D(q)^{2m+1,l}), \\ i_0: \widetilde{KO}(S^l L_q^{2m})\to \widetilde{KO}(S^l L_q^{2m+1}), \\ f_2=(p_1)^! \circ i_0: \widetilde{KO}(S^l L_q^{2m})\to \widetilde{KO}(D(q)^{2m+1,l}), \end{cases}$$

where f_1 is defined by $f_1(r(\eta_q-1))=\alpha(q)$, and i_0 is a right inverse of the restriction homomorphism $\widetilde{KO}(S^l L_q^{2m+1})\to \widetilde{KO}(S^l L_q^{2m})$.

Proposition 3.20 ([16]). *Suppose* $q\equiv 1 \pmod 2$ *and* $l>0$.

(1) *The homomorphism*

$$f\colon \widetilde{KO}(L_q^{2m})\oplus A(q,0,l)_1^{2m}\oplus \widetilde{KO}(RP(l))\oplus \widetilde{KO}(S^m RP_{m+1}^{m+l+1})\to \widetilde{KO}(D(q)^{2m+1,l})$$

defined by $f(x,y,z,w)=f_1(x)+f_2(y)+(p_0)^!(z)+(p_2)^!(w)$ *is an isomorphism.*

(2) *The homomorphism*

$$g: \tilde{J}(L_q^{2m})\oplus B(q,0,l)_1^{2m}\oplus \tilde{J}(RP(l))\oplus \tilde{J}(S^m RP_{m+1}^{m+l+1})\to \tilde{J}(D(q)^{2m+1,l})$$

defined by $g(J(x),J(y),J(z),J(w))=J(f_1(x)+f_2(y)+(p_0)^!(z)+(p_2)^!(w))$ *is an isomorphism.*

4. Proof of Theorems 1 and 2

The part (1) of Theorems 1 and 2 is a direct consequence of Lemma 3.16. It follows from Lemma 3.11 that there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{KO}(S^j D(q)_{2m+1,k}^{2m+2,l}) & \xrightarrow{f_1} & \widetilde{KO}(S^j D(q)_{2n,k}^{2m+2,l}) & \xrightarrow{f_2} & \widetilde{KO}(S^j D(q)_{2n,k}^{2m,l}) & \to & 0 \\ & & \downarrow{\scriptstyle f_3} & & \downarrow & & \| & & \\ 0 & \to & \widetilde{KO}(S^{j+m} RP_{m+k+1}^{m+l+1}) & \to & \widetilde{KO}(S^j D(q)_{2n,k}^{2m+1,l}) & \to & \widetilde{KO}(S^j D(q)_{2n,k}^{2m,l}) & \to & 0 \end{array}$$

with exact rows. Since $\widetilde{KO}(S^j D(q)_{2m+1,k}^{2m+2,l})$ has an odd order, $f_3=0$ and we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & \operatorname{Coker} f_1 & \xrightarrow{\cong} & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) \\
 & & & & \downarrow & & \| \\
0 \to & \widetilde{KO}(S^{j+m}RP^{m+l+1}_{m+k+1}) & \to & & \widetilde{KO}(S^jD(q)^{2m+1,l}_{2n,k}) & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) \to 0,
\end{array}$$

in which the row is exact. Thus we obtain

$$\widetilde{KO}(S^jD(q)^{2m+1,l}_{2n,k}) \cong \widetilde{KO}(S^{j+m}RP^{m+l+1}_{m+k+1}) \oplus \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

and $\tilde{J}(S^jD(q)^{2m+1,l}_{2n,k}) \cong \tilde{J}(S^{j+m}RP^{m+l+1}_{m+k+1}) \oplus \tilde{J}(S^jD(q)^{2m,l}_{2n,k})$. Similarly we obtain

$$\widetilde{KO}(S^jD(q)^{2m+1,l}_{2n+1,k}) \cong \widetilde{KO}(S^{j+m}RP^{m+l+1}_{m+k+1}) \oplus \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$$

and $\tilde{J}(S^jD(q)^{2m+1,l}_{2n+1,k}) \cong \tilde{J}(S^{j+m}RP^{m+l+1}_{m+k+1}) \oplus \tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})$.

Since the short exact sequence

$$0 \to \widetilde{KO}(S^jD(q)^{2m,l}_{1,k}) \to \widetilde{KO}(S^jD(q)^{2m,l}_{0,k}) \to \widetilde{KO}(S^jRP^l_k) \to 0$$

of ψ-groups splits, we obtain

$$\widetilde{KO}(S^jD(q)^{2m,l}_{0,k}) \cong \widetilde{KO}(S^jD(q)^{2m,l}_{1,k}) \oplus \widetilde{KO}(S^jRP^l_k)$$

and $\tilde{J}(S^jD(q)^{2m,l}_{0,k}) \cong \tilde{J}(S^jD(q)^{2m,l}_{1,k}) \oplus \tilde{J}(S^jRP^l_k)$.

Suppose $n>0$. There exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) & \to & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n-1,k}) & \to & \widetilde{KO}(S^jD(q)^{2n,l}_{2n-1,k}) & \to & 0
\end{array}$$

with exact rows. If $(j-l-2, j+2n+l) \not\equiv (0,0)$ (mod 4) and $(j-k, j+2n+k) \not\equiv (0,0)$ (mod 4), then $\widetilde{KO}(S^jD(q)^{2n,l}_{2n-1,k}) \cong 0$. Hence

$$\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) \cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \oplus \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})$$

and $\tilde{J}(S^jD(q)^{2m,l}_{2n,k}) \cong \tilde{J}(S^jD(q)^{2m,l}_{2n+1,k}) \oplus \tilde{J}(S^{j+n}RP^{n+l}_{n+k})$.

Suppose $m>n>0$ and $j-l-2 \equiv j+2n+l \equiv 0$ (mod 4). Then $j+n \equiv 1$ (mod 2) and we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n})) & \xrightarrow{f_1} & \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n-1})) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+2n+l}) & \to & 0 \\
 & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_3}\downarrow & & \\
0 & \to & \widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l}) & \xrightarrow{f_3} & \widetilde{KO}(S^jD(q)_{2n,k}^{2m,l}) & \xrightarrow{f_4} & \widetilde{KO}(S^{j+n}RP_{n+k}^{n+l}) & \to & 0 \\
 & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & \\
0 & \to & \widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l-1}) & \xrightarrow{f_5} & \widetilde{KO}(S^jD(q)_{2n,k}^{2m,l-1}) & \xrightarrow{f_6} & \widetilde{KO}(S^{j+n}RP_{n+k}^{n+l-1}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of exact sequences. Since $j-l-1\equiv 1\not\equiv 0 \pmod 4$ and $j+n\equiv 1\not\equiv 0 \pmod 2$, there exists a ψ-map

$$f_7:\widetilde{KO}(S^jD(q)_{2n,k}^{2m,l-1})\to\widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l-1})$$

with $f_7\circ f_5=1$. By Lemma 3.16, we obtain a ψ-map

$$h_4:\widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l-1})\to\widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l})$$

with $h_1\circ h_4=1$. If $\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l-1})\cong 0$, then

$$\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l})\cong\widetilde{KO}(S^{j+2n+l})\cong \boldsymbol{Z},$$

f_5 is an isomorphism,

$$\widetilde{KO}(S^jD(q)_{2n,k}^{2m,l})\cong\widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l})\oplus\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l})$$

and

$$\begin{aligned}\tilde{J}(S^jD(q)_{2n,k}^{2m,l})&\cong\tilde{J}(S^jD(q)_{2n+1,k}^{2m,l-1})\oplus\tilde{J}(S^{j+l}(L_q^{2m}/L_q^{2n-1}))\\&\cong B(q,j-1,k+1)_{2n+1}^{2m}\oplus B(q,j,l)_{2n}^{2m}.\end{aligned}$$

