



Title	On 3-fold irregular branched covering spaces of pretzel knots
Author(s)	Hosokawa, Fujitsugu; Nakanishi, Yasutaka
Citation	Osaka Journal of Mathematics. 1986, 23(1), p. 249-254
Version Type	VoR
URL	https://doi.org/10.18910/9996
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON 3-FOLD IRREGULAR BRANCHED COVERING SPACES OF PRETZEL KNOTS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

FUJITSUGU HOSOKAWA AND YASUTAKA NAKANISHI

(Received December 21, 1984)

It is well-known that any orientable closed 3-manifold is a 3-fold irregular branched covering space of a 3-sphere branched along a knot. It is an interesting problem to know which 3-manifold can be a 3-fold irregular branched covering space of a given knot. In this paper we consider those of pretzel knots.

For the permutation group S_3 on $\{0, 1, 2\}$, let $a=(01)$, $b=(02)$, $c=(12)$, $x=(012)$, $y=(021)$. Then there are relations $a^2=b^2=c^2=1$, $ab=bc=ca=x$, $ba=ac=cb=y$. Especially, we remark the following relations:

$$\begin{aligned} aba^{-1} &= c, &aca^{-1} &= b, &axa^{-1} &= y, &aya^{-1} &= x, \\ bab^{-1} &= c, &bcb^{-1} &= a, &bx b^{-1} &= y, &byb^{-1} &= x, \\ cac^{-1} &= b, &cbc^{-1} &= a, &cxc^{-1} &= y, &cyc^{-1} &= x, \\ xax^{-1} &= b, &xbx^{-1} &= c, &xcx^{-1} &= a, &xyx^{-1} &= y, \\ yay^{-1} &= c, &yby^{-1} &= a, &ycy^{-1} &= b, &yxy^{-1} &= x. \end{aligned}$$

A knot group G has a Wirtinger presentation:

$$(x_1, x_2, \dots, x_n; r_1, r_2, \dots, r_{n-1}) \quad (1)$$

where each relator r_i indicates the relation form $r_i = x_{j(i)}^{\varepsilon} x_i x_{j(i)}^{-\varepsilon} x_{i+1}^{-1}$ ($\varepsilon = \pm 1$) at a crossing as in Fig. 1.

Then a homomorphism from a knot group G to S_3 satisfies a condition as follows.

Proposition 1. *Let the above (1) be a Wirtinger presentation of a knot group G . Then a homomorphism h from G to S_3 satisfies one of the following.*

- (i) $h(x_i) = a$ (or b , c) $(i=1, 2, \dots, n)$,
- (ii) $h(x_i) = x$ (or y) $(i=1, 2, \dots, n)$.

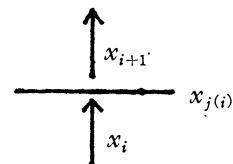


Fig. 1

Proposition 2. *Let $(x_{11}, \dots, x_{1n_1}, \dots, x_{m1}, \dots, x_{mn_m}; r_1, \dots, r_k)$ be a Wirtinger*

presentation of a link group of an m -component link, where $x_{i1}, x_{i2}, \dots, x_{in_i}$ represent meridians of the i -th component ($i=1, 2, \dots, m$). Then a homomorphism h from this link group to S_3 satisfies one of the followings.

- (i) $h(x_{ij})=a$ (or b, c) ($j=1, 2, \dots, n_i$),
- (ii) $h(x_{ij})=x$ (or y) ($j=1, 2, \dots, n_i$).

If all generators are mapped to a (or b, c) by h , then the branched covering space corresponding to h is the disjoint union of a 3-sphere and the 2-fold regular branched covering space of the knot. And if all generators are mapped to x (or y) by h , then that is the 3-fold regular branched covering space of the knot. So, the branched covering space corresponding to h is 3-fold irregular, iff h satisfies the condition (i) of Proposition 1 and there exist generators x_i and x_j with $h(x_i) \neq h(x_j)$.

First, we consider the image of meridians by h at twists, especially for typical cases as in Fig. 2.

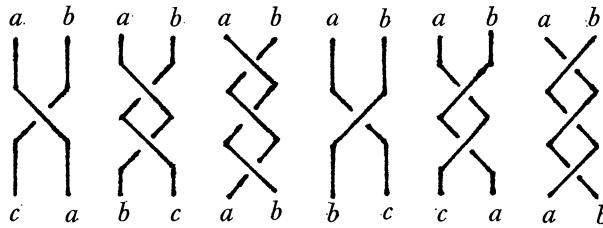


Fig. 2

Since $a^2=b^2=c^2=1$, we can ignore the orientation of a knot or a link.

If there is a block of three half-twists at the projection of a link L , deform L by the operation cancelling three half-twists as shown in Fig. 3; we have a new link L' . Then, determining the image of meridians of L' except at the three half-twists to be the same to that by h , we have a homomorphism h' from the link group of L' to S_3 . We call h' a homomorphism induced from h .



