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# ON 3-FOLD IRREGULAR BRANCHED COVERING SPACES OF PRETZEL KNOTS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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It is well-known that any orientable closed 3-manifold is a 3-fold irregular branched covering space of a 3-sphere branched along a knot. It is an interesting problem to know which 3-manifold can be a 3-fold irregular branched covering space of a given knot. In this paper we consider those of pretzel knots.

For the permutation group $S_3$ on $\{0, 1, 2\}$, let $a=(01)$, $b=(02)$, $c=(12)$, $x=(012)$, $y=(021)$. Then there are relations $a^2=b^2=c^2=1$, $ab=bc=ca=x$, $ba=ac=cb=y$. Especially, we remark the following relations:

$$\begin{array}{llll} aba^{-1}=c\,, & aca^{-1}=b\,, & axa^{-1}=y\,, & aya^{-1}=x\,, \\ bab^{-1}=c\,, & bcb^{-1}=a\,, & bxb^{-1}=y\,, & byb^{-1}=x\,, \\ cac^{-1}=b\,, & cbc^{-1}=a\,, & cxc^{-1}=y\,, & cyc^{-1}=x\,, \\ xax^{-1}=b\,, & xbx^{-1}=c\,, & xcx^{-1}=a\,, & xyx^{-1}=y\,, \\ yay^{-1}=c\,, & yby^{-1}=a\,, & ycy^{-1}=b\,, & yxy^{-1}=x\,. \end{array}$$

A knot group $G$ has a Wirtinger presentation:

$$(x_1, x_2, \cdots, x_n; r_1, r_2, \cdots, r_{n-1}) \tag{1}$$

where each relator $r_i$ indicates the relation form $r_i=x_{j(i)}^{\varepsilon}x_i x_{j(i)}^{-\varepsilon}x_{i+1}^{-1}$ ($\varepsilon=\pm 1$) at a crossing as in Fig. 1.

Then a homomorphism from a knot group $G$ to $S_3$ satisfies a condition as follows.

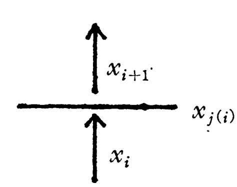


Fig. 1

**Proposition 1.** *Let the above* (1) *be a Wirtinger presentation of a knot group $G$. Then a homomorphism $h$ from $G$ to $S_3$ satisfies one of the followings.*
(i) $h(x_i)=a$ (*or* $b$, $c$) $\quad (i=1, 2, \cdots, n)$,
(ii) $h(x_i)=x$ (*or* $y$) $\quad (i=1, 2, \cdots, n)$.

**Proposition 2.** *Let* $(x_{11}, \cdots, x_{1n_1}, \cdots, x_{m1}, \cdots, x_{mn_m}; r_1, \cdots, r_k)$ *be a Wirtinger* *presentation of a link group of an m-component link, where* $x_{i1}, x_{i2}, \cdots, x_{in_i}$ *represent meridians of the i-th component* ($i=1, 2, \cdots, m$). *Then a homomorphism h from this link group to* $S_3$ *satisfies one of the followings.*

(i) $h(x_{ij})=a$ (*or* $b$, $c$) $\quad (j=1, 2, \cdots, n_i)$,

(ii) $h(x_{ij})=x$ (*or* $y$) $\quad (j=1, 2, \cdots, n_i)$.

If all generators are mapped to $a$ (or $b$, $c$) by $h$, then the branched covering space corresponding to $h$ is the disjoint union of a 3-sphere and the 2-fold regular branched covering space of the knot. And if all generators are mapped to $x$ (or $y$) by $h$, then that is the 3-fold regular branched covering space of the knot. So, the branched covering space corresponding to $h$ is 3-fold irregular, iff $h$ satisfies the condition (i) of Proposition 1 and there exist generators $x_i$ and $x_j$ with $h(x_i)\neq h(x_j)$.

First, we consider the image of meridians by $h$ at twists, especially for typical cases as in Fig. 2.