Suppose $m>n>0$, $j-l-2\equiv j+n-3\equiv 0 \pmod 4$, $j+l+2n\equiv 4 \pmod 8$ and $l>k+1$. Then

$$\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l-1})\cong \boldsymbol{Z}/2,$$

$$\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l})\cong\widetilde{KO}(S^{j+2n+l})\cong \boldsymbol{Z}$$

and $\widetilde{KO}(S^jD(q)_{2n,k}^{2m,l})\cong\widetilde{KO}(S^jD(q)_{2n+1,k}^{2m,l})\oplus\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l})$. Choose generators $\alpha\in\widetilde{KO}(S^{j+2n+l})$ and $\beta\in\widetilde{KO}(S^{j+n}RP_{n+k}^{n+l})$ with $g_3(\alpha)=2\beta$. Choose $z\in\widetilde{KO}(S^jD(q)_{2n,k}^{2m,l})$ with $f_4(z)=\beta$. Set

$$y=z-(f_3\circ h_4\circ f_7\circ h_2)(z).$$

Since $f_6(h_2(2z))=h_3(2f_4(z))=h_3(2\beta)=0$, there exists an element $u\in \widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k})$ with $f_5(u)=h_2(2z)$. Then

$$\begin{aligned}h_2(2y)&=h_2(2z)-f_5(f_7(h_2(2z)))\\&=f_5(u)-f_5(f_7(f_5(u)))\\&=f_5(u)-f_5(u)=0.\end{aligned}$$

So, there exists an element $x\in \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n-1}))$ with $g_2(x)=2y$. Then $f_2(x)=\alpha$. Since $\widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n}))$ has an odd order, the homomorphism

$$i_0:\widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n}))\to \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n}))$$

defined by $i_0(a)=2a$ is an isomorphism. Let

$$f_8:\widetilde{KO}(S^{j+2n+l})\to \widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n-1}))$$

be the homomorphism defined by $f_8(a\alpha)=ax$ for $a\in \boldsymbol{Z}$, and

$$f_9:\widetilde{KO}(S^{j+2n+l})\to \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

the homomorphism defined by $f_9(a\alpha)=ay$ for $a\in \boldsymbol{Z}$. Define the homomorphism

$$g_0:\widetilde{KO}(S^{j+l}(L_q^{2m}/L_q^{2n-1}))\to \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

by setting

$$g_0(a)=f_3(g_1(i_0^{-1}(f_1^{-1}(a-f_8(f_2(a))))))+f_9(f_2(a)).$$

Suppose $g_0(a)=0$. Then $f_4(g_0(a))=f_4(f_9(f_2(a)))=0$. This implies that $f_2(a)=0$. Hence $f_3(g_1(i_0^{-1}(f_1^{-1}(a))))=0$. Since f_3 and g_1 are monomorphisms, this implies that $a=0$. Thus g_0 is a monomorphism. Since g_2 is given by

$$\begin{aligned}g_2(a)&=g_2(a-f_8(f_2(a)))+g_2(f_8(f_2(a)))\\&=g_2(f_1(i_0^{-1}(2f_1^{-1}(a-f_8(f_2(a))))))+2f_9(f_2(a))\\&=2f_3(g_1(i_0^{-1}(f_1^{-1}(a-f_8(f_2(a))))))+2f_9(f_2(a))\\&=2g_0(a),\end{aligned}$$

$g_2=2g_0$. This implies that the homomorphism g_0 is a ψ-map. Consider the sequence

$$(4.1)\quad 0\to A(q,j,l)^{2m}_{2n}\xrightarrow{g_0}\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\xrightarrow{f_7\circ h_2}\widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k})\to 0.$$

Noting that $f_7\circ h_2\circ f_3\circ h_4=f_7\circ f_5=1$, it is not difficult to see that (4.1) is a split exact sequence of ψ-groups. Thus we obtain

$$\begin{aligned}\tilde{J}(S^jD(q)^{2m,l}_{2n,k}) &\cong \tilde{J}(S^jD(q)^{2m,l-1}_{2n+1,k})\oplus\tilde{J}(S^{j+l}(L^{2m}_q/L^{2n-1}_q))\\ &\cong B(q,j-1,k+1)^{2m}_{2n+1}\oplus B(q,j,l)^{2m}_{2n}.\end{aligned}$$

Suppose $m>n>0, j-l-2\equiv n+j-1\equiv 0 \pmod 4$ and $l>k+1$. In the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{KO}(S^jD(q)^{2m,l+1}_{2n+1,k}) & \to & \widetilde{KO}(S^jD(q)^{2m,l+1}_{2n,k}) & \to & \widetilde{KO}(S^{j+n}RP^{n+l+1}_{n+k}) & \to & 0\\ & & {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow & & \\ 0 & \to & \widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k}) & \xrightarrow{f_5} & \widetilde{KO}(S^jD(q)^{2m,l-1}_{2n,k}) & \xrightarrow{f_6} & \widetilde{KO}(S^{j+n}RP^{n+l-1}_{n+k}) & \to & 0\end{array}$$

with exact rows, k_1 and k_3 are isomorphisms. This implies that k_2 is an isomorphism. Using k_2, we obtain a ψ-map

$$h_5:\widetilde{KO}(S^jD(q)^{2m,l-1}_{2n,k})\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

with $h_2\circ h_5=1$. Thus we have

$$\begin{aligned}\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) &\cong \widetilde{KO}(S^{j+l}(L^{2m}_q/L^{2n-1}_q))\oplus\widetilde{KO}(S^jD(q)^{2m,l-1}_{2n,k})\\ &\cong A(q,j,l)^{2m}_{2n+1}\oplus \boldsymbol{Z}\oplus A(q,j-1,k+1)^{2m}_{2n+1}\oplus\widetilde{KO}(S^{j+n}RP^{n+l-1}_{n+k})\\ &\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})\end{aligned}$$

and

$$\begin{aligned}\tilde{J}(S^jD(q)^{2m,l}_{2n,k}) &\cong \tilde{J}(S^{j+l}(L^{2m}_q/L^{2n-1}_q))\oplus\tilde{J}(S^jD(q)^{2m,l-1}_{2n,k})\\ &\cong B(q,j,l)^{2m}_{2n}\oplus B(q,j-1,k+1)^{2m}_{2n+1}\oplus G_0(j+2n+k).\end{aligned}$$

