Mall, D.
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THE COHOMOLOGY OF LINE BUNDLES ON HOPF MANIFOLDS

DANIEL MALL

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0. Introduction

The purpose of the present paper is to give an elementary method for the computation of the cohomoloyg groups $H^q(X, \Omega^p(L))$, $(0 \leq q \leq n)$ of an n-dimensional Hopf manifold X, $(n \geq 2)$, where $\Omega^p(L)$ denotes the sheaf of germs of holomorphic p-forms with values in a holomorphic line bundle L on X.

Ise [10] has given a solution of this problem for homogeneous Hopf manifolds. He makes strong use of the fact that any homogeneous Hopf manifold of dimension n is a fibre bundle over the complex projective space P^{n-1}. His main tools are knowledge of the cohomology groups $H^q(P^{n-1}, \Omega^p(L))$ and the Leray spectral sequence.

Our approach to Ise's problem is a generalization of a method used by Douady [4] to study the deformation of Hopf manifolds.

We shall now describe the contents of this paper. Section 1 presents our method for the computation of the cohomology groups for flat line bundles and makes evident the dichotomy between Hopf manifolds of dimension $n>2$ and Hopf surfaces ($n=2$). Section 2 treats the case of Hopf manifolds of dimension $n>2$ and section 3 the case of Hopf surfaces.

Generalizing a result of Kodaira [11] for Hopf surfaces we show in section 4 that all line bundles on an arbitrary Hopf manifold are flat. To this end we calculate first the Hodge numbers of these manifolds. The flatness of line bundles shows that the method introduced in section 1 applies to the general case.

Applications to holomorphic vector bundles and foliations on Hopf manifolds will be published elsewhere.

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1. Hopf manifolds, flat line bundles and the main lemma

Let $W := \boldsymbol{C}^n - \{0\}$ and $n \geq 2$. We recall some facts about Hopf manifolds, for details we refer to [3], [8], [11] § 10 and [14]. By a contraction $f: (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}^n, 0)$ we understand an automorphism of $\boldsymbol{C}^n$ fixing 0 with the property that the eigenvalues $\mu_1, \cdots, \mu_n$ of $f'(0)$, the differential of f at 0, are inside the unit circle. We call a contraction of the form $f: (z_1, \cdots, z_n) \to (\mu_1 z_1, \cdots, \mu_n z_n)$ diagonal and always assume that $0 < |\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_n| < 1$.

The subgroup of automorphisms of $\boldsymbol{C}^n$ generated by a contraction f operates freely and properly discontinuously on W. The quotient X (or X_f if we want to emphasize the contraction f) is a compact, complex manifold of dimension n called a Hopf manifold. The canonical projection is denoted by $\pi: W \to X$. Given two contractions f_1, f_2 of $\boldsymbol{C}^n$ the Hopf manifolds X_{f_1} and X_{f_2} are isomorphic iff there is a $g \in \mathrm{Aut}(\boldsymbol{C}^n)$ fixing 0 such that $f_1 = g \circ f_2 \circ g^{-1}$ on $\boldsymbol{C}^n$. Thus the classification problem of Hopf manifolds up to isomorphisms corresponds to the normal form problem for conjugation classes of contractions in $\mathrm{Aut}(\boldsymbol{C}^n, 0)$, the group of automorphisms of $\boldsymbol{C}^n$ fixing 0. For dimension $n=2$ the solution to this problem is well known (cf. [11] p. 695). Reich (cf. [33] p. 248) gives a solution for dimension $n=3$. For dimension $n>3$ a complete solution is not to be expected (cf. [13] p. 247). However, there is a weaker result which is enough for our purposes (for details see [8] p. 249). Let G_μ be the group of invertible polynomial maps $f: \boldsymbol{C}^n \to \boldsymbol{C}^n$ which commute with the diagonal linear contraction $d_\mu: z = (z_1, \cdots, z_n) \mapsto \mu z = (\mu_1 z_1, \cdots, \mu_n z_n)$. The following statement is a consequence from the Poincaré-Dulac theorem (cf. [1] p. 187). Given a Hopf manifold X generated by a contraction f with $f'(0) = d_\mu$ then there is a contraction $g \in G_\mu$ such that X is isomorphic to X_g.

Assume $L \in H^1(X, \mathcal{O}^*)$ is a flat line bundle. Then it is the quotient of $W \times \boldsymbol{C}$, the trivial line bundle on the universal covering of X, by the operation of a representation of the fundamental group $\pi_1(X) \cong \boldsymbol{Z}$ of X

$$\begin{array}{rcl} \rho_L\colon \pi_1(X) & \to & \boldsymbol{C}^* := GL(1, \boldsymbol{C}) \\ \gamma & \mapsto & \rho_L(\gamma) \end{array}$$

in the following way:

$$\begin{array}{rcl} W \times \boldsymbol{C} & \to & W \times \boldsymbol{C} \\ (z, v) & \mapsto & (f(z), \rho_L(1) v)\,. \end{array}$$

NOTATION: We write L_b for the bundle induced by the representation ρ_L with $b := \rho_L(1)$. For the locally free sheaf $\mathcal{O}(L_b)$ of germs of holomorphic sections we write sometimes by abuse of notation L_b.

We want to calculate $h^q(X, \Omega_X^p(L_b)) := \dim H^q(X, \Omega_X^p(L_b))$, where $\Omega_X^p(L_b) := \Omega_X^p \otimes \mathcal{O}(L_b)$ and Ω_X^p denotes the sheaf of germs of holomorphic p-forms on X as usual.

Take a covering $\mathcal{A} = \{U_i\}$ of X such that the U_i are open and contractible Stein subsets of X and $\tilde{U}_i := \pi^{-1}(U_i)$ is a disjoint union of open Stein subsets

$\{U'_{ij}\}$ of W, each of them isomorphic to U_i:

$$\tilde{U}_i := \bigcup_{m\in\mathbb{Z}} f^m(U'_{i0});\ \tilde{\mathcal{A}} := \{\tilde{U}_i\}\,.$$

If $\varphi\in\Gamma(U_i, \Omega_X^p\otimes L_b)$ then $\tilde{\varphi}:=\pi^*(\varphi)$ lies in $\Gamma(\tilde{U}_i, \pi^*(\Omega_X^p\otimes L_b))\cong\Gamma(\tilde{U}_i, \Omega_W^p)$, because $\pi^*(\Omega_X^p\otimes L_b)\cong\pi^*(\Omega_X^p)\otimes\pi^*(L_b)$, $\pi^*(L_b)\cong\mathcal{O}_W$ and $\pi^*(\Omega_X^p)\cong\Omega_W^p$. Thus we can build a sequence of Čech-complexes:

$$(1)\qquad 0\to C^\cdot(\mathcal{A}, \Omega_X^p(L_b))\xrightarrow{\pi^*} C^\cdot(\mathcal{A}, \Omega_W^p)\xrightarrow{bId-f^*} C^\cdot(\tilde{\mathcal{A}}, \Omega_X^p)\to 0\,.$$