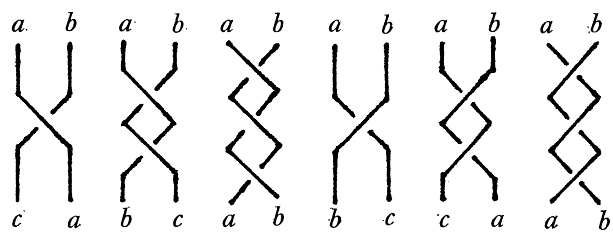



Fig. 2

Since $a^2=b^2=c^2=1$, we can ignore the orientation of a knot or a link.

If there is a block of three half-twists at the projection of a link $L$, deform $L$ by the operation cancelling three half-twists as shown in Fig. 3; we have a new link $L'$. Then, determining the image of meridians of $L'$ except at the three half-twists to be the same to that by $h$, we have a homomorphism $h'$ from the link group of $L'$ to $S_3$. We call $h'$ a homomorphism induced from $h$.

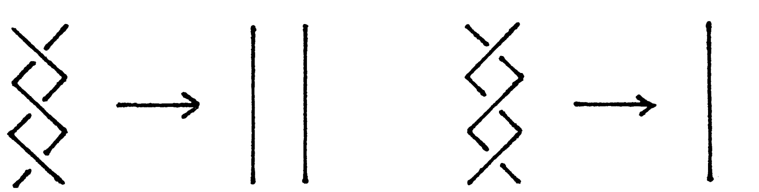


Fig. 3

Since it is easily seen that the inverse of the above operation is also an operation cancelling three half-twists after a slight deformation of the projection of $L'$ as Fig. 4, we have

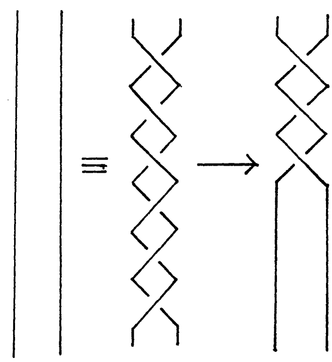


Fig. 4

**Proposition 3.** *Let $L'$ be a link obtained from a link $L$ by operation cancelling three half-twists, and $G$ and $G'$ be the link group of $L$ and $L'$. Then the followings are equivalent.*

(a) *There exists a homomorphism $h$ from $G$ to $S_3$ satisfying the condition* (i) *of Proposition* 2.

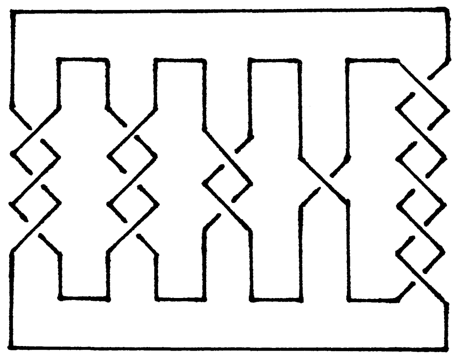
(b) *There exists a homomorphism $h'$ from $G'$ to $S_3$ satisfying the condition* (i) *of Proposition* 2.

We regard a part of the three half-twists as a *trivial tangle* i.e. a pair of a 3-ball and two proper arcs which are trivial and separated in the 3-ball. Since the irregular 3-fold branched covering space of a trivial tangle is a 3-ball (Burde [2]), we have

**Proposition 4** (Montesinos [5]). *Let $L'$ be a link obtained from $L$ by operation cancelling three half-twists, and $G$ and $G'$ be the link group of $L$ and $L'$. Suppose that a homomorphism $h$ from $G$ to $S_3$ exists and $h'$ is a homomorphism induced by $h$. Furthermore, at the three half-twists deformed by operation, we suppose that the images of medidians of the two arcs by $h$ are distinct and transpositions. Then the* 3*-fold irregular branched covering space of a* 3*-sphere branched along $L$ corresponding to $h$ is homeomorphic to the irregular* 3*-fold branched covering space of a* 3*-sphere branched along $L'$ corresponding to $h'$.*

From Proposition 4, we can deside *all* 3-fold irregular branched covering spaces of a 3-sphere branched along a pretzel knot. *A pretzel knot* is a knot consisting of a row of 2-strand braids of $q_1, q_2, \cdots, q_m$ half-twists, which we denote by $k(q_1, q_2, \cdots, q_m)$. We assume $q_i \neq 0$ for $i=1, 2, \cdots, m$. Fig. 5 shows $k(3, 3, -2, -1, -5)$.