Suppose $m>n>0$ and $j-k\equiv j+2n+k\equiv 0 \pmod 4$. Then $j+n\equiv 0 \pmod 2$. If $n+j\equiv 2 \pmod 4$ and $j+2n+k\equiv 4 \pmod 8$, then we obtain the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) & \xrightarrow{f_1} & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1}) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) & \to & 0\\ & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_3}\downarrow & & \\ 0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) & \xrightarrow{f_3} & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) & \xrightarrow{f_4} & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}) & \to & 0\\ & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & \\ 0 & \to & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) & \xrightarrow{f_5} & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) & \xrightarrow{f_6} & \widetilde{KO}(S^{j+2n+k}) & \to & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

of exact sequences. Choose $r \geq l$ with $r \not\equiv j+2 \pmod 4$ and $j+2n+r \equiv 3,4,5,6$ or $7 \pmod 8$. Then, in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KO}(S^jD(q)^{2m,r}_{2n+1,k}) \arrow[r] \arrow[d, "k_1"'] & \widetilde{KO}(S^jD(q)^{2m,r}_{2n,k}) \arrow[r] \arrow[d, "k_2"'] & \widetilde{KO}(S^{j+n}RP^{n+r}_{n+k}) \arrow[r] \arrow[d, "k_3"'] & 0 \\
0 \arrow[r] & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) \arrow[r, "f_5"] & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \arrow[r, "f_6"] & \widetilde{KO}(S^{j+2n+k}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, k_1 and k_3 are isomorphisms. This implies that k_2 is an isomorphism. Using k_2, we obtain a ψ-map

$$h_5 : \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \to \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

with $h_2 \circ h_5 = 1$. Since $j+n \equiv 0 \pmod 2$ and $j-k-1 \equiv 3 \not\equiv 0 \pmod 4$, there exists a ψ-map

$$f_7 : \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) \to \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1})$$

with $f_2 \circ f_7 = 1$. Thus we obtain

$$\begin{aligned}
\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) &\cong \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \oplus \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1}) \\
&\cong A(q,j-1,k+1)^{2m}_{2n+1} \oplus Z \oplus A(q,j,l)^{2m}_{2n+1} \oplus \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) \\
&\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \oplus \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{J}(S^jD(q)^{2m,l}_{2n,k}) &\cong \tilde{J}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \oplus \tilde{J}(S^jD(q)^{2m,l}_{2n,k+1}) \\
&\cong B(q,j-1,k+1)^{2m}_{2n} \oplus B(q,j,l)^{2m}_{2n+1} \oplus G_0(j+2n+l).
\end{aligned}$$

If $n+j \equiv 2 \pmod 4$ and $j+2n+k \equiv 0 \pmod 8$, then we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \widetilde{KO}(S^{j+k+1}(L^{2m}_q/L^{2n-1}_q)) \arrow[r] \arrow[d, "h_4"'] & \widetilde{KO}(S^{j+2n+k+1}) \arrow[r] \arrow[d, "h_5"'] & 0 \\
0 \arrow[r] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) \arrow[r, "f_1"] \arrow[d, "g_1"'] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1}) \arrow[r, "f_2"] \arrow[d, "g_2"'] & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) \arrow[r] \arrow[d, "g_3"'] & 0 \\
0 \arrow[r] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \arrow[r, "f_3"] \arrow[d, "h_1"'] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) \arrow[r, "f_4"] \arrow[d, "h_2"'] & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}) \arrow[r] \arrow[d, "h_3"'] & 0 \\
0 \arrow[r] & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) \arrow[r, "f_5"] \arrow[d] & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \arrow[r, "f_6"] & \widetilde{KO}(S^{j+2n+k}) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

of exact sequences. If $l=k+1$, then $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) \cong 0$ and there exists a homotopy equivalence

$$g : S^{j+n}RP^{n+l}_{n+k} \overset{\simeq}{\to} S^{j+2n+k+1} \vee S^{j+2n+k}.$$

Using g, we obtain a ψ-map

$$g_6 : \widetilde{KO}(S^{j+n}RP^{n+k+1}_{n+k}) \to \widetilde{KO}(S^{j+2n+k+1})$$

with $g_6 \circ g_3 = 1$. Define a ψ-map

$$g_5 : \widetilde{KO}(S^jD(q)^{2m,k+1}_{2n,k}) \to \widetilde{KO}(S^jD(q)^{2m,k+1}_{2n,k+1})$$

by $g_5(a) = f_2^{-1}(g_6(f_4(a)))$ for $a \in \widetilde{KO}(S^jD(q)^{2m,k+1}_{2n,k})$. Then

$$g_5 \circ g_2 = f_2^{-1} \circ g_6 \circ f_4 \circ g_2 = f_2^{-1} \circ g_6 \circ g_3 \circ f_2 = f_2^{-1} \circ f_2 = 1.$$

Thus we obtain

$$\begin{aligned}\widetilde{KO}(S^jD(q)^{2m,k+1}_{2n,k}) &\cong \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \oplus \widetilde{KO}(S^{j+2n+k+1}) \\ &\cong \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}/2 \\ &\cong \widetilde{KO}(S^jD(q)^{2m,k+1}_{2n+1,k}) \oplus \widetilde{KO}(S^{j+n}RP^{n+k+1}_{n+k})\end{aligned}$$

and

$$\begin{aligned}\tilde{J}(S^jD(q)^{2m,k+1}_{2n,k}) &\cong \tilde{J}(S^{j+k}(L^{2m}_q/L^{2n-1}_q)) \oplus \tilde{J}(S^{j+2n+k+1}) \\ &\cong B(q, j-1, k+1)^{2m}_{2n} \oplus \boldsymbol{Z}/2.\end{aligned}$$

If $l > k+1$, then $\operatorname{Im} h_3 = 2\widetilde{KO}(S^{j+2n+k})$, $\operatorname{Im} h_2 = 2\widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n-1}_q))$ and

$$\operatorname{Ker} g_2 \cong \operatorname{Ker} g_3 \cong \widetilde{KO}(S^{j+2n+k+1}) \cong \boldsymbol{Z}/2.$$

Thus we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) & \xrightarrow{\bar{f}_1} & \operatorname{Coker} h_4 & \xrightarrow{\bar{f}_2} & \operatorname{Coker} h_5 & \to & 0 \\
 & & {\scriptstyle g_1}\downarrow & & {\scriptstyle \bar{g}_2}\downarrow & & {\scriptstyle \bar{g}_3}\downarrow & & \\
0 & \to & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) & \xrightarrow{f_3} & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) & \xrightarrow{f_4} & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}) & \to & 0 \\
 & & {\scriptstyle h_1}\downarrow & & {\scriptstyle \bar{h}_2}\downarrow & & {\scriptstyle \bar{h}_3}\downarrow & & \\
0 & \to & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) & \xrightarrow{\bar{f}_5} & \operatorname{Im} h_2 & \xrightarrow{\bar{f}_6} & \operatorname{Im} h_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of exact swquences. Since $j+n\equiv 0 \pmod 2$ and $j-k-1\equiv 3\not\equiv 0 \pmod 4$, there exists a ψ-map

$$f_7:\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1})\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1})$$

with $f_7\circ f_1=1$. Since $\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1})$ has an odd order, f_7 induces a ψ-map

$$\bar{f}_7:\operatorname{Coker}h_4\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1})$$

with $\bar{f}_7\circ\bar{f}_1=1$. Choose an integer $r\geqq l$ with $j+2n+r\equiv 5 \pmod 8$. Then $j\not\equiv r+2 \pmod 4$ and using the isomorphism