Lemma 1. *The sequence* (1) *is exact.*

Proof. The only problem is to show the surjectivity of $bId-f^*$. Let without loss of generality γ be an element of $\Gamma(\tilde{U}_{i_0}\cap\cdots\cap\tilde{U}_{i_q})$. Our assumption on $\mathcal{A}$ implies that $\tilde{U}_{i_0}\cap\cdots\cap\tilde{U}_{i_q}=\bigcup f^m(U'_{i_00}\cap\cdots\cap U'_{i_q0})$. We put $\gamma=\gamma_1-\gamma_2$ with $\gamma_1=0$ on all $f^m(U'_{i_00}\cap\cdots\cap U'_{i_q0})$ for $m<0$ and $\gamma_2=0$ on $f^m(U'_{i_00}\cap\cdots\cap U'_{i_q0})$ with $m\geq 0$. The expression

$$\beta=\sum_{m\geq 0} b^{-m-1}(f^*)^m(\gamma_1)+\sum_{m<0} b^{-m-1}(f^*)^m(\gamma_2)$$

is well defined since both sums are locally finite. A straightforward calculation shows that $(bId-f^*)\beta=\gamma$. qed.

From (1) we derive a generalized long exact Douady sequence:

$$(2)\quad 0\to H^0(X, \Omega_X^p(L_b))\to H^0(W, \Omega_W^p)\xrightarrow{P_0} H^0(W, \Omega_W^p)\to H^1(X, \Omega_X^p(L_b))$$
$$\to H^1(W, \Omega_W^p)\xrightarrow{P_1} H^1(W,\Omega_W^p)\to H^2(X, \Omega_X^p(L_b))\to\cdots$$

where $P=bId-f^*$.

Because $H^q(W, \Omega^p)\cong H^q(W, \mathcal{O}^{\binom{n}{p}})$ and $H^q(W, \mathcal{O})\neq 0$ only for $q=0$, $n-1$ (cf. [8] p. 246), (2) decomposes for $n>2$ into the following exact sequences:

$$0\to H^0(X, \Omega_X^p(L_b))\to H^0(W, \Omega_W^p)\xrightarrow{P_0} H^0(W, \Omega_W^p)\to H^1(X, \Omega_X^p(L_b))\to 0$$

$$(3)\qquad 0\to H^{n-1}(X, \Omega_X^p(L_b))\to H^{n-1}(W, \Omega_W^p)\xrightarrow{P_{n-1}} H^{n-1}(W, \Omega_W^p)$$
$$\to H^n(X, \Omega_X^p(L_b))\to 0$$

With $h_b^{p,q}:=\dim H^q(X, \Omega_X^p(L_b))$ follows the

Main Lemma. *Let X be a Hopf manifold of dimension $n>2$ and let $L_b\in H^1(X, \mathcal{O}^*)$ be a flat line bundle on X. Then*

$$h_b^{p,0}=\dim\ker P_0\,,$$
$$h_b^{p,1}=\dim\operatorname{coker} P_0\,,$$

$$h_q^{p,q}=0 \qquad \text{for } 1<i<n-1\,,$$
$$h_b^{p,n-1}=\dim\ker P_{n-1}\,,$$
$$h_b^{p,n}=\dim\operatorname{coker} P_{n-1}\,.$$

2. The cohomology of flat line bundles for diagonal Hopf manifolds of dimension $n>2$

With the method described in § 1 one can calculate the cohomology of flat line bundles over Hopf manifolds. We give formulas for the classical and the generic cases. Explicit formulas for the resonant cases would involve the unsolved classification problem of Hopf manifolds.

A. The classical case

Assume that X is a Hopf manifold generated by a contraction f of the type $f\colon (z_1, \cdots, z_n)\to(\mu z_1, \cdots, \mu z_n)$ with $0<|\mu|<1$. This is the classical case introduced by Hopf [9] and studied by Ise [10].

We need some notations:

$\boldsymbol{N}_0:=\boldsymbol{N}\cup\{0\}$ where $\boldsymbol{N}$ is the set of positive integers.
$\Delta_\mu:=\langle\mu\rangle=$cyclic subgroup of $\boldsymbol{C}^*$ generated by the eigenvalue μ of f.
For $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{N}_0^n$ we introduce $|\alpha|:=\alpha_1+\alpha_2+\cdots+\alpha_n$.
$\chi(m, n):=\#\{\alpha\in\boldsymbol{N}_0^n\,|\,|\alpha|=m\}$ where $\# M$ denotes the cardinality of the set M.
$L_b\in H^1(X, \mathcal{O}^*)$, $b\in\boldsymbol{C}^*$ (cf. § 1).
$J\in I_p:=\{(i_1, \cdots, i_p)\in\boldsymbol{N}^p\,|\,1\le i_1<i_2<\cdots<i_p\le n\}$.

We now give a short proof of the main result of [10].

Theorem 1. [10] *Let X_f be a Hopf manifold of dimension $n>2$ with $f\colon (z_1, \cdots, z_n)\to(\mu z_1, \cdots, \mu z_n)$, $0<|\mu|<1$, and $L_b\in H^1(X, \mathcal{O}^*)$ a holomorphic flat line bundle. Then either $b\notin\Delta_\mu$, in which case $h_b^{p,q}=0$ for all p and q, or $b=\mu^m$, $m\in\boldsymbol{Z}$, and then we have the following result*:

$$h_b^{p,0}=h_b^{p,1}=\begin{cases}\dbinom{m-p+n-1}{n-1}\dbinom{n}{p} & \text{if } m\ge p\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

$$h_b^{p,q}=0 \qquad \text{if } 2\le q\le n-2\,.$$

$$h_b^{p,n-1}=h_b^{p,n}=\begin{cases}\dbinom{-m+p-1}{n-1}\dbinom{n}{n-p} & \text{if } m\le p-n\le 0\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Proof. The Main Lemma implies that it is enough to calculate the dimension of the kernels and the cokernels of P_0 and P_{n-1}. By Serre-duality we can avoid the calculation of P_{n-1} (see the remark at the end of section 2):

$$H^q(X, \Omega^p(L_b)) \cong H^{n-q}(X, \Omega^{n-p}(L_b^*)) \text{ with } L_b^* \cong L_{1/b}\,.$$

a) dim ker P_0: The section $\omega \in \Gamma(W, \Omega^p)$ can be written as

$$\omega = \sum_{J\in I_p} \sum_{\alpha\in N_0^n} a_\alpha^J z^\alpha dz_J = \sum_{(i_1,\cdots,i_p)\in I_p} \sum_{(\alpha_1,\cdots,\alpha_n)\in N_0^n} a_{\alpha_1,\cdots,\alpha_n}^{(i_1,\cdots,i_p)} z_1^{\alpha_1}\cdots z_n^{\alpha_n} dz_{i_1}\wedge\cdots\wedge dz_{i_p}.$$