**Theorem.** *Each* 3*-fold irregular branched covering space of a* 3*-sphere branched along a pretzel knot, if it exists, is isomorphic to a* 3*-sphere, a lens space of type* $(p, 1)$ *for some non-negative integer $p$, or a connected sum of those spaces.*



$k(3, 3, -2, -1, -5)$

Fig. 5

Proof. We note that a pretzel knot does not always have an irregular 3-fold cover.

First, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by $h$ are distinct. In this case, we see each $q_i = \pm 1$ (*mod* 3) from Fig. 2. So, by operations cancelling three half-twists, this pretzel knot can be deformed to $k(1, 1, \cdots, 1)$ or $k(-1, -1, \cdots, -1)$. Moreover, on each operation cancelling three half-twists, the condition of Proposition 4 is satisfied. Here the number of "1"'s or "$-1$"'s in the above $k(1, 1, \cdots, 1)$ or $k(-1, -1, \cdots, -1)$ is a multiple of three, and we can obtain a trivial link of 2-components from this link by operations cancelling three half-twists such that the image of meridians of components by $h$ are distinct. Since the 3-fold irregular branched covering space of a 3-sphere branched along a trivial link

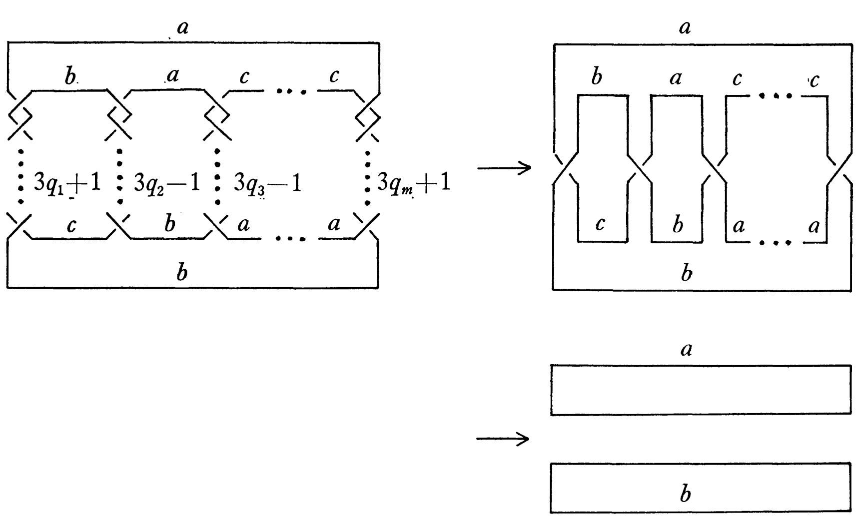



Fig. 6

of 2-components is a 3-sphere, those of the original pretzel knot is also a 3-sphere.

Secondly, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by $h$ are same. In this case, the image of meridians of 2-strand braids by $h$ must be (i) or (ii) shown as in Fig. 7.