$$f_8:\widetilde{KO}(S^jD(q)^{2m,r}_{2n,k})\to\operatorname{Im}h_2,$$

we obtain a ψ-map

$$h_6:\operatorname{Im}h_2\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

with $\bar{h}_2\circ h_6=1$. Thus we obtain

$$\begin{aligned}\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k}) &\cong \operatorname{Im}h_2\oplus\operatorname{Coker}h_4\\ &\cong \widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n-1}))\oplus\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1})\oplus\operatorname{Coker}h_5\\ &\cong A(q,j-1,k+1)^{2m}_{2n+1}\oplus \boldsymbol{Z}\oplus A(q,j,l)^{2m}_{2n+1}\oplus G_0(j+2n+l)\\ &\cong \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})\end{aligned}$$

and

$$\begin{aligned}\tilde{J}(S^jD(q)^{2m,l}_{2n,k}) &\cong J''(\operatorname{Im}h_2)\oplus J''(\operatorname{Coker}h_4)\\ &\cong B(q,j-1,k+1)^{2m}_{2n}\oplus B(q,j,l)^{2m}_{2n+1}\oplus G_0(j+2n+l).\end{aligned}$$

Suppose $j-k\equiv j+n\equiv 0 \pmod 4$. Then, there exists a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k+1}) \arrow[r, "f_1"] \arrow[d, "g_1"] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1}) \arrow[r, "f_2"] \arrow[d, "g_2"] & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k}) \arrow[r, "f_3"] \arrow[d, "h_1"] & \widetilde{KO}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k}) \arrow[r, "f_4"] \arrow[d, "h_2"] & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) \arrow[r] & 0\\
 & \widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n})) \arrow[r, equal] \arrow[d] & \widetilde{KO}(S^{j+k}(L_q^{2m}/L_q^{2n})) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

of exact sequences. Since $j+n\equiv 0 \pmod 4$ and $j-k-1\equiv 3\not\equiv 0 \pmod 4$, there exists a ψ-map

$$f_8:\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1})\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k+1})$$

with $f_2\circ f_8=1$. Thus we obtain

$$\widetilde{KO}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k})\cong\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1})\oplus\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})$$

and $\tilde{J}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k})\cong\tilde{J}(S^{j+n}RP^{n+l}_{n+k+1})\oplus\tilde{J}(S^jD(q)^{2m,l}_{2n+1,k})$. There exists an exact sequence

$$0\to\widetilde{KO}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k})\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\to\widetilde{KO}(S^{j+2n+k})\to 0.$$

Since $\widetilde{KO}(S^{j+2n+k})\cong \boldsymbol{Z}$, we obtain

$$\begin{aligned}\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})&\cong\widetilde{KO}(S^jD(q)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k})\oplus\boldsymbol{Z}\\&\cong\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1})\oplus\boldsymbol{Z}\\&\cong\widetilde{KO}(S^jD(q)^{2m,l}_{2n+1,k})\oplus\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}).\end{aligned}$$

If $j+n\equiv 0 \pmod 4$ and $l\equiv j+2 \pmod 4$, then there exists an exact sequence

$$0\to A(q,j,l)^{2m}_{2n+1}\xrightarrow{h_1}\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\xrightarrow{h_2}\widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\to 0.$$

In the commutative diagram

$$\begin{array}{ccccccc}0\to&\widetilde{KO}(S^jD(q)^{2m,l+1}_{2n+1,k})&\to&\widetilde{KO}(S^jD(q)^{2m,l+1}_{2n,k})&\to&\widetilde{KO}(S^{j+n}RP^{n+l+1}_{n+k})&\to 0\\ &\downarrow{\scriptstyle k_1}&&\downarrow{\scriptstyle k_2}&&\downarrow{\scriptstyle k_3}&\\0\to&\widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k})&\to&\widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})&\to&\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})&\to 0\end{array}$$

with exact rows, k_1 and k_3 are isomorphisms. This implies that k_2 is an isomorphism. Using k_2, we obtain a ψ-map

$$h_3:\widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\to\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})$$

with $h_2\circ h_3=1$. Thus we obtain

$$\widetilde{KO}(S^jD(q)^{2m,l}_{2n,k})\cong\widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\oplus\widetilde{KO}(S^{j+l}(L^{2m}_q/L^{2n}_q))$$

and

$$\begin{aligned}\tilde{J}(S^jD(q)^{2m,l}_{2n,k})&\cong\tilde{J}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\oplus\tilde{J}(S^{j+l}(L^{2m}_q/L^{2n}_q))\\&\cong\tilde{J}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n,k})\oplus B(q,j,l)^{2m}_{2n+1}.\end{aligned}$$

There exists a commutative diagram

$$\begin{array}{ccccccc} 0 \to & \widetilde{KO}(S^jD(q)^{2m,l-1}_{2n+1,k}) & \to & \widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n+1,k}) & \to & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1}) & \to 0 \\ & \cong \downarrow & & \downarrow & & & \\ & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) & = & \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)), & & & \end{array}$$

in which the row is exact. This implies that

$$\widetilde{KO}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n+1,k}) \cong \widetilde{KO}(S^{j+k}(L^{2m}_q/L^{2n}_q)) \oplus \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k+1})$$

and

$$\begin{aligned} \tilde{J}(S^jD(q)^{2m,l-1,2n,k-1}_{2n,k,2n+1,k}) &\cong \tilde{J}(S^{j+k}(L^{2m}_q/L^{2n}_q)) \oplus \tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}) \\ &\cong B(q, j-1, k+1)^{2m}_{2n+1} \oplus \tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}). \end{aligned}$$

This completes the proof of Theorems 1 and 2.

5. Proof of Theorem 3

Suppose $m>n>0$, $j-k\equiv j+n\equiv 0 \pmod 4$ and p is an odd prime. We set

$$X=\begin{cases} S^jD(p)^{2m,l-1,2n,k-1}_{2n,k,2n,k} & (l\equiv j+2 \pmod 4) \\ S^jD(p)^{2m,l}_{2n,k} & (\text{otherwise}) \end{cases}$$

and

$$Y=\begin{cases} S^jD(p)^{2m,l-1,2n,k-1}_{2n,k,2n+1,k} & (l\equiv j+2 \pmod 4) \\ S^jD(p)^{2m,l,2n-1,k-1}_{2n,k,2n+1,k} & (\text{otherwise}). \end{cases}$$