Fig. 3

Since it is easily seen that the inverse of the above operation is also an operation cancelling three half-twists after a slight deformation of the projection of L' as Fig. 4, we have

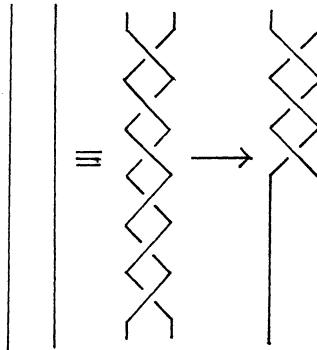


Fig. 4

Proposition 3. *Let L' be a link obtained from a link L by operation cancelling three half-twists, and G and G' be the link group of L and L' . Then the followings are equivalent.*

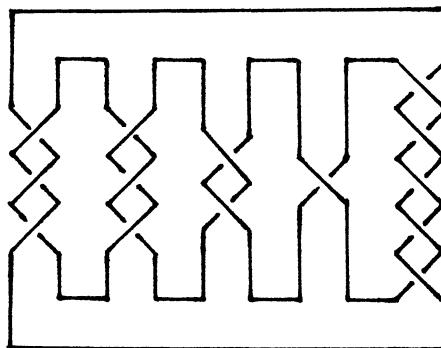
- (a) *There exists a homomorphism h from G to S_3 satisfying the condition (i) of Proposition 2.*
- (b) *There exists a homomorphism h' from G' to S_3 satisfying the condition (i) of Proposition 2.*

We regard a part of the three half-twists as a *trivial tangle* i.e. a pair of a 3-ball and two proper arcs which are trivial and separated in the 3-ball. Since the irregular 3-fold branched covering space of a trivial tangle is a 3-ball (Burde [2]), we have

Proposition 4 (Montesinos [5]). *Let L' be a link obtained from L by operation cancelling three half-twists, and G and G' be the link group of L and L' . Suppose that a homomorphism h from G to S_3 exists and h' is a homomorphism induced by h . Furthermore, at the three half-twists deformed by operation, we suppose that the images of medians of the two arcs by h are distinct and transpositions. Then the 3-fold irregular branched covering space of a 3-sphere branched along L corresponding to h is homeomorphic to the irregular 3-fold branched covering space of a 3-sphere branched along L' corresponding to h' .*

From Proposition 4, we can deside all 3-fold irregular branched covering spaces of a 3-sphere branched along a pretzel knot. A *pretzel knot* is a knot consisting of a row of 2-strand braids of q_1, q_2, \dots, q_m half-twists, which we denote by $k(q_1, q_2, \dots, q_m)$. We assume $q_i \neq 0$ for $i=1, 2, \dots, m$. Fig. 5 shows $k(3, 3, -2, -1, -5)$.

Theorem. *Each 3-fold irregular branched covering space of a 3-sphere branched along a pretzel knot, if it exists, is isomorphic to a 3-sphere, a lens space of type $(p, 1)$ for some non-negative integer p , or a connected sum of those spaces.*



$k(3, 3, -2, -1, -5)$

Fig. 5

Proof. We note that a pretzel knot does not always have an irregular 3-fold cover.

First, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by h are distinct. In this case, we see each $q_i = \pm 1$ (mod 3) from Fig. 2. So, by operations cancelling three half-twists, this pretzel knot can be deformed to $k(1, 1, \dots, 1)$ or $k(-1, -1, \dots, -1)$. Moreover, on each operation cancelling three half-twists, the condition of Proposition 4 is satisfied. Here the number of “1”’s or “-1”’s in the above $k(1, 1, \dots, 1)$ or $k(-1, -1, \dots, -1)$ is a multiple of three, and we can obtain a trivial link of 2-components from this link by operations cancelling three half-twists such that the image of meridians of components by h are distinct. Since the 3-fold irregular branched covering space of a 3-sphere branched along a trivial link

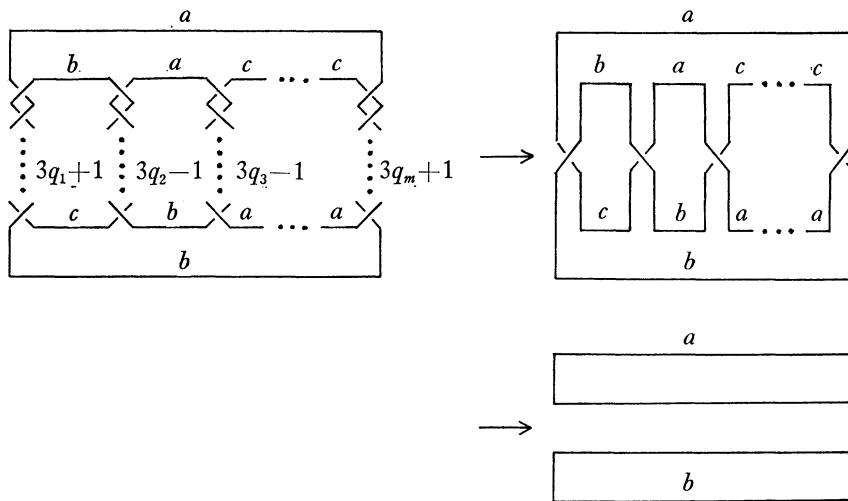


Fig. 6

of 2-components is a 3-sphere, those of the original pretzel knot is also a 3-sphere.