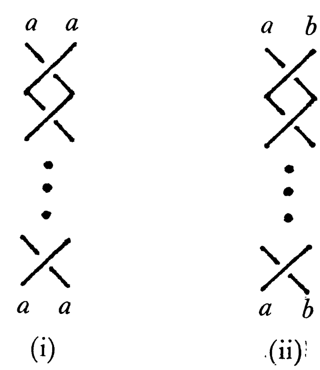



Fig. 7

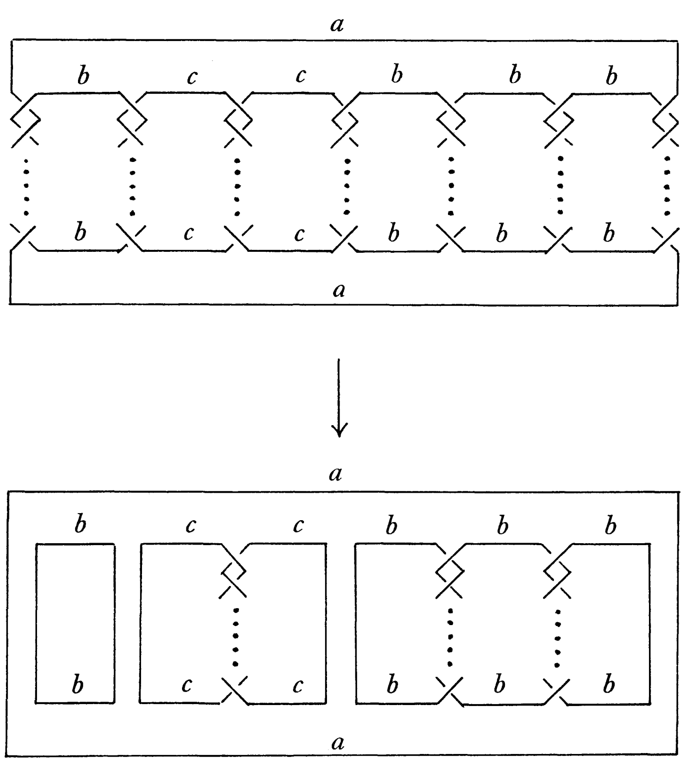



Fig. 8

Furthermore, for the case (i) the number of half-twists is arbitrary, but, for the case (ii) the number of half-twists is a multiple of three. For the case (ii), we can deform the 2-strand braid by operations cancelling three half-twists to the 2-strand braid with no half-twists and this deformation satisfies the condition of Proposition 4. But, for the case (i), we cannot do, since the condition of Proposition 4 is not satisfied. By operations cancelling three half-twists only for the case (ii), this pretzel knot can be deformed to a split link such that each component is a trivial knot, a $(p, 2)$-torus knot, or a connected sum of those knots and that every meridians of the same component are mapped by $h$ to the same element $a$, $b$, or $c$ in $S_3$.

Regarding $S^2\times S^1$ as a lens space of type $(0, 1)$, the 2-fold branched covering space branched along a $(p, 2)$-torus knot is isomorphic to a lens space of type $(p, 1)$. Since the above link is a split sum of trivial knots, $(p, 2)$-torus knots, and their connected sum, the 3-fold irregular branched covering space of the above link corresponding to $h$ is isomorphic to a 3-sphere, a lens space of type $(p, 1)$, or a connected sum of those spaces, and this covering space is isomorphic to the 3-fold irregular branched covering space branched along the original pretzel knot.

The proof is complete.

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## References


[1] J.W. Alexander: *Note on Riemann spaces,* Bull. Amer. Math. Soc. **26** (1920), 370–372.

[2] G. Burde: *On branched coverings of* $S^3$, Canad. J. Math. **23** (1971), 84–89.

[3] H.M. Hilden: *Every closed, orientable* 3*-manifold is a* 3*-fold branched covering space of* $S^3$, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[4] H.M. Hilden: *Three-fold branched coverings of* $S^3$, Amer. J. Math. **98** (1976), 989–997.

[5] J.M. Montesinos: *Reducciòn de la conjetura de Poincaré a otras conjecturas geométricas,* Rev. Mat. Hisp.-Amer. **32** (1972), 33–51.

[6] J.M. Montesinos: *Una nota a un Teorema de Alexander,* ibid. 167–187.

[7] J.M. Montesinos: *A representation of closed, orientable* 3*-manifolds as* 3*-fold branched coverings of* $S^3$, Bull. Amer. Math. Soc. **80** (1974), 845–846.

[8] J.M. Montesinos: *Three-manifolds as* 3*-fold branched covers of* $S^3$, Knots, Groups and 3-Manifolds, Ann. of Math. Studies ♯84, Princeton Univ. Pr., 1975, 227–259.


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