Applying the operator P_0: $\Gamma(W, \Omega^p) \to \Gamma(W, \Omega^p)$ gives

$$P_0\omega = \sum_J \sum_{\alpha\in N_0^n} a_\alpha^J (b-\mu^{|\alpha|+p}) z^\alpha dz_J$$

Hence: If $b-\mu^{|\alpha|+p} \neq 0$ then we get that $P_0\omega = 0$ iff $a_\alpha^J = 0$ for all $J \in I_p$ and $\alpha \in N_0^n$. This implies: either $b \notin \Delta_\mu$ and then ker $P_0 = 0$ or $b = \mu^m$ and we have

$$\begin{aligned} a_\alpha^J &\in C && \text{if } m = |\alpha| + p \\ a_\alpha^J &= 0 && \text{if } m \neq |\alpha| + p \end{aligned}$$

i.e. dim ker $P_0 = \#\{\alpha = (\alpha_1, \cdots, \alpha_n) \mid |\alpha| = m-p\}\binom{n}{p} = \chi(m-p, n)\binom{n}{p}$. It is well known that

$$\chi(m, n) = \binom{m+n-1}{n-1}$$

and hence the claim follows.

b) dim coker P_0: Assume $b = \mu^m$. We claim that ω is an element of coker P_0 iff it can be written in the from

$$\omega = \sum_J \sum_{|\alpha|+p=m} a_\alpha^J z^\alpha dz_J\,.$$

That this condition is sufficient follows from the considerations above. We show now that it is necessary.

Let $\omega = \sum_J \sum_{|\alpha|+p\neq m} a_\alpha^J z^\alpha dz_J$. The claim is that a τ exists with $P_0\tau = \omega$. Setting

$$\tau := \sum_J \sum_{|\alpha|+p\neq m} a_\alpha^J (b-\mu^{|\alpha|+p})^{-1} z^\alpha dz_J$$

it follows that $P_0\tau = \omega$ if τ converges. It is easy to see that a n_0 exists with

$$|b-\mu^{|\alpha|+p}| > \left|\frac{b}{2}\right| \quad \text{for all } \alpha \text{ with } |\alpha| > n_0$$

i.e.

$$|b-\mu^{|\alpha|+p}|^{-1} < \left|\frac{2}{b}\right| \quad \text{for all } \alpha \text{ with } |\alpha| > n_0\,.$$

There is a $t \in R_+^n$ such that $\sum |a_\alpha^J| t^\alpha$ converges. Hence

$$\sum |(b-\mu^{|\alpha|+p})^{-1} a_\alpha^J| t^\alpha \leq \sum_{|\alpha|\leq n_0} |b-\mu^{|\alpha|+p}|^{-1} |a_\alpha^J| t^\alpha + \frac{2}{|b|} \sum_{|\alpha|>n_0} |a_\alpha^J| t^\alpha < \infty\,.$$

We get dim coker P_0=dim ker P_0. qed.

B. The generic case

We say that a Hopf manifold X is of generic type if it is generated by a contraction of the type $f\colon (z_1, \cdots, z_n)\to(\mu_1 z_1, \cdots, \mu_n z_n)$ with $0<|\mu_1|\leq|\mu_2|\leq\cdots\leq|\mu_n|<1$, and there are no relations except trivial ones between the μ_i's of the form

$$\prod_{i\in A}\mu_i^{r_i}=\prod_{j\in B}\mu_j^{r_j},\ r_i,\ r_j\in \boldsymbol{N}_0,\ A\cap B=\emptyset,\ A\cup B=\{1, \cdots, n\}\ .$$

The following notations are used:

$$\mu:=(\mu_1, \mu_2, \cdots, \mu_n)\in \boldsymbol{C}^n,\ \alpha:=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{N}_0^n,\ m:=(m_1, \cdots, m_n)\in \boldsymbol{N}_0^n\,.$$

$$A_p^m:=\#\{K=(k_1, k_2, \cdots, k_n)\,|\,k_i\in\{0, 1\}\,;\ \textstyle\sum k_i=p;\ m-K\geq(0, \cdots, 0)\}$$

$$B_p^m:=\#\{K=(k_1, k_2, \cdots, k_n)\,|\,k_i\in\{0, 1\}\,;\ \textstyle\sum k_i=n-p;\ m+K\leq(0, \cdots, 0)\}$$

Let $\Delta_{\mu_1,\cdots,\mu_n}$ be the free commutative subgroup of $\boldsymbol{C}^*$ generated by the eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ of f.

Theorem 2. *If L_b is a holomorphic flat line bundle on a generic Hopf manifold of dimension $n>2$, then we have the following results:*
If $b\notin\Delta_{\mu_1,\cdots,\mu_n}$, then $h_b^{p,q}=0$.
If $b=\mu^m$, then

$$h_b^{p,0}=h_b^{p,1}=\begin{cases}\binom{n}{p} & \textit{if } m\geq(1, 1, \cdots, 1)\,,\\ A_p^m & \textit{if } m\geq(0, 0, \cdots, 0) \textit{ and there exists an } i \textit{ with } m_i=0\,,\\ 0 & \textit{otherwise}\,.\end{cases}$$

$$h_b^{p,q}=0 \qquad \textit{if } 2\leq q\leq n-2\,.$$

$$h_b^{p,n}=h_b^{p,n-1}=\begin{cases}\binom{n}{p} & \textit{if } m\leq(-1, -1, \cdots, -1)\,,\\ B_p^m & \textit{if } m\leq(0, 0, \cdots, 0) \textit{ and there exists an } i \textit{ with } m_i=0\,,\\ 0 & \textit{otherwise}\,.\end{cases}$$