Secondly, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by h are same. In this case, the image of meridians of 2-strand braids by h must be (i) or (ii) shown as in Fig. 7.

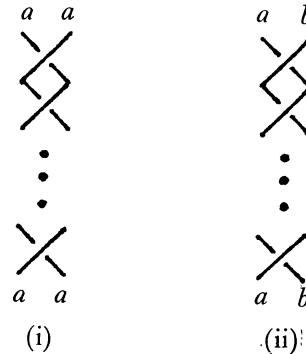


Fig. 7

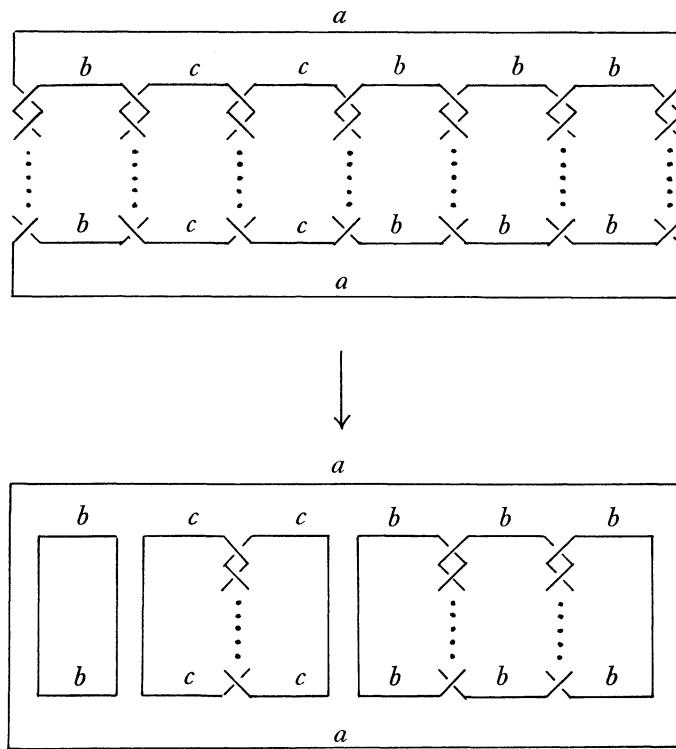


Fig. 8

Furthermore, for the case (i) the number of half-twists is arbitrary, but, for the case (ii) the number of half-twists is a multiple of three. For the case (ii), we can deform the 2-strand braid by operations cancelling three half-twists to the 2-strand braid with no half-twists and this deformation satisfies the condition of Proposition 4. But, for the case (i), we cannot do, since the condition of Proposition 4 is not satisfied. By operations cancelling three half-twists only for the case (ii), this pretzel knot can be deformed to a split link such that each component is a trivial knot, a $(p, 2)$ -torus knot, or a connected sum of those knots and that every meridians of the same component are mapped by h to the same element a , b , or c in S_3 .

Regarding $S^2 \times S^1$ as a lens space of type $(0, 1)$, the 2-fold branched covering space branched along a $(p, 2)$ -torus knot is isomorphic to a lens space of type $(p, 1)$. Since the above link is a split sum of trivial knots, $(p, 2)$ -torus knots, and their connected sum, the 3-fold irregular branched covering space of the above link corresponding to h is isomorphic to a 3-sphere, a lens space of type $(p, 1)$, or a connected sum of those spaces, and this covering space is isomorphic to the 3-fold irregular branched covering space branched along the original pretzel knot.

The proof is complete.

References

- [1] J.W. Alexander: *Note on Riemann spaces*, Bull. Amer. Math. Soc. **26** (1920), 370–372.
- [2] G. Burde: *On branched coverings of S^3* , Canad. J. Math. **23** (1971), 84–89.
- [3] H.M. Hilden: *Every closed, orientable 3-manifold is a 3-fold branched covering space of S^3* , Bull. Amer. Math. Soc. **80** (1974), 1243–1244.
- [4] H.M. Hilden: *Three-fold branched coverings of S^3* , Amer. J. Math. **98** (1976), 989–997.
- [5] J.M. Montesinos: *Reducción de la conjectura de Poincaré a otras conjecturas geométricas*, Rev. Mat. Hisp.-Amer. **32** (1972), 33–51.
- [6] J.M. Montesinos: *Una nota a un Teorema de Alexander*, ibid. 167–187.
- [7] J.M. Montesinos: *A representation of closed, orientable 3-manifolds as 3-fold branched coverings of S^3* , Bull. Amer. Math. Soc. **80** (1974), 845–846.
- [8] J.M. Montesinos: *Three-manifolds as 3-fold branched covers of S^3* , Knots, Groups and 3-Manifolds, Ann. of Math. Studies **#84**, Princeton Univ. Pr., 1975, 227–259.

Department of Mathematics
Kobe University
Nada-ku, Kobe 657, Japan