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ON 3-FOLD IRREGULAR BRANCHED COVERING SPACES OF PRETZEL KNOTS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

FUJITSUGU HOSOKAWA AND YASUTAKA NAKANISHI

(Received December 21, 1984)

It is well-known that any orientable closed 3-manifold is a 3-fold irregular branched covering space of a 3-sphere branched along a knot. It is an interesting problem to know which 3-manifold can be a 3-fold irregular branched covering space of a given knot. In this paper we consider those of pretzel knots.

For the permutation group $S_3$ on $\{0, 1, 2\}$, let $a = (01)$, $b = (02)$, $c = (12)$, $x = (012)$, $y = (021)$. Then there are relations $a^2 = b^2 = c^2 = 1$, $ab = ba = ca = cx = y$. Especially, we remark the following relations:

$$aba^{-1} = c, \quad aca^{-1} = b, \quad axa^{-1} = y, \quad ayax^{-1} = x,$$
$$bab^{-1} = c, \quad bcb^{-1} = a, \quad bxb^{-1} = y, \quad byb^{-1} = x,$$
$$cac^{-1} = b, \quad cbc^{-1} = a, \quad cxc^{-1} = y, \quad cyc^{-1} = x,$$
$$xax^{-1} = b, \quad xbx^{-1} = c, \quad xx^{-1} = a, \quad xxy^{-1} = y,$$
$$yay^{-1} = c, \quad yby^{-1} = a, \quad ycy^{-1} = b, \quad yxy^{-1} = x.$$

A knot group $G$ has a Wirtinger presentation:

$$(x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_{n-1})$$

where each relator $r_i$ indicates the relation form $r_i = x_j^\sigma x_i x_j^\tau x_{i+1}^\tau (\sigma = \pm 1)$ at a crossing as in Fig. 1.

Then a homomorphism from a knot group $G$ to $S_3$ satisfies a condition as follows.

**Proposition 1.** Let the above (1) be a Wirtinger presentation of a knot group $G$. Then a homomorphism $h$ from $G$ to $S_3$ satisfies one of the followings.

(i) $h(x_i) = a$ (or $b, c$) \hspace{1cm} (i = 1, 2, \ldots, n),$
(ii) $h(x_i) = x$ (or $y$) \hspace{1cm} (i = 1, 2, \ldots, n).

**Proposition 2.** Let $(x_{i1}, \ldots, x_{in}; r_1, \ldots, r_k)$ be a Wirtinger
presentation of a link group of an m-component link, where \(x_{i1}, x_{i2}, \ldots, x_{im}\) represent meridians of the \(i\)-th component \((i=1, 2, \ldots, m)\). Then a homomorphism \(h\) from this link group to \(S_3\) satisfies one of the followings.

(i) \(h(x_{ij}) = a\) (or \(b, c\)) \((j=1, 2, \ldots, n_i)\),

(ii) \(h(x_{ij}) = x\) (or \(y\)) \((j=1, 2, \ldots, n_i)\).

If all generators are mapped to \(a\) (or \(b, c\)) by \(h\), then the branched covering space corresponding to \(h\) is the disjoint union of a 3-sphere and the 2-fold regular branched covering space of the knot. And if all generators are mapped to \(x\) (or \(y\)) by \(h\), then that is the 3-fold regular branched covering space of the knot. So, the branched covering space corresponding to \(h\) is 3-fold irregular, iff \(h\) satisfies the condition (i) of Proposition 1 and there exist generators \(x_i\) and \(x_j\) with \(h(x_i) \neq h(x_j)\).

First, we consider the image of meridians by \(h\) at twists, especially for typical cases as in Fig. 2.

\[\begin{array}{cccccccccc}
& a & & b & & a & & b & & a \\n& c & & a & & b & & c & & a \\
& b & & c & & a & & b & & c \\n& a & & b & & a & & b & & a \\
\end{array}\]

Fig. 2

Since \(a^2 = b^2 = c^2 = 1\), we can ignore the orientation of a knot or a link.

If there is a block of three half-twists at the projection of a link \(L\), deform \(L\) by the operation cancelling three half-twists as shown in Fig. 3; we have a new link \(L'\). Then, determining the image of meridians of \(L'\) except at the three half-twists to be the same to that by \(h\), we have a homomorphism \(h'\) from the link group of \(L'\) to \(S_3\). We call \(h'\) a homomorphism induced from \(h\).

\[\begin{array}{cccccccccc}
\begin{array}{c}
\triangle \quad \rightarrow \quad \bigg| \bigg| \quad \rightarrow \quad \bigg| \bigg|
\end{array}
\end{array}\]

Fig. 3

Since it is easily seen that the inverse of the above operation is also an operation cancelling three half-twists after a slight deformation of the projection of \(L'\) as Fig. 4, we have
Proposition 3. Let \( L' \) be a link obtained from a link \( L \) by operation canceling three half-twists, and \( G \) and \( G' \) be the link group of \( L \) and \( L' \). Then the followings are equivalent,

(a) There exists a homomorphism \( h \) from \( G \) to \( S_3 \) satisfying the condition (i) of Proposition 2.

