



Title	The cohomology of line bundles on Hopf manifolds
Author(s)	Mall, Daniel
Citation	Osaka Journal of Mathematics. 1991, 28(4), p. 999-1015
Version Type	VoR
URL	<a href="https://doi.org/10.18910/9995">https://doi.org/10.18910/9995</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## THE COHOMOLOGY OF LINE BUNDLES ON HOPF MANIFOLDS

DANIEL MALL

(Received June 12, 1990)

### 0. Introduction

The purpose of the present paper is to give an elementary method for the computation of the cohomology groups  $H^q(X, \Omega^p(L))$ , ( $0 \leq q \leq n$ ) of an  $n$ -dimensional Hopf manifold  $X$ , ( $n \geq 2$ ), where  $\Omega^p(L)$  denotes the sheaf of germs of holomorphic  $p$ -forms with values in a holomorphic line bundle  $L$  on  $X$ .

Ise [10] has given a solution of this problem for homogeneous Hopf manifolds. He makes strong use of the fact that any homogeneous Hopf manifold of dimension  $n$  is a fibre bundle over the complex projective space  $P^{n-1}$ . His main tools are knowledge of the cohomology groups  $H^q(P^{n-1}, \Omega^p(L))$  and the Leray spectral sequence.

Our approach to Ise's problem is a generalization of a method used by Doody [4] to study the deformation of Hopf manifolds.

We shall now describe the contents of this paper. Section 1 presents our method for the computation of the cohomology groups for flat line bundles and makes evident the dichotomy between Hopf manifolds of dimension  $n > 2$  and Hopf surfaces ( $n = 2$ ). Section 2 treats the case of Hopf manifolds of dimension  $n > 2$  and section 3 the case of Hopf surfaces.

Generalizing a result of Kodaira [11] for Hopf surfaces we show in section 4 that all line bundles on an arbitrary Hopf manifold are flat. To this end we calculate first the Hodge numbers of these manifolds. The flatness of line bundles shows that the method introduced in section 1 applies to the general case.

Applications to holomorphic vector bundles and foliations on Hopf manifolds will be published elsewhere.

*Acknowledgement:* This paper is a part of the author's Ph.D. thesis [12] with Prof. A. Haefliger. My heartfelt thanks go to him for his help and encouragement.

### 1. Hopf manifolds, flat line bundles and the main lemma

Let  $W := \mathbb{C}^n - \{0\}$  and  $n \geq 2$ . We recall some facts about Hopf manifolds,

for details we refer to [3], [8], [11] § 10 and [14]. By a contraction  $f: (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$  we understand an automorphism of  $\mathbf{C}^n$  fixing 0 with the property that the eigenvalues  $\mu_1, \dots, \mu_n$  of  $f'(0)$ , the differential of  $f$  at 0, are inside the unit circle. We call a contraction of the form  $f: (z_1, \dots, z_n) \mapsto (\mu_1 z_1, \dots, \mu_n z_n)$  diagonal and always assume that  $0 < |\mu_1| \leq |\mu_2| \leq \dots \leq |\mu_n| < 1$ .

The subgroup of automorphisms of  $\mathbf{C}^n$  generated by a contraction  $f$  operates freely and properly discontinuously on  $W$ . The quotient  $X$  (or  $X_f$  if we want to emphasize the contraction  $f$ ) is a compact, complex manifold of dimension  $n$  called a Hopf manifold. The canonical projection is denoted by  $\pi: W \rightarrow X$ . Given two contractions  $f_1, f_2$  of  $\mathbf{C}^n$  the Hopf manifolds  $X_{f_1}$  and  $X_{f_2}$  are isomorphic iff there is a  $g \in \text{Aut}(\mathbf{C}^n)$  fixing 0 such that  $f_1 = g \circ f_2 \circ g^{-1}$  on  $\mathbf{C}^n$ . Thus the classification problem of Hopf manifolds up to isomorphisms corresponds to the normal form problem for conjugation classes of contractions in  $\text{Aut}(\mathbf{C}^n, 0)$ , the group of automorphisms of  $\mathbf{C}^n$  fixing 0. For dimension  $n=2$  the solution to this problem is well known (cf. [11] p. 695). Reich (cf. [33] p. 248) gives a solution for dimension  $n=3$ . For dimension  $n>3$  a complete solution is not to be expected (cf. [13] p. 247). However, there is a weaker result which is enough for our purposes (for details see [8] p. 249). Let  $G_\mu$  be the group of invertible polynomial maps  $f: \mathbf{C}^n \rightarrow \mathbf{C}^n$  which commute with the diagonal linear contraction  $d_\mu: z = (z_1, \dots, z_n) \mapsto \mu z = (\mu_1 z_1, \dots, \mu_n z_n)$ . The following statement is a consequence from the Poincaré-Dulac theorem (cf. [1] p. 187). Given a Hopf manifold  $X$  generated by a contraction  $f$  with  $f'(0) = d_\mu$  then there is a contraction  $g \in G_\mu$  such that  $X$  is isomorphic to  $X_g$ .

Assume  $L \in H^1(X, \mathcal{O}^*)$  is a flat line bundle. Then it is the quotient of  $W \times \mathbf{C}$ , the trivial line bundle on the universal covering of  $X$ , by the operation of a representation of the fundamental group  $\pi_1(X) \cong \mathbf{Z}$  of  $X$

$$\begin{aligned} \rho_L: \pi_1(X) &\rightarrow \mathbf{C}^* := GL(1, \mathbf{C}) \\ \gamma &\mapsto \rho_L(\gamma) \end{aligned}$$

in the following way:

$$\begin{aligned} W \times \mathbf{C} &\rightarrow W \times \mathbf{C} \\ (z, v) &\mapsto (f(z), \rho_L(1)v). \end{aligned}$$

NOTATION: We write  $L_b$  for the bundle induced by the representation  $\rho_L$  with  $b := \rho_L(1)$ . For the locally free sheaf  $\mathcal{O}(L_b)$  of germs of holomorphic sections we write sometimes by abuse of notation  $L_b$ .

We want to calculate  $h^q(X, \Omega_X^b(L_b)) := \dim H^q(X, \Omega_X^b(L_b))$ , where  $\Omega_X^b(L_b) := \Omega_X^b \otimes \mathcal{O}(L_b)$  and  $\Omega_X^b$  denotes the sheaf of germs of holomorphic  $p$ -forms on  $X$  as usual.