Proof. We introduce a map

$$\begin{aligned} I_p &\longrightarrow \{K=(k_1, \cdots, k_n)\,|\,k_i\in\{0, 1\}\}\\ J=(i_1, \cdots, i_p) &\mapsto \quad K_J=(k_1^J, \cdots, k_n^J)\end{aligned}$$

where

$$k_l^J=\begin{cases}1 & \text{if } l\in\{i_1, \cdots, i_p\}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Let

$$\omega = \sum_J \sum_\alpha a_\alpha^J z^\alpha dz_J$$
$$= \sum_{(i_1,\cdots,i_p)\in I_p} \sum_{(\alpha_1,\cdots,\alpha_n)\in N_0^n} a_{\alpha_1,\cdots,\alpha_n}^{(i_1,\cdots,i_p)} z_1^{\alpha_1} \cdots z_n^{\alpha_n} dz_{i_1}\wedge\cdots\wedge dz_{i_p} \in \Gamma(W, \Omega^p),$$

then

$$P_0\omega = \sum_J \sum_\alpha a_\alpha^J (b-\mu^{\alpha+K_J}) z^\alpha dz_J .$$

Now $P_0\omega=0$ iff for all α either $a_\alpha^J=0$ or $b=\mu^{\alpha+K_J}$. If $m\geq(1, \cdots, 1)$, then we have for all J some α such that $\alpha+K_J=m$ and it follows that $h_b^{p,0}=\binom{n}{p}$. If $m\geq(0, \cdots, 0)$ and $m-K_J$ contains at least one $m_i-k_i^J<0$, then there is no α for which $m_i=k_i^J+\alpha_i$, $\alpha_i\geq 0$. This implies $b\neq\mu^{\alpha+K_J}$ for all α and the given $J\in I_p$. Hence $h_b^{p,0}=A_p^m$. The other cases are treated in the same way and $h_b^{p,n}=h_b^{p,n-1}$ results from $h_b^{p,0}=h_b^{p,1}$ by Serre-duality. q.e.d.

REMARK. We could avoid Serre-duality and calculate $h_b^{p,n}$ in a similar way to the above $h_b^{p,n}$ applying P_{n-1} to $H^{n-1}(W, \Omega^p)\cong\Gamma(U_1\cap\cdots\cap U_n, \Omega^p)$ with $U_i=\{(z_1, \cdots, z_n)\in C^n \mid z_i\neq 0\}$.

3. The cohomology of flat line bundles on Hopf surfaces

The Hopf surfaces X are completely classified (see for example [3], [14] or [11] § 10). For our purposes it is convenient to distinguish the following cases corresponding to the contractions f: $C^2\to C^2$ inducing these surfaces.

1) The generic case

$$f=\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix} \text{ with no relations of the kind } \mu_1^{r_1}=\mu_2^{r_2},\ r_1, r_2\in N .$$

2) The classical case

$$f=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix}$$

3) The resonant case

$$f=\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix} \text{ with } \mu_1=\mu_2^r,\ r\in N \text{ and } r>1 .$$

4) The hyperresonant case

$$f=\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix} \text{ with } \mu_1^{r_1}=\mu_2^{r_2},\ r_1, r_2\in N \text{ and } 1<r_1\leq r_2 .$$

5) The exceptional case

$$f(z_1, z_2)=(z_1', z_2')=(\mu_1 z_1+z_2^r, \mu_2 z_2) \text{ with } \mu_1=\mu_2^r,\ r\in N \text{ and } r\geq 1 .$$

We are interested in determining the cohomology groups of the flat line bundles in all these cases.

To this end we show first, that it is enough to calculate $h^0(X, \Omega^p(L_b))$ for all $b\in \boldsymbol{C}^*$ and $p=0, 1, 2$:
By Serre-duality we obtain:

$$h^2(X, \Omega^p(L_b)) = h^0(X, \Omega^{2-p}(L_b^*)) = h^0(X, \Omega^{2-p}(L_{1/b}))\,.$$

Because $c_1(TX)=c_2(TX)=0$ (cf. [14] p. 23), $c_1(L)=0$ for all $L\in H^1(X, \mathcal{O}^*)$ and $\Omega^p(L)=\mathcal{O}(\Lambda^p\ T^*X\otimes L)$ it follows with the Riemann-Roch formula

$$\chi(X, \Omega^p(L)) = ch(\Omega^p(L)).\mathcal{T}(TX)[X] = 0\,.$$

From $\chi(X, \Omega^p(L))=h^0(X, \Omega^p(L))-h^1(X, \Omega^p(L))+h^2(X, \Omega^p(L))$ we get $h^1(X, \Omega^p(L_b))=h^0(X, \Omega^p(L_b))+h^0(X, \Omega^{2-p}(L_{1/b}))$. So it is enough to be able to compute $h^0(X, \Omega^p(L))$ which is equal to dim ker P_0 in the first part of the Douady-sequence:

$$0\to H^0(X, \Omega^p(L))\to H^0(W, \pi^*(\Omega^p(L)))\xrightarrow{P_0} H^0(W, \pi^*(\Omega^p(L)))\,.$$

These groups can be computed on the same lines as in § 2 for the generic, the classical, the resonant and the hyperresonant cases by counting admissible partitions of numbers. The exceptional case is somewhat different and we shall give a proof after having stated the results.

NOTATIONS: As in § 2 we need the following subgroups of $\boldsymbol{C}^*$: Δ_{μ_1,μ_2}=group generated by μ_1 and μ_2.

REMARK. The structure of these groups depends on the contraction f. For example if X is generic Δ_{μ_1,μ_2} is a free commutative group. If X is hyperresonant then $\Delta_{\mu_1,\mu_2}=\langle\mu_1, \mu_2\,|\,\mu_1\mu_2=\mu_2\mu_1,\ \mu_1^{r_1}=\mu_2^{r_2}\rangle$.

Proposition 2. *Let X be a generic Hopf surface and $L_b\in H^1(X, \mathcal{O}^*)$. Then either $b\notin\Delta_{\mu_1,\mu_2}$, in which case $h_b^{p,q}=0$ for all p and q or $b=\mu_1^{m_1}\mu_2^{m_2}\in\Delta_{\mu_1,\mu_2}$ and we have the following result:*

p	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	1 if $(m_1, m_2)\geq(0, 0)$, 0 otherwise.	2 if $(m_1, m_2)\geq(1, 1)$, 1 if $m_1=0$, $m_2>0$ or $m_1>0$, $m_2=0$, 0 otherwise.	1 if $(m_1, m_2)\geq(1, 1)$, 0 otherwise.
$h_b^{p,1}$	$h_b^{0,0}+h_b^{0,2}$	$h_b^{1,0}+h_b^{1,2}$	$h_b^{2,0}+h_b^{2,2}$

$h_b^{p,2}$	1 if $(m_1, m_2)\leq(-1, -1)$, 0 otherwise.	2 if $(m_1,m_2)\leq(-1,-1)$, 1 if $m_1=0$, $m_2<0$ or $m_1<0$, $m_2=0$, 0 otherwise.	1 if $(m_1, m_2)\leq(0, 0)$, 0 otherwise.