There exists a commutative diagram

$$\begin{array}{ccccccc} 0 \to & VO^{j+n}_{n+l,n+k}(2) & \xrightarrow{f_{2,1}} & \widetilde{KO}(S^{j+n}RP^{n+l}_{n+k}) & \xrightarrow{f_{2,2}} & \widetilde{KO}(S^{j+2n+k}) & \to 0 \\ & {\scriptstyle h_{2,1}}\uparrow & & {\scriptstyle h_{2,2}}\uparrow & & \| & \\ 0 \to & \widetilde{KO}(Y) & \xrightarrow{f_1} & \widetilde{KO}(X) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+2n+k}) & \to 0 \\ & {\scriptstyle h_{p,1}}\downarrow & & {\scriptstyle h_{p,2}}\downarrow & & \| & \\ 0 \to & VO^{j+k}_{2m,2n}(p) & \xrightarrow{f_{p,1}} & \widetilde{KO}(S^{j+k}(L^{2m}_p/L^{2n-1}_p)) & \xrightarrow{f_{p,2}} & \widetilde{KO}(S^{j+2n+k}) & \to 0 \end{array}$$

with exact rows. In the diagram, $h_{2,2}$ and $h_{p,2}$ are epimorphisms. There exist ψ-maps

$$g_2: VO^{j+n}_{n+l,n+k}(2) \to \widetilde{KO}(Y)$$

and $g_p: VO^{j+k}_{2m,2n}(p) \to \widetilde{KO}(Y)$ with $h_{2,1}\circ g_2=1$, $h_{p,1}\circ g_p=1$, $\operatorname{Im} g_2=\operatorname{Ker} h_{p,1}$ and $\operatorname{Im} g_p=\operatorname{Ker} h_{2,1}$. For each i prime to p (resp. 2), $N_p(i)$ (resp. $N_2(i)$) denote the integer chosen to satisfy the property

$$(5.1)\qquad iN_p(i)\equiv 1 \pmod{p^m} \text{ (resp. } iN_2(i)\equiv 1 \pmod{2^l}).$$

In order to state the next lemma, we set

$$(5.2)\quad \begin{cases} (1)\quad v=(p-1)([(j+k)/2(p-1)]+1)-(j+k)/2. \\ (2)\quad s=[(n-v)/(p-1)]. \\ (3)\quad u_p=\begin{cases} N_p(2)p^{s+1}c_v^{j+k,2m}(p) & (j+2n+k\equiv 0 \pmod 8) \\ p^{s+1}c_v^{j+k,2m}(p) & (j+2n+k\equiv 4 \pmod 8). \end{cases} \\ (4)\quad UO=\Sigma_i(\cap_e i^e(\psi^i-1)\widetilde{KO}(Y)). \end{cases}$$

Lemma 5.3. *There exists an element $x\in\widetilde{KO}(X)$ such that*

(1) *$f_2(x)$ generates the group $\widetilde{KO}(S^{j+2n+k})\cong \boldsymbol{Z}$.*

(2) *The Adams operations are given by*

$$\psi^i(x)\equiv i^u x+f_1(g_p(v_p)+g_2(v_2)) \pmod{f_1(UO)},$$

where $u=(j+2n+k)/2$,

$$v_2=\begin{cases} -(i^u/2)u_2 & (i\equiv 0 \pmod 2) \\ -((i^u-i^{(j+n)/2})/2)u_2 & (i\equiv 1 \pmod 2), \end{cases}$$

$$v_p=\begin{cases} -(i^u/p)0^{u-t(p-1)}u_p & (i\equiv 0 \pmod p) \\ -((i^u-1+((j+k)/2)(i^{p-1}-1))/p)0^{u-t(p-1)}u_p & (i\not\equiv 0 \pmod p), \end{cases}$$

$t=[u/(p\text{-}1)]$ and u_2 is a generator of the group $VO^{j+n}_{n+l,n+k}(2)$.

Proof. According to [14], there exists an element

$$x_p\in\widetilde{KO}(S^{j+k}(L_p^{2m}/L_p^{2n-1}))$$

such that

i) $f_{p,2}(x_p)$ generates the group $\widetilde{KO}(S^{j+2n+k})\cong \boldsymbol{Z}$.

ii) The Adams operations are given by

$$\psi^i(x_p)\equiv i^{(j+2n+k)/2}x_p+f_{p,1}(v_p) \pmod{f_{p,1}(UO^{j+k}_{2m,2n}(p))},$$

where

$$v_p=\begin{cases} -(i^u/p)0^{u-t(p-1)}u_p & (i\equiv 0 \pmod p) \\ -((i^u-1+((j+k)/2)(i^{p-1}-1))/p)0^{u-t(p-1)}u_p & (i\not\equiv 0 \pmod p), \end{cases}$$

$u=(j+2n+k)/2$ and $t=[u/(p\text{-}1)]$. Choose an element $\tilde{x}\in\widetilde{KO}(X)$ with $f_2(\tilde{x})$

$=f_{p,2}(x_p)$. Then, there exists an element $y_p \in VO^{j+k}_{2m,2n}(p)$ with $x_p - h_{p,2}(\tilde{x}) = f_{p,1}(y_p)$. Set $x = \tilde{x} + f_1(g_p(y_p))$ and $x_2 = h_{2,2}(x)$. Then, we have $h_{p,2}(x) = x_p$ and $f_{2,2}(x_2) = f_2(x)$ $= f_{p,2}(x_p)$. It follows from [13] that the Adams operations are given by

$$\psi^i(x_2) = i^u x_2 + f_{2,1}(v_2),$$

where

$$v_2 = \begin{cases} -(i^u/2)u_2 & (i \equiv 0 \pmod 2) \\ -((i^u - i^{(j+n)/2})/2)u_2 & (i \equiv 1 \pmod 2), \end{cases}$$

and u_2 is a generator of the group $VO^{j+n}_{n+l,n+k}(2)$. We necessarily have

$$\psi^i(x) = ax + f_1(g_2(b) + g_p(c))$$

for some integer a and an element $g_2(b) + g_p(c) \in \widetilde{KO}(Y)$. By using the ψ-map f_2, we see that $a = i^u$. Under $h_{2,2}$, $f_1(g_2(b) + g_p(c))$ maps into $f_{2,1}(b)$ and x maps into x_2, and we see that

$$\psi^i(x_2) = i^u x_2 + f_{2,1}(b).$$

This implies that $b = v_2$. Under $h_{p,2}$, $f_1(g_2(b) + g_p(c))$ maps into $f_{p,1}(c)$ and x maps into x_p, and we see that

$$\psi^i(x_p) = i^u x_p + f_{p,1}(c).$$

This implies that $c \equiv v_p \pmod{UO^{j+k}_{2m,2n}(p)}$. Since $g_p(UO^{j+k}_{2m,2n}(p))$ is contained in UO, we obtain

$$\psi^i(x) \equiv i^u x + f_1(g_p(v_p) + g_2(v_2)) \pmod{f_1(UO)}.$$

This completes the proof of the lemma. q.e.d.

We now recall some difinition in [3]. Let f be a function which assigns to each integer i a non-negative integer $f(i)$. Given such a funciton f, we define $\widetilde{KO}(X)_f$ to be the subgroup of $\widetilde{KO}(X)$ generated by

$$\{i^{f(i)}(\psi^i - 1)(y) | i \in \boldsymbol{Z},\ y \in \widetilde{KO}(X)\};$$

that is, $\widetilde{KO}(X)_f = \langle \{i^{f(i)}(\psi^i - 1)(y) | i \in \boldsymbol{Z},\ y \in \widetilde{KO}(X)\} \rangle$. According to [2], [3] and [17], the kernel of the homomorphism $J: \widetilde{KO}(X) \to \tilde{J}(X)$ coincides with $\bigcap_f \widetilde{KO}(X)_f$, where the intersection runs over all functions f. Set $w_2 = f_1(g_2(u_2))$ and $w_p = f_1(g_p(u_p))$. Suppose that f satisfies