Take a covering  $\mathcal{A} = \{U_i\}$  of  $X$  such that the  $U_i$  are open and contractible Stein subsets of  $X$  and  $\tilde{U}_i := \pi^{-1}(U_i)$  is a disjoint union of open Stein subsets

$\{U'_{ij}\}$  of  $W$ , each of them isomorphic to  $U_i$ :

$$\tilde{U}_i := \bigcup_{m \in \mathbb{Z}} f^m(U'_{i0}); \quad \tilde{\mathcal{A}} := \{\tilde{U}_i\}.$$

If  $\varphi \in \Gamma(U_i, \Omega_X^b \otimes L_b)$  then  $\tilde{\varphi} := \pi^*(\varphi)$  lies in  $\Gamma(\tilde{U}_i, \pi^*(\Omega_X^b \otimes L_b)) \cong \Gamma(\tilde{U}_i, \Omega_W^b)$ , because  $\pi^*(\Omega_X^b \otimes L_b) \cong \pi^*(\Omega_X^b) \otimes \pi^*(L_b)$ ,  $\pi^*(L_b) \cong \mathcal{O}_W$  and  $\pi^*(\Omega_X^b) \cong \Omega_W^b$ . Thus we can build a sequence of Čech-complexes:

$$(1) \quad 0 \rightarrow C^*(\mathcal{A}, \Omega_X^b(L_b)) \xrightarrow{\pi^*} C^*(\mathcal{A}, \Omega_W^b) \xrightarrow{bId - f^*} C^*(\tilde{\mathcal{A}}, \Omega_X^b) \rightarrow 0.$$

**Lemma 1.** *The sequence (1) is exact.*

*Proof.* The only problem is to show the surjectivity of  $bId - f^*$ . Let without loss of generality  $\gamma$  be an element of  $\Gamma(\tilde{U}_{i_0} \cap \dots \cap \tilde{U}_{i_q})$ . Our assumption on  $\mathcal{A}$  implies that  $\tilde{U}_{i_0} \cap \dots \cap \tilde{U}_{i_q} = \bigcup f^m(U'_{i_0} \cap \dots \cap U'_{i_q})$ . We put  $\gamma = \gamma_1 - \gamma_2$  with  $\gamma_1 = 0$  on all  $f^m(U'_{i_0} \cap \dots \cap U'_{i_q})$  for  $m < 0$  and  $\gamma_2 = 0$  on  $f^m(U'_{i_0} \cap \dots \cap U'_{i_q})$  with  $m \geq 0$ . The expression

$$\beta = \sum_{m \geq 0} b^{-m-1} (f^*)^m (\gamma_1) + \sum_{m < 0} b^{-m-1} (f^*)^m (\gamma_2)$$

is well defined since both sums are locally finite. A straightforward calculation shows that  $(bId - f^*)\beta = \gamma$ . qed.

From (1) we derive a generalized long exact Douady sequence:

$$(2) \quad 0 \rightarrow H^0(X, \Omega_X^b(L_b)) \rightarrow H^0(W, \Omega_W^b) \xrightarrow{P_0} H^0(W, \Omega_W^b) \rightarrow H^1(X, \Omega_X^b(L_b)) \\ \rightarrow H^1(W, \Omega_W^b) \xrightarrow{P_1} H^1(W, \Omega_W^b) \rightarrow H^2(X, \Omega_X^b(L_b)) \rightarrow \dots$$

where  $P = bId - f^*$ .

Because  $H^q(W, \Omega^b) \cong H^q(W, \mathcal{O}^{(b)})$  and  $H^q(W, \mathcal{O}) \neq 0$  only for  $q = 0, n-1$  (cf. [8] p. 246), (2) decomposes for  $n > 2$  into the following exact sequences:

$$(3) \quad 0 \rightarrow H^0(X, \Omega_X^b(L_b)) \rightarrow H^0(W, \Omega_W^b) \xrightarrow{P_0} H^0(W, \Omega_W^b) \rightarrow H^1(X, \Omega_X^b(L_b)) \rightarrow 0 \\ 0 \rightarrow H^{n-1}(X, \Omega_X^b(L_b)) \rightarrow H^{n-1}(W, \Omega_W^b) \xrightarrow{P^{n-1}} H^{n-1}(W, \Omega_W^b) \\ \rightarrow H^n(X, \Omega_X^b(L_b)) \rightarrow 0$$

With  $h_b^{p,q} := \dim H^q(X, \Omega_X^b(L_b))$  follows the

**Main Lemma.** *Let  $X$  be a Hopf manifold of dimension  $n > 2$  and let  $L_b \in H^1(X, \mathcal{O}^*)$  be a flat line bundle on  $X$ . Then*

$$h_b^{b,0} = \dim \ker P_0, \\ h_b^{b,1} = \dim \operatorname{coker} P_0,$$

$$\begin{aligned} h_q^{p,q} &= 0 \quad \text{for } 1 < i < n-1, \\ h_b^{p,n-1} &= \dim \ker P_{n-1}, \\ h_b^{p,n} &= \dim \operatorname{coker} P_{n-1}. \end{aligned}$$

## 2. The cohomology of flat line bundles for diagonal Hopf manifolds of dimension $n > 2$

With the method described in § 1 one can calculate the cohomology of flat line bundles over Hopf manifolds. We give formulas for the classical and the generic cases. Explicit formulas for the resonant cases would involve the unsolved classification problem of Hopf manifolds.

### A. The classical case

Assume that  $X$  is a Hopf manifold generated by a contraction  $f$  of the type  $f: (z_1, \dots, z_n) \rightarrow (\mu z_1, \dots, \mu z_n)$  with  $0 < |\mu| < 1$ . This is the classical case introduced by Hopf [9] and studied by Ise [10].

We need some notations:

$N_0 := N \cup \{0\}$  where  $N$  is the set of positive integers.

$\Delta_\mu := \langle \mu \rangle =$  cyclic subgroup of  $C^*$  generated by the eigenvalue  $\mu$  of  $f$ .

For  $\alpha = (\alpha_1, \dots, \alpha_n) \in N_0^n$  we introduce  $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$ .

$\chi(m, n) := \# \{ \alpha \in N_0^n \mid |\alpha| = m \}$  where  $\# M$  denotes the cardinality of the set  $M$ .

$L_b \in H^1(X, \mathcal{O}^*)$ ,  $b \in C^*$  (cf. § 1).

$J \in I_p := \{ (i_1, \dots, i_p) \in N^n \mid 1 \leq i_1 < i_2 < \dots < i_p \leq n \}$ .

We now give a short proof of the main result of [10].

**Theorem 1.** [10] *Let  $X_f$  be a Hopf manifold of dimension  $n > 2$  with  $f: (z_1, \dots, z_n) \rightarrow (\mu z_1, \dots, \mu z_n)$ ,  $0 < |\mu| < 1$ , and  $L_b \in H^1(X, \mathcal{O}^*)$  a holomorphic flat line bundle. Then either  $b \notin \Delta_\mu$ , in which case  $h_b^{p,q} = 0$  for all  $p$  and  $q$ , or  $b = \mu^m$ ,  $m \in \mathbb{Z}$ , and then we have the following result:*

$$\begin{aligned} h_b^{p,0} = h_b^{p,1} &= \begin{cases} \binom{m-p+n-1}{n-1} \binom{n}{p} & \text{if } m \geq p, \\ 0 & \text{otherwise.} \end{cases} \\ h_b^{p,q} &= 0 \quad \text{if } 2 \leq q \leq n-2. \\ h_b^{p,n-1} = h_b^{p,n} &= \begin{cases} \binom{-m+p-1}{n-1} \binom{n}{n-p} & \text{if } m \leq p-n \leq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