Proposition 3. *Let X be a classical Hopf surface and $L_b\in H^1(X, \mathcal{O}^*)$. Then either $b\notin\Delta_\mu$, in which case $h_b^{p,q}=0$ for all p and q or $b=\mu^m$ and we have the following result:*

p	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	$m+1$ if $m\geq 0$, 0 if $m<0$.	$2m$ if $m\geq 1$, 0 if $m<1$.	$m-1$ if $m\geq 2$, 0 if $m<2$.
$h_b^{p,1}$	$h_b^{0,0}+h_b^{0,2}$	$h_b^{1,0}+h_b^{1,2}$	$h_b^{2,0}+h_b^{2,2}$
$h_b^{p,2}$	$-m-1$ if $m\leq -2$, 0 if $m>-2$.	$-2m$ if $m\leq -1$, 0 if $m>-1$.	$-m+1$ if $m\leq 0$, 0 if $m>0$.

Proposition 4. *Let X be a resonant Hopf surface and $L_b\in H^1(X, \mathcal{O}^*)$. Then either $b\notin\Delta_{\mu_1,\mu_2}$, in which case $h_b^{p,q}=0$ for all p and q or $b=\mu_2^m\in\Delta_{\mu_1\mu_2}$ and we have the following result:*

p	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	n if $(n-1)r\leq m<n\cdot r$, 0 if $m<0$.	$2n$ if $m=n\cdot r$, $2n-1$ if $(n-1)r<m<n\cdot r$, 0 if $m\leq 0$.	n if $n\cdot r<m\leq(n+1)r$, 0 if $m\leq r$.
$h_b^{p,1}$	$h_b^{0,0}+h_b^{0,2}$	$h_b^{1,0}+h_b^{1,2}$	$h_b^{2,0}+h_b^{2,2}$
$h_b^{p,2}$	n if $-(n+1)r\leq m<-n\cdot r$, 0 if $m\geq -r$.	$2n$ if $m=-n\cdot r$, $2n-1$ if $-n\cdot r<m<-(n-1)r$, 0 if $m\geq 0$.	n if $-n\cdot r<m\leq -(n-1)r$, 0 if $m>0$.

Proposition 5. *Let X be a hyperresonant Hopf surface and $L_b\in H^1(X, \mathcal{O}^*)$. Then either $b\notin\Delta_{\mu_1,\mu_2}$, in which case $h_b^{p,q}=0$ for all p and q or $b=\mu_1^{m_1}\mu_2^{m_2}\in\Delta_{\mu_1,\mu_2}$. In this case $b=\mu_1^{m_1}\mu_2^{m_2}$ always has a representation such that $0\leq m_1<r_1$ or $0<m_1\leq r_1$ and we have the following result:*

p	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	$0\leq m_1<r_1$ n if $(n-1)r_2\leq$ $m_2<n\cdot r_2$, 0 if $m_2<0$.	$m_1=0$: $2n$ if $m_2=n\cdot r_2$, $2n+1$ if $n\cdot r_2<m_2<(n+1)r_2$, 0 if $m_2\leq 0$. $0<m_1<r_1$: $2n+1$ if $m_2=n\cdot r_2$, $2n$ if $(n-1)r_2<m_2<n\cdot r_2$, 0 if $m_2<0$.	$0<m_1\leq r_1$ n if $(n-1)r_2<m_2\leq n\cdot r_2$, 0 if $m_2\leq 0$.
$h_b^{p,1}$	$h_b^{0,0}+h_b^{0,2}$	$h_b^{1,0}+h_0^{1,2}$	$h_b^{2,0}+h_b^{2,2}$
$h_b^{p,2}$	$0\leq m_1<r_1$ n if $-(n+1)r_2\leq$ $m_2<-n\cdot r_2$, 0 if $m_2\geq -r_2$.	$m_1=0$: $2n$ if $m_2=-n\cdot r_2$, $2n+1$ if $-(n+1)r_2<m_2<$ $-n\cdot r_2$, 0 if $m_2\geq 0$. $0<m_1<r_1$: $2n+1$ if $m_2=-(n+1)r_2$, $2n$ if $-(n+1)r_2<m_2<$ $-n\cdot r_2$, 0 if $m_2>-r_2$.	$0<m_1\leq r_1$ n if $-(n+1)r_2<m_2\leq$ $-n\cdot r_2$, 0 if $m_2>-r_2$.

Proposition 6. *Let X be an exceptional Hopf surface and $L_b\in H^1(X,\mathcal{O}^*)$. Then either $b\notin\Delta_{\mu_1,\mu_2}$, in which case $h_b^{p,q}=0$ for all p and q or $b=\mu_2^m\in\Delta_{\mu_1,\mu_2}$ and we have the following result*:

p	$p=0$	$p=1$	$p=2$
$h_b^{p,0}$	1 if $m\geq 0$, 0 otherwise.	2 if $m>r$, 1 if $1\leq m\leq r$, 0 otherwise.	1 if $m>r$, 0 otherwise.
$h_b^{p,1}$	$h_b^{0,0}+h_b^{0,2}$	$h_b^{1,0}+h_b^{1,2}$	$h_b^{2,0}+h_b^{2,2}$
$h_b^{p,2}$	1 if $m<-r$, 0 otherwise.	2 if $m<-r$, 1 if $-r\leq m\leq -1$, 0 otherwise.	1 if $m\leq 0$, 0 otherwise.

Proof of Proposition 6.