(5.4) $f(i) \geqq m + l + \max\{v_r(\mathrm{m}(u)) | r$ is a prime divisor of $i\}$

for every $i \in \boldsymbol{Z}$. It follows from Lemma 5.3 that we have

$$
\begin{aligned}
& i^{f(i)}(\psi^i-1)(x) \\
& \equiv i^{f(i)}(i^u-1)x+(i^{f(i)}(i^{(j+n)/2}-i^u)/2)w_2 \\
& -(i^{f(i)}(i^u-1+((j+k)/2)(i^{p-1}-1))/p)0^{u-t(p-1)}w_p \qquad (\operatorname{mod} f_1(UO)) \\
& = i^{f(i)}(i^u-1)x+(i^{f(i)}N_2(u/2^{v_2(u)})(u(i^{(j+n)/2}-1)-u(i^u-1))/2^{v_2(2u)})w_2 \\
& -(i^{f(i)}N_p(u/p^{v_p(u)})(u(i^u-1)+u((j+k)/2)(i^{p-1}-1))/p^{v_p(pu)})0^{u-t(p-1)}w_p \\
& \equiv i^{f(i)}(i^u-1)x+(i^{f(i)}N_2(u/2^{v_2(u)})((j+n-2u)/2)(i^u-1)/2^{v_2(2u)})w_2 \\
& -(i^{f(i)}N_p(u/p^{v_p(u)})((2u-j-k)/2)(i^u-1)/p^{v_p(pu)})0^{u-t(p-1)}w_p \qquad (\operatorname{mod} f_1(UO)) \\
& = (i^{f(i)}(i^u-1)/(2^{v_2(2u)}p^{v_p(pu)}))(2^{v_2(2u)}p^{v_p(pu)}x \\
& -p^{v_p(pu)}N_2(u/2^{v_2(u)})((n+k)/2)w_2-2^{v_2(2u)}N_p(u/p^{v_p(u)})n0^{u-t(p-1)}w_p).
\end{aligned}
$$

By virtue of [3; II, Theorem (2.7) and Lemma (2.12)], we have

$$\langle f_1(UO)\cup\{i^{f(i)}(\psi^i-1)(x)|i\in \boldsymbol{Z}\}\rangle = f_1(UO)\cup\{\mathfrak{m}(u)x-M_2w_2-M_pw_p\}\rangle,$$

where $M_2=(\mathfrak{m}(u)/2^{v_2(4u)})N_2(u/2^{v_2(u)})(n+k)$ and

$$M_p=\begin{cases}(\mathfrak{m}(u)/p^{v_p(pu)})N_p(u/p^{v_p(u)})n & (u\equiv 0 \pmod{p-1}) \\ 0 & (\text{otherwise}).\end{cases}$$

Since this is true for every function f which satisfies (5.4), we obtain

(5.5) $\tilde{J}(X)\cong \widetilde{KO}(X)/\langle f_1(UO)\cup\{\mathfrak{m}((j+2n+k)/2)x-M_2w_2-M_pw_p\}\rangle$,

where $w_2=f_1(g_2(u_2))$, $w_p=f_1(g_p(u_p))$, $v_2(M_2)=v_2(n+k)$ *and*

$$\begin{cases}v_p(M_p)=v_p(n) & (j+2n+k\equiv 0 \pmod{2(p-1)}) \\ M_p=0 & (\textit{otherwise}).\end{cases}$$

It follows from [13], [14] and the proof of Theorem 2 that we have

$$\tilde{J}(X)\cong F(z)/\langle\{B_0,B_2,B_p\}\rangle,$$

where $F(z)$ is a free abelian group generated by $\{z_0,z_2,z_p\}$,

$$B_2=2^{b_2(j+n,n+l,n+k)}z_2,$$

$$B_p=p^{b_p(j+k,2m,2n)}z_p,$$

$$B_0=M_0z_0-M_2z_2-M_pz_p$$

and $M_0=\mathfrak{m}((j+2n+k)/2)$. Set

$$\begin{cases} i_2 = \min\{b_2(j+n, n+l, n+k),\ v_2(n+k)\} \\ i_p = \min\{b_p(j+k, 2m, 2n),\ v_p\}, \end{cases} \tag{5.6}$$

where

$$v_p = \begin{cases} v_p(n) & (M_p \neq 0) \\ m & (M_p = 0). \end{cases}$$

For the sake of simplicity, we put $b_2 = b_2(j+n, n+l, n+k)$ and $b_p = b_p(j+k, 2m, 2n)$ in the following calculation. Choose integers e_1, e_2, e_3 and e_4 with $e_1 2^{b_2} - e_2 p^{b_p - i_p} M_2 = 2^{i_2}$ and $e_3 p^{b_p} - e_4 2^{b_2 - i_2} M_p = p^{i_p}$. We assume $e_4 = 0$ if $M_p = 0$. Then we have

$$A \begin{pmatrix} B_0 \\ B_2 \\ B_p \end{pmatrix} = \begin{pmatrix} 2^{b_2 - i_2} p^{b_p - i_p} M_0 z_0 \\ e_2 p^{b_p - i_p} M_0 z_0 + 2^{i_2} z_2 \\ e_4 2^{b_2 - i_2} M_0 z_0 + p^{i_p} z_p \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2^{b_2 - i_2} p^{b_p - i_p} & p^{b_p - i_p} M_2 / 2^{i_2} & 2^{b_2 - i_2} M_p / p^{i_p} \\ e_2 p^{b_p - i_p} & e_1 & e_2 M_p / p^{i_p} \\ e_4 2^{b_2 - i_2} & e_4 M_2 / 2^{i_2} & e_3 \end{pmatrix}$$

and $\det A = 1$. This implies that

$$\tilde{J}(X) \cong \boldsymbol{Z}/2^{b_2 - i_2} p^{b_p - i_p} M_0 \oplus \boldsymbol{Z}/2^{i_2} \oplus \boldsymbol{Z}/p^{i_p}.$$

This completes the proof of Theorem 3.

6. Proof of Theorems 4 and 5

By Proposition 3.20, $J(h(q,m)\alpha(q)) = J(2^{\varphi(l)}(\xi(q)-1)) = 0$. It follows from [5, Proposition (2.6)] that

$$(D(q)^{m,l})^{(n+s)\eta(q) \oplus (k-s+t+s)\xi(q)} \underset{S}{\simeq} (D(q)^{m,l})^{n\eta(q) \oplus (k-s+s)\xi(q)}.$$

Theorem 4 follows from Lemma 3.10.