**Proof.** The Main Lemma implies that it is enough to calculate the dimension of the kernels and the cokernels of  $P_0$  and  $P_{n-1}$ . By Serre-duality we can avoid the calculation of  $P_{n-1}$  (see the remark at the end of section 2):

$$H^q(X, \Omega^p(L_b)) \cong H^{n-q}(X, \Omega^{n-p}(L_b^*)) \text{ with } L_b^* \cong L_{1/b}.$$

a)  $\dim \ker P_0$ : The section  $\omega \in \Gamma(W, \Omega^p)$  can be written as

$$\omega = \sum_{J \in I_p} \sum_{\alpha \in N_0^n} a_\alpha^J z^\alpha dz_J = \sum_{(i_1, \dots, i_p) \in I_p} \sum_{(\alpha_1, \dots, \alpha_n) \in N_0^n} a_{\alpha_1, \dots, \alpha_n}^{(i_1, \dots, i_p)} z_1^{\alpha_1} \dots z_n^{\alpha_n} dz_{i_1} \wedge \dots \wedge dz_{i_p}.$$

Applying the operator  $P_0: \Gamma(W, \Omega^p) \rightarrow \Gamma(W, \Omega^p)$  gives

$$P_0 \omega = \sum_J \sum_{\alpha \in N_0^n} a_\alpha^J (b - \mu^{|\alpha|+p}) z^\alpha dz_J$$

Hence: If  $b - \mu^{|\alpha|+p} \neq 0$  then we get that  $P_0 \omega = 0$  iff  $a_\alpha^J = 0$  for all  $J \in I_p$  and  $\alpha \in N_0^n$ . This implies: either  $b \notin \Delta_\mu$  and then  $\ker P_0 = 0$  or  $b = \mu^m$  and we have

$$\begin{aligned} a_\alpha^J &\in \mathcal{C} & \text{if } m = |\alpha| + p \\ a_\alpha^J &= 0 & \text{if } m \neq |\alpha| + p \end{aligned}$$

i.e.  $\dim \ker P_0 = \# \{ \alpha = (\alpha_1, \dots, \alpha_n) \mid |\alpha| = m - p \} \binom{n}{p} = \chi(m - p, n) \binom{n}{p}$ . It is well known that

$$\chi(m, n) = \binom{m+n-1}{n-1}$$

and hence the claim follows.

b)  $\dim \operatorname{coker} P_0$ : Assume  $b = \mu^m$ . We claim that  $\omega$  is an element of  $\operatorname{coker} P_0$  iff it can be written in the form

$$\omega = \sum_J \sum_{|\alpha|+p=m} a_\alpha^J z^\alpha dz_J.$$

That this condition is sufficient follows from the considerations above. We show now that it is necessary.

Let  $\omega = \sum_J \sum_{|\alpha|+p \neq m} a_\alpha^J z^\alpha dz_J$ . The claim is that a  $\tau$  exists with  $P_0 \tau = \omega$ . Setting

$$\tau := \sum_J \sum_{|\alpha|+p \neq m} a_\alpha^J (b - \mu^{|\alpha|+p})^{-1} z^\alpha dz_J$$

it follows that  $P_0 \tau = \omega$  if  $\tau$  converges. It is easy to see that a  $n_0$  exists with

$$|b - \mu^{|\alpha|+p}| > \left| \frac{b}{2} \right| \text{ for all } \alpha \text{ with } |\alpha| > n_0$$

i.e.

$$|b - \mu^{|\alpha|+p}|^{-1} < \left| \frac{2}{b} \right| \text{ for all } \alpha \text{ with } |\alpha| > n_0.$$

There is a  $t \in \mathbf{R}_+^n$  such that  $\sum |a_\alpha^J| t^\alpha$  converges. Hence

$$\sum |a_\alpha^J| (b - \mu^{|\alpha|+p})^{-1} t^\alpha \leq \sum_{|\alpha| \leq n_0} |b - \mu^{|\alpha|+p}|^{-1} |a_\alpha^J| t^\alpha + \frac{2}{|b|} \sum_{|\alpha| > n_0} |a_\alpha^J| t^\alpha < \infty.$$

We get  $\dim \operatorname{coker} P_0 = \dim \ker P_0$ .

qed.

### B. The generic case

We say that a Hopf manifold  $X$  is of generic type if it is generated by a contraction of the type  $f: (z_1, \dots, z_n) \rightarrow (\mu_1 z_1, \dots, \mu_n z_n)$  with  $0 < |\mu_1| \leq |\mu_2| \leq \dots \leq |\mu_n| < 1$ , and there are no relations except trivial ones between the  $\mu_i$ 's of the form

$$\prod_{i \in A} \mu_i^{r_i} = \prod_{j \in B} \mu_j^{r_j}, \quad r_i, r_j \in \mathbb{N}_0, \quad A \cap B = \emptyset, \quad A \cup B = \{1, \dots, n\}.$$

The following notations are used:

$$\mu := (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{C}^n, \quad \alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \quad m := (m_1, \dots, m_n) \in \mathbb{N}_0^n.$$

$$A_p^m := \# \{K = (k_1, k_2, \dots, k_n) \mid k_i \in \{0, 1\}; \sum k_i = p; m - K \geq (0, \dots, 0)\}$$

$$B_p^m := \# \{K = (k_1, k_2, \dots, k_n) \mid k_i \in \{0, 1\}; \sum k_i = n - p; m + K \leq (0, \dots, 0)\}$$

Let  $\Delta_{\mu_1, \dots, \mu_n}$  be the free commutative subgroup of  $\mathbb{C}^*$  generated by the eigenvalues  $\mu_1, \mu_2, \dots, \mu_n$  of  $f$ .

**Theorem 2.** *If  $L_b$  is a holomorphic flat line bundle on a generic Hopf manifold of dimension  $n > 2$ , then we have the following results:*

*If  $b \notin \Delta_{\mu_1, \dots, \mu_n}$ , then  $h_b^{p,q} = 0$ .*

*If  $b = \mu^m$ , then*

$$h_b^{p,0} = h_b^{p,1} = \begin{cases} \binom{n}{p} & \text{if } m \geq (1, 1, \dots, 1), \\ A_p^m & \text{if } m \geq (0, 0, \dots, 0) \text{ and there exists an } i \text{ with } m_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$h_b^{p,q} = 0 \quad \text{if } 2 \leq q \leq n-2.$$

$$h_b^{p,n} = h_b^{p,n-1} = \begin{cases} \binom{n}{p} & \text{if } m \leq (-1, -1, \dots, -1), \\ B_p^m & \text{if } m \leq (0, 0, \dots, 0) \text{ and there exists an } i \text{ with } m_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We introduce a map

$$\begin{aligned} I_p &\longrightarrow \{K = (k_1, \dots, k_n) \mid k_i \in \{0, 1\}\} \\ J = (i_1, \dots, i_p) &\mapsto K_J = (k_1^J, \dots, k_n^J) \end{aligned}$$

where

$$k_l^J = \begin{cases} 1 & \text{if } l \in \{i_1, \dots, i_p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned}\omega &= \sum_J \sum_{\alpha} a_{\alpha}^J z^{\alpha} dz_J \\ &= \sum_{(i_1, \dots, i_p) \in I_p} \sum_{(\alpha_1, \dots, \alpha_n) \in N_0^n} a_{\alpha_1, \dots, \alpha_n}^{(i_1, \dots, i_p)} z_1^{\alpha_1} \dots z_n^{\alpha_n} dz_{i_1} \wedge \dots \wedge dz_{i_p} \in \Gamma(W, \Omega^p),\end{aligned}$$

then

$$P_0 \omega = \sum_J \sum_{\alpha} a_{\alpha}^J (b - \mu^{\alpha + K_J}) z^{\alpha} dz_J.$$