(b) There exists a homomorphism \( h' \) from \( G' \) to \( S_3 \) satisfying the condition (i) of Proposition 2.

We regard a part of the three half-twists as a trivial tangle i.e. a pair of a 3-ball and two proper arcs which are trivial and separated in the 3-ball. Since the irregular 3-fold branched covering space of a trivial tangle is a 3-ball (Burde [2]), we have

Proposition 4 (Montesinos [5]). Let \( L' \) be a link obtained from \( L \) by operation canceling three half-twists, and \( G \) and \( G' \) be the link group of \( L \) and \( L' \). Suppose that a homomorphism \( h \) from \( G \) to \( S_3 \) exists and \( h' \) is a homomorphism induced by \( h \). Furthermore, at the three half-twists deformed by operation, we suppose that the images of medians of the two arcs by \( h \) are distinct and transpositions. Then the 3-fold irregular branched covering space of a 3-sphere branched along \( L \) corresponding to \( h \) is homeomorphic to the irregular 3-fold branched covering space of a 3-sphere branched along \( L' \) corresponding to \( h' \).

From Proposition 4, we can deside all 3-fold irregular branched covering spaces of a 3-sphere branched along a pretzel knot. A pretzel knot is a knot consisting of a row of 2-strand braids of \( q_1, q_2, \ldots, q_m \) half-twists, which we denote by \( k(q_1, q_2, \ldots, q_m) \). We assume \( q_i \neq 0 \) for \( i=1, 2, \ldots, m \). Fig. 5 shows \( k(3, 3, -2, -1, -5) \).

Theorem. Each 3-fold irregular branched covering space of a 3-sphere branched along a pretzel knot, if it exists, is isomorphic to a 3-sphere, a lens space of type \((p, 1)\) for some non-negative integer \( p \), or a connected sum of those spaces.
Proof. We note that a pretzel knot does not always have an irregular 3-fold cover.

First, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by $h$ are distinct. In this case, we see each $q_i = \pm 1 \pmod{3}$ from Fig. 2. So, by operations cancelling three half-twists, this pretzel knot can be deformed to $k(1, 1, \ldots, 1)$ or $k(-1, -1, \ldots, -1)$. Moreover, on each operation cancelling three half-twists, the condition of Proposition 4 is satisfied. Here the number of "1"s or "-1"s in the above $k(1, 1, \ldots, 1)$ or $k(-1, -1, \ldots, -1)$ is a multiple of three, and we can obtain a trivial link of 2-components from this link by operations cancelling three half-twists such that the image of meridians of components by $h$ are distinct. Since the 3-fold irregular branched covering space of a 3-sphere branched along a trivial link
of 2-components is a 3-sphere, those of the original pretzel knot is also a 3-sphere.

Secondly, we consider the case that the image of meridians of the top and bottom lines of a pretzel knot by \( h \) are same. In this case, the image of meridians of 2-strand braids by \( h \) must be (i) or (ii) shown as in Fig. 7.

Fig. 7

![Diagram](image1.png)

Fig. 8

![Diagram](image2.png)
Furthermore, for the case (i) the number of half-twists is arbitrary, but, for the case (ii) the number of half-twists is a multiple of three. For the case (ii), we can deform the 2-strand braid by operations cancelling three half-twists to the 2-strand braid with no half-twists and this deformation satisfies the condition of Proposition 4. But, for the case (i), we cannot do, since the condition of Proposition 4 is not satisfied. By operations cancelling three half-twists only for the case (ii), this pretzel knot can be deformed to a split link such that each component is a trivial knot, a \((p, 2)\)-torus knot, or a connected sum of those knots and that every meridians of the same component are mapped by \(h\) to the same element \(a, b,\) or \(c\) in \(S_3\).

Regarding \(S^2 \times S^1\) as a lens space of type \((0, 1)\), the 2-fold branched covering space branched along a \((p, 2)\)-torus knot is isomorphic to a lens space of type \((p, 1)\). Since the above link is a split sum of trivial knots, \((p, 2)\)-torus knots, and their connected sum, the 3-fold irregular branched covering space of the above link corresponding to \(h\) is isomorphic to a 3-sphere, a lens space of type \((p, 1)\), or a connected sum of those spaces, and this covering space is isomorphic to the 3-fold irregular branched covering space branched along the original pretzel knot.

The proof is complete.

References