Suppose $D(q)^{2m+1,l}_{2n,k}$ and $D(q)^{2m+2s+1,l+t}_{2n+2s,k+t}$ are of the same stable homotopy type, $s \geqq 0$ and $k+t \geqq 0$. There exists an integer $j > 2s+t$ and a cellular homotopy equivalence

$$h: S^{j-2s-t} D(q)^{2m+2s+1,l+t}_{2n+2s,k+t} \to S^j D(q)^{2m+1,l}_{2n,k},$$

which induces isomorphisms

$$h^*: \tilde{H}^*(S^jD(q)_{2n,k}^{2m+1,l}; Z/2) \to \tilde{H}^*(S^{j-2s-t}D(q)_{2n+2s,k+t}^{2m+2s+1,l+t}; Z/2),$$
$$h^!: \widetilde{KO}(S^jD(q)_{2n,k}^{2m+1,l}) \to \widetilde{KO}(S^{j-2s-t}D(q)_{2n+2s,k+t}^{2m+2s+1,l+t})$$

and $J(h): \tilde{J}(S^jD(q)_{2n,k}^{2m+1,l}) \to \tilde{J}(S^{j-2s-t}D(q)_{2n+2s,k+t}^{2m+2s+1,l+t})$. If $n+k \equiv 0 \pmod 2$, then h induces a homotopy equivalence

$$\bar{h}: S^{j-2s-t}D(q)_{2n+2s,k+t,2n+2s+1,k+t}^{2m+2s+1,l+t,2n+2s-1,k+t-1} \to S^jD(q)_{2n,k,2n+1,k}^{2m+1,l,2n-1,k-1}.$$

By Lemma 3.11, we obtain

$$\mathrm{Sq}^i(\sigma^{j-2s-t}([(c^{2n+2s}, d^{k+t})])) = \binom{n+k+s+t}{i}\sigma^{j-2s-t}([(c^{2n+2s}, d^{k+t+i})])$$

and $\mathrm{Sq}^i(\sigma^j([(c^{2n}, d^k)])) = \binom{n+k}{i}\sigma^j([(c^{2n}, d^{k+i})])$ for $1 \leqq i \leqq l-k$, where $\sigma: \tilde{H}^*(X; Z/2) \to \tilde{H}^{*+1}(SX; Z/2)$ is the suspension isomorphism. Since $h^*(\sigma^j([(c^{2n}, d^k)])) = \sigma^{j-2s-t}([(c^{2n+2s}, d^{k+t})])$, we obtain

$$\binom{n+k}{i} \equiv \binom{n+k+s+t}{t} \pmod 2$$

for $1 \leqq i \leqq l-k$. It follows from [12, Lemma 2.1] that $v \geqq [\log_2(l-k)]+1$, where $v = v_2(|s+t|+2^t)$. This completes the proof of the part i) of (1) of Theorem 5.

To prove the parts ii) and iii) of (1) of Theorem 5, we may assume $l \geqq k+9$. So, assume $l \geqq k+9$ and $v \geqq 4$. If $m=n$, then

$$\tilde{J}(S^{j-2s-t}D(q)_{2n+2s,k+t}^{2m+2s+1,l+t}) \cong \tilde{J}(S^{j+n-s-t}RP_{n+k+s+t}^{n+l+s+t}) \oplus \tilde{J}(S^{j+n-s-t}RP_{n+k+s+t+1}^{n+l+s+t+1})$$

and $\tilde{J}(S^jD(q)_{2n,k}^{2m+1,l}) \cong \tilde{J}(S^{j+n}RP_{n+k}^{n+l}) \oplus \tilde{J}(S^{j+n}RP_{n+k+1}^{n+l+1})$. Suppose $v_2(j+n) > \varphi(l-k)$. By the isomorphism $J(h)$, we see

$$v+1 \geqq \max\{a_2(n+l, n+k-1),\ a_2(n+l+1, n+k)\}.$$

If $n+k \equiv 0 \pmod{2^{\varphi(l-k)-1}}$, then $a_2(n+l, n+k-1) = \varphi(l-k)$ and

$$n+k+s+t \equiv n+k \pmod{2^{\varphi(l-k)}}.$$

This implies that $v \geqq \varphi(l-k)$. If $n+k+1 \equiv 0 \pmod{2^{\varphi(l-k)-1}}$, then $a_2(n+l+1, n+k) = \varphi(l-k)$ and

$$n+k+1+s+t \equiv n+k+1 \pmod{2^{\varphi(l-k)}}.$$

This implies that $v \geqq \varphi(l-k)$. Thus the parts ii) and iii) of (1) of Theorem 5 for the case $m=n$ are obtained by using Lemma 3.13.

Suppose $m > n$. If $m \equiv n \pmod 4$, then

$$h(i_0(S^{j+n-s-t}RP^{n+k+s+t+8}_{n+k+s+t})) \subset S^jD(q)^{2m,l}_{2n,k}$$

and $i_0^! \circ h^! \circ p_2^! = 0$, where

$$\begin{aligned} i_0 : S^{j+n-s-t}RP^{n+k+s+t+8}_{n+k+s+t} &\approx S^{j-2s-t}D(q)^{2n+2s,k+t+8}_{2n+2s,k+t} \\ &\subset S^{j-2s-t}D(q)^{2m+2s+1,l+t}_{2n+2s,k+t} \end{aligned}$$

is an inclusion map and

$$p_2 : S^jD(q)^{2m+1,l}_{2n,k} \to S^jD(q)^{2m+1,l}_{2m+1,k} \approx S^{j+m}RP^{m+l+1}_{m+k+1}$$

is an identification. Let

$$\begin{aligned} &i_1 : S^{j+n-s-t}RP^{n+l+s+t}_{n+k+s+t} \to S^{j-2s-t}D(q)^{2m+2s+1,l+t}_{2n+2s,k+t}, \\ &i_2 : S^{j+n}RP^{n+l}_{n+k} \to S^jD(q)^{2m+1,l}_{2n,k}, \\ &i_3 : S^{j+2n+k} \to S^{j-2s-t}D(q)^{2m+2s+1,l+t}_{2n+2s,k+t} \end{aligned}$$

and $i_4 : S^{j+2n+k} \to S^jD(q)^{2m+1,l}_{2n,k}$ be inclusion maps, and

$$p_1 : S^{j-2s-t}D(q)^{2m+2s+1,l+t}_{2n+2s,k+t} \to S^{j+m-s-t}RP^{m+l+s+t+1}_{m+k+s+t+1}$$

an identification. Suppose $v_2(j+n) \geqq \varphi(l-k)$. If $n+k \not\equiv 0 \pmod 4$, then $J(h)$ induces an isomorphism

$$\tilde{J}(S^{j+n}RP^{n+l}_{n+k}) \overset{\cong}{\to} \tilde{J}(S^{j+n-s-t}RP^{n+l+s+t}_{n+k+s+t}).$$

This implies that $v_2(j+n-s-t)+1 \geqq a_2(n+l, n+k-1)$ and $v \geqq a_2(n+l, n+k-1)-1$. If $n+k \equiv 0 \pmod 4$, then $J(\bar{h})$ induces an isomorphism

$$\tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}) \overset{\cong}{\to} \tilde{J}(S^{j+n-s-t}RP^{n+l+s+t}_{n+k+s+t+1}).$$