Now  $P_0 \omega = 0$  iff for all  $\alpha$  either  $a_{\alpha}^J = 0$  or  $b = \mu^{\alpha + K_J}$ . If  $m \geq (1, \dots, 1)$ , then we have for all  $J$  some  $\alpha$  such that  $\alpha + K_J = m$  and it follows that  $h_b^{p,0} = \binom{n}{p}$ . If  $m \geq (0, \dots, 0)$  and  $m - K_J$  contains at least one  $m_i - k_i^J < 0$ , then there is no  $\alpha$  for which  $m_i = k_i^J + \alpha_i$ ,  $\alpha_i \geq 0$ . This implies  $b \neq \mu^{\alpha + K_J}$  for all  $\alpha$  and the given  $J \in I_p$ . Hence  $h_b^{p,0} = A_p^m$ . The other cases are treated in the same way and  $h_b^{p,n} = h_b^{p,n-1}$  results from  $h_b^{p,0} = h_b^{p,1}$  by Serre-duality. q.e.d.

REMARK. We could avoid Serre-duality and calculate  $h_b^{p,n}$  in a similar way to the above  $h_b^{p,n}$  applying  $P_{n-1}$  to  $H^{n-1}(W, \Omega^p) \cong \Gamma(U_1 \cap \dots \cap U_n, \Omega^p)$  with  $U_i = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq 0\}$ .

### 3. The cohomology of flat line bundles on Hopf surfaces

The Hopf surfaces  $X$  are completely classified (see for example [3], [14] or [11] § 10). For our purposes it is convenient to distinguish the following cases corresponding to the contractions  $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  inducing these surfaces.

1) The generic case

$$f = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \text{ with no relations of the kind } \mu_1^{r_1} = \mu_2^{r_2}, r_1, r_2 \in \mathbb{N}.$$

2) The classical case

$$f = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

3) The resonant case

$$f = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \text{ with } \mu_1 = \mu_2^r, r \in \mathbb{N} \text{ and } r > 1.$$

4) The hyperresonant case

$$f = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \text{ with } \mu_1^{r_1} = \mu_2^{r_2}, r_1, r_2 \in \mathbb{N} \text{ and } 1 < r_1 \leq r_2.$$

5) The exceptional case

$$f(z_1, z_2) = (z_1', z_2') = (\mu_1 z_1 + z_2^r, \mu_2 z_2) \text{ with } \mu_1 = \mu_2^r, r \in \mathbb{N} \text{ and } r \geq 1.$$

We are interested in determining the cohomology groups of the flat line bundles in all these cases.



To this end we show first, that it is enough to calculate  $h^0(X, \Omega^p(L_b))$  for all  $b \in \mathcal{C}^*$  and  $p=0, 1, 2$ :

By Serre-duality we obtain:

$$h^2(X, \Omega^p(L_b)) = h^0(X, \Omega^{2-p}(L_b^*)) = h^0(X, \Omega^{2-p}(L_{1/b})) .$$

Because  $c_1(TX) = c_2(TX) = 0$  (cf. [14] p. 23),  $c_1(L) = 0$  for all  $L \in H^1(X, \mathcal{O}^*)$  and  $\Omega^p(L) = \mathcal{O}(\Lambda^p T^*X \otimes L)$  it follows with the Riemann-Roch formula

$$\chi(X, \Omega^p(L)) = ch(\Omega^p(L)) \cdot \mathcal{I}(TX)[X] = 0 .$$

From  $\chi(X, \Omega^p(L)) = h^0(X, \Omega^p(L)) - h^1(X, \Omega^p(L)) + h^2(X, \Omega^p(L))$  we get  $h^1(X, \Omega^p(L_b)) = h^0(X, \Omega^p(L_b)) + h^0(X, \Omega^{2-p}(L_{1/b}))$ . So it is enough to be able to compute  $h^0(X, \Omega^p(L))$  which is equal to  $\dim \ker P_0$  in the first part of the Douady-sequence:

$$0 \rightarrow H^0(X, \Omega^p(L)) \rightarrow H^0(W, \pi^*(\Omega^p(L))) \xrightarrow{P_0} H^0(W, \pi^*(\Omega^p(L))) .$$

These groups can be computed on the same lines as in § 2 for the generic, the classical, the resonant and the hyperresonant cases by counting admissible partitions of numbers. The exceptional case is somewhat different and we shall give a proof after having stated the results.

NOTATIONS: As in § 2 we need the following subgroups of  $\mathcal{C}^*$ :  $\Delta_{\mu_1, \mu_2}$  = group generated by  $\mu_1$  and  $\mu_2$ .

REMARK. The structure of these groups depends on the contraction  $f$ . For example if  $X$  is generic  $\Delta_{\mu_1, \mu_2}$  is a free commutative group. If  $X$  is hyperresonant then  $\Delta_{\mu_1, \mu_2} = \langle \mu_1, \mu_2 \mid \mu_1 \mu_2 = \mu_2 \mu_1, \mu_1^2 = \mu_2^2 \rangle$ .

**Proposition 2.** *Let  $X$  be a generic Hopf surface and  $L_b \in H^1(X, \mathcal{O}^*)$ . Then either  $b \notin \Delta_{\mu_1, \mu_2}$ , in which case  $h_b^{p,q} = 0$  for all  $p$  and  $q$  or  $b = \mu_1^{m_1} \mu_2^{m_2} \in \Delta_{\mu_1, \mu_2}$  and we have the following result:*

$p$	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	1 if $(m_1, m_2) \geq (0, 0)$ , 0 otherwise.	2 if $(m_1, m_2) \geq (1, 1)$ , 1 if $m_1=0, m_2>0$ or $m_1>0, m_2=0$ , 0 otherwise.	1 if $(m_1, m_2) \geq (1, 1)$ , 0 otherwise.
$h_b^{p,1}$	$h_b^{0,0} + h_b^{0,2}$	$h_b^{1,0} + h_b^{1,2}$	$h_b^{2,0} + h_b^{2,2}$

$h_b^{k,2}$	1 if $(m_1, m_2) \leq (-1, -1)$ , 0 otherwise.	2 if $(m_1, m_2) \leq (-1, -1)$ , 1 if $m_1=0, m_2<0$ or $m_1<0, m_2=0$ , 0 otherwise.	1 if $(m_1, m_2) \leq (0, 0)$ , 0 otherwise.
-------------	---	---	---

**Proposition 3.** *Let  $X$  be a classical Hopf surface and  $L_b \in H^1(X, \mathcal{O}^*)$ . Then either  $b \in \Delta_\mu$ , in which case  $h_b^{k,q} = 0$  for all  $p$  and  $q$  or  $b = \mu^m$  and we have the following result:*

$p$	$p=0$	$p=1$	$p=2$
$h_b^{k,0}$	$m+1$ if $m \geq 0$ , 0 if $m < 0$ .	$2m$ if $m \geq 1$ , 0 if $m < 1$ .	$m-1$ if $m \geq 2$ , 0 if $m < 2$ .
$h_b^{k,1}$	$h_b^{0,0} + h_b^{0,2}$	$h_b^{1,0} + h_b^{1,2}$	$h_b^{2,0} + h_b^{2,2}$
$h_b^{k,2}$	$-m-1$ if $m \leq -2$ , 0 if $m > -2$ .	$-2m$ if $m \leq -1$ , 0 if $m > -1$ .	$-m+1$ if $m \leq 0$ , 0 if $m > 0$ .