A. *Calculation of $h^0(X, \mathcal{O}(L_b))$.*

Recall the following definitions: f: $(z_1, z_2)\mapsto(z_1', z_2')$ with $z_1'=\mu_1 z_1+z_2^r$, $z_2'=\mu_2 z_2$; $\mu_1=\mu_2^r$, $r\in \boldsymbol{N}$ and

$$\ker P_0 \cong \{\omega\in\Gamma(W, \mathcal{O}) = \Gamma(\boldsymbol{C}^2, \mathcal{O})\,|\, b\omega-f^*\omega = 0\}.$$

We get for $\omega=\sum_{k,l} a_{kl}\, z_1^k z_2^l$

$$f^*\omega = \sum_{k,l} a_{kl}\, z_1'^k z_2'^l = \sum_{k,l}\sum_t a_{kl}\binom{k}{t}\mu_1^t \mu_2^l z_1^t z_2^{r(k-t)+l} =: \sum \tilde{a}_{\alpha,\beta} z_1^\alpha z_2^\beta .$$

We calculate $\tilde{a}_{\alpha,\beta}$: comparison of the exponents of the variables z_1, z_2 gives the condition:

$$(\alpha, \beta) = (t, r(k-t)+l) .$$

It follows that $\alpha=t$, $k\geq\alpha$ and $l=\beta-r(k-t)=\beta-jr$ where $j=k-\alpha$ and j can take the values $0, \cdots, [\beta/r]$ ($[x]$ being the greatest integer less than x). Hence:

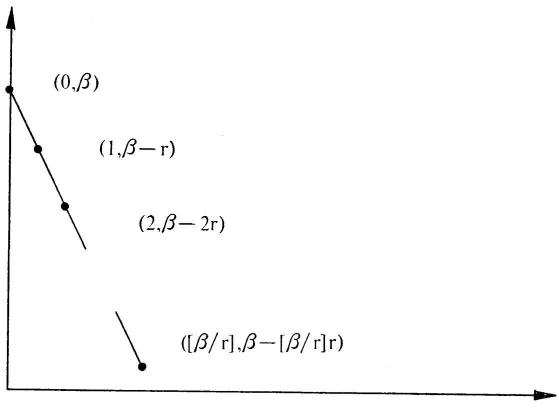
$$P_0\omega = \sum_{\alpha,\beta\geq 0}(ba_{\alpha\beta}-\tilde{a}_{\alpha\beta})z_1^\alpha z^\beta = \sum_{\alpha,\beta\geq 0}\Big(ba_{\alpha\beta}-\sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr}\binom{\alpha+j}{\alpha}\mu_2^{\beta+r(\alpha-j)}\Big)z_1^\alpha z_2^\beta .$$

Thus we get the following conditions for the coefficients $a_{\alpha\beta}$ using the relation $P_0\omega=0$:

$$(4)\qquad a_{\alpha\beta}(b-\mu_2^{\beta+r\alpha}) = \sum_{j=1}^{[\beta/r]} a_{\alpha+j,\beta-jr}\binom{\alpha+j}{\alpha}\mu_2^{\beta+r(\alpha-j)} \qquad \text{for all } (\alpha, \beta)\geq(0, 0) .$$

Picturally the conditions (4) impose a constraint on those coefficients $a_{\alpha\beta}$ the indices (α, β) of which lie on a "ladder"

$$Ld(\beta) = \{(0, \beta); (1, \beta-r); (2, \beta-2r); \cdots; ([\beta/r], \beta-[\beta/r]r)\}:$$



These conditions can be written as a system of linear equations:

$$(5) \qquad C^{\beta}\circ A^{\beta}=0$$

where

$$C^{\beta}:=\begin{pmatrix} b-\mu_2^{\beta} & -\binom{0+1}{0}\mu_2^{\beta-r} & -\binom{0+2}{0}\mu_2^{\beta-2r} & \cdots & -\binom{0+[\beta/r]}{0}\mu_2^{\beta-[\beta/r]r} \\ & b-\mu_2^{\beta} & -\binom{1+1}{1}\mu_2^{\beta-r} & \cdots & -\binom{1+([\beta/r]-1)}{1}\mu_2^{\beta-([\beta/r]-1)r} \\ & & b-\mu_2^{\beta} & \cdots & -\binom{2+([\beta/r]-2)}{2}\mu_2^{\beta-([\beta/r]-2)r} \\ & & & \ddots & \vdots \\ & & & & b-\mu_2^{\beta} \end{pmatrix}$$

and

$$A^{\beta}:=\begin{pmatrix} a_{0,\beta} \\ a_{1,\beta-r} \\ \vdots \\ a_{[\beta/r],\beta-[\beta/r]r} \end{pmatrix}$$

The system (5) has a non trivial solution A^{β} iff $b-\mu_2^{\beta}=0$. In this case the space of solutions has dimension one.
Hence:

$$\dim\ker P_0=\begin{cases}1 & \text{if } b\in\{1,\ \mu_2,\ \mu_2^2,\ \cdots\}\ , \\ 0 & \text{otherwise.}\end{cases}$$

B. *Calculation of $h^0(X,\ \Omega_X^1(L_b))$.*

$$\ker P_0=\{\omega(z_1,\ z_2)=\sum a_{kl}^1 z_1^k z_2^l dz_1+\sum a_{kl}^2 z_1^k z_2^l dz_2\,|\,b\omega-f^*\omega=0\}$$
$$dz_1'=\mu_1 dz_1+rz_2^{r-1}dz_2\,,\quad dz_2'=\mu_2 dz_2\,.$$

If $P_0\omega=0$, then the following equations for the coefficients must hold:

$$(6) \qquad ba_{\alpha\beta}^1=\sum_{j=0}^{[\beta/r]}a_{\alpha+j,\beta-jr}^1\binom{\alpha+j}{\alpha}\mu_2^{\beta+r(\alpha+1-j)}$$

and

$$(7) \qquad ba_{\alpha\beta}^2=\sum_{j=0}^{[\beta/r]}a_{\alpha+j,\beta-jr}^2\binom{\alpha+j}{\alpha}\mu_2^{\beta+1+r(\alpha-j)}$$
$$+r\sum_{j=0}^{[\beta/r]}a_{\alpha+j,\beta+1-(j+1)r}^1\binom{\alpha+j}{\alpha}\mu_2^{\beta+1+r(\alpha-j-1)}\,.$$

We proceed as in section A: to satisfy the equations (6), it is necessary that either $a_{\alpha\beta}^1=0$ for all $(\alpha,\ \beta)$ or else there are integers $s\in[0,\ r)$ and $t\in\boldsymbol{N}_0$ such that $b=\mu_2^{s+r(t+1)}$, $a_{\alpha\beta}^1=0$, if $(\alpha,\ \beta)\neq(0,\ s+rt)$ and $a_{0,s+rt}^1\in\boldsymbol{C}$.
Then the equations (7) become:

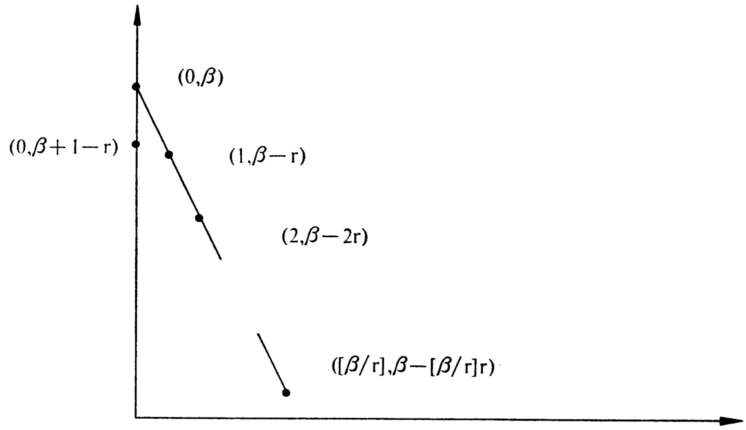
$$(8) \qquad ba^2_{\alpha\beta} = \sum_{j=0}^{[\beta/r]} a^2_{\alpha+j,\beta-jr}\binom{\alpha+j}{\alpha}\mu_2^{\beta+1+r(\alpha-j)} \qquad \text{if } \alpha>0$$

and

$$(9) \qquad ba^2_{0\beta} = \sum_{j=0}^{[\beta/r]} a^2_{j,\beta-jr}\,\mu_2^{\beta+1-rj} + ra^1_{0,\beta+1-r}\mu_2^{\beta+1-r}\,.$$

Again the conditions (8), (9) impose a constraint on those coefficients $a_{\alpha\beta}$ the indices of which lie on the (somewhat different) ladder

$$Ld_2(\beta) = \{(0,\ \beta+1-r);\ (0,\ \beta);(1,\ \beta-r);\ \cdots;\ ([\beta/r],\ \beta-[\beta/r]r)\}:$$



We express these conditions by means of the following inhomogeneous linear systems:

$$(10) \qquad C_2^\beta \circ A_2^\beta = B^\beta$$

where

$$C_2^\beta := \begin{pmatrix} b-\mu_2^{\beta+1} & -\binom{0+1}{0}\mu_2^{\beta+1-r} & -\binom{0+2}{0}\mu_2^{\beta+1-2r} & \cdots & -\binom{0+[\beta/r]}{0}\mu_2^{\beta+1-[\beta/r]r} \\ & b-\mu_2^{\beta+1} & -\binom{1+1}{1}\mu_2^{\beta+1-r} & \cdots & \vdots \\ & & b-\mu_2^{\beta+1} & \cdots & \\ & & & \ddots & \\ & & & & b-\mu_2^{\beta+1} \end{pmatrix},$$

$$A_2^\beta := \begin{pmatrix} a^2_{0,\beta} \\ a^2_{1,\beta-r} \\ \vdots \\ a^2_{[\beta/r],\beta-[\beta/r]r} \end{pmatrix}$$

and

$$B^{\beta} := \begin{pmatrix} r\mu_2^{\beta+1-r} & a^1_{0,\beta+1-r} \\ & 0 \\ & \vdots \\ & 0 \end{pmatrix}.$$

Of course $B^{\beta}=0$, if $\beta<r-1$. It follows: if $B^{\beta}=0$, then the space of solutions of (10) has dimension 1 iff $b-\mu_2^{\beta+1}=0$.
If $B^{\beta}\neq 0$, then $a^1_{0,\beta+1-r}\neq 0$ and (6) implies $b=\mu_2^{\beta+1}$. If $\beta>r$ then the space of solutions of (10) has dimension 1 and $a^1_{0,\beta+1-r}$ can be chosen arbitrarily. If $\beta=r$ then (6) and (10) imply $B^{\beta}=0$.
Collecting the results we obtain:

$$\dim\ker P_0 = \begin{cases} 2 & \text{if } b=\mu_2^m \text{ and } m>r\,, \\ 1 & \text{if } b=\mu_2^m \text{ and } 1\leq m\leq r\,, \\ 0 & \text{otherwise.} \end{cases}$$

C. *Calculation of* $h^0(X, \Omega^2_X(L_b))$.

We have:

$$\ker P_0 = \{\omega(z_1, z_2) = \sum_{k,l} a_{k,l} z_1^k z_2^l dz_1\wedge dz_2 \,|\, b\omega - f^*\omega = 0\}$$

and

$$dz_1'\wedge dz_2' = \mu_1\mu_2 dz_1\wedge dz_2\,.$$

This implies:

$$f^*\omega = \sum_{\alpha,\beta}\sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr}\binom{\alpha+j}{\alpha}\mu_2^{\beta+1+r(\alpha-j+1)} z_1^{\alpha} z_2^{\beta} dz_1\wedge dz_2\,,$$

which gives conditions for the coefficients:

$$ba_{\alpha,\beta} = \sum_{j=0}^{[\beta/r]} a_{\alpha+j,\beta-jr}\binom{\alpha+j}{\alpha}\mu_2^{\beta+1+r(\alpha-j+1)}\,.$$

In the same manner as above we get the following result:

1) $a_{\alpha,\beta}=0$ for all (α, β) with $\alpha>0$
2) $a_{0,\beta}=0$ if $b\neq\mu_2^{\beta+1+r}$
   $a_{0,\beta}\in \boldsymbol{C}$ if $b=\mu_2^{\beta+1+r}$.

Hence:

$$\dim\ker P_0 = \begin{cases} 1 & \text{if } b=\mu_2^m,\ m\geq r+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

qed.

4. Hodge numbers and flatness of line bundles

Theorem 3. *The Hodge numbers of an arbitrary Hopf manifold of dimension n are:*

$$h^{0,0} = h^{0,1} = h^{n,n} = h^{n,n-1} = 1$$
$$h^{p,q} = 0 \quad \textit{in all other cases.}$$

Proof. Obviously $h^{0,0}=h^{n,n}=1$. For a Hopf manifold generated by a diagonal contraction $f\colon (z_1, \cdots, z_n) \mapsto (\mu_1 z_1, \cdots, \mu_n z_n)$ the rest of the assertion follows by arguing as in the proof of lemma 9.4 in [2] p. 216 and as in § 2 and § 3, if we choose $L_b=\mathcal{O}$, i.e. $b=1$. For a general contraction f we can assume without loss of generality that $f'(0)=d_\mu\colon (z_1, \cdots, z_n) \mapsto (\mu_1 z_1, \cdots, \mu_n z_n)$ and $f \in G_\mu$ (Poincaré-Dulac form), see section 1. The group G_μ operates on itself by conjugation. A slice through the orbit of f in G_μ represents the Kuranishi space K_{X_f} of X_f. The orbit of f in G_μ has d_μ in its closure (cf. [8] p. 242, 248). From Grauert's semicontinuity theorem, (cf. [7] p. 210), follows: $h^{p,q}(X_{d_\mu}) \geq h^{p,q}(X_f)$. This implies

$$h^{n,n-1}(X_f) = h^{0,1}(X_f) \leq h^{0,1}(X_{d_\mu}) \leq 1 \quad \text{and} \quad h^{p,q}(X_f) = 0$$

if $(p, q) \neq (0, 1); (0, 0); (n, n); (n, n-1)$.
By the Fröhlicher spectral sequence we get $b_r(X_f) \leq \sum_{p+q=r} h^{p,q}(X_f)$ with $b_r(X_f) = \operatorname{rank} H^r(X_f, \boldsymbol{Z})$, see [5]. However

$$1 = b_1(X_f) \leq h^{0,1}(X_f) + h^{1,0}(X_f) = h^{0,1}(X_f) \leq 1\,. \qquad \text{qed.}$$

Armed with this result we can tackle our next theorem. For Hopf surfaces this statement was proven by Kodaira (cf. [11] p. 699). We follow his arguments closely.