This implies that $v_2(j+n-s-t)+1 \geqq a_2(n+l, n+k) = a_2(n+l, n+k-1)$ and $v \geqq a_2(n+l, n+k-1)-1$. If $n+k \equiv 0 \pmod{2^{\varphi(l-k)-1}}$, then

$$\widetilde{KO}(S^{j+2n+k}) \cong Z.$$

Let x be an element of $\widetilde{KO}(S^jD(q)^{2m+1,l}_{2n,k})$ with $(i_4)^!(x)$ generates the group $\widetilde{KO}(S^{j+2n+k})$. Then $(i_3)^!(h^!(x))$ generates the group $\widetilde{KO}(S^{j+2n+k})$. It follows from [13] that

$$(i_1)^!(\psi^3(y)) = 3^{(j+2n+k)/2}(i_1)^!(y) + ((3^{(j+n-s-t)/2} - 3^{(j+2n+k)/2})/2)v$$

and

$$(i_2)^!(\psi^3(x)) = 3^{(j+2n+k)/2}(i_2)^!(x) + ((3^{(j+n)/2} - 3^{(j+2n+k)/2})/2)u,$$

where $y = h^!(x)$, v is a generator of torsion subgroup of

$$\widetilde{KO}(S^{j+n-s-t}RP^{n+l+s+t}_{n+k+s+t})$$

and u is a generator of torsion subgroup of $\widetilde{KO}(S^{j+n}RP^{n+l}_{n+k})$. It follows from [15, Lemma 3.1] that

$$(3^{(j+n)/2}-3^{(j+2n+k)/2})/2 \equiv -(n+k) \pmod{2^{\varphi(l-k)}}$$

and $(3^{(j+n-s-t)/2}-3^{(j+2n+k)/2})/2 \equiv -(s+t+n+k) \pmod{2^{\varphi(l-k)}}$. Since $J(\bar{h})$ induces an isomorphism

$$\tilde{J}(S^{j+n}RP^{n+l}_{n+k+1}) \overset{\cong}{\to} \tilde{J}(S^{j+n-s-t}RP^{n+l+s+t}_{n+k+s+t+1}),$$

this implies that $v \geqq \varphi(l-k)$. Suppose $v_2(j+m) \geqq \varphi(l-k)$. Then $J(h)$ induces an isomorphism

$$\tilde{J}(S^{j+m}RP^{m+l+1}_{m+k+1}) \overset{\cong}{\to} \tilde{J}(S^{j+m-s-t}RP^{m+l+s+t+1}_{m+k+s+t+1}).$$

This implies that $v+1 \geqq a_2(m+l+1, m+k)$. If $m+k+1 \equiv 0 \pmod{2^{\varphi(l-k)-1}}$, then $m+k+s+t+1 \equiv m+k+1 \pmod{2^{\varphi(l-k)}}$ and $v \geqq \varphi(l-k)$. Thus the parts ii) and iii) of (1) of Theorem 5 are obtained by using Lemma 3.13. This completes the proof of the part (1) of Theorem 5.

Let q be an odd prime.. By the part i) of (1) of Theorem 5, $s+t \equiv 0 \pmod 2$. Suppose $j+k \equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$ and $j+k \equiv 2(-2+k-2[(n+k)/2]) \pmod{2(q-1)}$. Then $j \equiv k \pmod 4$, $j-2s-t \equiv k+t \pmod 4$,

$$B(q, j-1, k+1)^{2m}_{2n+1} \cong Z/q^{a_q(j+k+2m, j+k+2n)},$$

$$B(q, j-2s-t-1, k+t+1)^{2m+2s}_{2n+2s+1} \cong Z/q^{b_q(j+k-2s, 2m+2s, 2n+2s)},$$

$$b_q(j+k-2s, 2m+2s, 2n+2s) = \min\{v_q(j+k-2s)+1,\ a_q(j+k+2m, j+k+2n)\}$$

and $a_q(j+k+2m, j+k+2n) = [(m+k-2[(n+k)/2]-2)/(q-1)]+1$. Suppose $j+l \equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$ and $j+l \equiv 2(-1+l-2[(n+l+1)/2]) \pmod{2(q-1)}$. Then $j \equiv l+2 \pmod 4$, $j-2s-t \equiv l+t+2 \pmod 4$,

$$B(q, j, l)^{2m}_{2n+1} \cong Z/q^{a_q(j+l+2m, j+l+2n)},$$

$$B(q, j-2s-t, l+t)^{2m+2s}_{2n+2s+1} \cong Z/q^{b_q(j+l-2s, 2m+2s, 2n+2s)},$$

$$b_q(j+l-2s, 2m+2s, 2n+2s) = \min\{v_q(j+l-2s)+1,\ a_q(j+l+2m, j+l+2n)\}$$

and $a_q(j+l+2m, j+l+2n) = [(m+l-2[(n+l+1)/2]-1)/(q-1)]+1$. This implies that

$$v_q(s+q^m) \geqq [(m+k-2[(n+k)/2]-2)/(q-1)]$$

and $v_q(s+q^m) \geqq [(m+l-2[(n+l+1)/2]-1)/(q-1)]$ except for the case $l \equiv k+2$

(mod 4),

$$d=[(m+k-2[(n+k)/2]-2)/(q-1)]=[(m+l-2[(n+l+1)/2]-1)/(q-1)]>0,$$
$$l-k-2s\equiv 0 \pmod{q^d}$$

and $l-k+2s\equiv 0 \pmod{q^d}$. If $l\equiv k+2 \pmod 4$,

$$d=[(m+k-2[(n+k)/2]-2)/(q-1)]=[(m+l-2[(n+l+1)/2]-1)/(q-1)]>0,$$
$$l-k-2s\equiv 0 \pmod{q^d}$$

and $l-k+2s\equiv 0 \pmod{q^d}$, then $l\equiv k \pmod{2q^d}$, $l\geqq k+2q^d\geqq k+2q$,

$$h(\bar{\imath}_0(S^{j+k-2s}(L_q^{2n+2s+2q-2}/L_q^{2n+2s-1})))\subset S^jD(q)_{2n,k}^{2m+1,l-1}$$

and $\bar{\imath}_0^! \circ h^! \circ \bar{p}_2^! = 0$, where

$$\begin{aligned}\bar{\imath}_0: S^{j+k-2s}(L_q^{2n+2s+2q-2}/L_q^{2n+2s-1}) &\approx S^{j-2s-t}D(q)_{2n+2s,k+t}^{2n+2s+2q-2,k+t}\\ &\subset S^{j-2s-t}D(q)_{2n+2s,k+t}^{2m+2s+1,l+t}\end{aligned}$$

is an inclusion map and

$$\bar{p}_2: S^jD(q)_{2n,k}^{2m+1,l}\to S^jD(q)_{2n,l}^{2m+1,l}\approx S^{j+l}(L_q^{2m+1}/L_q^{2n-1})$$

is an identification. This implies that $h^!$ induces isomorphisms

$$\tilde{J}(S^{j+k-2s}(L_q^{2m+2s}/L_q^{2n+2s}))\cong \tilde{J}(S^{j+k}(L_q^{2m}/L_q^{2n}))$$

and $\tilde{J}(S^{j+l-2s}(L_q^{2m+2s}/L_q^{2n+2s}))\cong \tilde{J}(S^{j+l}(L_q^{2m}/L_q^{2n}))$. Thus we obtain the part i) of (2) of Theorem 5. If $n\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$, $n+k\equiv 0 \pmod 2$, $j+k\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$ and $j+k\equiv -2n \pmod{2(q-1)}$, then $j\equiv k \pmod 4$ and the isomorphism $J(h)$ implies

$$n+s\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$$

and $s\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$. If $n\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$, $n+l\equiv 1 \pmod 2$, $j+l\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$ and $j+l\equiv -2n \pmod{2(q-1)}$, then $j\equiv l-2 \pmod 4$ and the isomorphism $J(h)$ implies

$$n+s\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$$

and $s\equiv 0 \pmod{q^{[(m-n)/(q-1)]}}$. Thus the part ii) of (2) of Theorem 5 is obtained by using Lemma 3.13. This completes the proof of Theorem 5.

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