**Proposition 4.** *Let  $X$  be a resonant Hopf surface and  $L_b \in H^1(X, \mathcal{O}^*)$ . Then either  $b \in \Delta_{\mu_1, \mu_2}$ , in which case  $h_b^{k,q} = 0$  for all  $p$  and  $q$  or  $b = \mu_2^m \in \Delta_{\mu_1, \mu_2}$  and we have the following result:*

$p$	$p=0$	$p=1$	$p=2$
$h_b^{k,0}$	$n$ if $(n-1)r \leq m < n \cdot r$ , 0 if $m < 0$ .	$2n$ if $m = n \cdot r$ , $2n-1$ if $(n-1)r < m < n \cdot r$ , 0 if $m \leq 0$ .	$n$ if $n \cdot r < m \leq (n+1)r$ , 0 if $m \leq r$ .
$h_b^{k,1}$	$h_b^{0,0} + h_b^{0,2}$	$h_b^{1,0} + h_b^{1,2}$	$h_b^{2,0} + h_b^{2,2}$
$h_b^{k,2}$	$n$ if $-(n+1)r \leq m < -n \cdot r$ , 0 if $m \geq -r$ .	$2n$ if $m = -n \cdot r$ , $2n-1$ if $-n \cdot r < m < -(n-1)r$ , 0 if $m \geq 0$ .	$n$ if $-n \cdot r < m \leq -(n-1)r$ , 0 if $m > 0$ .

**Proposition 5.** *Let  $X$  be a hyperresonant Hopf surface and  $L_b \in H^1(X, \mathcal{O}^*)$ . Then either  $b \in \Delta_{\mu_1, \mu_2}$ , in which case  $h_b^{k,q} = 0$  for all  $p$  and  $q$  or  $b = \mu_1^{m_1} \mu_2^{m_2} \in \Delta_{\mu_1, \mu_2}$ . In this case  $b = \mu_1^{m_1} \mu_2^{m_2}$  always has a representation such that  $0 \leq m_1 < r_1$  or  $0 < m_1 \leq r_1$  and we have the following result:*

$p$	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	$0 \leq m_1 < r_1$ $n$ if $(n-1)r_2 \leq m_2 < n \cdot r_2$ , $0$ if $m_2 < 0$ .	$m_1=0$ : $2n$ if $m_2 = n \cdot r_2$ , $2n+1$ if $n \cdot r_2 < m_2 < (n+1)r_2$ , $0$ if $m_2 \leq 0$ . $0 < m_1 < r_1$ : $2n+1$ if $m_2 = n \cdot r_2$ , $2n$ if $(n-1)r_2 < m_2 < n \cdot r_2$ , $0$ if $m_2 < 0$ .	$0 < m_1 \leq r_1$ $n$ if $(n-1)r_2 < m_2 \leq n \cdot r_2$ , $0$ if $m_2 \leq 0$ .
$h_b^{p,1}$	$h_b^{0,0} + h_b^{0,2}$	$h_b^{1,0} + h_b^{1,2}$	$h_b^{2,0} + h_b^{2,2}$
$h_b^{p,2}$	$0 \leq m_1 < r_1$ $n$ if $-(n+1)r_2 \leq m_2 < -n \cdot r_2$ , $0$ if $m_2 \geq -r_2$ .	$m_1=0$ : $2n$ if $m_2 = -n \cdot r_2$ , $2n+1$ if $-(n+1)r_2 < m_2 < -n \cdot r_2$ , $0$ if $m_2 \geq 0$ . $0 < m_1 < r_1$ : $2n+1$ if $m_2 = -(n+1)r_2$ , $2n$ if $-(n+1)r_2 < m_2 < -n \cdot r_2$ , $0$ if $m_2 > -r_2$ .	$0 < m_1 \leq r_1$ $n$ if $-(n+1)r_2 < m_2 \leq -n \cdot r_2$ , $0$ if $m_2 > -r_2$ .

**Proposition 6.** Let  $X$  be an exceptional Hopf surface and  $L_b \in H^1(X, \mathcal{O}^*)$ . Then either  $b \in \Delta_{\mu_1, \mu_2}$ , in which case  $h_b^{p,q} = 0$  for all  $p$  and  $q$  or  $b = \mu_2^m \in \Delta_{\mu_1, \mu_2}$  and we have the following result:

$p$	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	$1$ if $m \geq 0$ , $0$ otherwise.	$2$ if $m > r$ , $1$ if $1 \leq m \leq r$ , $0$ otherwise.	$1$ if $m > r$ , $0$ otherwise.
$h_b^{p,1}$	$h_b^{0,0} + h_b^{0,2}$	$h_b^{1,0} + h_b^{1,2}$	$h_b^{2,0} + h_b^{2,2}$
$h_b^{p,2}$	$1$ if $m < -r$ , $0$ otherwise.	$2$ if $m < -r$ , $1$ if $-r \leq m \leq -1$ , $0$ otherwise.	$1$ if $m \leq 0$ , $0$ otherwise.

Proof of Proposition 6.

A. *Calculation of  $h^0(X, \mathcal{O}(L_b))$ .*

Recall the following definitions:  $f: (z_1, z_2) \mapsto (z'_1, z'_2)$  with  $z'_1 = \mu_1 z_1 + z_2^r$ ,  $z'_2 = \mu_2 z_2$ ;  $\mu_1 = \mu_2^r$ ,  $r \in \mathbb{N}$  and

$$\ker P_0 \cong \{\omega \in \Gamma(W, \mathcal{O}) = \Gamma(\mathcal{C}^2, \mathcal{O}) \mid b\omega - f^*\omega = 0\}.$$

We get for  $\omega = \sum_{k,l} a_{kl} z_1^k z_2^l$

$$f^*\omega = \sum_{k,l} a_{kl} z_1'^k z_2'^l = \sum_{k,l} \sum_t a_{kl} \binom{k}{t} \mu_1^t \mu_2^l z_1^t z_2^{r(k-t)+l} =: \sum \tilde{a}_{\alpha,\beta} z_1^\alpha z_2^\beta.$$

We calculate  $\tilde{a}_{\alpha,\beta}$ : comparison of the exponents of the variables  $z_1, z_2$  gives the condition:

$$(\alpha, \beta) = (t, r(k-t)+l).$$

It follows that  $\alpha = t$ ,  $k \geq \alpha$  and  $l = \beta - r(k-t) = \beta - jr$  where  $j = k - \alpha$  and  $j$  can take the values  $0, \dots, [\beta/r]$  ( $[x]$  being the greatest integer less than  $x$ ). Hence:

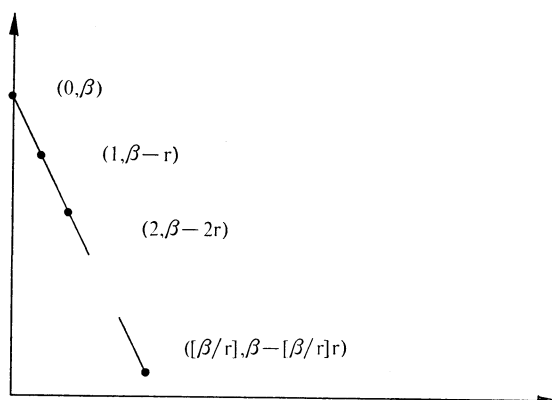
$$P_0\omega = \sum_{\alpha,\beta \geq 0} (ba_{\alpha\beta} - \tilde{a}_{\alpha\beta}) z_1^\alpha z_2^\beta = \sum_{\alpha,\beta \geq 0} (ba_{\alpha\beta} - \sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr} \binom{\alpha+j}{\alpha} \mu_2^{\beta+r(\alpha-j)}) z_1^\alpha z_2^\beta.$$

Thus we get the following conditions for the coefficients  $a_{\alpha\beta}$  using the relation  $P_0\omega = 0$ :

$$(4) \quad a_{\alpha\beta}(b - \mu_2^{\beta+ra}) = \sum_{j=1}^{[\beta/r]} a_{\alpha+j,\beta-jr} \binom{\alpha+j}{\alpha} \mu_2^{\beta+r(\alpha-j)} \quad \text{for all } (\alpha, \beta) \geq (0, 0).$$

Pictorially the conditions (4) impose a constraint on those coefficients  $a_{\alpha\beta}$  the indices  $(\alpha, \beta)$  of which lie on a "ladder"

$$Ld(\beta) = \{(0, \beta); (1, \beta-r); (2, \beta-2r); \dots; ([\beta/r], \beta - [\beta/r]r)\}:$$



These conditions can be written as a system of linear equations:

$$(5) \quad C^\beta \circ A^\beta = 0$$

where

$$C^\beta := \begin{pmatrix} b - \mu_2^\beta & -\binom{0+1}{0} \mu_2^{\beta-r} & -\binom{0+2}{0} \mu_2^{\beta-2r} & \dots & -\binom{0+[\beta/r]}{0} \mu_2^{\beta-[\beta/r]r} \\ & b - \mu_2^\beta & -\binom{1+1}{1} \mu_2^{\beta-r} & \dots & -\binom{1+([\beta/r]-1)}{1} \mu_2^{\beta-([\beta/r]-1)r} \\ & & b - \mu_2^\beta & \dots & -\binom{2+([\beta/r]-2)}{2} \mu_2^{\beta-([\beta/r]-2)r} \\ & & & \ddots & \vdots \\ & & & & b - \mu_2^\beta \end{pmatrix}$$

and

$$A^\beta := \begin{pmatrix} a_{0,\beta} \\ a_{1,\beta-r} \\ \vdots \\ a_{[\beta/r],\beta-[\beta/r]r} \end{pmatrix}$$

The system (5) has a non trivial solution  $A^\beta$  iff  $b - \mu_2^\beta = 0$ . In this case the space of solutions has dimension one.

Hence:

$$\dim \ker P_0 = \begin{cases} 1 & \text{if } b \in \{1, \mu_2, \mu_2^2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

B. *Calculation of  $h^0(X, \Omega_X^1(L_b))$ .*

$$\ker P_0 = \{\omega(z_1, z_2) = \sum a_{k1}^1 z_1^k z_2^l dz_1 + \sum a_{k1}^2 z_1^k z_2^l dz_2 \mid b\omega - f^*\omega = 0\}$$

$$dz_1' = \mu_1 dz_1 + r z_2^{r-1} dz_2, \quad dz_2' = \mu_2 dz_2.$$

If  $P_0\omega=0$ , then the following equations for the coefficients must hold:

$$(6) \quad b a_{\alpha\beta}^1 = \sum_{j=0}^{[\beta/r]} a_{\alpha+j, \beta-jr}^1 \binom{\alpha+j}{\alpha} \mu_2^{\beta+r(\alpha+1-j)}$$

and

$$(7) \quad b a_{\alpha\beta}^2 = \sum_{j=0}^{[\beta/r]} a_{\alpha+j, \beta-jr}^2 \binom{\alpha+j}{\alpha} \mu_2^{\beta+1+r(\alpha-j)} \\ + r \sum_{j=0}^{[\beta/r]} a_{\alpha+j, \beta+1-(j+1)r}^1 \binom{\alpha+j}{\alpha} \mu_2^{\beta+1+r(\alpha-j-1)}.$$

We proceed as in section A: to satisfy the equations (6), it is necessary that either  $a_{\alpha\beta}^1=0$  for all  $(\alpha, \beta)$  or else there are integers  $s \in [0, r)$  and  $t \in \mathbb{N}_0$  such that  $b = \mu_2^{s+r(t+1)}$ ,  $a_{\alpha\beta}^1=0$ , if  $(\alpha, \beta) \neq (0, s+rt)$  and  $a_{0, s+rt}^1 \in \mathbb{C}$ .

Then the equations (7) become:

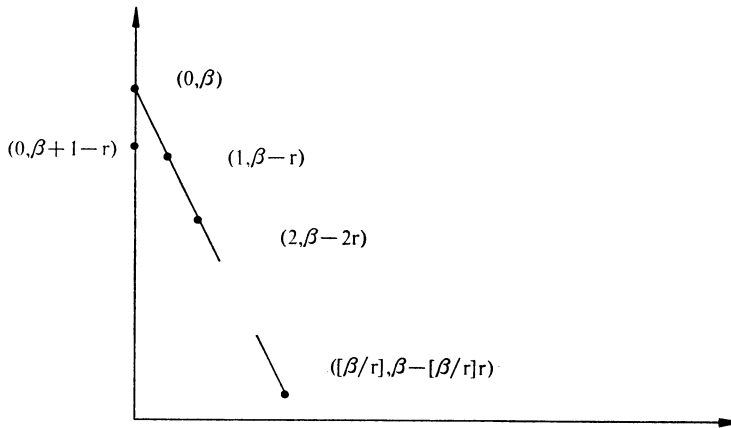
$$(8) \quad ba_{\alpha\beta}^2 = \sum_{j=0}^{\lfloor \beta/r \rfloor} a_{\alpha+j, \beta-jr}^2 \binom{\alpha+j}{\alpha} \mu_2^{\beta+1+r(\alpha-j)} \quad \text{if } \alpha > 0$$

and

$$(9) \quad ba_{0\beta}^2 = \sum_{j=0}^{\lfloor \beta/r \rfloor} a_{j, \beta-jr}^2 \mu_2^{\beta+1-rj} + ra_{0, \beta+1-r}^1 \mu_2^{\beta+1-r}.$$