Theorem 4. *All line bundles over a Hopf manifold X are flat.*

Proof. a) Since X is homeomorphic to $S^{2n-1} \times S^1$ we have the following Betti numbers $b_1(X) = b_{2n-1}(X) = 1$ and $b_2(X) = \cdots = b_{2n-2}(X) = 0$, (cf. [6] p. 144).
b) The diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \boldsymbol{Z} \arrow[r] \arrow[d, equal] & \boldsymbol{C} \arrow[r, "\exp"] \arrow[d, "i"] & \boldsymbol{C}^* \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \boldsymbol{Z} \arrow[r] & \mathcal{O} \arrow[r, "\exp"] \arrow[d, "d"] & \mathcal{O}^* \arrow[r] & 0 \\
 & & \Omega & &
\end{tikzcd}$$

is exact with d taken to be the de Rahm derivative.
X is compact and connected: $H^0(X, \boldsymbol{C}) \cong H^0(X, \mathcal{O}) \cong \boldsymbol{C}$, $H^0(X, \boldsymbol{C}^*) \cong H^0(X, \mathcal{O}^*) \cong \boldsymbol{C}^*$. This implies:

$$0 \to H^0(X, \boldsymbol{Z}) \to H^0(X, \boldsymbol{C}) \to H^0(X, \boldsymbol{C}^*) \xrightarrow{0} H^1(X, \boldsymbol{Z}) \to \cdots$$

Since $\dim H^2(X, \mathcal{O}) = h^{0,2} = 0$ and $\dim H^2(X, \boldsymbol{C}) = b_2 = 0$ we get:

$$\begin{array}{ccccccccc} 0 \to & H^1(X, \boldsymbol{Z}) & \to & H^1(X, \boldsymbol{C}) & \to & H^1(X, \boldsymbol{C}^*) & \to & H^2(X, \boldsymbol{Z}) & \to 0 \\ & \| & & \downarrow i_* & & \downarrow j_* & & \| & \\ 0 \to & H^1(X, \boldsymbol{Z}) & \to & H^1(X, \mathcal{O}) & \to & H^1(X, \mathcal{O}^*) & \to & H^2(X, \boldsymbol{Z}) & \to 0 \end{array}$$

c) By the five-lemma it is enough to show that i_* is an isomorphism. First we will show that i_* is injective. Let $d\mathcal{O}$ be the sheaf of germs of 1-forms which are locally the derivative of a holomorphic function $\omega = df$. We take the exact sequence

$$0 \to \boldsymbol{C} \xrightarrow{i} \mathcal{O} \xrightarrow{d} d\mathcal{O} \to 0$$

to get the long exact sequence

$$0 \to H^0(X, \boldsymbol{C}) \to H^0(X, \mathcal{O}) \xrightarrow{0} H^0(X, d\mathcal{O}) \to H^1(X, \boldsymbol{C}) \xrightarrow{i_*} H^1(X, \mathcal{O}) \to \cdots$$

To prove the injectivity of i_* it suffices to show that $H^0(X, d\mathcal{O}) = 0$. Assume the contrary. Then there is a $\varphi \in H^0(X, d\mathcal{O})$ such that $\varphi \neq 0$ and $d\varphi = 0$. Let $\mathcal{D}$ be the sheaf of germs of differentiable complex valued functions on X. There is a natural inclusion mapping $\mathcal{O} \hookrightarrow \mathcal{D}$ and by the de Rham theorem we get: $H^1(X, \boldsymbol{C}) \cong H^0(X, d\mathcal{D})/dH^0(X, \mathcal{D})$. This establishes a map

$$\begin{aligned} [\ ]: H^0(X, d\mathcal{O}) &\to H^1(X, \boldsymbol{C}) \\ \varphi \quad &\mapsto \quad [\varphi]\,. \end{aligned}$$

Now suppose that there exist $a, b \in \boldsymbol{C}$, such that $a[\varphi] + b[\bar{\varphi}] = 0$ in $H^1(X, \boldsymbol{C})$. The above remarks imply that there exists a $\mu \in H^0(X, \mathcal{D})$ with $a\varphi + b\bar{\varphi} = d\mu$. Locally we have:

$$\varphi = d\lambda,\ \lambda \in \mathcal{O}(U),\ U \subset X \text{ open, and } \mu = a\lambda + b\bar{\lambda} + \textit{some constant}.$$

It follows that μ has the mean-value-property for sufficiently small balls in X. Hence μ is constant, because X is compact. But then we have $a\varphi + b\bar{\varphi} = 0$ in $H^0(X, d\mathcal{D})$ which implies $a = b = 0$, because $\varphi \in H^0(X, d\mathcal{O})$, i.e. $H^1(X, \boldsymbol{C}) \geq 2$. However, this last inequality contradicts $H^1(X, \boldsymbol{C}) = 1$.
The surjectivity follows now from $\dim H^1(X, \boldsymbol{C}) = 1$ and $\dim H^1(X, \mathcal{O}) = h^{0,1}(X) = 1$. qed.

REMARK 1. Theorem 4 implies that we know the cohomology of *all* line bundles in the cases calculated in § 2 and § 3.

REMARK 2. Let TX be the tangent bundle of X and $\Theta=\mathcal{O}(TX)$. The spaces $H^q(X, \Theta)$ are of interest in deformation theory. From our calculations we get their dimensions by the simple observation that

$$H^q(X, \Theta) = H^q(X, \mathcal{O}(TX)) \cong H^{n-q}(X, \Omega^n(T^*X))^* \cong$$
$$H^{n-q}(X, K \otimes \Omega^1)^* = H^{n-q}(X, \Omega^1(K))^*$$

where K is the canonical line bundle of X.

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Dept. Math.
ETH-Zentrum
CH-8092 Zürich
Switzerland