Again the conditions (8), (9) impose a constraint on those coefficients  $a_{\alpha\beta}$  the indices of which lie on the (somewhat different) ladder

$$Ld_2(\beta) = \{(0, \beta+1-r); (0, \beta); (1, \beta-r); \dots; (\lfloor \beta/r \rfloor, \beta - \lfloor \beta/r \rfloor r)\}:$$



We express these conditions by means of the following inhomogeneous linear systems:

$$(10) \quad C_2^\beta \circ A_2^\beta = B^\beta$$

where

$$C_2^\beta := \begin{pmatrix} b - \mu_2^{\beta+1} & -\binom{0+1}{0} \mu_2^{\beta+1-r} & -\binom{0+2}{0} \mu_2^{\beta+1-2r} & \dots & -\binom{0+\lfloor \beta/r \rfloor}{0} \mu_2^{\beta+1-\lfloor \beta/r \rfloor r} \\ & b - \mu_2^{\beta+1} & -\binom{1+1}{1} \mu_2^{\beta+1-r} & \dots & \\ & & b - \mu_2^{\beta+1} & \dots & \\ & & & \ddots & \\ & & & & b - \mu_2^{\beta+1} \end{pmatrix},$$

$$A_2^\beta := \begin{pmatrix} a_{0,\beta}^2 \\ a_{1,\beta-r}^2 \\ \vdots \\ a_{\lfloor \beta/r \rfloor, \beta - \lfloor \beta/r \rfloor r}^2 \end{pmatrix}$$

and

$$B^\beta := \begin{pmatrix} r\mu_2^{\beta+1-r} & a_{0,\beta+1-r}^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Of course  $B^\beta=0$ , if  $\beta < r-1$ . It follows: if  $B^\beta=0$ , then the space of solutions of (10) has dimension 1 iff  $b-\mu_2^{\beta+1}=0$ .

If  $B^\beta \neq 0$ , then  $a_{0,\beta+1-r}^1 \neq 0$  and (6) implies  $b=\mu_2^{\beta+1}$ . If  $\beta > r$  then the space of solutions of (10) has dimension 1 and  $a_{0,\beta+1-r}^1$  can be chosen arbitrarily. If  $\beta=r$  then (6) and (10) imply  $B^\beta=0$ .

Collecting the results we obtain:

$$\dim \ker P_0 = \begin{cases} 2 & \text{if } b = \mu_2^m \text{ and } m > r, \\ 1 & \text{if } b = \mu_2^m \text{ and } 1 \leq m \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

### C. Calculation of $h^0(X, \Omega_X^2(L_b))$ .

We have:

$$\ker P_0 = \{\omega(z_1, z_2) = \sum_{k,l} a_{k,l} z_1^k z_2^l dz_1 \wedge dz_2 \mid b\omega - f^*\omega = 0\}$$

and

$$dz_1' \wedge dz_2' = \mu_1 \mu_2 dz_1 \wedge dz_2.$$

This implies:

$$f^*\omega = \sum_{\alpha,\beta} \sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr} \binom{\alpha+j}{\alpha} \mu_2^{\beta+1+r(\alpha-j+1)} z_1^\alpha z_2^\beta dz_1 \wedge dz_2,$$

which gives conditions for the coefficients:

$$ba_{\alpha,\beta} = \sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr} \binom{\alpha+j}{\alpha} \mu_2^{\beta+1+r(\alpha-j+1)}.$$

In the same manner as above we get the following result:

- 1)  $a_{\alpha,\beta}=0$  for all  $(\alpha, \beta)$  with  $\alpha > 0$
- 2)  $a_{0,\beta}=0$  if  $b \neq \mu_2^{\beta+1+r}$   
 $a_{0,\beta} \in \mathbb{C}$  if  $b = \mu_2^{\beta+1+r}$ .

Hence:

$$\dim \ker P_0 = \begin{cases} 1 & \text{if } b = \mu_2^m, m \geq r+1, \\ 0 & \text{otherwise.} \end{cases}$$

qed.

#### 4. Hodge numbers and flatness of line bundles

**Theorem 3.** *The Hodge numbers of an arbitrary Hopf manifold of dimension  $n$  are :*

$$\begin{aligned} h^{0,0} &= h^{0,1} = h^{n,n} = h^{n,n-1} = 1 \\ h^{p,q} &= 0 \quad \text{in all other cases.} \end{aligned}$$

Proof. Obviously  $h^{0,0}=h^{n,n}=1$ . For a Hopf manifold generated by a diagonal contraction  $f: (z_1, \dots, z_n) \mapsto (\mu_1 z_1, \dots, \mu_n z_n)$  the rest of the assertion follows by arguing as in the proof of lemma 9.4 in [2] p. 216 and as in § 2 and § 3, if we choose  $L_b = \mathcal{O}$ , i.e.  $b=1$ . For a general contraction  $f$  we can assume without loss of generality that  $f'(0) = d_\mu: (z_1, \dots, z_n) \mapsto (\mu_1 z_1, \dots, \mu_n z_n)$  and  $f \in G_\mu$  (Poincaré-Dulac form), see section 1. The group  $G_\mu$  operates on itself by conjugation. A slice through the orbit of  $f$  in  $G_\mu$  represents the Kuranishi space  $K_{X_f}$  of  $X_f$ . The orbit of  $f$  in  $G_\mu$  has  $d_\mu$  in its closure (cf. [8] p. 242, 248). From Grauert's semicontinuity theorem, (cf. [7] p. 210), follows:  $h^{p,q}(X_{d_\mu}) \geq h^{p,q}(X_f)$ . This implies

$$h^{n,n-1}(X_f) = h^{0,1}(X_f) \leq h^{0,1}(X_{d_\mu}) \leq 1 \quad \text{and} \quad h^{p,q}(X_f) = 0$$

if  $(p, q) \neq (0, 1); (0, 0); (n, n); (n, n-1)$ .

By the Fröhlicher spectral sequence we get  $b_r(X_f) \leq \sum_{p+q=r} h^{p,q}(X_f)$  with  $b_r(X_f) = \text{rank } H^r(X_f, \mathbb{Z})$ , see [5]. However

$$1 = b_1(X_f) \leq h^{0,1}(X_f) + h^{1,0}(X_f) = h^{0,1}(X_f) \leq 1. \quad \text{qed.}$$

Armed with this result we can tackle our next theorem. For Hopf surfaces this statement was proven by Kodaira (cf. [11] p. 699). We follow his arguments closely.

**Theorem 4.** *All line bundles over a Hopf manifold  $X$  are flat.*

Proof. a) Since  $X$  is homeomorphic to  $S^{2n-1} \times S^1$  we have the following Betti numbers  $b_1(X) = b_{2n-1}(X) = 1$  and  $b_2(X) = \dots = b_{2n-2}(X) = 0$ , (cf. [6] p. 144).

b) The diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & \text{exp} & & \\ 0 & \rightarrow & \mathbf{Z} & \rightarrow & \mathbf{C} & \rightarrow & \mathbf{C}^* \rightarrow 0 \\ & & \parallel & & \downarrow i & & \downarrow j \\ 0 & \rightarrow & \mathbf{Z} & \rightarrow & \mathcal{O} & \xrightarrow{\text{exp}} & \mathcal{O}^* \rightarrow 0 \\ & & & & \downarrow d & & \\ & & & & \Omega & & \end{array}$$



is exact with  $d$  taken to be the de Rahm derivative.

$X$  is compact and connected:  $H^0(X, \mathbf{C}) \cong H^0(X, \mathcal{O}) \cong \mathbf{C}$ ,  $H^0(X, \mathbf{C}^*) \cong H^0(X, \mathcal{O}^*) \cong \mathbf{C}^*$ . This implies:

$$0 \rightarrow H^0(X, \mathbf{Z}) \rightarrow H^0(X, \mathbf{C}) \rightarrow H^0(X, \mathbf{C}^*) \xrightarrow{0} H^1(X, \mathbf{Z}) \rightarrow \dots$$

Since  $\dim H^2(X, \mathcal{O}) = h^{0,2} = 0$  and  $\dim H^2(X, \mathbf{C}) = b_2 = 0$  we get:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(X, \mathbf{Z}) & \rightarrow & H^1(X, \mathbf{C}) & \rightarrow & H^1(X, \mathbf{C}^*) \rightarrow H^2(X, \mathbf{Z}) \rightarrow 0 \\ & & \parallel & & \downarrow i_* & & \downarrow j_* & & \parallel \\ 0 & \rightarrow & H^1(X, \mathbf{Z}) & \rightarrow & H^1(X, \mathcal{O}) & \rightarrow & H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbf{Z}) \rightarrow 0 \end{array}$$

c) By the five-lemma it is enough to show that  $i_*$  is an isomorphism. First we will show that  $i_*$  is injective. Let  $d\mathcal{O}$  be the sheaf of germs of 1-forms which are locally the derivative of a holomorphic function  $\omega = df$ . We take the exact sequence

$$0 \rightarrow \mathbf{C} \xrightarrow{i} \mathcal{O} \xrightarrow{d} d\mathcal{O} \rightarrow 0$$

to get the long exact sequence

$$0 \rightarrow H^0(X, \mathbf{C}) \rightarrow H^0(X, \mathcal{O}) \xrightarrow{0} H^0(X, d\mathcal{O}) \rightarrow H^1(X, \mathbf{C}) \xrightarrow{i_*} H^1(X, \mathcal{O}) \rightarrow \dots$$

To prove the injectivity of  $i_*$  it suffices to show that  $H^0(X, d\mathcal{O}) = 0$ . Assume the contrary. Then there is a  $\varphi \in H^0(X, d\mathcal{O})$  such that  $\varphi \neq 0$  and  $d\varphi = 0$ . Let  $\mathcal{D}$  be the sheaf of germs of differentiable complex valued functions on  $X$ . There is a natural inclusion mapping  $\mathcal{O} \hookrightarrow \mathcal{D}$  and by the de Rham theorem we get:  $H^1(X, \mathbf{C}) \cong H^0(X, d\mathcal{D})/dH^0(X, \mathcal{D})$ . This establishes a map

$$\begin{array}{ccc} [\ ]: H^0(X, d\mathcal{O}) & \rightarrow & H^1(X, \mathbf{C}) \\ \varphi & \mapsto & [\varphi]. \end{array}$$

Now suppose that there exist  $a, b \in \mathbf{C}$ , such that  $a[\varphi] + b[\bar{\varphi}] = 0$  in  $H^1(X, \mathbf{C})$ . The above remarks imply that there exists a  $\mu \in H^0(X, \mathcal{D})$  with  $a\varphi + b\bar{\varphi} = d\mu$ . Locally we have:

$$\varphi = d\lambda, \lambda \in \mathcal{O}(U), U \subset X \text{ open, and } \mu = a\lambda + b\bar{\lambda} + \text{some constant.}$$

It follows that  $\mu$  has the mean-value-property for sufficiently small balls in  $X$ . Hence  $\mu$  is constant, because  $X$  is compact. But then we have  $a\varphi + b\bar{\varphi} = 0$  in  $H^0(X, d\mathcal{D})$  which implies  $a = b = 0$ , because  $\varphi \in H^0(X, d\mathcal{O})$ , i.e.  $H^1(X, \mathbf{C}) \geq 2$ . However, this last inequality contradicts  $H^1(X, \mathbf{C}) = 1$ .

The surjectivity follows now from  $\dim H^1(X, \mathbf{C}) = 1$  and  $\dim H^1(X, \mathcal{O}) = h^{0,1}(X) = 1$ . qed.

REMARK 1. Theorem 4 implies that we know the cohomology of *all* line bundles in the cases calculated in § 2 and § 3.

REMARK 2. Let  $TX$  be the tangent bundle of  $X$  and  $\Theta = \mathcal{O}(TX)$ . The spaces  $H^q(X, \Theta)$  are of interest in deformation theory. From our calculations we get their dimensions by the simple observation that

$$\begin{aligned} H^q(X, \Theta) &= H^q(X, \mathcal{O}(TX)) \cong H^{*-q}(X, \Omega^*(T^*X))^* \cong \\ &H^{*-q}(X, K \otimes \Omega^1)^* = H^{*-q}(X, \Omega^1(K))^* \end{aligned}$$

where  $K$  is the canonical line bundle of  $X$ .

---

### References

- [1] V.I. Arnol'd: chapitres supplémentaires de la théorie des équations différentielles ordinaires, éditions mir 1980.
- [2] A. Borel: in F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Berlin 1978.
- [3] K. Dabrowski: *Moduli Spaces for Hopf Surfaces*, Math. Ann. **259** (1982) 201–225.
- [4] A. Douady: Séminaire Cartan **13** (1960/61) Exposé 3.
- [5] A. Fröhlicher: *Relations between the cohomology groups of Dolbeault and topological invariants*, Proc. Nat. Acad. Sci. USA **41** (1955), 641–644.
- [6] H. Grauert-K. Fritzsche: *Several Complex Variables*, Berlin 1976.
- [7] H. Grauert-R. Remmert: *Coherent Analytic Sheaves*, Berlin 1984.
- [8] A. Haefliger: *Deformations of transversely holomorphic flows on spheres and deformations of Hopf manifolds*, Compositio Mathematica **55** (1985), 241–251.
- [9] H. Hopf: *Zur Topologie der komplexen Mannigfaltigkeiten*, Courant Anniversary Volume, 167–185, New York 1948.
- [10] M. Ise: *On the geometry of Hopf manifolds*, Osaka Math. J. **12** (1960), 387–402.
- [11] K. Kodaira: *On the structure of compact complex analytic surfaces*, I, II, Am. J. Math. **86** (1964) 751–798, **88** (1966), 682–721.
- [12] D. Mall: *Holomorphic vector bundles and foliations on Hopf manifolds*, Ph.D. thesis, Geneva 1988.
- [13] L. Reich: *Das Typenproblem bei formal-biholomorphen Abbildungen mit anziehendem Fixpunkt*, Math. Ann. **179** (1969), 227–250.
- [14] J. Wehler: *Versal deformation of Hopf surfaces*, J. reine angew. Math. **328** (1981), 22–32.

Dept. Math.  
ETH-Zentrum  
CH-8092 Zürich  
Switzerland

